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# ON THE STRUCTURE OF THE MINIMUM TIME FUNCTION*

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**Abstract.** A minimum time problem with a nonlinear smooth dynamics and a target satisfying an internal sphere condition is considered. Under the assumption that the minimum time $T$ be continuous and the normal cone to the hypograph of $T$, $N_{hypo(T)}$, be pointed, we show that $hypo(T)$ is $\varphi$-convex, i.e., satisfies a strong external sphere condition. Consequently, $T$ is a.e. twice differentiable and satisfies some further regularity properties. Our results are based on a representation of Clarke generalized gradient of $T$. An example is provided, showing that if $N_{hypo(T)}$ is not pointed, then the result may fail.

**Key words.** normal vectors, $\varphi$-convex (prox-regular, positive reach) sets, internal sphere condition, small time controllability, adjoint flow

**AMS subject classifications.** 49K15, 49N60, 49J52

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## 1. Introduction.

This paper is concerned with a rather general class of minimum time problems with a nonlinear dynamics and with a target which is not a singleton. More precisely, the dynamics

$$\begin{cases} \dot{y}(t) \ = \ f(y(t), u(t)) & \text{a.e.,} \\ u(t) \ \in \mathcal{U} & \text{a.e.,} \\ y(0) \ = \ x \end{cases}$$

is considered, where the function $f : \mathbb{R}^N \times \mathcal{U} \longrightarrow \mathbb{R}^N$ is (for simplicity) $\mathcal{C}^2$ and the control set $\mathcal{U}$ is a compact nonempty subset of $\mathbb{R}^m$. The target $\mathcal{S}$ is assumed to be a closed subset of the state space $\mathbb{R}^N$, and the minimum time to reach $\mathcal{S}$ subject to the above dynamics is denoted by $T$.

Minimum time problems are widely studied from several viewpoints (see, e.g., [1, Chapter IV], [2], and [7, Chapter 8] and references therein). In particular, it is well known that, in general, the minimum time function $T$ is not everywhere differentiable. It is also well known that suitable controllability conditions imply the Hölder continuity of $T$ (see, e.g., [1, Chapter IV] and references therein), which, however, provides no information on differentiability and on the structure of the nondifferentiability set.

In a 1995 paper (see [6] and also Chapter 8 in [7]), Cannarsa and Sinestrari found a connection between the dynamics and the target which actually implies the semiconcavity (or the semiconvexity) of $T$. Semiconcave functions are essentially $\mathcal{C}^2$-perturbations of concave functions and therefore inherit several regularity properties from convexity. Several features of semiconcavity were thoroughly studied (see Chapters 3, 4, and 5 in [7] and references therein), thus providing a rich set of information on the structure of the minimum time function and suggesting semiconcavity/semiconvexity as a good regularity class for such value functions. The main result in [6] shows that if the target satisfies a *uniform internal ball condition* (see Definition

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2.2 below) and the dynamics is smooth enough, then $T$ is semiconcave, provided a strong *controllability assumption*, called Petrov's condition, holds. A partially symmetric result, also contained in [6], states that if the target is convex and the dynamics is linear, then $T$ is semiconvex, provided, again, Petrov's condition holds. The latter requires that the minimized Hamiltonian at all boundary points of $\mathcal{S}$, computed along unit normal vectors, be bounded away from zero locally uniformly, i.e., for all $R > 0$, there exists $\mu > 0$, such that

$$(1.1)\quad \min_{u\in\mathcal{U}} \langle f(x,u), \zeta\rangle < -\mu \quad \text{for all } x \in \text{bdry}\mathcal{S} \cap B(0,R) \text{ for all } \zeta \in N_{\mathcal{S}}(x),\ \|\zeta\| = 1.$$

It is well known that Petrov's condition is equivalent to the local Lipschitz continuity of $T$ (see, e.g., [7, section 8.2]).

Other semiconcavity results for $T$ appear in [8, 5, 25]. In these papers the regularity assumption on the target is substituted by suitable properties on the dynamics. In particular, in [8] no controllability assumption is made, yet a kind of semiconcavity (implying local Lipschitz continuity) is established on an open dense subset of the complement of the target.

In an entirely different setting, a class which includes both convex and $\mathcal{C}^2$-sets was studied independently by several authors (including Federer [15], Canino [4], Clarke, Stern and Wolenski [9], and Poliquin, Rockafellar, and Thibault [22]) under different names, for example, *sets with positive reach* [15], *$\varphi$-convex sets* [4], *proximally smooth sets* [9], and *prox-regular sets* [22]. This class, which we prefer to name with $\varphi$-convexity, is characterized by a *strong external sphere condition* (see Definition 2.1 below): every normal vector must be realized by a locally uniform ball. By observing that a convex set satisfies the same type of external sphere condition with an arbitrarily large radius, it is natural to expect that $\varphi$-convex sets enjoy locally several properties that are enjoyed globally by convex sets. In particular, this is the case for the metric projection, which is unique in a neighborhood of a $\varphi$-convex set $K$. This fact is used in proving all the regularity properties which are satisfied by $\varphi$-convex sets (see, e.g., [15, section 4]).

Semiconcave functions and $\varphi$-convex sets are linked together through the hypograph (see, e.g., Theorem 5.2 in [9], where semiconvex functions are called *lower–$\mathcal{C}^2$*): a locally Lipchitz function is semiconcave if and only if there exists $\varphi_0 > 0$ such that its hypograph is $\varphi_0$-convex. Of course, an entirely symmetric characterization for semiconvex functions can be expressed using the epigraph. Trying to generalize to functions with a $\varphi$-convex hypograph/epigraph some regularity properties enjoyed by semiconcave/convex functions was therefore a natural challenge. Some results on this line were obtained in [11, 12], including the a.e. twice differentiability (see Theorem 2.2 below) together with an analysis of the structure of singularities (i.e., nondifferentiability points).

In several minimum time problems, controllability assumptions weaker than Petrov's condition hold, and therefore $T$ is not locally Lipschitz. A natural question therefore is trying to understand whether the structure of the minimum time function remains unchanged if, in the above setting, the controllability assumptions are weakened. In other words it is natural to investigate whether the hypograph/epigraph of $T$ is $\varphi$-convex if $T$ is supposed to be only continuous. For a linear dynamics and a convex target, the answer is positive, as proved in [13]. This paper is devoted to the nonlinear analogue: we assume that the dynamics is (essentially) $\mathcal{C}^2$, the target $\mathcal{S}$ satisfies an internal sphere condition, and $T$ is continuous, and we study the hypograph of $T$ in the complement of $\mathcal{S}$. Here the situation is more complicated than in the Lipschitz case: the results depend on the pointedness of the normal cone to the hypograph. More precisely, we show (Theorem 3.3) that if the normal cone to the hypograph is pointed in the complement of $\mathcal{S}$, then the hypograph of $T$ is $\varphi$-convex for a suitable $\varphi$ (Theorem 3.3) so that the minimum time function satisfies the regularity properties contained in Theorem 2.2 below and in [12]. Our result is based on an analysis of how proximal normals (to the complement of the target) are transported by the adjoint flow, which in turn permits a representation of the generalized gradient of $T$ in terms of suitable adjoint vectors (Theorems 3.1 and 3.2). Here the pointedness assumption plays a major role: actually *exposed rays* of the normal cone to the hypograph are special normals as they can be approximated by normals at differentiability points of $T$ (Lemma 4.7). Moreover, pointedness is used in Theorem 3.3 in order to obtain a uniform estimate for radii of the balls realizing proximal normals to the hypograph. We show also through an example (Example 2 in section 7) that if the normal cone is not pointed, then Theorem 3.3 may fail. However, a weaker external sphere condition to the hypograph of $T$ still holds (see Proposition 3.1). An analysis of this general case is contained in the papers [18, 19], where topological and measure theoretic results on the set where the normal cone is not pointed are given. In particular, in [18, Theorem 3.1] this set is shown to be closed with Lebesgue $n$-dimensional measure zero.

The paper is organized as follows: section 2 is devoted to definitions and basic facts, while section 3 contains assumptions and statement of the main results. Detailed arguments begin in section 4, which contains several lemmas whose geometrical meaning is illustrated, and ends with a result (Theorem 4.1) giving a representation of the normal cone to the hypograph of $(T)$ under the pointedness assumption. Section 5 is devoted to the conclusion of the proof of the main theorems, which is now only a simple use of the lemmas contained in section 4. Section 6 is dedicated to an improvement of Theorems 3.1 and 3.2 for an *optimal point*, i.e., a point which is crossed by a time-optimal trajectory, while section 7 contains examples and section 8 some general basic estimates.

## 2. Preliminaries.

**2.1. Nonsmooth analysis.** Let $C \subset \mathbb{R}^N$ be a cone (i.e., if $x \in C$ and $\lambda \geq 0$, then $\lambda x \in C$). We say that $C$ is *pointed* if $C \cap (-C) = \{0\}$. In [23, Corollary 18.7.1, p. 169] it is proved that

(2.1) if $C$ is closed, convex, and pointed, then it is the closed convex hull of its exposed rays.

We recall (see [23, p. 163]) that an exposed ray $\mathbb{R}^+ v$ of a convex cone $C$ is defined by the property that there exists a linear functional $h$ which is zero on it and is such that if $h(p) = 0$ and $p \in C$, then $p \in \mathbb{R}^+ v$.

Let $K \subset \mathbb{R}^N$ be closed. Its boundary will be denoted by bdry$K$ and its interior by int$K$. Let now $x \in K$ and $v \in \mathbb{R}^N$. We say that $v$ is a *proximal normal* to $K$ at $x$ (and we will denote this fact by $v \in N_K^P(x)$) if there exists $\sigma = \sigma(v, x) \geq 0$ such that

$$\langle v, y - x\rangle \leq \sigma \|y - x\|^2 \quad \text{for all } y \in K; \tag{2.2}$$

equivalently $v \in N_K^P(x)$ if and only if there exists $\lambda > 0$ such that $\pi_K(x + \lambda v) = \{x\}$. We say that the proximal normal $v$ *is realized by a ball of radius* $\rho > 0$ if $\rho$ is the supremum of all $\lambda$ such that $\pi_K(x + \lambda v) = \{x\}$. In this case the best constant $\sigma$ such that (2.2) holds true is $\|v\| / (2\rho)$. The following further concepts of normal vectors will be used (see [10, Chapter 1] and [24, Chapter 6]). Let $x \in K$ and $v \in \mathbb{R}^N$. We say that

(1) $v$ is a *Fréchet normal* (or Bouligand normal) to $K$ at $x$ ($v \in N_K^F(x)$) if

$$\limsup_{K \ni y \to x} \left\langle v, \frac{y-x}{\|y-x\|} \right\rangle \le 0;$$

(2) $v$ is a *limiting, or Mordukhovich, normal* to $K$ at $x$ ($v \in N_K^L(x)$) if

$$v \in \{w |\, w = \lim w_n,\, w_n \in N_K^P(x_n),\, x_n \to x\}$$

and is a *Clarke normal* ($v \in N_K^C(x)$) if $v \in \overline{\text{co}} N_K^L(x)$. It is well known that $N_K^P(x)$ is convex.

Let $f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. By using $\text{epi}(f) := \{(x,\xi) |\, \xi \ge f(x)\}$ and $\text{hypo}(f) := \{(x,\xi) |\, \xi \le f(x)\}$, one can define some concepts of generalized differential for $f$ at $x \in \text{dom}(f) = \{x \in \mathbb{R}^N |\, f(x) < +\infty\}$. Let $x \in \text{dom}(f)$, $v \in \mathbb{R}^N$. We say that

(1) $v$ is a *proximal subgradient* of $f$ at $x$ ($v \in \partial_P f(x)$) if $(v,-1) \in N^P_{\text{epi}(f)}(x, f(x))$; equivalently (see [10, Theorem 1.2.5]) $v \in \partial_P f(x)$ if and only if there exist $\sigma$, $\eta > 0$ such that

$$f(y) \ge f(x) + \langle v, y-x \rangle - \sigma \|y-x\|^2 \qquad \text{for all } y \in B(x,\eta) \cap \text{dom}\,(f); \tag{2.3}$$

(2) $v$ is a *proximal supergradient* of $f$ at $x$ ($v \in \partial^P f(x)$) if $(-v,1) \in N^P_{hypo(f)}(x, f(x))$; equivalently $v \in \partial^P f(x)$ if and only if $-v \in \partial_P(-f)(x)$, i.e., if and only if there exist $\sigma$, $\eta > 0$ such that

$$f(y) \le f(x) + \langle v, y-x \rangle + \sigma \|y-x\|^2 \qquad \text{for all } y \in B(x,\eta) \cap \text{dom}\,(f); \tag{2.4}$$

(3) $v$ is a *Fréchet subgradient* of $f$ at $x$ ($v \in \partial_F f(x)$) if $(v,-1) \in N^F_{epi(f)}(x, f(x))$, i.e.,

$$\liminf_{y \to x} \frac{f(y) - f(x) - \langle v, y-x \rangle}{\|y-x\|} \ge 0;$$

(4) $v$ is a *Fréchet supergradient* of $f$ at $x$ ($v \in \partial^F f(x)$) if $(-v,1) \in N^F_{hypo(f)}(x, f(x))$;

(5) $v$ is a *limiting subgradient* of $f$ at $x$ ($v \in \partial_L f(x)$) if $(v,-1) \in N^L_{epi(f)}(x, f(x))$;

(6) $v$ is a *limiting supergradient* of $f$ at $x$ ($v \in \partial^L f(x)$) if $(-v,1) \in N^L_{hypo(f)}(x, f(x))$;

(7) $v$ is a *Clarke generalized gradient* of $f$ at $x$ ($v \in \partial f(x)$) if $(v,-1) \in N^C_{\text{epi}(f)}(x, f(x))$.

We recall that *if $f$ is Lipschitz continuous in a neighborhood of $x$*, then $v \in \partial f(x)$ if and only if $v \in \overline{\text{co}}\{\zeta |\, \zeta = \lim Df(x_i), x_i \in \text{dom}(Df), x_i \to x\}$ (see [10, Theorem 8.1]).

It follows readily from the definitions that the inclusions

$$N_K^P(x) \subseteq N_K^F(x) \subseteq N_K^L(x) \subseteq N_K^C(x)$$

hold, together with their analogues for the sub- and supergradient. Moreover, if a vector $v$ belongs to both the Fréchet sub- and supergradient of $f$ at $x$, then $f$ is Fréchet differentiable at $x$ and $Df(x) = v$.

For a not necessarily Lipschitz function $f$, the *horizon subgradient* $\partial_\infty f$ plays an important role. This is defined as

$$\partial_\infty f(x) = \{v \in \mathbb{R}^N \mid (v, 0) \in N^C_{\mathrm{epi}(f)}(x, f(x))\}$$

and is clearly a closed convex cone. In particular, if $f$ is not locally Lipschitz in a neighborhood of $x$, then $\partial f(x)$ may be represented using $\partial_\infty$, namely (see [17, Proposition 2.6] or [24, Theorem 8.49]),

$$\partial f(x) = \mathrm{cl}\,(\mathrm{co}\,\partial_L f(x) + \mathrm{co}\,\partial_\infty f(x))\,. \tag{2.5}$$

Finally we also consider a notion of *proximal* horizon supergradient, namely, the convex cone

$$\partial^\infty f(x) = \{v \in \mathbb{R}^N \mid (-v, 0) \in N^P_{hypo(T)}(x, f(x))\}.$$

We introduce now two classes of sets which will be used throughout the paper.

DEFINITION 2.1. *Let $K \subset R^N$ be closed, and let $\varphi : K \to [0, \infty]$ be continuous. We say that $K$ is $\varphi$-convex if for all $x, y \in K, v \in N^P_K(x)$, the inequality*

$$\langle v, y - x\rangle \le \varphi(x)\,\|v\|\,\|x - y\|^2$$

*holds. By $\varphi_0$-convexity we mean $\varphi$-convexity with $\varphi \equiv \varphi_0$.*

It is clear that every closed and convex set is $\varphi_0$-convex with $\varphi_0 = 0$, and every closed set with a $\mathcal{C}^{1,1}$-boundary is $L/2$-convex, where $L$ is the Lipschitz constant of a suitable parametrization of bdry$K$. Some properties of the distance from a $\varphi$-convex set $K$ and the metric projection onto $K$ are important features of this class of sets.

THEOREM 2.1. *Let $K \subset \mathbb{R}^N$ be a $\varphi$-convex set. Then there exists an open set $U \supset K$ such that*

1. *$d_K \in \mathcal{C}^{1,1}(U \setminus K)$ and $Dd_K(y) = \frac{y - \pi_K(y)}{d_K(y)}$ for every $y \in U \setminus K$;*
2. *$\pi_K : U \to K$ is a locally Lipschitz single-valued map. In particular, if $K$ is $\varphi_0$-convex (with $\varphi_0 > 0$), then $\pi_K : \{x \in \mathbb{R}^N \mid d(x, K) < 1/(4\varphi_0)\} \to K$ is Lipschitz with Lipschitz ratio 2.*

*Moreover,*

3. *$K$ has finite perimeter in $\mathbb{R}^N$ (provided it is compact);*
4. *for every $x \in K$, $N^P_K(x) = N^C_K(x)$;*
5. *the set-valued map $N^P_K(\cdot)$ has closed graph in* bdry$K \times \mathbb{R}^N$.

*Proof.* The proof of (1) and (2) can be found in [4, Propositions 2.6 and 2.9, Remark 2.10] or in [15, section 4]. The proof of (3) is in [11, section 5], while (4) and (5) can be found in several papers, including [22]. ∎

*Remark* 2.1. Conditions (1) and (2) in Theorem 2.1 are actually equivalent to $\varphi$-convexity as it is proved, e.g., in [15, section 4]. Examples of finite dimensional $\varphi$-convex sets can be found, e.g., in [15]. In both optimal control and partial differential equations theory, *semiconcave functions* play an important role (see, e.g., [1, 7]). Let $\Omega \subset \mathbb{R}^N$ be open: a function $f : \Omega \longrightarrow \mathbb{R}$ is said to be semiconcave if for every $x \in \Omega$ and every $\delta > 0$ there exists a constant $C > 0$ such that

$$f(x) - C\,\|x\|^2 \ \text{ is concave in } B(x, \delta).$$

Semiconcave functions are therefore locally Lipschitz. Moreover, thanks to Theorem 5.2 in [9], the hypograph of such functions is $\varphi_0$-convex for a suitable $\varphi_0 \ge 0$.

More in general, upper semicontinuous functions with $\varphi$-convex hypograph (or lower semicontinuous functions with $\varphi$-convex epigraph) enjoy several of the regularity properties, except Lipschitz continuity, that semiconcave functions satisfy. Such functions identify the class to which we want to show that our minimum time belongs. To this aim, we state a result which collects the main properties. We denote by $\mathcal{L}^N$ the Lebesgue $N$-dimensional measure in $\mathbb{R}^N$ and by $\mathcal{H}^k$ the $k$-dimensional Hausdorff measure, $0 \le k \le N$. For basic concepts of geometric measure theory, we refer to [14].

THEOREM 2.2. *Let $\Omega \subset \mathbb{R}^N$ be open, and let $f : \overline{\Omega} \to \mathbb{R} \cup \{+\infty\}$ be proper, upper semicontinuous, and such that hypo$(f)$ is $\varphi$-convex for a suitable continuous $\varphi$. Then there exists a sequence of sets $\Omega_h \subseteq \Omega$ such that $\Omega_h$ is compact in* $\mathrm{dom}(f)$ *and*

(1) *the union of $\Omega_h$ covers $\mathcal{L}^N$-almost all* $\mathrm{dom}(f)$*;*

(2) *for all $x \in \bigcup_h \Omega_h$ there exist $\delta = \delta(x) > 0$, $L = L(x) > 0$ such that*

$$f \text{ is Lipschitz on } B(x,\delta) \text{ with ratio } L \text{ and hence semiconcave on } B(x,\delta). \tag{2.6}$$

*Consequently,*

(3) *$f$ is a.e. Fréchet differentiable and admits a second order Taylor expansion around a.e. point of its domain.*

*Moreover, the set of points where the graph of $f$ is nonsmooth has a small Hausdorff dimension. More precisely, for every $k = 1, \ldots, N$, the set $\{x \in \mathrm{int}\,\mathrm{dom}(f) \mid$ the dimension of $\partial^P f(x)$ is $\ge k\}$ is countably $\mathcal{H}^{N-k}$-rectifiable.*

This result is essentially Theorem 5.1 in [11].

For any set $K \subset \mathbb{R}^N$, we denote by $K^c$ the complement of $K$, i.e., $\mathbb{R}^N \setminus K$.

DEFINITION 2.2. *Let $K \subset \mathbb{R}^N$ be closed, and let $\rho > 0$ be given. We say that $K$ satisfies the external sphere condition with radius $\rho$ if for all $x \in \mathrm{bdry}K$ there exists $v \in N^P_K(x)$, $v \ne 0$, which is realized by a ball of radius $\rho$. We say also that $K$ satisfies the internal sphere condition with radius $\rho$ if $\overline{K^c}$ satisfies the external sphere condition with radius $\rho$, namely, for all $x \in \mathrm{bdry}K$ there exists $v \in N^P_{\overline{K^c}}(x)$, $v \ne 0$, which is realized by a ball of radius $\rho$.*

Obviously the complement of an open convex set satisfies the internal sphere condition of radius $\rho$ for any $\rho > 0$. A comparison between the external sphere condition and $\varphi$-convexity was performed in [20, 21].

**2.2. Control theory: Generalities.** We consider throughout the paper a nonlinear control system of the form

$$\begin{cases} \dot{y}(t) \;=\; f(y(t), u(t)) & \text{a.e.,} \\ u(t) \;\in\; \mathcal{U} & \text{a.e.,} \\ y(0) \;=\; x, \end{cases} \tag{2.7}$$

where the Lipschitz function $f : \mathbb{R}^N \times \mathcal{U} \longrightarrow \mathbb{R}^N$ and the control set $\mathcal{U}$, a compact nonempty subset of $\mathbb{R}^m$, are given. We denote by $\mathcal{U}_{\mathrm{ad}}$ the set of admissible controls, i.e., the measurable functions $u : \mathbb{R} \to \mathbb{R}^m$ such that $u(t) \in \mathcal{U}$ a.e. For any $u(\cdot) \in \mathcal{U}_{\mathrm{ad}}$, the unique Carathéodory solution of (2.7) is denoted by $y^{x,u}(\cdot)$.

The adjoint vectors associated with a trajectory $y^{x,u}(\cdot)$ can be represented using the fundamental solution matrix $M(\cdot, x, u)$ of the linear equation

$$\dot{p}(t) = D_x f(y^{x,u}(t), u(t))\; p(t), \quad p(0) = \mathbb{I}^{N\times N}. \tag{2.8}$$

We also define $M^{-1}(\cdot, x, u)$ to be the fundamental solution matrix of the time reversed adjoint equation

$$\dot{q}(t) = -q(t)\; D_x f(y^{x,u}(t), u(t)), \quad q(0) = \mathbb{I}^{N\times N}. \tag{2.9}$$

Suppose we are now given a closed nonempty set $\mathcal{S} \subset \mathbb{R}^N$, which is called the target. For a fixed $x \in \mathbb{R}^N \setminus \mathcal{S}$, we define

$$\theta(x, u) := \min \{t \geq 0 \mid y^{x,u}(t) \in \mathcal{S}\}.$$

Of course, $\theta(x, u) \in (0, +\infty]$, and $\theta(x, u)$ is the time taken for the trajectory $y^{x,u}(\cdot)$ to reach $\mathcal{S}$, provided $\theta(x, u) < +\infty$. The *minimum time* $T(x)$ to reach $\mathcal{S}$ from $x$ is defined by

$$T(x) := \inf \{\theta(x, u) \mid u(\cdot) \in \mathcal{U}_{\text{ad}}\}. \tag{2.10}$$

In general, an *optimal trajectory*, i.e., a trajectory which attains the infimum in (2.10), does not exist. Therefore, we need also to consider *minimizing sequences* and *limiting optimal trajectories* steering $x$ to the target $\mathcal{S}$. In particular, we will consider the limits of the endpoints (thus belonging to $\mathcal{S}$) of minimizing sequences of trajectories. More precisely,

$$\begin{aligned}\mathcal{S}_x \;=\; \{\bar{x} \in \mathcal{S} \mid & \text{ there exist sequences } \{x_n\} \subset \mathcal{S}^c \text{ and } \{\bar{u}_n(\cdot)\} \subset \mathcal{U}_{\text{ad}} \text{ such that} \\ & x_n \to x,\ \theta(x_n, \bar{u}_n) \to T(x),\ y^{x_n,\bar{u}_n}(\theta(x_n, \bar{u}_n)) \to \bar{x}\}.\end{aligned}$$

Observe that if $T(x) < +\infty$, then $\emptyset \neq \mathcal{S}_x \subseteq \text{bdry}\mathcal{S}$.

For any $\bar{x} \in \mathcal{S}_x$, we define also

$$\begin{aligned}\mathcal{U}_{\bar{x}} = \{\{\bar{u}_n(\cdot)\} \subset \mathcal{U}_{ad} \mid & \text{ there exists a sequence } \{x_n\} \text{ satisfying} \\ & x_n \to x,\, \theta(x_n, \bar{u}_n) \to T(x) \text{ and } y^{x_n,\bar{u}_n}(\theta(x_n, \bar{u}_n)) \to \bar{x}\},\end{aligned}$$

i.e., the set of *minimizing sequences of controls* steering $x$ to $\bar{x}$. Together with $\mathcal{U}_{\bar{x}}$, we define also

$$\begin{aligned}\mathcal{T}_{\bar{x}} = \{\{y^{x_n,\bar{u}_n}(\cdot)\} \mid & \ x_n \to x,\ \bar{u}_n \in \mathcal{U}_{\text{ad}}, \\ & \theta(x_n, \bar{u}_n) \to T(x), \text{ and } y^{x_n,\bar{u}_n}(\theta(x_n, \bar{u}_n)) \to \bar{x}\},\end{aligned}$$

i.e., the set of trajectories corresponding to minimizing sequences of controls steering $x$ to $\bar{x}$.

Correspondingly, the *limiting adjoint trajectories* related to *minimizing sequences of controls* are defined by the following:

$$\begin{aligned}\mathcal{M}_{\bar{x}} = \ \{M : [0, T(x)] \to \mathbb{M}^{N\times N} \mid & \text{ there exists } \{y^{x_n,\bar{u}_n}(\cdot)\} \subset \mathcal{T}_{\bar{x}} \text{ such that} \\ & M(\cdot) \text{ is the uniform limit on } [0, T(x)] \text{ of } M(\cdot, x_n, \bar{u}_n)\}.\end{aligned} \tag{2.11}$$

*Remark* 2.2. If $T(\cdot)$ is everywhere finite, both $S_x$ and $\mathcal{T}_{\bar{x}}$ are nonempty. By compactness, $\mathcal{M}_{\bar{x}}$ is nonempty as well for all $\bar{x} \in \mathcal{S}_x$. Moreover, if $F(x) := \{f(x, u) | u \in \mathcal{U}\}$ is convex for all $x$, then the infimum is attained and the sets $\mathcal{S}_x$, $\mathcal{U}_{\bar{x}}$, and $\mathcal{T}_{\bar{x}}$ can be substituted by the simpler sets

$$\begin{aligned}\mathcal{S}_x &= \{\bar{x} \in \mathcal{S} \mid \text{there exists } \bar{u} \in \mathcal{U}_{ad} \text{ such that} \\ &\qquad \theta(x, \bar{u}) = T(x), \bar{x} = y^{x,\bar{u}}(T(x))\}, \\ \mathcal{U}_{\bar{x}} &= \{\bar{u} \in \mathcal{U}_{ad} \mid \theta(x, \bar{u}) = T(x), y^{x,\bar{u}}(T(x)) = \bar{x}\}, \\ \mathcal{T}_{\bar{x}} &= \{y^{x,\bar{u}} \mid \bar{u} \in \mathcal{U}_{\bar{x}}\}.\end{aligned}$$

Finally, the *maximized Hamiltonian*, namely, the function

$$H : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}, \quad H(x, p) = \max_{u\in\mathcal{U}} \langle f(x, u), p\rangle\,,$$

will be important in our analysis.

**3. Statement of the main results.** We repeat first the setting we are concerned with and specify our assumptions.

We consider the nonlinear system (2.7) under the following assumptions:

(H1) $\mathcal{U} \subset \mathbb{R}^N$ is compact.

(H2) $f : \mathbb{R}^N \times \mathcal{U} \to \mathbb{R}^N$ is continuous and satisfies

$$\|f(x,u) - f(y,u)\| \leq L\,\|x-y\| \quad \text{for all } x, y \in \mathbb{R}^N,\ u \in \mathcal{U}$$

for a positive constant $L$. Moreover, the differential of $f$ with respect to the $x$ variable $D_x f$ exists everywhere, is continuous with respect to both $x$ and $u$, and satisfies the following Lipschitz condition:

$$\|D_x f(x,u) - D_x f(y,u)\| \leq L_1\,\|x-y\| \quad \text{for all } x, y \in \mathbb{R}^N,\ u \in \mathcal{U}$$

for a positive constant $L_1$.

(H3) The minimum time function $T : \mathbb{R}^N \longrightarrow [0, +\infty)$ is everywhere finite and continuous, (i.e., controllability and small time controllability hold).

(H4) The target $\mathcal{S}$ is nonempty, closed, and satisfies the internal sphere condition of radius $\rho > 0$.

*Remark* 3.1. Conditions ensuring small time controllability when the target is not necessarily a singleton can be found, e.g., in [1, Chapter IV], [7, Chapter 8], and [16].

Our analysis will be based on the transportation of certain vectors, normal to the closure of the complement of the target $\mathcal{S}$, by means of the (limiting) adjoint flow. More precisely, two sets of transported normals will be considered, according to the Hamiltonian:

$$N_0(x) = \{M^T(r)v \mid M(\cdot) \in \mathcal{M}_{\bar{x}}, v \in N^P_{\overline{\mathcal{S}^c}}(\bar{x}), \bar{x} \in \mathcal{S}_x,\ \text{and } H(M^T(r)v, x) = 0\},$$
$$N_1(x) = \{M^T(r)v \mid M(\cdot) \in \mathcal{M}_{\bar{x}}, v \in N^P_{\overline{\mathcal{S}^c}}(\bar{x}), \bar{x} \in \mathcal{S}_x,\ \text{and } H(M^T(r)v, x) = 1\}.$$

Our main results are the following three theorems, together with the corollary.

THEOREM 3.1. *Let $x \in \mathcal{S}^c$ and $r = T(x)$. Under the conditions* (H1), (H2), (H3)*, and* (H4)*, together with the further assumption*

$$N^P_{hypo(T)}(x, T(x)) \text{ is pointed,} \tag{3.1}$$

*the (proximal) horizontal supergradient of the minimum time function $T(\cdot)$ at the point $x$ can be computed as follows:*

$$\partial^\infty T(x) \;=\; -\text{co}(N_0(x)). \tag{3.2}$$

THEOREM 3.2. *Let $x \in \mathcal{S}^c$ and $r = T(x)$. Under the same assumptions of Theorem* 3.1*, the proximal supergradient of the minimum time function at the point $x$ can be computed as follows:*

$$\partial^P T(x) \;=\; -[\text{co}(N_1(x)) + \text{co}(N_0(x))]. \tag{3.3}$$

THEOREM 3.3. *Let the assumptions of Theorem* 3.1 *hold for all $x \in \mathcal{S}^c$. Then there exists a continuous function $\varphi : hypo(T) \cap (\mathcal{S}^c \times \mathbb{R}) \longrightarrow [0, +\infty)$ such that, for every closed set $\mathcal{S}' \subset \mathcal{S}^c$, $hypo(T) \cap (\mathcal{S}' \times \mathbb{R})$ is $\varphi$-convex.*

COROLLARY 3.1. *Let the assumptions of Theorem* 3.1 *hold. Then the minimum time function $T$ satisfies all the properties listed in Theorem* 2.2.

The last result is concerned with the case where the pointedness assumption (3.1) does not hold. We will present here, for the sake of brevity, only a partial result together with two examples, a thorough analysis being postponed to a forthcoming paper.

PROPOSITION 3.1. *Let the assumptions* (H1), (H2), (H3), *and* (H4) *hold. Then the hypograph of the minimum time function $T$ satisfies the external sphere condition with a locally uniform radius, namely, for every $x \in \mathcal{S}^c$ there exists a unit proximal normal $v$ to hypo$(T)$ at $(x, T(x))$ which is realized by a sphere with a locally constant radius $\sigma > 0$.*

*Remark* 3.2. Both $\varphi$ and $\sigma$ can be explicitly computed and depend only on $x$, on $f$ and $\mathcal{U}$, and on the constants $L$, $L_1$, and $\rho$ appearing in the assumptions (H2) and (H4).

**4. Some preparatory lemmas.** This section is devoted to several partial results which are needed to prove Theorems 3.1 and 3.2. In particular, the proof of "$\supseteq$" inclusions in (3.1) and (3.2) will be based on Lemmas 4.2 and 4.3 below.

In the first three lemmas we do not assume that $\mathcal{S}$ satisfies the internal sphere condition nor that the normal cone to the hypograph of $T(\cdot)$ at $(x, T(x))$ is pointed.

The following notation for sublevels of the minimum time function will be used: for $r > 0$, we set

$$\mathcal{S}(r) := \{x \in \mathbb{R}^N \mid T(x) < r\},$$
$$\mathcal{S}^c(r) := \{x \in \mathbb{R}^N \mid T(x) \geq r\}.$$

We state first a technical lemma, showing that the limiting adjoint flow transports proximal normals to the complement of the target to proximal normals to the complement of sublevels of $T$. Moreover, the radius of the ball which realizes the transported normal can be explicitly estimated.

LEMMA 4.1. *Assume that $\mathcal{S}$ is closed, and let the assumptions* (H1), (H2), *and* (H3) *hold. Let $x \in \mathcal{S}^c$, and set $r = T(x) > 0$. Fix $\bar{x} \in \mathcal{S}_x$, $v \in N^P_{\overline{\mathcal{S}^c}}(\bar{x})$, and $M(\cdot) \in \mathcal{M}_{\bar{x}}$. Then*

$$M^T(r)v \in N^P_{\mathcal{S}^c(r)}(x).$$

*More precisely, assume that $v$ is realized by a ball of radius $\rho > 0$. Then there exists an explicitly computable continuous function $K$ depending only on $r$, $\|x\|$, $\rho$ such that for all $z \in \mathcal{S}^c(r)$, we have*

$$\left\langle M^T(r)v, z - x\right\rangle \leq \; K(r, \|x\|, \rho) \; \left\|M^T(r)v\right\| \|z - x\|^2 . \tag{4.1}$$

*Proof.* Let $x_n \to x$, $\bar{x} \in \mathcal{S}_x$, and $\{\bar{u}_n\} \subset \mathcal{U}_{\text{ad}}$ be such that $\{y^{x_n,\bar{u}_n}(\cdot)\} \in \mathcal{T}_{\bar{x}}$ and $M(\cdot, x_n, \bar{u}_n)$ converges to $M(\cdot)$ uniformly on $[0, T(x)]$. By definition of the proximal normal realized by a $\rho$-ball,

$$\langle v \;,\; \bar{z} - \bar{x}\rangle \;\leq\; \frac{\|v\|}{2\rho}\; \|\bar{z} - \bar{x}\|^2 \quad \text{for all } \bar{z} \in \overline{\mathcal{S}^c}.$$

Fix $z \in \mathcal{S}^c(r)$. We define

$$\bar{x}_n \;=\; y^{x_n,\bar{u}_n}(\theta(x_n, \bar{u}_n)),\; \bar{z}_n \;=\; y^{z,\bar{u}_n}(\theta(x_n, \bar{u}_n)),$$

and observe that $\bar{x}_n \in \mathcal{S}$, $\bar{x}_n \to \bar{x}$, and we can assume without loss of generality that $\bar{z}_n$ converges to a point $\bar{z}$ which belongs to $\overline{\mathcal{S}^c}$ since $\theta(x_n, \bar{u}_n) \to r \leq T(z)$.

We set, for simplicity, $\alpha_n(\cdot) = y^{x_n,\bar{u}_n}(\cdot)$, $\beta_n(\cdot) = y^{z,\bar{u}_n}(\cdot)$, $t_n = \theta(x_n, \bar{u}_n)$ so that

$$\bar{x}_n = x_n + \int_0^{t_n} f(\alpha_n(s), \bar{u}_n(s))ds \ , \ \bar{z}_n \ = \ z + \int_0^{t_n} f(\beta_n(s), \bar{u}_n(s))ds;$$

whence

$$\bar{z}_n - \bar{x}_n = z - x_n + \int_0^{t_n}\left(\int_0^1 D_x f(\alpha_n(s) + \tau(\beta_n(s) - \alpha_n(s)), \bar{u}_n(s))d\tau\right)(\beta_n(s) - \alpha_n(s))ds.$$

We define now

$$A_n^1(s) = D_x f(\alpha_n(s), \bar{u}_n(s)),$$
$$A_n^2(s) = \int_0^1 D_x f(\alpha_n(s) + \tau(\beta_n(s) - \alpha_n(s)), \bar{u}_n(s))d\tau$$

and observe that, thanks to (H2), for all $s \in [0, t_n]$, we have

$$\left\|A_n^2(s) \ - \ A_n^1(s)\right\| \le \frac{L_1}{2} \ \|\beta_n(s) \ - \ \alpha_n(s)\| \, . \tag{4.2}$$

Using (iv) in Lemma 8.1 and the definition of $L_2$ in (8.1), we obtain

$$\left\|A_n^1(s)\right\| \le L_2(s, \|x_n\|) \tag{4.3}$$

for all $s \in [0, t_n]$. Thus,

$$\left\|A_n^2(s)\right\| \le L_2(s, \|x_n\|) + \frac{L_1}{2} \ \|\beta_n(s) \ - \ \alpha_n(s)\| \tag{4.4}$$

for all $s \in [0, t_n]$. Now Gronwall's lemma yields

$$\|\beta_n(s) - \alpha_n(s)\| \le e^{Ls} \|z - x_n\| \tag{4.5}$$

so that by combining (4.4) and (4.5), we obtain

$$\left\|A_n^2(s)\right\| \le L_2(s, \|x_n\|) + \frac{L_1}{2} e^{Ls} \|z - x_n\| \, . \tag{4.6}$$

Define $M_n^2(\cdot)$ to be the solution of the problem

$$\dot{p}(s) = A_n^2(t)p(s), \quad p(0) = \mathbb{I}^{N\times N}.$$

Recalling that $M(\cdot, x, u)$ is the fundamental solution of (2.8), set $M_n^1(\cdot) = M(\cdot, x_n, \bar{u}_n)$, $z_n^1(s) = M_n^1(s)(z - x_n)$, and $z_n^2(s) = M_n^2(s)(z - x_n)$ for all $s \in [0, t_n]$. Using these notations, we can write

$$\begin{aligned} \langle v, \bar{z}_n - \bar{x}_n\rangle &= \left\langle v, z_n^2(t_n)\right\rangle \\ &= \left\langle v, z_n^1(t_n)\right\rangle + \left\langle v, z_n^2(t_n) - z_n^1(t_n)\right\rangle \\ &= \left\langle v, M_n^1(t_n)(z - x_n)\right\rangle + \left\langle v, (M_n^2(t_n) - M_n^1(t_n))(z - x_n)\right\rangle \\ &\ge \left\langle v, M_n^1(t_n)(z - x_n)\right\rangle - \|v\| \left\|(M_n^2(t_n) - M_n^1(t_n))(z - x_n)\right\| . \end{aligned} \tag{4.7}$$

To simplify our writing, we set, for all $s \geq 0$ and $y, z \in \mathbb{R}^N$, $L_3(s, y, z) = \frac{L_1}{2} e^{Ls} \|z - y\|$. By (4.3), (4.6), Lemma 8.3, and (4.2), we have

$$\begin{aligned}
&\left\|(M_n^2(t_n) - M_n^1(t_n))(z - x_n)\right\| \leq \\
&\qquad \leq e^{[2L_2(t_n, \|x_n\|) + L_3(t_n, x_n, z)]t_n} \int_0^{t_n} \left\|A_n^2(s) - A_n^1(s)\right\| ds \ \ \|z - x_n\| \\
&\qquad \leq \frac{L_1}{2} e^{[2L_2(t_n, \|x_n\|) + L_3(t_n, x_n, z)]t_n} \int_0^{t_n} \|\beta_n(s) - \alpha_n(s)\| \, ds \ \ \|z - x_n\| .
\end{aligned}$$

Recalling (4.5), we obtain

$$\begin{aligned}
&\left\|(M_n^2(t_n) - M_n^1(t_n))(z - x_n)\right\| \\
&\qquad \leq \frac{L_1}{2} e^{[2L_2(t_n, \|x_n\|) + L_3(t_n, x_n, z) + L]t_n} \|z - x_n\|^2 .
\end{aligned} \tag{4.8}$$

Therefore, by passing to the limit in (4.7) and (4.8) (recall that $M_n^1(\cdot) \to M(\cdot)$ uniformly), we have

$$\begin{aligned}
\left\langle M^T(r)v, z - x\right\rangle &\leq \langle v, \bar{z} - \bar{x}\rangle + \|v\| \frac{L_1}{2} e^{[2L_2(r, \|x\|) + L_3(r, x, z) + L]r} \|z - x\|^2 \\
&\leq \frac{\|v\|}{2\rho} \ \|\bar{z} - \bar{x}\|^2 + \|v\| \frac{L_1}{2} e^{[2L_2(r, \|x\|) + L_3(r, x, z) + L]r} \ \|z - x\|^2 .
\end{aligned}$$

Moreover, from (4.5) we have $\|\bar{z} - \bar{x}\| \leq e^{Lr} \ \|z - x\|$. Therefore,

$$\left\langle M^T(r)v, z - x\right\rangle \leq \left( \frac{L_1}{2} e^{[2L_2(r, \|x\|) + L_3(r, x, z) + L]r} + \frac{e^{2Lr}}{2\rho} \right) \|v\| \, \|z - x\|^2 \tag{4.9}$$

for all $z \in \mathcal{S}^c(r)$.

Observe that

$$\begin{aligned}
\|v\| &= \left\|(M^T(r))^{-1} M^T(r)v\right\| \\
&\leq \left\|M(r)^{-1}\right\| \left\|M^T(r)v\right\| .
\end{aligned}$$

By (ii) in Lemma 8.2, we obtain

$$\left\|M(r)^{-1}\right\| \ \leq \ e^{L_2(r, \|x\|)r}.$$

Combining the above inequalities with (4.9), we thus have

$$\begin{aligned}
&\left\langle M^T(r)v, z - x\right\rangle \\
&\qquad \leq \left( \frac{L_1}{2} e^{[3L_2(r, \|x\|) + L_3(r, x, z) + L]r} + \frac{e^{2Lr + L_2(r, \|x\|)}}{2\rho} \right) \ \left\|M^T(r)v\right\| \|z - x\|^2 .
\end{aligned} \tag{4.10}$$

In order to complete the proof, we consider two cases.

If $\|z - x\| < 1$, then $L_3(r, x, z) \leq \frac{L_1}{2} e^{Lr}$. Thus, by (4.10) we have

$$\begin{aligned}
\left\langle M^T(r)v, z - x\right\rangle \leq &\left( \frac{L_1}{2} e^{[3L_2(r, \|x\|) + \frac{L_1}{2} e^{Lr} + L]r} + \frac{e^{2Lr + L_2(r, \|x\|)}}{2\rho} \right) \\
&\cdot \left\|M^T(r)v\right\| \|z - x\|^2 .
\end{aligned} \tag{4.11}$$

If instead $\|z-x\| \geq 1$, then $\langle M^T(r)v, z-x\rangle \leq \|M^T(r)v\| \, \|z-x\|^2$.

Therefore, in both cases we have that

$$\langle M^T(r)v, z-x\rangle \leq K(r, \|x\|, \rho) \; \|M^T(r)v\| \; \|z-x\|^2 \quad \text{for all } z \in \mathcal{S}^c(r), \tag{4.12}$$

where the continuous function $K$, defined for $r, \delta \geq 0$ and $\rho > 0$ as

$$K(r,\delta,\rho) := \max\left\{1, \frac{L_1}{2} e^{[3L_2(r,\delta)+\frac{L_1}{2}e^{Lr}+L]r} + \frac{e^{2Lr+L_2(r,\delta)}}{2\rho}\right\}, \tag{4.13}$$

depends only on the variables $r$, $\delta$, $\rho$ and on the constants $L$, $L_1$, $K_1$, $K_2$.

The proof is complete. □

*Remark* 4.1. It follows from (4.13) that $K(r,\delta,\rho)$ is nondecreasing with respect to both $r$ and $\delta$.

The next lemma establishes that normals transported along the *limiting adjoint flow* generate horizontal proximal normals to the hypograph of $T(\cdot)$, provided their Hamiltonian is zero. Moreover, the radius of the ball realizing them can be explicitly estimated.

LEMMA 4.2. *Let $\mathcal{S}$ be closed, and let the assumptions* (H1), (H2), *and* (H3) *hold. Let $x \in \mathcal{S}^c$, set $r := T(x) > 0$, and let $\xi \in N_0(x)$. Then $-\xi \in \partial^\infty T(x)$, or, equivalently, $(\xi, 0) \in N^P_{hypo(T(x))}(x, T(x))$.*

*More precisely, let $\bar{x} \in \mathcal{S}_x$, and let $v \in N^P_{\overline{\mathcal{S}^c}}(\bar{x})$, $M(\cdot) \in \mathcal{M}_{\bar{x}}$ be such that $H(M^T(r)v,$
$x) = 0$. Assume that $v$ is realized by a ball of radius $\rho$. Then there exists an explicitly computable continuous function $K_3(r,x,\rho)$, depending only on $r$, $x$, $\rho$, such that for all $z \in \overline{\mathcal{S}^c}$ and all $\beta \leq T(z)$, we have*

$$\langle M^T(r)v, z-x\rangle \leq \; K_3(r,x,\rho) \; \|M^T(r)v\| \left(\|z-x\|^2 + |\beta - T(x)|^2\right). \tag{4.14}$$

*Proof.* Let $v \in N^P_{\overline{\mathcal{S}^c}}(\bar{x})$ be such that

$$\langle v, \bar{z} - \bar{x}\rangle \leq \frac{\|v\|}{2\rho} \|\bar{z} - \bar{x}\|^2 \quad \text{for all } \bar{z} \in \overline{\mathcal{S}^c}. \tag{4.15}$$

Recalling Lemma 4.1, for all $z \in \mathcal{S}^c(r)$, we have

$$\langle M^T(r)v, z-x\rangle \leq K(r, \|x\|, \rho) \|M^T(r)v\| \, \|z-x\|^2 . \tag{4.16}$$

Let $z \in \overline{\mathcal{S}^c}$. Two cases may occur:

(i) $T(z) \geq T(x)$,

(ii) $T(z) < T(x)$.

In the first case, (4.14) follows immediately from (4.16).

In the second case, define $r_1 = T(z)$, and take sequences $\{x_n\}$ with $x_n \to x$, $\{\bar{u}_n\} \subset \mathcal{U}_{\rm ad}$, and $\{\alpha_n(\cdot) := y^{x_n,\bar{u}_n}(\cdot)\}$ corresponding to $M(\cdot)$, according to the definition given in (2.11). For all $n$ large enough, there exists $r_n^1 < r$ for which

$$\bar{x}_n^1 := \alpha_n(r - r_n^1) = x_n + \int_0^{r-r_n^1} f(\alpha_n(s), \bar{u}_n(s))ds$$

is such that $T(\bar{x}_n^1) = r_1$. We can assume without loss of generality that $\alpha_n(\cdot)$ converges uniformly to some $\alpha(\cdot)$ and that $r_n^1 \to \bar{r}_1$. Observe that $\bar{r}_1 < r$.

Setting $\bar{x}^1 = \alpha(r - \bar{r}_1)(= \lim \bar{x}_n^1)$, one can easily see that $T(\bar{x}^1) = r_1$ by the continuity of $T(x)$. Then by Lemma 4.1, we obtain that

$$\left\langle M^T(r_1)v, z - \bar{x}^1\right\rangle \leq K(r_1, \left\|\bar{x}^1\right\|, \rho) \left\|M^T(r_1)v\right\| \left\|z - \bar{x}^1\right\|^2. \tag{4.17}$$

We write

$$\left\langle M^T(r)v, z - x\right\rangle = \left\langle M^T(r)v, z - \bar{x}^1\right\rangle + \left\langle M^T(r)v, \bar{x}^1 - x\right\rangle$$

and perform some estimates.

First we consider

$$\left\langle M^T(r)v, z - \bar{x}^1\right\rangle = \left\langle M^T(r_1)v, z - \bar{x}^1\right\rangle + \left\langle (M^T(r) - M^T(r_1))v, z - \bar{x}^1\right\rangle.$$

By (4.17) we have

$$\begin{aligned} \left\langle M^T(r)v, z - \bar{x}^1\right\rangle \leq\ & K(r_1, \left\|\bar{x}^1\right\|, \rho) \left\|M^T(r_1)v\right\| \left\|z - \bar{x}^1\right\|^2 \\ & + \left\|(M^T(r) - M^T(r_1))v\right\| \left\|z - \bar{x}^1\right\|. \end{aligned}$$

Moreover, from (ii) in Lemma 8.2, we have

$$\begin{aligned} \left\|M^T(r_1)v\right\| &\leq \left\|(M^T(r - r_1))^{-1}\right\| \left\|M^T(r)v\right\| \\ &\leq e^{L_2(r - r_1, \|x\|)(r - r_1)} \left\|M^T(r)v\right\| \\ &\leq e^{L_2(r, \|x\|)r} \left\|M^T(r)v\right\|. \end{aligned}$$

Also, by using (iv) in Lemma 8.1, we obtain

$$\begin{aligned} \left\|(M^T(r) - M^T(r_1))v\right\| &\leq \int_{r_1}^{r} \left\|\dot{M}^T(s)v\right\| ds \\ &\leq \int_{r_1}^{r} e^{L_2(r, \|x\|)r} \left\|M^T(r)v\right\| ds \\ &= e^{L_2(r, \|x\|)r} \left\|M^T(r)v\right\| |r - r_1|. \end{aligned}$$

Therefore,

$$\begin{aligned} \left\langle M^T(r)v, z - \bar{x}^1\right\rangle \leq\ & K(r_1, \left\|\bar{x}^1\right\|, \rho)\, e^{L_2(r, \|x\|)r} \left\|M^T(r)v\right\| \left\|z - \bar{x}^1\right\|^2 \\ & + e^{L_2(r, \|x\|)r} \left\|(M^T(r)v\right\| |r - r_1| \left\|z - \bar{x}^1\right\|. \end{aligned}$$

Recalling (i) in Lemma 8.1 for $\alpha(\cdot) = y^{x_n, \bar{u}_n(\cdot)}$ and $t = r - r_1$ and then taking $n \to \infty$, we obtain

$$\left\|\bar{x}^1 - x\right\| \leq \frac{(L\left\|x\right\| + K_1)(e^{L(r - r_1)} - 1)}{L} \leq \frac{(L\left\|x\right\| + K_1)(e^{Lr} - 1)}{L}, \tag{4.18}$$

from which it follows that $\left\|\bar{x}^1\right\| \leq e^{Lr}\left\|x\right\| + \frac{(e^{Lt} - 1)K_1}{L}$. Hence,

$$\begin{aligned} \left\langle M^T(r)v, z - \bar{x}^1\right\rangle\ \leq\ & R_1(r, \left\|x\right\|, \rho)\ e^{L_2(r, \|x\|r} \left\|M^T(r)v\right\| \left\|z - \bar{x}^1\right\|^2 \\ & +\ e^{L_2(r, \|x\|)r} \left\|M^T(r)v\right\|\ |r - r_1|\ \left\|z - \bar{x}^1\right\|, \end{aligned} \tag{4.19}$$

where

$$R_1(r,\delta,\rho) = K\left(r, e^{Lr}\delta + \frac{(e^{Lt}-1)K_1}{L}, \rho\right) \qquad \text{for } r,\delta \geq 0, \rho > 0.$$

Observe also that we obtain from (iii) in Lemma 8.1 that

$$\begin{aligned}\left\|z - \bar{x}^1\right\| &\leq \lim_{n\to\infty}\left(\|z - x_n\| + \int_0^{r-r_n^1} \|f(\alpha_n(s), \bar{u}_n(s))\|\, ds\right)\\ &\leq \lim_{n\to\infty}\left(\|z - x_n\| + \int_0^{r-r_n^1} \left(Le^{Ls}\|x_n\| + e^{Ls}K_1\right) ds\right)\\ &\leq \|z - x\| + L_4(r, \|x\|)\, |r - r_1|,\end{aligned}$$

where $L_4(s,\delta) = Le^{Ls}\delta + e^{Ls}K_1$ for $s,\delta \geq 0$.

Combining the above inequality and (4.19), we obtain

$$\left\langle M^T(r)v, z - \bar{x}^1\right\rangle \leq R_2(r, \|x\|, \rho)\left\|M^T(r)v\right\|(\|z - x\|^2 + |r - r_1|^2), \tag{4.20}$$

where we have defined, for $r,\delta \geq 0$, $\rho > 0$,

$$R_2(r,\delta,\rho) = e^{L_2(r,\delta)r}\left(2R_1(r,\delta,\rho)\left(\frac{3}{2} + L_4^2(r,\delta)\right) + L_4(r,\delta)\right). \tag{4.21}$$

Second we consider

$$\begin{aligned}\left\langle M^T(r)v, \bar{x}_n^1 - x\right\rangle &= \left\langle M^T(r)v, x_n - x\right\rangle + \left\langle M^T(r)v, \int_0^{r-r_n^1} f(\alpha_n(s), \bar{u}_n(s))ds\right\rangle\\ &= \left\langle M^T(r)v, x_n - x\right\rangle + \left\langle M^T(r)v, \int_0^{r-r_n^1} f(x, \bar{u}_n(s))ds\right\rangle\\ &\quad + \left\langle M^T(r)v, \int_0^{r-r_n^1} (f(\alpha_n(s), \bar{u}_n(s)) - f(x, \bar{u}_n(s)))\, ds\right\rangle.\end{aligned}$$

Recalling that $H(M^T(r)v, x) = 0$, we obtain from the above expression that

$$\begin{aligned}\left\langle M^T(r)v, \bar{x}_n^1 - x\right\rangle &\leq \left\langle M^T(r)v, x_n - x\right\rangle\\ &\quad + \left\langle M^T(r)v, \int_0^{r-r_n^1} (f(\alpha_n(s), \bar{u}_n(s)) - f(x, \bar{u}_n(s)))\, ds\right\rangle\\ &\leq \left\|M^T(r)v\right\|\Bigg(\|x_n - x\|\\ &\quad + \int_0^{r-r_n^1} \|f(\alpha_n(s), \bar{u}_n(s)) - f(x, \bar{u}_n(s))\|\, ds\Bigg)\\ &\leq \left\|M^T(r)v\right\|\left(\|x_n - x\| + L\int_0^{r-r_n^1} \|\alpha_n(s) - x\|\, ds\right)\\ &\leq \left\|M^T(r)v\right\|\Bigg(\|x_n - x\| + L\|x_n - x\|\\ &\quad + L\int_0^{r-r_n^1}\int_0^s \|f(\alpha_n(\tau), \bar{u}_n(\tau))\|\, d\tau ds\Bigg).\end{aligned}$$

By (iii) in Lemma 8.1 and recalling that $\bar r_1 < r$, we now obtain that

$$\left\langle M^T(r)v, \bar x^1_n - x\right\rangle \leq \left\|M^T(r)v\right\| \left((L+1)\left\|x_n - x\right\| + L\int_0^{r-r_1}\int_0^s \left(Le^{Lr}\left\|x_n\right\| + e^{Lr}K_1\right)d\tau ds\right);$$

whence, taking $n\to\infty$,

$$\left\langle M^T(r)v, \bar x^1 - x\right\rangle \leq \frac{L(Le^{Lr}\left\|x\right\| + e^{Lr}K_1)}{2}\left\|M^T(r)v\right\| |r-r_1|^2. \tag{4.22}$$

Set now, for $r,\delta\geq 0$, $\rho>0$,

$$K_3(r,\delta,\rho) = R_2(r,\delta,\rho) + \frac{L(Le^{Lr}\delta + e^{Lr}K_1)}{2}. \tag{4.23}$$

Recalling (4.20) and (4.22), the proof is complete. ∎

Now we prove a similar result for normals such that the Hamiltonian along the *limiting adjoint flow* is 1. Actually, if $\xi$ is such a vector, we show that $(\xi, 1)$ is a proximal normal to the hypograph of $T(\cdot)$, and again the radius of the sphere which realizes it can be explicitly estimated.

Lemma 4.3. *Let $\mathcal{S}$ be closed, and let the assumptions* (H1), (H2), *and* (H3) *hold. Let $x\in\mathcal{S}^c$, set $r := T(x) > 0$, and let $\xi\in N_1(x)$. Then $-\xi\in\partial^P T(x)$, or, equivalently, $(\xi,1)\in N^P_{hypo(T(x))}(x, T(x))$.*

*More precisely, let $\bar x\in\mathcal{S}_x$, and let $v\in N^P_{\overline{\mathcal{S}^c}}(\bar x)$, $M(\cdot)\in\mathcal{M}_{\bar x}$ be such that $H(M^T(r)v, x) = 1$, and assume that $v$ is realized by a ball of radius $\rho>0$. Then there exists an explicitly computable continuous function $K_6(r,\|x\|,\rho)$ depending only on $r$, $\|x\|$, $\rho$ such that for all $z\in\overline{\mathcal{S}^c}$ and all $\beta\leq T(z)$, we have*

$$\left\langle M^T(r)v, z-x\right\rangle + \beta - r \leq K_6(r,\|x\|,\rho)\left\|(M^T(r)v,1)\right\|(\|z-x\|^2 + |\beta-r|^2). \tag{4.24}$$

*Proof.* Let $v\in N^P_{\overline{\mathcal{S}^c}}(\bar x)$ be such that

$$\langle v, \bar z - \bar x\rangle \leq \frac{\|v\|}{2\rho}\left\|\bar z - \bar x\right\|^2 \quad \text{for all } \bar z\in\overline{\mathcal{S}^c}.$$

Let $z\in\mathcal{S}^c$. Two cases may occur:

(i) $T(z)\geq T(x)$,

(ii) $T(z) < T(x)$.

*First case.* Recalling that $H(M^T(r)v, x) = 1$, one can find $\bar u\in\mathcal{U}$ such that

$$\left\langle M^T(r)v, f(x,\bar u)\right\rangle = 1.$$

Set $z_{\bar u}(\cdot) := y^{z,\bar u}(\cdot)$ to be the trajectory starting from $z$ with the constant control $\bar u$, namely, $z_{\bar u}(t) = z + \int_0^t f(z_{\bar u}(s),\bar u)ds$.

Taking $T(x)\leq r_1\leq T(z)$, we have that $z_{\bar u}(r_1 - r)\in\mathcal{S}^c(r)$. Recalling Lemma 4.1, we obtain that

$$\left\langle M^T(r)v, z_{\bar u}(r_1-r) - x\right\rangle \leq K(r,\|x\|,\rho)\ \left\|M^T(r)v\right\|\left\|z_{\bar u}(r_1-r) - x\right\|^2. \tag{4.25}$$

We estimate

$$\begin{aligned}\left\langle M^T(r)v, z - z_{\bar u}(r_1 - r)\right\rangle &= \left\langle M^T(r)v, -\int_0^{r_1-r} f(z_{\bar u}(t), \bar u)dt\right\rangle \\ &= \left\langle M^T(r)v, -\int_0^{r_1-r} f(x, \bar u)dt\right\rangle \\ &\quad + \left\langle M^T(r)v, \int_0^{r_1-r} \left(f(x,\bar u) - f(z_{\bar u}(t), \bar u)\right) dt\right\rangle \\ &\le r - r_1 + L\left\|M^T(r)v\right\| \int_0^{r_1-r} \|z_{\bar u}(t) - x\| \; dt.\end{aligned}$$

Combining the above inequality with (4.25), we get

$$\begin{aligned}\left\langle M^T(r)v, z - x\right\rangle \le r - r_1 + L\left\|M^T(r)v\right\| \int_0^{r_1-r} \|z_{\bar u}(t) - x\| \, dt \\ + K(r, \|x\|, \rho)\left\|M^T(r)v\right\| \|z_{\bar u}(r_1 - r) - x\|^2 .\end{aligned} \tag{4.26}$$

Moreover,

$$\begin{aligned}\|z_{\bar u}(s) - x\| &\le \|z - x\| + \int_0^s \|f(z_{\bar u}(\tau), \bar u)\| \, dt \\ &\le \|z - x\| + \tilde K(\|x\|)s + L\int_0^s \|z_{\bar u}(\tau) - x\| \, d\tau,\end{aligned}$$

where we set for $\delta \ge 0$, $\tilde K(\delta) := L\delta + K_1$. Thus, Gronwall's inequality yields, for all $0 \le s \le r_1 - r$,

$$\|z_{\bar u}(s) - x\| \le e^{Ls}\|z - x\| + \tilde K(\|x\|)\left(s + \frac{e^{Ls} - Ls - 1}{L}\right). \tag{4.27}$$

Since $e^{Ls} - Ls - 1 \le L(e^L - 1)s$ for all $s \in [0, 1]$, we obtain from (4.27)

$$\|z_{\bar u}(s) - x\| \le e^L\|z - x\| + \tilde K(\|x\|)e^L s \quad \text{for all } s \in [0, 1]. \tag{4.28}$$

Now we consider two subcases.

*First subcase*: $0 \le r_1 - r \le 1$. Combining (4.28) with (4.26), we obtain

$$\left\langle M^T(r)v, z - x\right\rangle + r_1 - r \le K_5(r, \|x\|, \rho)\left\|M^T(r)v\right\| (\|z - x\|^2 + |r_1 - r|^2), \tag{4.29}$$

where for $r, \delta \ge 0$, $\rho > 0$, we set

$$K_5(r, \delta, \rho) = e^L\left(\frac{L}{2} + 2e^L K(r, \delta, \rho)\big(1 + \tilde K(\delta)^2\big) + \frac{\tilde K(\delta)}{2}\right). \tag{4.30}$$

*Second subcase*: $r_1 - r > 1$. Recalling Lemma 4.1, we obtain

$$\begin{aligned}&\left\langle M^T(r)v, z - x\right\rangle + r_1 - r \\ &\qquad \le (K(r, \|x\|, \rho) + 1)\left\|(M^T(r)v, 1)\right\| (\|z - x\|^2 + |r_1 - r|^2).\end{aligned} \tag{4.31}$$

Observe now that if $\beta \le T(x)$, recalling Lemma 4.1, we have

$$\left\langle M^T(r)v, z - x\right\rangle + \beta - T(x) \le K(r, \|x\|, \rho) \; \left\|M^T(r)v\right\| (\|z - x\|^2 + |\beta - T(x)|^2). \tag{4.32}$$

We are now ready to conclude the first case. Indeed, it suffices to combine (4.29), (4.32), and (4.31) and recall (4.30), obtaining, for all $z \in \mathcal{S}^c(r)$ and $\beta \le T(z)$,

$$\begin{aligned} &\left\langle M^T(r)v, z-x \right\rangle + \beta - T(x) \\ &\quad \le (K_5(r, \|x\|, \rho)+1) \left\|(M^T(r)v, 1)\right\| (\|z-x\|^2 + |\beta - T(x)|^2). \end{aligned} \tag{4.33}$$

*Second case.* It is entirely similar to the proof of the second case of Lemma (4.2). Indeed, by using the condition $H(M^T(r)v, x) = 1$, we can replace (4.22) with

$$\left\langle M^T(r)v, \bar{x}^1 - x \right\rangle \le T(x) - T(z) + \frac{L(Le^{Lr}\|x\| + e^{Lr}K_1)}{2}|r - r_1|^2. \tag{4.34}$$

Then by combining (4.20) and (4.34), we obtain

$$\left\langle M^T(r)v, z-x \right\rangle + \beta - T(x) \le K_3(r, \|x\|, \rho) \left\|M^T(r)v\right\| (\|z-x\|^2 + |\beta - T(x)|^2) \tag{4.35}$$

for all $\beta \le T(z)$, $z \in \overline{\mathcal{S}^c}$, and $T(z) \le T(x)$.

To conclude the proof of the lemma, we recall (4.35), (4.33), and (4.30) and we set, for $r, \delta \ge 0$, $\rho > 0$,

$$K_6(r, \delta, \rho) = \max\{K_5(r, \delta, \rho) + 1, K_3(r, \delta, \rho)\}. \qquad \square \tag{4.36}$$

The next result is a crucial step in order to show that singularities of $T$ may be only of the "*upwards type*." Assuming that the target satisfies the internal sphere condition of radius $\rho$, we show that if $\xi$ belongs to the proximal subgradient of $T(\cdot)$ at $x$, then it belongs also to the proximal supergradient. Moreover, $-\xi$ is the transported vector by the limiting adjoint flow of a normal to $\mathcal{S}^c$, which is realized by $\rho$, and the radius of the sphere realizing $(-\xi, 1)$ as a proximal normal to the hypograph of $T(\cdot)$ can be explicitly estimated. In this lemma, the internal sphere condition (H4) is used for the first time.

In order to simplify our writing, we will replace the functions $K$, $K_3$, and $K_6$ appearing in Lemmas 4.1, 4.2, and 4.3, respectively, by the explicit (continuous) function

$$k(r, \|x\|, \rho) = \max\{K_6(r, \|x\|, \rho), K(r, \|x\|, \rho)\}. \tag{4.37}$$

LEMMA 4.4. *Let the assumptions* (H1)–(H4) *hold, let $x \in \mathcal{S}^c$, and let $\xi \in \partial_P T(x)$. Then*

(i) $\xi \in \partial^P T(x)$, *and therefore $T$ is differentiable at $x$;*

(ii) $-\xi \in N_1(x)$.

*Moreover, for all $z \in \overline{\mathcal{S}^c}$ and for all $\beta \le T(z)$,*

$$\langle -\xi, z-x \rangle + \beta - T(x) \le k(T(x), \|x\|, \rho) \|(-\xi, 1)\| (\|z-x\|^2 + |\beta - T(x)|^2). \tag{4.38}$$

*Proof.* Set $r = T(x)$, and let $\xi \in \partial_P T(x)$. By Proposition IV.2.3 in [1], $H(x, -\xi) \ge 1$ so that $\xi \ne 0$. It follows from the definition of proximal subgradient that there exists $\sigma \ge 0$ such that

$$\langle \xi, z-x \rangle \le \sigma \|z-x\|^2 \quad \text{for all } z \in \mathcal{S}(r). \tag{4.39}$$

Let $\bar{x} \in \mathcal{S}_x$ and $M(\cdot) \in \mathcal{M}_{\bar{x}}$, and take a sequence $\{y^{x_n, \bar{u}_n}(\cdot)\} \subset \mathcal{T}_{\bar{x}}$ such that $M(\cdot)$ is the uniform limit of $M(\cdot, x_n, \bar{u}_n)$. We claim that $(M^T(r))^{-1}\xi \in N^P_{\mathcal{S}}(\bar{x})$.

Indeed, take $\bar{z} \in \mathcal{S}$, and set $\bar{z}_n^-(\cdot) = y^-(\cdot, \bar{z}, \bar{u}_n)$, where $y^-(\cdot, \bar{z}, \bar{u}_n)$ is the solution of

$$\begin{cases} \dot{y}(t) = -f(y(t), \bar{u}_n(r-t)) \text{ a.e.,} \\ y(0) = \bar{z}. \end{cases}$$

We set $z_n = z_n^-(\theta(x_n, \bar{u}_n))$ and consider $\bar{z}_n = y^{z_n, \bar{u}_n}(\theta(x_n, \bar{u}_n))$. We can assume without loss of generality that $\{z_n\}$ converges to some $z$, which is easily seen belonging to $\mathcal{S}(r)$.

To simplify our writing, we set $t_n = \theta(x_n, \bar{u}_n)$, $\alpha_n(\cdot) = y^{x_n, \bar{u}_n}(\cdot)$, $\bar{x}_n = \alpha_n(t_n)$, and $M_n^1(\cdot) = M(\cdot, x_n, \bar{u}_n)$. Let also $\beta_n(\cdot) = y^{z_n, \bar{u}_n}(\cdot)$ and $A_n(t) = \int_0^1 D_x f(\alpha_n(t) + \tau(\beta_n(t) - \alpha_n(t)), \bar{u}_n(t))\, d\tau$, and let $M_n^2(\cdot)$ be the fundamental solution of $\dot{p}(t) = A_n(t)p(t)$, $p(0) = \mathbb{I}^{N \times N}$. Finally, we set $w_n^i(t) = M_n^i(z_n - x_n)$ for $i \in \{1, 2\}$.

Using Lemma 8.2 and the same argument leading to (4.8), we can perform the following estimate:

$$\begin{aligned} \left\langle M^T(r)^{-1}\xi, \bar{z}_n - \bar{x}_n \right\rangle &= \left\langle M^T(r)^{-1}\xi, w_n^2(t_n) \right\rangle \\ &= \left\langle M^T(r)^{-1}\xi, w_n^1(t_n) \right\rangle + \left\langle M^T(r)^{-1}\xi, w_n^2(t_n) - w_n^1(t_n) \right\rangle \\ &\le \left\langle M^T(r)^{-1}\xi, w_n^1(t_n) \right\rangle + \left\| M^T(r)^{-1} \right\| \; \|\xi\| \left\| w_n^2(t_n) - w_n^1(t_n) \right\| \\ &\le \left\langle M^T(r)^{-1}\xi, w_n^1(t_n) \right\rangle + \tilde{K}_0 \left\| z_n - x_n \right\|^2 \\ &\le \left\langle M^T(r)^{-1}\xi, w_n^1(t_n) \right\rangle + \tilde{K}_1 \left\| \bar{z} - \bar{x}_n \right\|^2 , \end{aligned}$$

where $\tilde{K}_0$ and $\tilde{K}_1$ are suitable constants. Taking $n \to \infty$ in the above inequalities, we obtain

$$\begin{aligned} \left\langle M^T(r)^{-1}\xi, \bar{z} - \bar{x} \right\rangle &\le \left\langle M^T(r)^{-1}\xi, M^T(r)(z - x) \right\rangle + \tilde{K}_1 \left\| \bar{z} - \bar{x} \right\|^2 \\ &= \langle \xi, z - x \rangle + \tilde{K}_1 \left\| \bar{z} - \bar{x} \right\|^2 . \end{aligned}$$

Recalling (4.39) and Lemma 8.2, we thus obtain

$$\begin{aligned} \left\langle M^T(r)^{-1}\xi, \bar{z} - \bar{x} \right\rangle &\le \sigma \|\xi\| \left\| z - x \right\|^2 + \tilde{K}_1 \left\| \bar{z} - \bar{x} \right\|^2 \\ &\le \tilde{K}_2 \left\| \bar{z} - \bar{x} \right\|^2 \end{aligned}$$

for a suitable constant $\tilde{K}_2$. The above inequality, in turn, implies that

$$(M^T(r))^{-1}\xi \in N_{\mathcal{S}}^P(\bar{x}). \tag{4.40}$$

Thanks to (H4), there exists $0 \neq \zeta \in N_{\overline{\mathcal{S}^c}}^P(\bar{x})$. Therefore, both $\mathcal{S}$ and $\overline{\mathcal{S}^c}$ admit at $\bar{x}$ an external nonzero proximal normal. This means that $\mathcal{S}$ is smooth at $\bar{x}$, and so, by (H4), the unique external normal to $\overline{\mathcal{S}^c}$ at $\bar{x}$, namely, $-M^T(r)^{-1}\xi$, must be realized by a ball of radius $\rho$.

Using Proposition IV.2.3 in [1], we see that $H(x, -\xi) \ge 1$, and so we can choose $\lambda \in (0, 1)$ such that $H(-\lambda\xi, x) = 1$. Applying Lemma 4.3 for $v = \lambda M^T(r)^{-1}\xi$, we obtain that $\lambda\xi \in \partial^P T(x)$. Therefore, $T$ is differentiable at $x$, and so $\lambda\xi = \xi$. Thus, both (i) and (ii) are proved.

In order to complete the proof, we apply the last statement of Lemma 4.3. ∎

The next lemma classifies limiting normals and shows that limiting subgradients generate proximal normals to the hypograph which are *horizontal/nonhorizontal* according to the *unboundedness/boundedness* of the corresponding sequence of proximal subgradients. Also, the radius of the sphere realizing the limiting vector can be explicitly estimated.

LEMMA 4.5. *Let the assumptions* (H1)–(H4) *hold, and let* $\{x_n\}$ *be a sequence converging to* $x \in \mathcal{S}^c$. *Assume that there exists a sequence* $\{\xi_n\}$ *satisfying* $\xi_n \in \partial_P T(x_n)$.

*Then the following alternatives hold true:*

(i) *If* $\limsup_{n\to\infty} \|\xi_n\| < +\infty$, *then there exists a subsequence* $\{\xi_{n_k}\}$ *converging to a vector* $\xi$ *such that* $-\xi \in N_1(x)$. *Moreover,* $(-\xi, 1) \in N^P_{hypo(T)}(x, T(x))$, *and, for all* $z \in \mathcal{S}^c$ *and all* $\beta \le T(z)$, *the inequality*

$$\langle -\xi, z - x\rangle + \beta - T(x) \ \le\ k(T(x), \|x\|, \rho)\, \|(-\xi, 1)\| \left(\|z - x\|^2 + |\beta - T(x)|^2\right) \tag{4.41}$$

*holds.*

(ii) *If* $\limsup_{n\to\infty} \|\xi_n\| = +\infty$, *then there exists a subsequence of* $\{\xi_n / \|\xi\|_n\}$ *converging to a vector* $\xi$ *such that* $-\xi \in N_0(x)$. *Moreover,* $(-\xi, 0) \in N^P_{hypo(T)}(x, T(x))$, *and for all* $z \in \mathcal{S}^c$ *and all* $\beta \le T(z)$, *the inequality*

$$\langle -\xi, z - x\rangle \ \le\ k(T(x), \|x\|, \rho)\left(\|z - x\|^2 + |\beta - T(x)|^2\right) \tag{4.42}$$

*holds.*

*Proof.* Set $r = T(x)$. Recalling Lemma (4.4), the function $T(\cdot)$ is differentiable at $x_n$. Taking $\bar{x}_n \in \mathcal{S}_{x_n}$ and $M_n(\cdot) \in \mathcal{M}_{\bar{x}_n}$, it follows from Lemma 4.4 that for all $n \in \mathbb{N}$,

(a) $-M_n^T(T(x_n))^{-1}\xi_n \in N^P_{\overline{\mathcal{S}^c}}(\bar{x}_n)$, and each $-M_n^T(T(x_n))^{-1}\xi_n$ is realized by a ball of radius $\rho$, namely,

(4.43)
$$\left\langle -M_n^T(T(x_n))^{-1}\xi_n, \bar{z} - \bar{x}_n\right\rangle \le \frac{\left\|M_n^T(T(x_n))^{-1}\xi_n\right\|}{2\rho}\, \|\bar{z} - \bar{x}_n\|^2 \quad \text{for all } \bar{z} \in \overline{\mathcal{S}^c};$$

(b) $H(-\xi_n, x_n) = 1$.

If $\limsup_{n\to\infty} \|\xi_n\| < +\infty$, we choose subsequences $\{\bar{x}_{n_k}\}$ and $\{\xi_{n_k}\}$ converging, respectively, to $\bar{x} \in \mathcal{S}$ and $\bar{\xi}$. By compactness and without loss of generality, we can assume that $\{M_{n_k}(\cdot)\}$ converges uniformly to $M(\cdot)$. We now take $n_k \to \infty$ in (4.43) and obtain

$$\left\langle -M^T(r)^{-1}\bar{\xi}, \bar{z} - \bar{x}\right\rangle \le \frac{\left\|M^T(r)^{-1}\bar{\xi}\right\|}{2\rho}\, \|\bar{z} - \bar{x}\|^2. \tag{4.44}$$

Thus, $-M^T(r)^{-1}\bar{\xi} \in N^P_{\overline{\mathcal{S}^c}}(\bar{x})$, and $-M^T(r)^{-1}\bar{\xi}$ is realized by a ball of radius $\rho$ .

On the other hand, we also take $n_k \to \infty$ in b) and obtain $H(-\bar{\xi}, x) = 1$.

One can also easily show that $M^T(\cdot) \in \mathcal{M}_{\bar{x}}$ so that $-\bar{\xi} \in N_1(x)$. Recalling Lemma 4.3 and setting $\xi := \bar{\xi}$, the proof of (i) is concluded.

Analogously, if $\limsup_{n\to\infty} \|\xi_n\| = +\infty$, we can assume that $-\bar{\xi} = -\lim_{n_k\to\infty} \frac{\xi_{n_k}}{\|\xi_{n_k}\|}$, together with $-M^T(r)^{-1}\bar{\xi} \in N^P_{\overline{\mathcal{S}^c}}(\bar{x})$ and $H(-\bar{\xi}, x) = 0$. Thus, $-\bar{\xi} \in N_0(x)$. Finally, recalling Lemma 4.2 and setting $\xi := \bar{\xi}$, we conclude the proof of (ii). ∎

The final results of this section use for the first time the pointedness assumption for the normal cone $N^P_{hypo(T)}(x, T(x))$. They show essentially that $N^P_{hypo(T)}(x, T(x))$ is a closed cone and that horizontal (respectively, nonhorizontal) exposed rays of $N^P_{hypo(T)}(x, T(x))$ belong to $N_0(x)$ (respectively, $N_1(x)$). As a by-product of our argument, we obtain a representation of $N^P_{hypo(T)}(x, T(x))$ through $N_0(x)$ and $N_1(x)$ (see Theorem 4.1).

LEMMA 4.6. *Let $s \in \mathcal{S}^c$, and let the assumptions* (H1)–(H4) *hold. Assume that $N^P_{hypo(T)}(x, T(x))$ is pointed, and set*

$$\begin{aligned} \tilde{N}_0(x) &= \{(\xi, 0) \mid \xi \in N_0(x)\}, \\ \tilde{N}_1(x) &= \{\lambda(\xi, 1) \mid \xi \in N_1(x),\ \lambda \geq 0\}, \\ N(x) &= \text{co}\tilde{N}_0(x) + \text{co}\tilde{N}_1(x). \end{aligned}$$

*Then $N(x)$ is a closed, convex, and pointed cone contained in $N^P_{hypo(T)}(x, T(x))$.*

*Proof.* Thanks to Lemmas 4.2 and 4.3 and the definition of $k$ in (4.37), every $\zeta \in \tilde{N}_0(x) \cup \tilde{N}_1(x)$ satisfies the following property: for every $y \in \mathcal{S}^c$ and every $\beta \leq T(y)$, the inequality

$$\langle \zeta, (y - x, \beta - T(x)) \rangle \leq k(T(x), \|x\|, \rho)\, \|\zeta\| \left( \|y - x\|^2 + |\beta - T(x)|^2 \right) \tag{4.45}$$

holds. It follows immediately from the above property that both $\tilde{N}_0(x)$ and $\tilde{N}_1(x)$ are cones contained in $N^P_{hypo(T)}(x, T(x))$. Thus, co$\tilde{N}_0(x)$ and co$\tilde{N}_1(x)$ are contained in $N^P_{hypo(T)}(x, T(x))$, and so they are pointed. Set $N^1_0 = \{\xi \in \mathbb{R}^N \mid \xi \in N_0(x), \|\xi\| = 1\}$, and observe that on one hand, $\tilde{N}_0(x) = \{\lambda(\xi, 0) \mid \xi \in N^1_0, \lambda \geq 0\}$, and on the other, $N^1_0$ (by the continuity of the Hamiltonian) is compact and $0 \notin N^1_0$. Analogously, observe that $N_1(x)$ is compact and does not contain zero. Therefore, using Lemma 8.4, we obtain that both co$\tilde{N}_0(x)$ and co$\tilde{N}_1(x)$ are closed, and the proof is concluded. ∎

LEMMA 4.7. *Let $x \in \mathcal{S}^c$, and let the assumptions of Theorem* 3.1 *hold. Let $\tilde{N}$ be a closed convex cone in $\mathbb{R}^{N+1}$ with the property*

$$N(x) \subseteq \tilde{N} \subseteq N^P_{hypo(T)}(x, T(x)). \tag{4.46}$$

*Let $\zeta$ belong to an exposed ray of $\tilde{N}$. The following statements hold true:*

(i) *if $\zeta = (\xi, 0)$ with $\xi \in \mathbb{R}^N$, then $\xi \in N_0(x)$;*

(ii) *if $\zeta = (\xi, \lambda)$ with $\xi \in \mathbb{R}^N$ and $\lambda > 0$, then $\xi/\lambda \in N_1(x)$.*

*Moreover, $\zeta$ satisfies* (4.45) *for all $y \in \mathcal{S}^c$ and all $\beta \leq T(y)$.*

*Proof.* By our assumption on $\zeta$, there exists $\bar{v} = (v_0, \lambda_0)$ satisfying $v_0 \in \mathbb{R}^N$, $\|v_0\| = 1$, and $\lambda_0 \in \mathbb{R}$ such that

$$\begin{cases} \langle (v_0, \lambda_0), \zeta \rangle = 0, \\ \langle (v_0, \lambda_0), w \rangle \leq 0 \quad \text{for all } w \in \tilde{N}, \\ \langle (v_0, \lambda_0), w \rangle = 0, \ \text{and } 0 \neq w \in \tilde{N} \Rightarrow \frac{w}{\|w\|} = \frac{\zeta}{\|\zeta\|}. \end{cases} \tag{4.47}$$

We now begin proving (i). Since $\zeta = (\xi, 0) \in N^P_{hypo(T)}(x, T(x))$, there exists a constant $\sigma \geq 0$ such that, for all $z \in \overline{\mathcal{S}^c}$ and all $\beta \leq T(z)$, the inequality

$$\langle \xi, z - x \rangle \ \leq\ \sigma(\|z - x\|^2 + |\beta - T(x)|^2) \tag{4.48}$$

holds. Set now $x_n = x + \frac{v_0}{n} + \frac{\xi}{n\sqrt{n}}$. Then by the density theorem (see [10, Theorem 1.3.1]), for each $n$ there exists $z_n$ such that

$$\partial_P T(z_n) \neq \emptyset, \tag{4.49}$$

$$\|z_n - x_n\| \leq \frac{1}{n^2}. \tag{4.50}$$

First we show that

$$T(z_n) \le T(x) \quad \text{for all } n \text{ large enough.} \tag{4.51}$$

Indeed, assume by contradiction that $T(z_n) > T(x)$. Taking $z = z_n$ and $\beta = T(x)$ in (4.48), we obtain

$$\langle \xi, z_n - x\rangle \ \le\ \sigma\ \|z_n - x\|^2 .$$

It follows from the above inequality, (4.47), and (4.50) that there exists a suitable constant $\sigma_1$ for which

$$\frac{\|\xi\|^2}{n\sqrt{n}} \ \le\ \frac{\sigma_1}{n^2}$$

for all $n$ large enough, a contradiction.

Second we claim that there exists $\sigma_2$ such that

$$|T(z_n) - T(x)| > \sigma_2 n^{-\frac{3}{4}} \quad \text{for all } n \text{ large enough.} \tag{4.52}$$

Indeed, by taking $z = z_n$ and $\beta = T(z_n)$ in (4.48), we obtain

$$\langle \xi, z_n - x\rangle \le \sigma\ (\|z_n - x\|^2 + |T(z_n) - T(x)|^2).$$

From the above inequality, (4.47), and (4.50), one can easily see that (4.52) holds.

On the other hand, by (4.49) and Lemma 4.4, we know that $T$ is differentiable at $z_n$, and we write $\xi_n = DT(z_n)$. Recalling (4.38), for all $z \in \mathcal{S}^c$ and all $\beta \le T(z)$, the inequality

$$\langle -\xi_n, z - z_n\rangle + \beta - T(z_n) \le k(T(z_n), \|z_n\|, \rho)\, \|(-\xi_n, 1)\|\, (\|z - z_n\|^2 + |\beta - T(z_n)|^2) \tag{4.53}$$

holds.

We claim that $\|\xi_n\| \to +\infty$.

Assume by contradiction that there exists a constant $Q$ such that $\|\xi_n\| \le Q$ for all $n$. Taking $z = x$, $\beta = T(x)$ in (4.53) and recalling (4.51), we obtain that

$$\begin{aligned}(T(x) - T(z_n))\Big(1 - k(T(z_n), \|z_n\|, \rho)\sqrt{Q^2+1}|T(x) - T(z_n)|\Big)\\ \le \|x - z_n\|\, \Big(Q + k(T(z_n), \|z_n\|, \rho)\sqrt{Q^2+1}\, \|x - z_n\|\, \Big).\end{aligned}$$

By the continuity of $T(\cdot)$ and $k(\cdot)$ and by (4.51), (4.50), and (4.52), there exists a constant $Q_1 > 0$ such that

$$\frac{Q_1}{n^{\frac{3}{4}}} \le \frac{1}{n} \quad \text{for all } n \text{ large enough,}$$

a contradiction.

Now recalling (ii) in Lemma 4.5 and assuming without loss of generality that $\lim_{n\to\infty} -\frac{\xi_n}{\|\xi_n\|} = -\bar{\xi}$, we see that $(-\bar{\xi}, 0) \in \tilde{N}_0(x) \subseteq \tilde{N}$. By (4.51) we can take $z = x$ and $\beta = T(z_n)$ in (4.53), obtaining

$$\left\langle -\frac{\xi_n}{\|(-\xi_n, 1)\|}, \frac{x - z_n}{\|x - z_n\|}\right\rangle \ \le\ k(T(z_n), \|z_n\|, \rho)\, \|x - z_n\|\, .$$

Taking $n \to \infty$ in the above inequality and recalling (4.50), we obtain

$$\left\langle -\bar{\xi}, -v_0 \right\rangle \leq 0$$

or, equivalently, $\left\langle (-\bar{\xi}, 0), (v_0, \lambda_0) \right\rangle \geq 0$. Therefore, we obtain from (4.47) that $(-\bar{\xi}, 0) = \frac{(\xi,0)}{\|\xi\|}$. Thus, $\xi = -\bar{\xi}$, and the proof of claim (i) is concluded .

We now begin proving (ii), and take $\zeta = (\xi, 1)$ and $\bar{v} = (v_0, \lambda_0)$ satisfying (4.47). Set $x_n = x + \frac{v_0}{n}$. Then by the density theorem (see Theorem 1.3.1 in [10]), for each $n$ there exists $z_n$ such that

$$\partial_P T(z_n) \neq \emptyset, \tag{4.54}$$

$$\|z_n - x_n\| \leq \frac{1}{n^2}. \tag{4.55}$$

Recalling Lemma 4.4, (4.54) implies that $T(\cdot)$ is differentiable at $z_n$. Moreover, if we set $\xi_n = DT(z_n)$, then $-\xi_n \in N_1(z_n)$, and for all $z \in \mathcal{S}^c$ and $\beta \leq T(z)$, we have

$$\begin{aligned} &\langle -\xi_n, z - z_n \rangle + \beta - T(z_n) \\ &\quad \leq\ k(T(z_n), \|z_n\|, \rho)\, \|(-\xi_n, 1)\| \left(\|z - z_n\|^2 + |\beta - T(z_n)|^2\right). \end{aligned} \tag{4.56}$$

We claim that the sequence $\{\xi_n\}$ is bounded.

Suppose by contradiction that $\limsup_{n\to\infty} \|\xi_n\| = +\infty$. Then assuming without loss of generality that $-\frac{\xi_n}{\|\xi_n\|} \to -\bar{\xi}$, (ii) of Lemma 4.5 yields that $-\bar{\xi} \in N_0(x)$ and $(-\bar{\xi}, 0) \in \tilde{N}_0(x)$.

In order to obtain a contradiction, we consider two cases:

(a) $T(x) \geq T(z_n)$ for infinitely many $n$;

(b) $T(x) < T(z_n)$ for infinitely many $n$.

In the first case, we can choose $z = x$, $\beta = T(z_n)$ in (4.56), obtaining

$$\left\langle -\frac{\xi_n}{\|(-\xi_n, 1)\|}\ ,\ \frac{x - z_n}{\|x - z_n\|} \right\rangle\ \leq\ k(T(z_n), z_n, \rho)\, \|x - z_n\|\,.$$

Taking $n \to \infty$ and recalling (4.55), we get

$$\left\langle -\bar{\xi}\,,\, -v_0 \right\rangle\ \leq\ 0, \tag{4.57}$$

which implies $\left\langle (-\bar{\xi}, 0), (v_0, \lambda_0) \right\rangle \geq 0$. Thus, by combining $(-\bar{\xi}, 0) \in \tilde{N}_0(x)$ with (4.47), we obtain $\frac{(-\bar{\xi},0)}{\|-\bar{\xi}\|} = \frac{(\xi,1)}{\|(\xi,1)\|}$, a contradiction.

In the second case, since $(\xi, 1) \in N^P_{hypo(T)}(x, T(x))$, there exists $\sigma \geq 0$ such that

$$\langle \xi, z_n - x \rangle + T(z_n) - T(x)\ \leq\ \sigma\ \left(\|z_n - x\|^2 + |T(z_n) - T(x)|^2\right) \quad \text{for all } n. \tag{4.58}$$

The above inequality implies that there exists $\sigma_1$ such that, for all $n$ large enough,

$$T(z_n) - T(x)\ =\ |T(z_n) - T(x)|\ \leq\ \sigma_1\, \|z_n - x\|\,. \tag{4.59}$$

Recalling (4.56) and taking $z = x$, $\beta = T(x)$, we have, for all $n$ large enough,

$$\begin{aligned} &\left\langle \frac{-\xi_n}{\|(-\xi_n, 1)\|}, \frac{x - z_n}{\|z_n - x\|} \right\rangle + \frac{T(x) - T(z_n)}{\|(-\xi_n, 1)\|\, \|z_n - x\|} \\ &\qquad \leq k(T(z_n), \|z_n\|, \rho) \left( \|x - z_n\| + \frac{|T(x) - T(z_n)|^2}{\|x - z_n\|} \right). \end{aligned} \tag{4.60}$$

Taking $n \to \infty$ in both (4.59) and (4.60), we obtain

$$\left\langle -\bar{\xi}\ ,\ v_0 \right\rangle \ \geq\ 0, \tag{4.61}$$

which implies in turn that $\left\langle (-\bar{\xi}, 0), (v_0, \lambda_0) \right\rangle \geq 0$. Thus, by combining the condition $(-\bar{\xi}, 0) \in \tilde{N}_0(x)$ with (4.47), we obtain $\frac{(-\bar{\xi},0)}{\|\bar{\xi}\|} = \frac{(\xi,1)}{\|(\xi,1)\|}$, a contradiction.

We can now assume that

$$\|\xi_n\| \leq Q \quad \text{for all } n, \tag{4.62}$$

for a suitable constant $Q$, and without loss of generality that

$$\lim_{n\to\infty} \xi_n \ =\ \bar{\xi}. \tag{4.63}$$

From (i) of Lemma 4.5, we have that $-\bar{\xi} \in N_1(x)$ and $(-\bar{\xi}, 1) \in \tilde{N}_1(x)$, and (4.41) with $\bar{\xi}$ in place of $\xi$ holds.

We claim that there exists a constant $\sigma_2$ such that

$$|T(z_n) - T(x)| \ \leq\ \sigma_2 \|z_n - x\| \quad \text{for all } n. \tag{4.64}$$

In the case $T(x) < T(z_n)$, this was already proved (see (4.59)).

Assume now $T(x) \geq T(z_n)$. Then, by using (4.56) with $z = x$ and $\beta = T(x)$, we obtain, for all $n$ large enough,

$$\begin{aligned} &\langle -\xi_n, x - z_n\rangle + T(x) - T(z_n) \\ &\quad \leq k(T(z_n), \|z_n\|, \rho)\, \|(-\xi_n, 1)\| \left(\|x - z_n\|^2 + |T(x) - T(z_n)|^2\right). \end{aligned} \tag{4.65}$$

The above inequality and (4.62) imply, for all $n$ large enough,

$$T(x) - T(z_n) \leq k(T(z_n), \|z_n\|, \rho)\sqrt{Q^2+1}\left(\|x - z_n\|^2 + |T(x) - T(z_n)|^2\right) + Q\|z_n - x\|,$$

from which, by the local boundedness of $k$, the inequality (4.64) follows.

Summing (4.58) and (4.65) we obtain, for a suitable constant $\sigma_3 \geq 0$, that for all $n$ large enough,

$$\left\langle \xi_n + \xi, \frac{z_n - x}{\|z_n - x\|} \right\rangle \leq \sigma_3 \left( \|z_n - x\| + \frac{|T(z_n) - T(x)|^2}{\|z_n - x\|} \right).$$

Taking $n \to \infty$ in the above inequality and using (4.64) and (4.55), we obtain

$$\left\langle \bar{\xi} + \xi\ ,\ v_0 \right\rangle \ \leq\ 0$$

or, equivalently,

$$\langle (\xi, 1)\ ,\ (v_0, \lambda_0) \rangle \ \leq\ \left\langle (-\bar{\xi}, 1)\ ,\ (v_0, \lambda_0) \right\rangle.$$

Recalling (4.47), we have $\langle (\xi, 1)\ ,\ (v_0, \lambda_0)\rangle = 0$; whence $\left\langle (-\bar{\xi}, 1)\ ,\ (v_0, \lambda_0) \right\rangle \geq 0$. Note that $(-\bar{\xi}, 1) \in \tilde{N}_1(x)$, so that $\left\langle (-\bar{\xi}, 1)\ ,\ (v_0, \lambda_0) \right\rangle = 0$ by (4.47). Moreover, using again (4.47), we finally arrive to

$$\frac{(-\bar{\xi}, 1)}{\left\|(-\bar{\xi}, 1)\right\|} \ =\ \frac{(\xi, 1)}{\|(-\xi, 1)\|}.$$

Therefore, we see that $\xi = -\bar{\xi} \in N_1(x)$, and the proof is concluded. ∎

The lemmas contained in this section yield immediately the following result.

THEOREM 4.1. *Let $x \in \mathcal{S}^c$, and let the assumptions of Theorem 3.1 hold. Then*

$$N^P_{hypo(T)}(x, T(x)) = N(x),$$

*where $N(x)$ was defined in the statement of Lemma 4.6 so that $N^P_{hypo(T)}(x, T(x))$ is a closed (convex) cone.*

*Proof.* Assume by contradiction that there exists $\zeta \in N^P_{hypo(T)}(x, T(x)) \setminus N(x)$. Set

$$\tilde{N} = \mathrm{co}\left(N(x) \cup \{\lambda\zeta \mid \lambda \geq 0\}\right),$$

and observe that $\tilde{N}$ is a closed convex cone which satisfies (4.46). Clearly, $\zeta$ belongs to an exposed ray of $\tilde{N}$ so that, by Lemma 4.7, $\zeta \in \tilde{N}_0(x) \cup \tilde{N}_1(x)$, a contradiction. ∎

## 5. Proof of the main results.

*Proof of Theorem* 3.1. It is clear that the "$\supseteq$" inclusion in (3.2) follows from Lemma 4.2 and the convexity of $\partial^\infty T(x)$.

In order to prove the "$\subseteq$" inclusion, take $\xi \in \partial^\infty T(x)$, i.e., $(-\xi, 0) \in N^P_{hypo(T)}(x, T(x))$. Since $N^P_{hypo(T)}(x, T(x))$ is pointed and closed (see Theorem 4.1), recalling (2.1) we can find numbers $\alpha_i,\ \beta_i \geq 0$ and vectors $\xi_i,\ \zeta_i \in \mathbb{R}^N$, $i \in \{1, \ldots, N+2\}$ such that

$$\begin{cases} (-\xi_i, 1) & \text{belongs to an exposed ray of } N^P_{hypo(T)}(x, T(x)), \\ (-\zeta_i, 0) & \text{belongs to an exposed ray of } N^P_{hypo(T)}(x, T(x)), \\ (-\xi, 0) = \sum_{i=1}^{N+2} \alpha_i(-\xi_i, 1) + \sum_{i=1}^{N+2} \beta_i(-\zeta_i, 0). & \end{cases} \tag{5.1}$$

From the above equality we deduce that $\alpha_i = 0$ for all $i \in \{1, \ldots, N+2\}$. Thus, we have

$$(-\xi, 0) \ = \ \sum_{i=1}^{N+2} \beta_i(-\zeta_i, 0). \tag{5.2}$$

Recalling (i) in Lemma 4.7, we obtain $-\zeta_i \in N_0(x)$. Setting $\bar{\zeta}_i = (\sum_{j=1}^{N+2} \beta_j)\zeta_i$ and $\bar{\beta}_i = \frac{\beta_i}{\sum_{i=1}^{N+2} \beta_i}$, one can easily see $-\bar{\zeta}_i \in N_0(x)$ and $\sum_{i=1}^{N+2} \bar{\beta}_i = 1$.

From (5.2), we obtain

$$\xi \ = \ -\sum_{i=1}^{N+2} \bar{\beta}_i(-\bar{\zeta}_i).$$

The proof is concluded. ∎

*Proof of Theorem* 3.2. Observe that from the definition, it follows that if $\xi \in \partial^P T(x)$ and $\zeta \in \partial^\infty T(x)$, then $\xi + \zeta \in \partial^P T(x)$. Thus, the "$\supseteq$" inclusion in (3.3) follows from Lemmas 4.3 and 4.2 and the above observation.

In order to prove the "$\subseteq$" inclusion, take $\xi \in \partial^P T(x)$, i.e., $(-\xi, 1) \in N^P_{hypo(T)}(x, T(x))$. Since $N^P_{hypo(T)}(x, T(x))$ is pointed and closed (see Theorem 4.1) and by recalling (2.1), we can find numbers $\alpha_i,\ \beta_i \geq 0$ and vectors $\xi_i,\ \zeta_i \in \mathbb{R}^N$, $i \in \{1, \ldots, N+2\}$, such that

$$\begin{cases} (-\xi_i, 1) & \text{belongs to an exposed ray of } N^P_{hypo(T)}(x, T(x)), \\ (-\zeta_i, 0) & \text{belongs to an exposed ray of } N^P_{hypo(T)}(x, T(x)), \\ (-\xi, 1) = \sum_{i=1}^{N+2} \alpha_i(-\xi_i, 1) + \sum_{i=1}^{N+2} \beta_i(-\zeta_i, 0). & \end{cases} \tag{5.3}$$

From the above equality we deduce that $\sum_{i=1}^{N+2} \alpha_i = 1$. Thus, recalling (ii) in Lemma 4.7, we obtain that $\sum_{i=1}^{N+2} \alpha_i(-\xi_i) \in \mathrm{co}(N_1(x))$.

On the other hand, arguing similarly to the above proof, we see that $\sum_{i=1}^{N+2} \beta_i(-\zeta_i) \in \mathrm{co}(N_0(x))$. Therefore,

$$\xi = -\left(\sum_{i=1}^{N+2} \alpha_i(-\xi_i) + \sum_{i=1}^{N+2} \bar{\beta}_i(-\bar{\zeta}_i)\right) \in -[\mathrm{co}(N_1(x)) + \mathrm{co}(N_0(x))].$$

The proof is concluded. ∎

**Proof of Theorem 3.3.**

We need the following technical lemma.

LEMMA 5.1. *Assume that* $N^P_{hypo(T)}(x, T(x))$ *is pointed for all* $x \in \mathcal{S}^c$. *Then for each continuous function* $\theta : \mathcal{S}^c \to [0, \infty)$, *there exists a continuous function* $\psi_\theta : \mathcal{S}^c \to (0, 1]$ *such that*

$$\langle \zeta_1, \zeta_2 \rangle \geq \psi_\theta(x) - 1 \tag{5.4}$$

*for all* $x \in \mathcal{S}^c$ *and for all* $\zeta_1,\ \zeta_2 \in N^P_{hypo(T)}(x, T(x))$ *satisfying both* $\|\zeta_1\| = \|\zeta_2\| = 1$ *and*

$$\langle \zeta_j, (z - x, \beta - T(x)) \rangle \leq \theta(x)(\|z - x\|^2 + |\beta - T(x)|^2) \tag{5.5}$$

*for all* $z \in \mathcal{S}^c$, $\beta \leq T(x)$, *and* $j = 1, 2$.

*Proof.* We need to show only that for every $n \in \mathbb{N}$, there exists a continuous function $\psi_n : \overline{B(0, n)} \cap \mathcal{S}^c \to (0, 1]$ satisfying (5.4) with $\psi_\theta(x)$ replaced by $\psi_n(x)$. It is easy to see that the following statement is sufficient to this aim.

Let, for all $m, n \in \mathbb{N}$, $\mathcal{K}^m_n = \overline{B(0, n)} \cap \mathcal{S}^c(\frac{1}{m})$, and observe that, by the continuity of $T(\cdot)$, $\mathcal{K}^m_n$ is compact. Fix $n$. We claim that for each $m \in \mathbb{N}$, there exists a constant $k_m \in (0, 1]$ such that

$$\langle \zeta_1\ ,\ \zeta_2 \rangle \geq k_m - 1 \tag{5.6}$$

for all $x \in \mathcal{K}^m_n$, $\zeta_1,\ \zeta_2 \in N^P_{hypo(T)}(x, T(x))$ satisfying $\|\zeta_1\| = \|\zeta_2\| = 1$ and (5.5).

Indeed, assume by contradiction that there exists a sequence $\{x_i\} \subset \mathcal{K}^m_n$ together with vectors $\zeta^i_1, \zeta^i_2 \in N^P_{hypo(T)}(x_i, T(x_i))$ satisfying $\|\zeta^i_1\| = \|\zeta^i_2\| = 1$ and

$$\left\langle \zeta^i_j, (z - x_i, \beta - T(x_i)) \right\rangle \leq \theta(x_i)(\|z - x_i\|^2 + |\beta - T(x_i)|^2) \tag{5.7}$$

for all $z \in \mathcal{S}^c$, $\beta \leq T(x_i)$, and $j \in \{1, 2\}$ but such that

$$\lim_{i\to\infty} \left\langle \zeta^i_1\ ,\ \zeta^i_2 \right\rangle = -1. \tag{5.8}$$

We can assume without loss of generality that $\{x_i\}$, $\{\zeta^i_1\}$, and $\{\zeta^i_2\}$ converge, respectively, to $\bar{x} \in \mathcal{K}^m_n$, $\bar{\zeta}_1$, and $\bar{\zeta}_2$. By the continuity of $T(\cdot)$, $\theta(\cdot)$, and (5.7), we obtain

$$\bar{\zeta}_i \in N^P_{hypo(T)}(\bar{x}, T(\bar{x})) \quad \text{for } i \in \{1, 2\}.$$

On the other hand, from $\|\zeta^i_1\| = \|\zeta^i_2\| = 1$ and (5.8), we get

$$\bar{\zeta}_1 = -\bar{\zeta}_2.$$

But then the normal cone $N^P_{hypo(T)}(\bar{x}, T(\bar{x}))$ contains a line, and this is a contradiction. ∎

*End of the proof of Theorem* 3.3.

We need to find a continuous function $\varphi : \mathcal{S}^c \to [0, \infty)$ such that for all $x \in \mathcal{S}^c$, $\zeta \in N^P_{hypo(T)}(x, T(x))$ and for all $z \in \mathcal{S}^c$, $\beta \le T(z)$, we have

$$\langle \zeta, (z - x, \beta - T(x)) \rangle \le \varphi(x) \, \|\zeta\| \, (\|z - x\|^2 + |\beta - T(x)|^2). \tag{5.9}$$

Observe that for every $\zeta \in N^P_{hypo(T)}(x, T(x))$, by the pointedness assumption and recalling Theorem 4.1, we have

$$\zeta = \sum_{i=1}^{N+2} \zeta_i, \tag{5.10}$$

where each $\zeta_i$ belongs to an exposed ray of $N^P_{hypo(T)}(x, T(x))$. For $k \in \{1, 2, \dots, N+2\}$, we set

$$N^P_k(x) = \Big\{ \zeta \mid \zeta = \sum_{i=1}^{k} \zeta_i, \quad \text{where } \zeta_i \text{ belongs to an exposed ray of } N^P_{hypo(T)}(x, T(x)) \Big\}. \tag{5.11}$$

Of course, $N^P_k(x) \subseteq N^P_{hypo(T)}(x, T(x))$, and $N^P_{N+2}(x) = N^P_{hypo(T)}(x, T(x))$.

Now we are going to construct by induction a continuous function $\varphi_k(\cdot)$ such that

$$\left\langle \zeta^k, (z - x, \beta - T(x)) \right\rangle \le \varphi_k(x) \left\| \zeta^k \right\| (\|z - x\|^2 + |\beta - T(x)|^2) \tag{5.12}$$

for all $x \in \mathcal{S}^c$, $\zeta^k \in N^P_k(x)$ and for all $z \in \mathcal{S}^c$, $\beta \le T(z)$.

For $k = 1$, we choose $\varphi_1(x) := k(T(x), \|x\|, \rho)$. Recalling Lemmas 4.7, 4.3, and 4.2, we obtain that for all $\zeta^1 \in N^P_1(x)$ and for all $z \in \mathcal{S}^c$, $\beta \le T(z)$,

$$\left\langle \zeta^1, (z - x, \beta - T(x)) \right\rangle \le \varphi_1(x) \left\| \zeta^1 \right\| (\|z - x\|^2 + |\beta - T(x)|^2). \tag{5.13}$$

Thus, (5.12) holds.

Assume now that (5.12) is satisfied for $k = h \ge 1$. We want to show that (5.12) holds for $k = h + 1$ with

$$\varphi_{h+1}(x) = \sqrt{\frac{\varphi_h(x)^2 + \varphi_1(x)^2}{\psi_{\max\{\varphi_1, \varphi_h\}}(x)}}, \tag{5.14}$$

where the function $\psi_{\max\{\varphi_1, \varphi_h\}}(\cdot)$ is given by Lemma 5.1 for $\theta(\cdot) = \max\{\varphi_1(\cdot), \varphi_h(\cdot)\}$. Indeed, given $\zeta^{h+1} \in N^P_{h+1}(x)$, one can write

$$\zeta^{h+1} \;=\; \zeta^h + \zeta^1,$$

where $\zeta^h \in N^P_h(x)$ and $\zeta^1 \in N^P_1(x)$. From (5.13) and the inductive assumption, one can easily see that

$$\left\langle \zeta^{h+1}, (z - x, \beta - T(x)) \right\rangle \le \Big( \varphi_1(x) \left\| \zeta^1 \right\| + \varphi_h(x) \left\| \zeta^h \right\| \Big) (\|z - x\|^2 + |\beta - T(x)|^2) \tag{5.15}$$

for all $z \in \mathcal{S}^c$, $\beta \le T(z)$.

On the other hand, by inductive assumption, (5.13), and Lemma 5.1 applied for $\theta(\cdot) = \max\{\varphi_1(\cdot), \varphi_h(\cdot)\}$, we obtain

$$\left\langle \frac{\zeta^h}{\|\zeta^h\|}, \frac{\zeta^1}{\|\zeta^1\|} \right\rangle \geq \psi_{\max\{\varphi_1,\varphi_h\}}(x) - 1.$$

Thus, since $\psi(x) \in (0, 1]$, we see that

$$\left\|\zeta^h + \zeta^1\right\|^2 \geq \psi_{\max\{\varphi_1,\varphi_h\}}(x)\Big(\left\|\zeta^h\right\|^2 + \left\|\zeta^1\right\|^2\Big).$$

Therefore,

$$\left\|\zeta^h + \zeta^1\right\|^2 \geq \frac{\psi_{\max\{\varphi_1,\varphi_h\}}(x)}{\varphi_h(x)^2 + \varphi_1(x)^2}\Big(\varphi_h(x)\left\|\zeta^h\right\| + \varphi_1(x)\left\|\zeta^1\right\|\Big)^2.$$

Combining the above inequality, (5.14), and (5.15), we obtain that

$$\left\langle \zeta^{h+1}, (z - x, \beta - T(x)) \right\rangle \leq \varphi_{h+1}(x)\left\|\zeta^{h+1}\right\|\Big(\|z - x\|^2 + |\beta - T(x)|^2\Big)$$

for all $z \in \mathcal{S}^c$, $\beta \leq T(z)$.
To conclude the proof, we choose $\varphi(\cdot) = \varphi_{N+2}(\cdot)$. ◻

*Proof of Proposition* 3.1. It is a straightforward consequence of Lemmas 4.2 and 4.3. ◻

**6. The case of optimal points.** This section is devoted to the representation of *supergradient* and *horizontal gradient* at optimal points. The corresponding formulas are easier than in the general case, and the structure of the Hamiltonian exhibits special properties.

The definition of optimal points here is based on the classical definition (see, e.g., Definition 2.24, p. 119 in [1]) but is adapted to *limiting optimal trajectories* since optimal trajectories may not exist.

DEFINITION 6.1. *Let $x \in \mathcal{S}^c$, and set $r = T(x)$. The point $x$ is called an optimal point if there exist $\tau > 0$ and $x_\tau \in \mathcal{S}^c$ such that*

(i) *$T(x_\tau) = r + \tau$;*
(ii) *there exist $\bar{x}_\tau \in \mathcal{S}_{x_\tau}$ and $\{\bar{u}_n\} \subset \mathcal{U}_{\bar{x}_\tau}$, together with the corresponding sequence $x_n \to x_\tau$, such that $y^{x_n,\bar{u}_n}(\tau) \to x$.*

At optimal points, the Hamiltonian has a special behavior. More precisely, let $x$ be an optimal point with $T(x) = r > 0$. Then the Hamiltonian $H(x, \cdot)$ is additive on the proximal normal cone to $\mathcal{S}^c(r)$. It follows from this property that the *supergradient* and *horizontal supergradient* of $T$ are contained, respectively, in the 1-*level set* and the 0-*level set* of the Hamiltonian.

THEOREM 6.1. *Let $x \in \mathcal{S}^c$ be an optimal point. Under the same assumptions of Theorem* 3.1, *the (proximal) horizontal gradient and the supergradient of the minimum time function $T(\cdot)$ at the point $x$ can be computed as follows:*

(a) $\partial^\infty T(x) = [-\mathrm{co}(N(x))] \cap \{-\xi \mid H(\xi, x) = 0\}$,
(b) $\partial^P T(x) = [-\mathrm{co}(N(x))] \cap \{-\xi \mid H(\xi, x) = 1\}$,

*where*

$$N(x) = \{M^T(r)v \mid M(\cdot) \in \mathcal{M}_{\bar{x}}, v \in N^P_{\overline{\mathcal{S}^c}}(\bar{x}), \bar{x} \in \mathcal{S}_x\}. \tag{6.1}$$

The proof of Theorem 6.1 requires some preliminary lemmas. The first one gives information on a lower bound of the Hamiltonian computed at a proximal normal of the sublevel of $T$ at an optimal point.

LEMMA 6.1. *Let $x \in \mathcal{S}^c$ be an optimal point, and let $\xi \in N^P_{\mathcal{S}^c(T(x))}(x)$. Then $H(x,\xi) \geq 0$.*

*Proof.* Set $r = T(x)$. Let $\tau$, $x_\tau$, $\bar{x}_\tau$, $\bar{u}_n$, and $x_n$ be with the properties stated in Definition 6.1. To simplify our writing, we set $\gamma_n(\cdot) = y^{x_n,\bar{u}_n}(\cdot)$. Assuming without loss of generality that $\gamma_n(\cdot)$ converges uniformly to $\gamma(\cdot)$, one can easily check that $\gamma(t) \in \mathcal{S}^c(r)$ for all $t \in [0,\tau]$. In fact should $\bar{t} \in (0,\tau]$ exist such that $T(\gamma(\bar{t})) < r$, then one would have $T(x_\tau) < r + \tau$, a contradiction. Now since $\xi \in N^P_{\mathcal{S}^c(r)}(x)$, there exists $\sigma > 0$ such that for all $t \in [0,\tau]$, we have

$$\langle \xi \ , \ \gamma(t) - x \rangle \ \leq \ \sigma \ \|\gamma(t) - x\|^2 \,, \tag{6.2}$$

namely, for all $t \in [0,\tau]$,

$$\lim_{n\to\infty} \langle \xi, \gamma_n(t) - x \rangle \leq \sigma \lim_{n\to\infty} \|\gamma_n(t) - x\|^2 \,.$$

Equivalently, for all $t \in [0,\tau]$,

$$\lim_{n\to\infty} \left\langle \xi, \gamma_n(\tau) - \int_{\tau-t}^{\tau} f(\gamma_n(s),\bar{u}_n(s))ds - x \right\rangle \\ \leq \sigma \lim_{n\to\infty} \left\| \gamma_n(\tau) - \int_{\tau-t}^{\tau} f(\gamma_n(s),\bar{u}_n(s))ds - x \right\|^2 .$$

Recalling (ii) in Definition 6.1, we obtain that for all $t \in [0,\tau]$,

$$\lim_{n\to\infty} \left\langle \xi \ , \ -\int_{\tau-t}^{\tau} f(\gamma_n(s),\bar{u}_n(s))ds \right\rangle \ \leq \ \sigma \ \lim_{n\to\infty} \left\| \int_{\tau-t}^{\tau} f(\gamma_n(s),\bar{u}_n(s))ds \right\|^2 .$$

From (iii) of Lemma 8.1 and (i) and (ii) in Definition 6.1, one can see that

$$\lim_{n\to\infty} \left\langle \xi \ , \ -\int_{\tau-t}^{\tau} f(\gamma_n(s),\bar{u}_n(s))ds \right\rangle \ \leq \ O(t^2) \text{ for } t \to 0^+.$$

Thus, for $t \to 0^+$,

$$\limsup_{n\to\infty} \left\langle \xi, -\int_{\tau-t}^{\tau} f(x,\bar{u}_n(s))ds \right\rangle \\ \leq O(t^2) + \limsup_{n\to\infty} \left\langle \xi, \int_{\tau-t}^{\tau} \big(f(\gamma_n(s),\bar{u}_n(s)) - f(x,\bar{u}_n(s))\big)ds \right\rangle .$$

Applying the Lipschitz condition of the function $f(\cdot,\cdot)$ and (iii) of Lemma 8.1, we easily obtain that

$$\limsup_{n\to\infty} \left\langle \xi, -\int_{\tau-t}^{\tau} f(x,\bar{u}_n(s))ds \right\rangle \ \leq \ O(t^2) \qquad \text{for } t \to 0^+.$$

Therefore, there exists a constant $Q > 0$ such that for each $t \in [0,\tau]$, one can find $n_t \in \mathbb{N}$ with the property

$$\left\langle \xi, -\frac{\int_{\tau-t}^{\tau} f(x,\bar{u}_{n_t}(s))ds}{t} \right\rangle \ \leq \ Qt.$$

Set $\bar{f}_t = \frac{\int_{\tau-t}^{\tau} f(x,\bar{u}_{n_t}(s))ds}{t}$. Since $\bar{f}_t \in \mathrm{co}(f(x,\mathcal{U}))$, by the compactness of $\mathcal{U}$, there exists a sequence $\{t_n\} \subseteq [0,\tau]$ converging to 0 and $\bar{f} \in \mathrm{co}f(x,\mathcal{U})$ such that both

$$\bar{f} = \lim_{n\to\infty} \bar{f}_{t_n}$$

and

$$\langle \xi \ , \ \bar{f} \rangle \ \geq \ 0$$

hold. Since

$$H(x,\xi) \ = \ \max\{\langle \xi \ , \ f\rangle \ | \ f \in \mathrm{co}f(x,\mathcal{U})\},$$

the proof is concluded. □

The next lemma is the key point in order to obtain the additivity of the Hamiltonian.

LEMMA 6.2. *Let $x \in \mathcal{S}^c$ be an optimal point, and set $T(x) = r$. Then there exists $\bar{f} \in \mathrm{co}f(x,\mathcal{U})$ such that for all $\xi \in N^P_{\mathcal{S}^c(r)}(x)$,*

$$H(x,\xi) \ = \ \langle \xi \ , \ \bar{f} \rangle .$$

*Proof.* Let $\tau$, $x_\tau$, $\bar{x}_\tau$, $\bar{u}_n$, and $x_n$ be with the properties stated in Definition 6.1.

To simplify our writing, we set $\gamma_n(\cdot) = y^{x_n,\bar{u}_n}(\cdot)$. Assuming without loss of generality that $\gamma_n(\cdot)$ converges uniformly to $\gamma(\cdot)$, we see that $\gamma(\tau) = x$ and $T(\gamma(\tau - t)) = r + t$ for all $t \in [0,\tau]$. Pick $v \in \mathcal{U}$, and define, for each $t \in [0,\tau]$, $\beta_{v,t}(\cdot) = y^{\gamma(\tau-t),v}(\cdot)$, where $v(\cdot)$ is the constant control $v(t) \equiv v$. Observe that $\beta_{v,t}(t) \in \mathcal{S}^c(r)$ for all $t \in [0,\tau]$.

Let now $\xi \in N^P_{\mathcal{S}^c(r)}$, together with a constant $\sigma \geq 0$ such that for all $t \in [0,\tau]$,

$$\langle \xi \ , \ \beta_{v,t}(t) - x\rangle \ \leq \ \sigma \ \|\beta_{v,t}(t) - x\|^2 .$$

Recalling (ii) in Definition 6.1, the latter is equivalent to

$$\begin{aligned} &\lim_{n\to\infty} \left\langle \xi, \int_0^t \left(f(\beta_{v,t}(s),v) - f(\gamma_n(\tau-t+s),\bar{u}_n(\tau-t+s))\right) ds \right\rangle \\ &\quad \leq \ \sigma \ \lim_{n\to\infty} \left\| \int_0^t \left(f(\beta_{v,t}(s),v) - f(\gamma_n(\tau-t+s),\bar{u}_n(\tau-t+s))\right) ds \right\|^2 \end{aligned} \tag{6.3}$$

for all $t \in [0,\tau]$. Moreover, by (iii) of Lemma 8.1, there exists a constant $M$ such that for all $n \in \mathbb{N}$, $t \in [0,\tau]$, and $s \in [0,t]$,

$$\|\gamma_n(\tau - t + s) - \gamma_n(\tau)\| \ \leq \ Mt$$

so that for all $t \in [0,\tau]$ and $s \in [0,t]$,

$$\lim_{n\to\infty} \|\gamma_n(\tau - t + s) - x\| \ \leq \ Mt.$$

Combining the above inequality with (6.3) and recalling the Lipschitz condition on $f$, we obtain that, for $t \to 0^+$,

$$\limsup_{n\to\infty} \left\langle \xi, \int_0^t \big(f(x,v) - f(x,\bar{u}_n(r-t+s))\big) ds \right\rangle \leq O(t^2)$$

or, equivalently,

$$\limsup_{n\to\infty} \left\langle \xi, f(x,v) - \frac{\int_0^t f(x,\bar{u}_n(r-t+s))ds}{t} \right\rangle \ \le\ O(t).$$

By arguing as in the proof of Lemma 6.1, we can find $\bar{f} \in \text{co}(f(x,U))$ independent of $\xi$ and $v$ such that

$$\langle \xi, f(x,v)\rangle \le \langle \xi, \bar{f}\rangle .$$

The proof is therefore complete. ∎

The desired additivity property follows immediately from the above lemma.

COROLLARY 6.1. *Let $x \in \mathcal{S}^c$ be an optimal point, and set $T(x) = r$. Then for all $\xi_1,\ \xi_2 \in N^P_{\mathcal{S}^c(r)}(x)$, the property*

$$H(x, \xi_1 + \xi_2) = H(x, \xi_1) + H(x, \xi_2)$$

*holds.*

We are now ready to prove Theorem 6.1.

*Proof of Theorem* 6.1. *Proof of part* (a). It is clear that the "$\subseteq$" inclusion of the equality in (a) follows from Theorem 3.1 and Corollary 6.1.

To prove the "$\supseteq$" inclusion, take $\xi \in\ [-\text{co}(N(x))] \cap \{-\xi \mid H(x,\xi) = 0\}$, namely,

$$\xi = -\sum_{i=1}^{m} M_i^T(r)v_i, \quad \text{where} \ \ M_i^T(r)v_i \in N(x) \tag{6.4}$$

and

$$H\left(x, \sum_{i=1}^{m} M_i^T(r)v_i, \right) = 0. \tag{6.5}$$

Applying Lemma 4.1 we get that $M_i^T(r)v_i \in N^P_{\mathcal{S}^c(r)}(x)$ for all $i \in \{1, 2, \ldots, m\}$. Thus, it follows from Lemma 6.1 that

$$H(x, M_i^T(r)v_i) \ge 0 \quad \text{for all } i \in \{1, 2, \ldots, m\}. \tag{6.6}$$

Combining (6.5) and (6.6), we obtain from Corollary 6.1 that $H(x, M_i^T(r)v_i) = 0$ for all $i \in \{1, 2, \ldots, m\}$. Therefore, $M_i^T(r)v_i \in N_0(x)$ for all $i \in \{1, 2, \ldots, m\}$. We conclude the proof using (6.4) and Theorem 3.1.

*Proof of part* (b). Similarly to part (a), the "$\subseteq$" inclusion of the equality in (b) follows from Theorem 3.2 and Corollary 6.1.

To show the "$\supseteq$" inclusion, let $\xi \in\ [-\text{co}(N(x))] \cap \{-\xi \mid H(x,\xi) = 1\}$. Recalling Lemma 6.1, $\xi$ can be represented as

$$\xi = -\sum_{i=1}^{m} \alpha_i M_{0i}^T(r)v_i - \sum_{j=1}^{m} \beta_j M_{1j}^T(r)w_j, \tag{6.7}$$

where $\alpha_i \ge 0$, $\beta_j \ge 0$, $M_{0i}^T(r)v_i \in N_0(x)$, and $M_{1j}^T(r)w_j \in N_1(x)$.

From $M_{0i}^T(r)v_i \in N^P_{\mathcal{S}^c(r)(x)}$, $M_{1j}^T(r)w_j \in N^P_{\mathcal{S}^c(r)(x)}$, and Corollary 6.1, we have

$$H(x,\xi) = \sum_{i=1}^{m} \alpha_i H(x, M_{0i}^T(r)v_i) + \sum_{j=1}^{m} \beta_j H(x, M_{1j}^T(r)w_j) = \sum_{j=1}^{m} \beta_j \tag{6.8}$$

so that $\sum_{j=1}^{m} \beta_j = 1$. The proof is concluded by using (6.8), (6.7), and Theorem 3.2. ∎

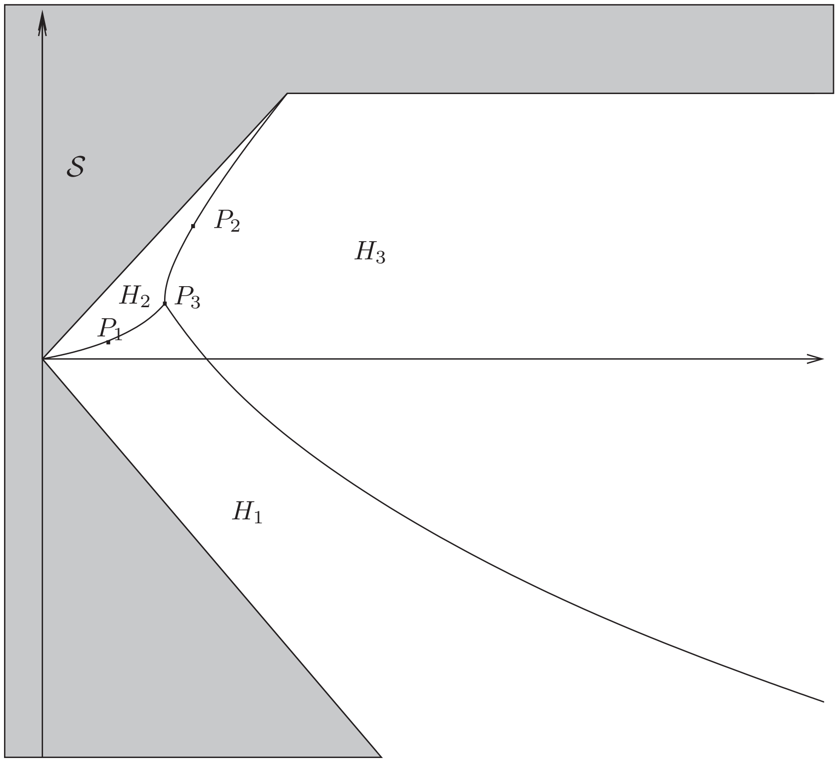


Fig. 1.

**7. Examples.** In this section we present some examples which illustrate our results. In particular, we provide an example showing that Theorem 3.3 is no longer valid if the pointedness assumption (3.1) is dropped.

*Example* 1. Consider the dynamics $x''(\cdot) \in [-1,1] =: \mathcal{U}$, i.e.,

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = A \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ u \end{pmatrix}, \; u \in \mathcal{U}, \text{ where } A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{7.1}$$

with the initial conditions $x_1(0) = x_1^0$, $x_2(0) = x_2^0$. The target is the set (see Figure 1)

$$\begin{aligned} \mathcal{S} \; = &\{ \, (x_1, x_2) \in \mathbb{R}^2 \mid x_1 \le 0 \, \} \cup \{ \, (x_1, x_2) \in \mathbb{R}^2 \mid x_1 \ge 0, \; x_2 \le -x_1 \} \\ \cup &\; \{ \, (x_1, x_2) \in \mathbb{R}^2 \mid 0 \le x_1 \le 1, \; x_2 \ge x_1 \} \\ \cup &\; \{ \, (x_1, x_2) \in \mathbb{R}^2 \mid x_1 \ge 1, \; x_2 \ge 1 \}. \end{aligned}$$

Optimal trajectories are arcs of parabolas

$$x_1 = \frac{1}{2}(x_2)^2 - \frac{1}{2}(x_2^0)^2 + x_1^0 \quad \text{(corresponding to the control } u \equiv 1)$$

and

$$x_1 = -\frac{1}{2}(x_2)^2 + \frac{1}{2}(x_2^0)^2 + x_1^0 \quad \text{(corresponding to the control } u \equiv -1).$$

By direct computation, the minimum time function $T$ is everywhere finite, continuous on the whole of $\mathbb{R}^2$, and the open set $S^c$ can be divided into three regions, say, $H_1$,

$H_2$, and $H_3$, where $T$ has a different explicit expression. More precisely, consider the curves

$$\gamma_1(t) = \left(\sqrt{2t}(1-t), t\right), \qquad 0 < t \le 2-\sqrt{3},$$
$$\gamma_2(t) = \left(\frac{1+t^2}{2}, t\right), \qquad 2-\sqrt{3} < t < 1,$$
$$\gamma_3(t) = \left(\frac{3-8t+3t^2}{2}, t\right), \qquad t \ge 2-\sqrt{3}.$$

Observe that $\gamma_1(2-\sqrt{3}) = \gamma_2(2-\sqrt{3}) = \gamma_3(2-\sqrt{3}) = 4-2\sqrt{3}$ and, moreover, that all points $\gamma_2(t)$, with $2-\sqrt{3} < t < 1$, are optimal (according to Definition 6.1), while all points $\gamma_1(t)$, $\gamma_2(t)$ are not optimal. Set

$$H_1 = \{(x_1,x_2) \in \mathcal{S}^c \mid 0 \le x_2 \le 2-\sqrt{3},\ \gamma_1(x_2) \le x_1 \le \gamma_3(x_2)\}$$
$$\cup \{(x_1,x_2) \in \mathcal{S}^c \mid x_2 \le 0,\ -x_2 \le \gamma_3(x_2)\},$$

$$H_2 = \{(x_1,x_2) \in \mathcal{S}^c \mid 0 \le x_2 \le 2-\sqrt{3},\ x_2 \le x_1 \le \gamma_1(x_2)\}$$
$$\cup \{(x_1,x_2) \in \mathcal{S}^c \mid 2-\sqrt{3} \le x_2 \le 1,\ x_2 \le x_1 \le \gamma_2(x_2)\},$$

$$H_3 = \{(x_1,x_2) \in \mathcal{S}^c \mid 2-\sqrt{3} \le x_2 \le 1,\ x_1 \ge \gamma_2(x_2)\}$$
$$\cup \{(x_1,x_2) \in \mathcal{S}^c \mid x_2 \le 2-\sqrt{3},\ x_1 \ge \gamma_3(x_2)\}.$$

The minimum time function $T : \mathcal{S}^c \to \mathbb{R}$ can be explicitly computed as

$$T(x_1,x_2) = \begin{cases} x_2 - 1 + \sqrt{1+2x_1+(x_2)^2} & := \theta_1(x_1,x_2), \quad (x_1,x_2) \in H_1, \\ 1 - x_2 - \sqrt{1-2x_1+(x_2)^2} & := \theta_2(x_1,x_2), \quad (x_1,x_2) \in H_2, \\ 1 - x_2 & := \theta_3(x_1,x_2), \quad (x_1,x_2) \in H_3. \end{cases}$$

In the interior of each region $H_i$, $i = 1,2,3$, T is differentiable. Singularities appear at each point of the curves $\gamma_i$, $i = 1,2,3$. Moreover, $T$ is Hölder continuous with exponent $\frac{1}{2}$.

In order to appreciate the role of nonsmoothness of the target, as well as optimality/nonoptimality of a point and failure of Petrov's condition (see (1.1)), we compute the generalized differential of $T$ at the three points

$$P_1 = \left(\frac{7}{16}, \frac{1}{8}\right), \quad P_2 = \left(\frac{5}{8}, \frac{1}{2}\right), \quad P_3 = \left(4-2\sqrt{3}, 2-\sqrt{3}\right).$$

Observe that $T(P_1) = \frac{1}{2}$, $T(P_2) = \frac{1}{2}$, and $T(P_3) = \sqrt{3}-1$.

To this aim we compute the adjoint flow

$$e^{A^T t} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

and the Hamiltonian

$$H\left((x_1,x_2),(\xi_1,\xi_2)\right) = x_2\xi_1 + |\xi_2|.$$

The point $P_1$ belongs to the curve $\gamma_1$ and is steered optimally in time $\frac{1}{2}$ to both $(\frac{5}{8}, \frac{5}{8})$ and $(\frac{3}{8}, -\frac{3}{8})$, where the normal cones to $\mathcal{S}^c$ are, respectively, $\mathbb{R}^+(-1,1)$ and

$\mathbb{R}^+(-1,-1)$, while $P_2$ belongs to the curve $\gamma_2$ and is steered optimally to $(1,1)$ in time $\frac{1}{2}$, where the normal cone to $\overline{\mathcal{S}^c}$ is $\mathbb{R}^+\text{co}\{(-1,1),\ (0,1)\}$. $P_2$ is an optimal point. Finally $P_3$ is steered optimally to both $(2\sqrt{3}-3, 3-2\sqrt{3})$ and $(1,1)$ in time $\sqrt{3}-1$. Observe that $H((1,1),(-1,1)) = 0$, i.e., Petrov's condition fails, while at all other (nonzero) points $P$ of the boundary of $\mathcal{S}$, we have $H(P,\zeta) > 0$ for all $\zeta \in N^P_{\overline{\mathcal{S}^c}}(P)$, $\zeta \neq 0$.

According to Theorem 3.2 and, of course, also to explicit computations from the expression of $T$, we have

$$\begin{aligned}
\partial^c T(P_1) &= \partial^P T(P_1)\\
&= -\text{co}\left\{ e^{A^T\frac{1}{2}}v \mid v = \begin{pmatrix} -\lambda \\ \lambda \end{pmatrix} \text{ or } v = \begin{pmatrix} -\lambda \\ -\lambda \end{pmatrix},\ H(P_1, e^{A^T\frac{1}{2}}) = 1 \right\}\\
&= -\text{co}\left\{ \begin{pmatrix} \frac{8}{3} \\ -\frac{4}{3} \end{pmatrix}, \begin{pmatrix} \frac{8}{11} \\ -\frac{12}{11} \end{pmatrix} \right\},
\end{aligned}$$

$$\partial^\infty T(P_1) = \{0\};$$

$$\begin{aligned}
\partial^c T(P_2) &= \partial^P T(P_2)\\
&= -\text{co}\left\{ e^{A^T\frac{1}{2}}v \mid v \in N^P_{\mathcal{S}^C}(1,1), H(P_2, e^{A^T\frac{1}{2}v}) = 1 \right\}\\
&\quad -\text{co}\left\{ e^{A^T\frac{1}{2}}v \mid v \in N^P_{\mathcal{S}^C}(1,1), H(P_2, e^{A^T\frac{1}{2}v}) = 0 \right\}\\
&= \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \left\{ \lambda \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} \mid \lambda \geq 0 \right\} = \left\{ \begin{pmatrix} \lambda \\ -1-\frac{\lambda}{2} \end{pmatrix} \mid \lambda \geq 0 \right\}\\
&= [-N(P_2)]\ \cap \left\{ \zeta \mid H(P_2, -\zeta) = 1 \right\}\\
&\quad \text{(where } N(P_2) \text{ was defined in (6.1)),}
\end{aligned}$$

$$\partial^\infty T(P_2) = \left\{ \lambda \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} \mid \lambda \geq 0 \right\};$$

$$\begin{aligned}
\partial^c T(P_3) &= \partial^P T(P_3)\\
&= -\text{co}\Big\{ e^{A^T(\sqrt{3}-1)v} \mid v \in N^P_{\overline{\mathcal{S}^c}}(1,1) \text{ or } v \in N^P_{\overline{\mathcal{S}^c}}(2\sqrt{3}-3, 3-2\sqrt{3}),\\
&\qquad\qquad \text{and } H(P_3, v) = 1 \Big\}\\
&= -\text{co}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{-1}{2(\sqrt{3}-1)} \\ \frac{-\sqrt{3}}{2(\sqrt{3}-1)} \end{pmatrix} \right\} - \left\{ \begin{pmatrix} \lambda \\ (2-\sqrt{3})\lambda \end{pmatrix} \mid \lambda \geq 0 \right\},
\end{aligned}$$

$$\partial^\infty T(P_3) = \left\{ \begin{pmatrix} \lambda \\ (2-\sqrt{3})\lambda \end{pmatrix} \mid \lambda \geq 0 \right\}.$$

Observe that the vector $\bar{f} \in \text{co}(f(P_2, \mathcal{U}))$ appearing in the statement of Lemma 6.2 is given by $\bar{f} = (1/2, -1)$.

If the target is modified to become $\mathcal{S}' := \mathcal{S} \setminus \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 1, x_1 \geq (x_2)^2/2 + 1/2 + (x_2-1)^4\}$ (note that the boundary of $\mathcal{S}'$ is $\mathcal{C}^2$ at $(1,1)$ (see Figure 2)), then the graph of the new minimum time function $T'$ is smooth at all points of $\gamma_2$, but the unique normal is horizontal so that $T'$ is not differentiable at those points. The next two examples deal with the case where the normal cone to the hypograph of $T$ is not pointed. We show first that Theorem 3.3 does not hold in general. Next we provide an example where—although the normal cone is not pointed—the situation is entirely analogous to the case where the cone is pointed.

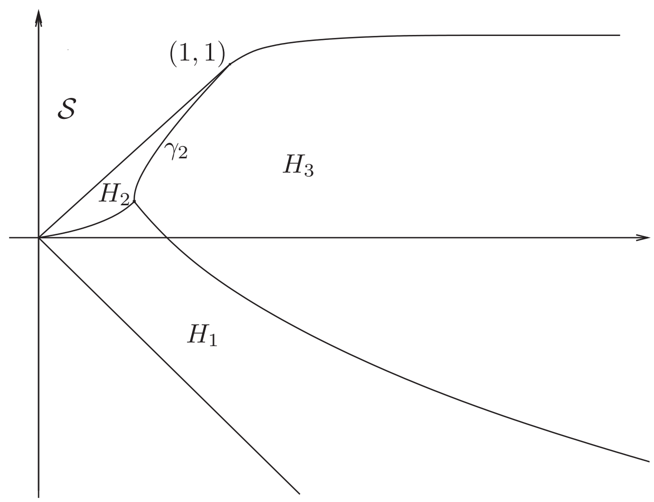


FIG. 2.

*Example* 2. Set

$$\gamma_1(y) = \begin{cases} (1 - \sqrt{-y^2 - 2y}, y), & -2 \le y \le -1, \\ (-1 + \sqrt{-y^2 - 2y}, y), & -1 \le t \le 0, \\ (-1 - \sqrt{-y^2 + 4y}, y), & 0 \le y \le 3, \end{cases}$$

and

$$\gamma_2(y) = \begin{cases} (1 + \sqrt{-y^2 - 2y}, y), & -2 \le y \le 0, \\ (1 - \sqrt{-y^2 + 2y}, y), & 0 \le y \le 1, \\ (0, y), & 1 \le y \le 2, \\ (-1 + \sqrt{-y^2 + 4y}, y), & 2 \le y \le 3. \end{cases}$$

Observe now that the concatenation of $\gamma_1$ with $\gamma_2$ defines a $\mathcal{C}^{1,1}$-curve $\gamma$. We set the target $\mathcal{S}$ to be the unbounded component of $\mathbb{R}^2 \setminus \{\gamma\}$ (see Figure 3) and the dynamics to be

$$\begin{cases} \dot{x}(t) = u, \\ \dot{y}(t) = 0, \\ \quad u \in \mathcal{U} = [-1, 1]. \end{cases}$$

It is readily verified that the minimum time function is everywhere defined and continuous. Observe furthermore that Petrov's condition (1.1) holds at no points of the segment $[-1, 1] \times \{0\}$.

Consider now the curve

$$\Gamma(t) = \frac{\gamma_1(t) + \gamma_2(t)}{2}, \qquad t \in [0, 1]$$

and define $T(t)$ to be the first coordinate of $\gamma_2(t) - \Gamma(t)$, $t \in [0, 1]$. Observe that $T$ is the minimum time to reach $\mathcal{S}$ from the point $\Gamma(t)$.

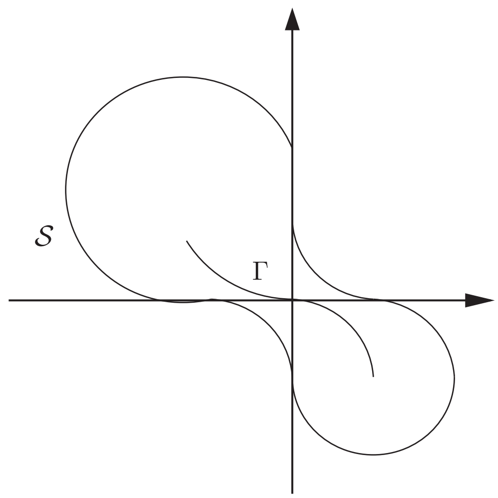


Fig. 3.

Observe that $T(t)$ is constantly equal to 1 for $-1 \le t \le 0$, and in this interval all points of $\Gamma$ are maximum points for $T$. Therefore, $(0, 0, 1)$ is a unit limiting normal vector to the hypograph of $T$ at $(0, 0, 1)$.

On the other hand, it can be easily computed that a unit tangent vector to the graph of $T$ at $(0, 0, 1)$ is

$$\left(-\frac{2+\sqrt{2}}{2\sqrt{3}}, 0, \frac{2-\sqrt{2}}{2\sqrt{3}}\right).$$

Since the latter has a positive scalar product with the limiting normal $(0, 0, 1)$, it is clear that the hypograph of $T$ is not regular at $(0, 0, 1)$. In particular, the normal vector $(0, 0, 1)$ is not proximal, thus showing that $hypo(T)$ is not $\varphi$-convex (see (4) in Theorem 2.1).

Observe that both $(1, 0, 0)$ and $(-1, 0, 0)$ are unit proximal normals to hypo($T$) at $(0, 0, 1)$ so that $N^C_{hypo(T)}(0, 0, 1)$ contains a line. Therefore, the assumption (3.1) used in Theorem 3.1 cannot be dropped in general.

Observe finally that the hypograph of $T$ satisfies the external sphere condition with radius $\rho$ for a suitable $\rho > 0$. Therefore, this is a simple example showing that this condition is weaker than $\varphi$-convexity.

*Example* 3. We consider again the dynamics (7.1) appearing in Example 1 and modify the target in order to allow lines in the normal cone to the hypograph of $T$.

The target is the set (see Figure 4)

$$\begin{aligned}\mathcal{S} \ &= \{\ (x_1, x_2) \in \mathbb{R}^2 \mid x_1 \le 0\ \} \ \cup\ \{\ (x_1, x_2) \in \mathbb{R}^2 \mid 0 \le x_1 \le 1,\ x_2 \le\ x_1 - 1\} \\ &\cup\ \ \{\ (x_1, x_2) \in \mathbb{R}^2 \mid x_1 \ge 1\} \ \cup\ \{\ (x_1, x_2) \in \mathbb{R}^2 \mid 0 \le x_1 \le 1,\ x_2 \ge x_1\}.\end{aligned}$$

The minimum time function is everywhere finite and continuous, but Petrov's condition (1.1) does not hold. Computations of the same type of Example 1 show that the normal cone to the hypograph of $T$ at $(1/2, 0, 1)$ is not pointed; however, $N^C_{hypo(T)}(1/2, 0, 1)$ can be represented exactly as in (3.3), and the hypograph of $T$ is

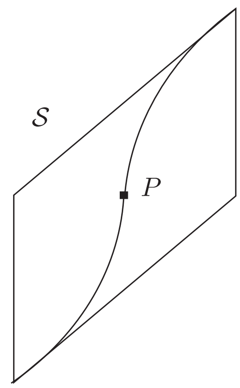


FIG. 4.

$\varphi$-convex. More precisely,

$$N^P_{hypo(T)}(1/2,0,1) = N^C_{hypo(T)}(1/2,0,1) = \mathbb{R}\begin{pmatrix}1\\0\\0\end{pmatrix} + \mathbb{R}^+\mathrm{co}\left\{\begin{pmatrix}\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{pmatrix}, \begin{pmatrix}-\frac{1}{\sqrt{3}}\\-\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{pmatrix}\right\}$$

and

$$\partial^P T(1/2,0) = \partial^C T(1/2,0) = \mathbb{R}\begin{pmatrix}1\\0\end{pmatrix} + \mathrm{co}\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\end{pmatrix}\right\}.$$

Observe that $H((1/2,0),(1,0)) = 0$, while $H((1/2,0),(1,1)) = H((1/2,0),(-1,-1)) = 1$ so that the conclusion of Theorem 3.2 holds. An explicit computation of the minimum time function shows also that the conclusion of Theorem 3.3 holds as well.

**8. Appendix.** In this section, under the assumptions $(H_1)$ and $(H_2)$ on (2.7), we prove first some elementary estimates which are needed in Lemmas 4.1, 4.2, and 4.3. At the end we state a result on the closedness of the convex hull of a closed pointed cone.

For future use, we set

$$K_1 = \max_{u\in\mathcal{U}} \|f(0,u)\|,$$
$$K_2 = \max_{u\in\mathcal{U}} \|D_x f(0,u)\|,$$
$$L_2(s,\delta) = L_1 e^{Ls}\delta + \frac{L_1(e^{Ls}-1)K_1}{L} + K_2 \quad \text{for all} \quad s,\delta \geq 0. \tag{8.1}$$

LEMMA 8.1. *Let* $\alpha(\cdot) := y^{x,u}(\cdot)$ *be the solution of* (2.7)*. The following estimates hold true for all* $t > 0$*:*

(i) $\|\alpha(t) - x\| \leq \frac{(L\|x\|+K_1)(e^{Lt}-1)}{L}$.

(ii) $\|\alpha(t)\| \leq e^{Lt}\|x\| + \frac{(e^{Lt}-1)K_1}{L}$.

(iii) $\|f(\alpha(t),u(t))\| \leq Le^{Lt}\|x\| + e^{Lt}K_1$.

(iv) $\|D_x f(\alpha(t),u(t))\| \leq L_2(t,\|x\|)$.

*Proof.* Since $\alpha(\cdot)$ is the solution of (2.7), for all $t > 0$ we have

$$\begin{aligned}\|\alpha(t) - x\| &= \left\|\int_0^t f(\alpha(s), u(s))ds\right\| \leq \int_0^t \|f(\alpha(s), u(s))\|\, ds \\ &\leq \int_0^t \|f(\alpha(s), u(s)) - f(x, u(s))\|\, ds \\ &\quad + \int_0^t \|f(x, u(s)) - f(0, u(s))\|\, ds + \int_0^t \|f(0, u(s))\|\, ds \\ &\leq L \int_0^t \|\alpha(s) - x\|\, ds + L\, \|x\|\, t + K_1 t.\end{aligned}$$

Applying Gronwall's inequality we obtain

$$\|\alpha(t) - x\| \leq \frac{(L\, \|x\| + K_1)(e^{Lt} - 1)}{L}; \tag{8.2}$$

whence

$$\|\alpha(t)\| \leq e^{Lt}\, \|x\| + \frac{(e^{Lt} - 1)K_1}{L}. \tag{8.3}$$

Recalling the condition $(H_2)$, we obtain

$$\|f(\alpha(t), u(t))\| \leq Le^{Lt}\, \|x\| + e^{Lt} K_1 \tag{8.4}$$

and also

$$\|D_x f(\alpha(t), u(t))\| \leq L_1 e^{Lt}\, \|x\| + \frac{L_1(e^{Lt} - 1)K_1}{L} + K_2. \tag{8.5}$$

The proof is concluded. ∎

In the next lemma, we will give some estimates related to the limiting adjoint trajectories $M^T(\cdot)$.

LEMMA 8.2. *Let $x \in \mathcal{S}^c$, set $r = T(x) > 0$, and take $\bar{x} \in \mathcal{S}_x$ and $M(\cdot) \in \mathcal{M}_{\bar{x}}$. Then*

(i) $\|M(t)\| \leq e^{L_2(t, \|x\|)t}$ *for all $t \in [0, r]$,*

(ii) $\left\|M(t)^{-1}\right\| \leq e^{L_2(t, \|x\|)t}$ *for all $t \in [0, r]$.*

*Proof.* Let $x_n \to x$, $\{\bar{u}_n\} \subset \mathcal{U}_{\text{ad}}$ be such that $\{y^{x_n, \bar{u}_n}(\cdot)\} \subset \mathcal{T}_{\bar{x}}$ and $M(\cdot, x_n, \bar{u}_n)$ converges to $M(\cdot)$ uniformly on $[0, T(x)]$. By (iv) in Lemma 8.1 and Theorem 2.2.1, p. 23, in [3], we obtain that for all $w \in \mathbb{R}^N$,

$$\|M(t, x_n, \bar{u}_n) w\| \leq e^{[L_1 e^{Lt} \|x\| + \frac{L_1(e^{Lt} - 1)K_1}{L} + K_2]t}\ \|w\|.$$

Taking $n \to \infty$, we conclude the proof of (i).

The proof of (ii) proceeds exactly as the proof of (i) by replacing $M(\cdot, x_n, \bar{u}_n)$ with $M(\cdot, x_n, \bar{u}_n)^{-1}$. ∎

The following result is essentially Theorem 2.2.4, pp. 25 and 26 in [3].

LEMMA 8.3. *Let $A_1, A_2 : [0, T] \to \mathcal{M}^{N \times N}$ be matrices with $L^\infty$-entries, and set $\|A_i\| = L_i$, $i = 1, 2$. Let $M_1, M_2$ be the fundamental solution of, respectively,*

$$\begin{aligned}\dot{p}(t) &= A_1(t)p(t), \quad p(0) = \mathbb{I}^{N \times N}, \\ \dot{p}(t) &= A_2(t)p(t), \quad p(0) = \mathbb{I}^{N \times N}.\end{aligned}$$

*Then for every* $t \in [0,T]$ *and every unit vector* $v \in \mathbb{R}^N$, *we have*

$$\|(M_2(t) - M_1(t))v\| \le e^{(L_1+L_2)t} \int_0^t \|A_2(s) - A_1(s)\|\, ds.$$

The last result is concerned with pointed cones in general.

LEMMA 8.4. *Let* $K \subset \mathbb{R}^N$ *be compact, and assume that* $0 \notin K$. *Set*

$$C := \{\lambda x \mid \lambda \ge 0, x \in K\},$$

*and assume that* $\operatorname{co} C$ *is a pointed cone. Then* $\operatorname{co} C$ *is closed.*

*Proof.* Let sequences $\{\alpha_k^n \in \mathbb{R} \mid k = 0,\dots,N, n \in \mathbb{N}\}$, $\{v_k^n \in \mathbb{R}^N \mid k = 0,\dots,N, n \in \mathbb{N}\}$ be such that $\alpha_k^n \ge 0$ and $v_k^n \in K$ for all $k = 0,\dots,N$, $n \in \mathbb{N}$. Assuming that

$$\lim_{n\to\infty} \sum_{k=0}^{N} \alpha_k^n v_k^n = v, \tag{8.6}$$

we wish to show that $v \in \operatorname{co} C$, i.e., there exist $\alpha_k \ge 0$, $v_k \in K$, $k = 0,\dots,N$ such that $v = \sum_{k=0}^N \alpha_k v_k$. Since $K$ is compact, there is no loss of generality in assuming that $v_k^n \to v_k \in K$ for all $k = 0,\dots,N$. We claim that the sequences $\alpha_k^n$ are bounded. Indeed, assume by contradiction that $\alpha_n := \sum_{k=0}^{\infty} \alpha_k^n \to +\infty$, and set, for $k = 0,\dots,N$, $\beta_k^n = \alpha_k^n/\alpha_n$. By (8.6) we obtain that

$$\sum_{k=0}^{N} \beta_k v_k = 0,$$

where $\beta_k \ge 0$ and $\sum_{k=1}^N \beta_k = 1$. Since $v_k \ne 0$ for all $k = 0,\dots,N$, we deduce from the above equality that $\operatorname{co} C$ is not pointed, a contradiction. Therefore, without loss of generality we can assume that $\alpha_k^n \to \alpha_k$ for all $k = 0,\dots,N$, and so the proof is concluded. ∎

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