P H Y S I C A L   R E V I E W   L E T T E R S

Compactified Strings as Quantum Statistical Partition Function on the Jacobian Torus

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We show that the solitonic contribution of toroidally compactified strings corresponds to the quantum statistical partition function of a free particle living on higher dimensional spaces. In the simplest case of compactification on a circle, the Hamiltonian is the Laplacian on the $2g$-dimensional Jacobian torus associated with the genus g Riemann surface corresponding to the string world sheet. T duality leads to a symmetry of the partition function mixing time and temperature. Such a classical-quantum correspondence and T duality shed some light on the well-known interplay between time and temperature in quantum field theory and classical statistical mechanics.

A puzzling feature of quantum field theory (QFT) concerns the crucial analogy with classical statistical mechanics where the inverse temperature plays the role of imaginary time. Time and temperature also mix by performing analytic continuation along complex paths in the path integral. Another well-known analogy concerns the transition amplitude for a particle for the time it that coincides with the classical partition function for a string of length t at $\beta = 1/\hbar$. Even if such analogies follow as technical properties, it is widely believed that they are deeply related to the properties of space-time and should emerge in the string context.

A step toward a better understanding of temperature-time and classical-quantum dualities, would be finding a physical model where these emerge naturally. In proposing such a model, we will start by showing that string theory possesses a basic surprising classical-quantum duality. It turns out that the solitonic sector of toroidally compactified strings has a *dual* description as quantum statistical partition function on higher dimensional spaces, built in terms of the Jacobian torus of the string world sheet and of the compactified space. More precisely, in this Letter we will see that in the case of compactification on a circle

$$\sum_{m,n\in\mathbb{Z}^g} e^{-\beta S_{m,n}} = \operatorname{Tr} e^{-\beta H}, \tag{1}$$

where $\beta = 2R^2/\alpha'$, with R the compactification radius, and the Hamiltonian H is $\Delta_{J_\Omega}/2\pi$, with Δ_{J_Ω} the Laplacian on the Jacobian torus J_Ω of the world sheet. Equation (1) is just the direct consequence of the stronger identity we will prove. Namely,

$$H\Psi_{m,n} = S_{m,n}\Psi_{m,n}, \tag{2}$$

that is the set $\{S_{m,n}|m, n \in \mathbb{Z}^g\}$ coincides with the spectrum of H. As will become clear from the construction, such a result seems to capture a general property of string theories which is due to the underlying geometrical properties of Riemann surfaces. In particular, the correspondence considered in this Letter admits natural generalizations to compactifications on higher dimensional tori.

Remarkably, temperature-time duality naturally emerges as a consequence of the complexified version of T duality, a fundamental feature of string theory [1]. By (1) the standard T duality corresponds to the invariance, up to a multiplicative term given by powers of β, of the partition under inversion of the temperature

$$\beta \to 1/\beta. \tag{3}$$

Complexification of β has basic motivations which interplay between physics and geometry. Let us first consider the time translate of an observable A, $\alpha_t(A) = e^{iHt}Ae^{-iHt}$, and the expectation value

$$\omega_\beta(A) = \operatorname{Tr}\rho_\beta A = Z_{\text{stat}}(\beta)^{-1}\operatorname{Tr}Ae^{-\beta H}.$$

By invariance of the trace under cyclic permutations we get $\omega_\beta(\alpha_t(A)B) = \omega_\beta(Be^{iH(t+i\beta)}Ae^{-iH(t+i\beta)})$, so that

$$\omega_\beta((\alpha_t A)B) = \omega_\beta(B\alpha_{t+i\beta}(A)),$$

that is time evolution is invariant, upon commutation, under an imaginary shift of the time which is inversely proportional to the temperature. In such a context the complexification of β naturally appears in globally conformal invariant QFT [2]. It is worth noticing that both in [2] and in the BC system [3] the Kubo-Martin-Schwinger (KMS) states [4–6] play a crucial role. In particular, in the limit of 0 temperature the KMS states may be used to define the concept of point in noncommutative space.

A simple but important property of the temperature is its positive definiteness. This becomes transparent once time and temperature are naturally combined in a unique variable τ as suggested by the above analogies, namely

$$\beta = \frac{1}{k_B T} = \operatorname{Im}\tau, \qquad \frac{t}{\hbar} = -\operatorname{Re}\tau. \tag{4}$$

As we will discuss later, complexification of β is intimately related to the positivity of the temperature.

Complexifying a variable taking positive values has a geometrical motivation that may lead to a geometrical understanding of the emergence of time. In particular, as will be illustrated, positivity of the temperature combined with the time variable to build the complex τ, leads us to consider τ as the torus modular parameter.

An apparently unrelated topic concerns the Riemann mapping theorem, which plays a central role in uniformization theory and in the theory of univalent functions. According to such a theorem there is a unique analytic function $w = f(z)$ mapping a simply connected region of $\mathbb{C}$ one-to-one onto the disk $|w| < 1$ such that $f(z_0) = 0$ and $f'(z_0) > 0$. As a consequence, any two simply connected regions except $\mathbb{C}$ itself can be mapped conformally onto each other. While $\mathbb{C}$ is the universal covering of the torus, any simply connected domain of $\mathbb{C}$ can be mapped by a locally univalent function to the upper half-plane, the universal covering of negatively curved Riemann surfaces. Positivity of the temperature and its combination with time to build τ leads to the geometry of the upper half-plane that, unlike the case of the Riemann sphere, is characterized by the identification of its automorphism group, $\mathrm{PSL}(2, \mathbb{R})$, with the isometry group of its natural metric, the Poincaré metric. Such a group acts by linear fractional transformations $\tau \rightarrow (A\tau + B)/(C\tau + D)$ that, according to (4), correspond to mixing the role of temperature and time.

Similar transformations appear in several physical models, for example, in Seiberg-Witten theory. Again, positivity, that this time is due to the coupling constant, plays a crucial role.

Geometrically τ can be seen as the modular parameter of a torus, and then related to a flat geometry. On the other hand, it can be seen as the inverse of the uniformizing map from a negatively curved Riemann surface to the upper half-plane. In this way to each point on a negatively curved Riemann surface one may associate an elliptic curve. It should be stressed that negatively curved Riemann surfaces are much more reach in nature and it may be convenient to consider punctured versions of the torus rather than the torus itself. Such a dual role for τ, connecting flat and negatively curved Riemann surfaces, is particularly transparent once one considers the ζ function

$$\zeta(s) = \sum_{m,n \in \mathbb{Z}} \frac{(\mathrm{Im}\tau)^s}{|m + n\tau|^{2s}}.$$

The elements of the set $\{|m + n\tau|^2/\mathrm{Im}\tau \mid m, n \in \mathbb{Z}\}$ are the eigenvalues of the Laplacian on the torus, so that $e^{-\zeta(0)'}$ defines the determinant of the Laplacian on the torus. On the other hand, $(\mathrm{Im}\tau)^s$ can be seen as an eigenfunction of the Poincaré Laplacian. The summation then can be seen as a sum on all $\mathrm{PSL}(2, \mathbb{Z})$ transformations of $(\mathrm{Im}\tau)^s$ guaranteeing the invariance under the uniformizing group $\mathrm{SL}(2, \mathbb{Z})$. This dual nature of τ is at the heart of basic mathematical structures also involving number theory,

and appears in several topics of physical interest, not only in string theory.

Let Σ be a genus g Riemann surface with a fixed basis $\{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g\}$ of the first homology group $H_1(\Sigma, \mathbb{Z})$ with intersection matrix $\alpha_i \cdot \beta_j = \delta_{ij}, \alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j = 0$. A dual basis $\{\omega_1, \ldots, \omega_g\}$, of holomorphic 1-differentials on Σ can be chosen with normalization $\int_{\alpha_i} \omega_j = \delta_{ij}, i, j = 1, \ldots, g$, whereas the Riemann period matrix $\Omega_{ij} = \int_{\beta_i} \omega_j$ can be proved to be symmetric with $\mathrm{Im}\Omega > 0$, and depends on the complex structure of Σ. Conversely, the Riemann period matrix completely determines the complex structure of the corresponding Riemann surface (although for $g \geq 4$ not every symmetric matrix with positive-definite imaginary part is the period matrix of a Riemann surface).

Let us consider a scalar field theory on Σ with the one-dimensional target space compactified to a circle $\mathbb{S}^1 = \mathbb{R}/2\pi R\mathbb{Z}$ of radius R. The partition function is defined as a path-integral

$$Z(\beta) = \int_{(\Sigma, \mathbb{S}^1)} dX e^{-\beta S},$$

where $(\Sigma, \mathbb{S}^1)$ is the space of maps from Σ to $\mathbb{S}^1$ and

$$S[X] = \frac{1}{4\pi R^2} \int_\Sigma \partial X \bar\partial X, \tag{5}$$

with the normalization chosen for later reference. By setting $\beta = 2R^2/\alpha'$, this path-integral corresponds to the g-loop contribution of a string theory with target space $\mathbb{S}^1$. Each function $X: \Sigma \rightarrow \mathbb{S}^1$ satisfies the condition

$$X(z + p^t\alpha + q^t\beta) = X(z) + 2\pi R(m^t p - n^t q),$$

$p, q \in \mathbb{Z}^g$, where the winding numbers $m, n \in \mathbb{Z}^g$ label the different solitonic sectors of $(\Sigma, \mathbb{S}^1)$. Let us split X into classical and quantum parts $X = X^{\mathrm{cl}} + X^q$, where X^{cl} satisfies the classical equation of motion

$$\Delta X^{\mathrm{cl}} = 0,$$

so that X^{cl} is a harmonic function on Σ. It is worth noticing that there exists a unique harmonic function for each solitonic sector, i.e., for each pair $(m, n) \in \mathbb{Z}^{2g}$. Therefore, the splitting can be performed in such a way that the quantum contribution X_q is a real single-valued function on Σ, that is $X_q(z + p^t\alpha + q^t\beta) = X_q(z)$. It follows that the path integral splits into a sum over the inequivalent sectors labeled by solitonic numbers $(m, n) \in \mathbb{Z}^{2g}$ times the functional integral over the space of real single-valued functions on Σ

$$Z(\beta) = \sum_{m,n \in \mathbb{Z}^g} e^{-\beta S_{m,n}} \int_{(\Sigma, \mathbb{R})} dX e^{-\beta S},$$

whereas the mixed terms with classical times quantum part vanish. The harmonic function in the sector $(m, n) \in \mathbb{Z}^{2g}$ is

$$X_{m,n}^{\mathrm{cl}}(z,\bar{z}) = \frac{\pi R}{i}(m+\bar{\Omega}n)^t(\mathrm{Im}\Omega)^{-1}\int^z \omega + \mathrm{c.c.},$$

so that, by using the Riemann bilinear relations

$$\int_\Sigma \omega_i \wedge \bar{\omega}_j = -2i\mathrm{Im}\Omega_{ij},$$

we obtain

$$S_{m,n} = S[X_{m,n}^{\mathrm{cl}}] = \pi(m+\bar{\Omega}n)^t(\mathrm{Im}\Omega)^{-1}(m+\Omega n).$$

Therefore, the partition function is $Z = Z_{\mathrm{sol}}Z_q$, where

$$Z_{\mathrm{sol}}(\beta) = \sum_{m,n\in\mathbb{Z}^g} e^{-\beta\pi(m+\bar{\Omega}n)^t(\mathrm{Im}\Omega)^{-1}(m+\Omega n)},$$

$$Z_q(\beta) = \left(\frac{A_\Sigma}{\det'(-\beta\Delta)}\right)^{1/2}, \qquad (6)$$

and $A_\Sigma = \int_\Sigma \sqrt{g}$.

On the other hand, let us consider the quantum statistical partition function at the temperature T on a g-dimensional complex torus $J_\Omega = \mathbb{C}^g/(\mathbb{Z}^g+\Omega\mathbb{Z}^g)$, for some symmetric $g\times g$ matrix Ω, with $\mathrm{Im}\Omega > 0$,

$$Z_{\mathrm{stat}}(\beta) = \mathrm{Tr}\,e^{-\beta H},$$

where $\beta = 1/k_B T$, $H \equiv \Delta_{J_\Omega}/2\pi$ and

$$\Delta_{J_\Omega} = -2\mathrm{Im}\Omega_{ij}\frac{\partial}{\partial z_i}\frac{\partial}{\partial \bar{z}_j},$$

is the Laplacian on J_Ω with respect to the natural metric

$ds^2 = (2\mathrm{Im}\Omega)_{ij}^{-1}dz^i d\bar{z}^j$. Here and in the following, β and H are rescaled by some fixed length L and thought of as dimensionless quantities. A complete orthogonal basis of eigenfunctions for H is $\{\Psi_{m,n}\}_{m,n\in\mathbb{Z}^g}$, with

$$\Psi_{m,n}(z,\bar{z}) = e^{\pi(m+\bar{\Omega}n)^t(\mathrm{Im}\Omega)^{-1}z-\mathrm{c.c.}}.$$

Indeed, a trivial computation shows that

$$H\Psi_{m,n} = \lambda_{m,n}\Psi_{m,n},$$

with eigenvalues

$$\lambda_{m,n} = \pi(m+\bar{\Omega}n)^t(\mathrm{Im}\Omega)^{-1}(m+\Omega n),$$

so that $S_{m,n} = \lambda_{m,n}$ and

$$Z_{\mathrm{stat}}(\beta) = \sum_{m,n\in\mathbb{Z}^g} e^{-\beta\pi(m+\bar{\Omega}n)^t(\mathrm{Im}\Omega)^{-1}(m+\Omega n)}. \qquad (7)$$

Comparing Eqs. (6) and (7), we obtain the remarkable identity Eq. (1) between the classical contribution to the partition function in a 2-dimensional field theory and the quantum statistical partition function of a free particle in a $2g$-dimensional space. Note that the space in the statistical theory is exactly the Jacobian torus of the Riemann surface where the first theory is defined. In particular, in the case of genus 1, Σ coincides with its Jacobian and we obtain a duality on the same space.

By applying the Poisson summation formula

$$\sum_{m\in\mathbb{Z}^d} e^{-\pi(m+a)^t A(m+a)+2\pi im^t b} = (\det A)^{-1/2}\sum_{m\in\mathbb{Z}^d} e^{-\pi(m+b)^t A^{-1}(m+b)-2\pi i(m+b)^t a},$$

to the sum over m in $Z_{\mathrm{sol}}(\beta)$, one obtains

$$Z_{\mathrm{sol}}(\beta) = \left(\frac{\det\mathrm{Im}\Omega}{\beta^g}\right)^{1/2}$$
$$\times \sum_{m,n\in\mathbb{Z}^g} e^{\pi[-\beta^{-1}m^t(\mathrm{Im}\Omega)m-\beta n^t(\mathrm{Im}\Omega)n+2im^t(\mathrm{Re}\Omega)n]}$$
$$= \beta^{-g}Z_{\mathrm{sol}}(1/\beta),$$

where the correspondence $\beta \to 1/\beta$ implies $R \to R' = \alpha'/2R$, which is the standard T duality for string theory on a circle. In the case of $Z_{\mathrm{stat}}(\beta)$, the same calculation leads to

$$Z_{\mathrm{stat}}(\beta) = \beta^{-g}Z_{\mathrm{stat}}(1/\beta),$$

which has to be interpreted as a hot-cold duality $T \to T' = T_{\mathrm{sd}}^2/T$, where T_{sd} is the self-dual temperature $T_{\mathrm{sd}} = \hbar c/k_B L$. Note that, by setting $L \sim (\alpha'/2)^{1/2}$, a correspondence can be established between the energy scales in the string theory model and in the quantum statistical one, relating the fixed points (self-dual radius and temperature, respectively) under the duality $\beta \to 1/\beta$.

From the point of view of statistical mechanics, it is natural to consider the complexification

$$\tau = -t + i\beta, \qquad (8)$$

where t denotes the time, so that we can define

$$Z_{\mathrm{stat}}(\tau) = \mathrm{Tr}\,e^{i\tau H}.$$

By Poisson summation, we obtain the duality

$$Z_{\mathrm{stat}}(\tau) = (-i\tau)^{-g}Z_{\mathrm{stat}}(-1/\tau),$$

that mixes time and temperature, namely

$$t \to \frac{-L^2}{c^2[t^2+(\hbar/k_B T)^2]}t, \qquad \frac{1}{T} \to \frac{L^2}{c^2[t^2+(\hbar/k_B T)^2]}\frac{1}{T},$$

where the dimensional parameters have been recovered.

We have seen that the classical sector of the compactified string on $\mathbb{S}^1$ has a quantum statistical description on the Jacobian torus. This corresponds, like the original string, to a first quantized theory. Also note that restricting the eigenfunctions of H to the image of Σ in J_Ω provides a direct link between $\Psi_{m,n}$ and $X_{m,n}^{\mathrm{cl}}$, namely

$$\Psi_{m,n}\left(\int^z \omega, \overline{\int^z \omega}\right) = e^{(i/R)X_{m,n}^{\mathrm{cl}}}, \qquad (9)$$

resembling a classical vertex. This may indicate that the

dual description of the classical contribution is the facet of a more general dual description of the full string. For example, the fact that the Laplacian acting on sections of the torus and the one acting on sections of its Jacobian coincide, implies a functional relation between the classical and quantum sectors of the string (coming from the heat equation). This suggests that even the quantum sector of the string admits a dual description which may extend to a higher genus as well. Understanding such an extension means to investigate the possible relations, for any g, between the spectrum of Δ_{J_Ω} and the one of Δ, a problem that, as we will comment below, is of considerable mathematical interest. For the time being we note that, due to the appearance of $(Im\Omega)^{-1}_{ij}$, a distinguished role may be played by the Bergman metric

$$ds^2 = \frac{1}{g} \sum_{i,j=1}^{g} \omega_i(z)(Im\Omega)^{-1}_{ij}\overline{\omega_j(z)}.$$

We note that the equality (1) between the sum over the topologically nontrivial states of a two-dimensional sigma model and the trace of operators over the Hilbert space of a free particle on the Jacobian torus, is reminiscent of electric-magnetic duality in $\mathcal{N} = 2$ SYM theory. In the latter, the solitonic objects are the 't Hooft–Polyakov monopoles and Julia-Zee dyons, and the elementary objects are gluons. In our model Eq. (8) can be thought of as the complex coupling constant. Its analog in SYM theory is $\tau = \frac{\Theta}{2\pi} + \frac{4\pi i}{g^2}$. However, while electric-magnetic duality is between weakly coupled and strongly coupled regimes of two different theories (or of the same theory, if the theory is self-dual), in our model the duality is established between the same regimes of different theories. The way out is to note that by the composition of Eq. (1) and T duality (3) precisely reproduces what we expect for an analog of the electric-magnetic duality.

Let us further comment the complex combination of time and temperature. For the quantum theory to be unitary the action must be real. The imaginary part of the action leads to nonconservation of probability, as it is clear from the form of the Feynman weight $e^{(i/\hbar)S}$. The probability grows with the entropy, so the imaginary part of the action should be negative. Moreover, its presence leads to decreasing of the information, and therefore to increasing of the entropy. A possible interpretation is to consider the imaginary part of the action as the entropy: $S = \mathcal{A} - i\mathcal{E}$. If we write the action $\mathcal{A}$ in the momentum space as a function of momentum and energy E (which can be achieved after taking its Legendre transform) and use $\partial\mathcal{A}/\partial E = t$, we obtain

$$\partial S/\partial E = t - i\beta = -\tau.$$

This suggests that the combination of time and temperature at hands is the variable conjugated to energy in the case when the action is complex.

Let us conclude by observing that an intriguing outcome is that in the case of $g = 1$, the Jacobian corresponds to the torus itself. In this way the quantum and solitonic contributions both are expressed in terms of the same Laplacian. An old problem in the theory of Riemann surfaces is to find an analytic expression for the determinant of Laplacian acting on degree zero bundles in which the dependence on the period Riemann matrix appears explicitly. In [7] it was conjectured a relation between the determinant of the Laplacian on the Jacobian and the one on the Riemann surface. This would provide a functional relation between $Z_{stat}(\beta)$ and the quantum contribution to the string partition function also in the higher genus case, generalizing the relation for $g = 1$. The eigenvalues $\lambda_{m,n}$ also appear in considering the Laplacian with respect to degenerate metrics [7], for which ramified covering of the torus play a crucial role [8]. Ramified covering of the torus correspond to a particular kind of CM (complex multiplication) satisfied by the Riemann period matrix [8]. Remarkably, such special Riemann surfaces also appear in the null compactification of type-IIA string perturbation theory at finite temperature [9]. It is worth noticing that CM, which is a lattice condition, also appears in the study of sigma models on Calabi-Yau manifolds [10].

We note that the dimensionality of our model suggests an intriguing relation with the string theory compactified on a Riemann surface of unitary volume in string units, where the effective degrees of freedom are still the ones of a $2g$-dimensional theory [11]. Presumably this is connected to the relation between Fuchsian groups and Liouville theory and to the fact that, in particular regimes, the commutators between the Fuchsian generators may be negligible so leading to a homological description.

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