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Homogenization and Vanishing Viscosity in Fully Nonlinear Elliptic Equations: Rate of Convergence Estimates[1]

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Abstract

This paper is devoted to studying the behavior as $\varepsilon \to 0$ of the equations

$$u_\varepsilon + H(x, x/\varepsilon, Du_\varepsilon, \varepsilon^\gamma D^2 u_\varepsilon) = 0$$

with $\gamma > 0$. It is known that, under some periodicity and ellipticity or coercivity assumptions, the solution u_ε converges to the solution u of an effective equation $u + \overline{H}(x, Du) = 0$, with an effective Hamiltonian $\overline{H}$ dependent on the value of γ. The main purpose of this paper is to estimate the rate of convergence of u_ε to u. Moreover we discuss some examples and model problems.

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1 Introduction

This paper is devoted to studying the behavior as $\varepsilon \to 0$ of the fully nonlinear elliptic equations of Hamilton-Jacobi-Bellman type

$$u_\varepsilon + H\left(x, \frac{x}{\varepsilon}, Du_\varepsilon, \varepsilon^\gamma D^2 u_\varepsilon\right) = 0 \qquad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $\gamma > 0$ is a fixed parameter, under the basic structural assumption that the Hamiltonian H is periodic in the second variable (see assumption (H1) in Section 2). It is clear that the limit is affected by two different effects: the *homogenization* and the *vanishing viscosity* which are due respectively to the oscillation term x/ε and to the coefficient ε^γ of the Hessian matrix.

The simultaneous effect of vanishing viscosity and homogenization has been studied in [8] for linear problem, in [16] by means of probabilistic methods and [12] for parabolic scalar conservation laws. For periodic fully nonlinear equations, [13] considered the case $\gamma = 1$ for uniformly elliptic Hamiltonians; afterwards, [15] analyzed the different limit behaviour according to the parameter γ. Finally in [19] the almost-periodic setting is considered under different assumptions (in particular H is just degenerate elliptic, but satisfies an appropriate coercivity condition).

The solution u_ε to (1.1) converges to the solution of a first order Hamilton-Jacobi equation

$$u + \overline{H}(x, Du) = 0 \tag{1.2}$$

where the effective Hamiltonian $\overline{H}$ is identified by means of an appropriate additive eigenvalue problem, also called *cell problem*, which is different according to the value of γ. For $\gamma > 1$, the vanishing viscosity term is too 'fast' and it does not give any contribution to the cell problem which amounts to a first order equation: for x and p fixed, find the unique value $\overline{H}(x, p)$ such that there exists a periodic solution to

$$H(x, y, p + D\chi(y), 0) = \overline{H}(x, p).$$

For $\gamma < 1$, the vanishing viscosity term dominates and the cell problem involves only the Hessian matrix of the corrector χ:

$$H(x, y, p, D^2\chi(y)) = \overline{H}(x, p).$$

This equation is a particular case of cell problems arising in the homogenization of second order equations, i.e. $\gamma = 0$ in (1.1) (see [13], [9]). Finally in the critical case $\gamma = 1$, an intermediate situation appears with an interaction between the vanishing viscosity effect and homogenization one giving the presence of a first and a second order term in the cell problem:

$$H(x, y, p + D\chi(y), D^2\chi(y)) = \overline{H}(x, p).$$

The previous equation can be interpreted as a particular case of cell problems for singular perturbation of second order equations (see [3]-[4]). We recall that a very heuristic motivation for these constructions of $\overline{H}$ is obtained by plugging the usual formal asymptotic expansion of u_ε

$$u_\varepsilon(x) = u(x) + \varepsilon^{\max(1, 2-\gamma)}\chi(x/\varepsilon) + \ldots \tag{1.3}$$

in the initial equation (1.1) and identifying the terms in front of powers of ε.

The aim of this paper is to estimate the rate of convergence of u_ε solutions to (1.1) to u solution to (1.2), namely $\|u_\varepsilon - u\|_\infty$. Let us emphasize that the class of problems we consider encompasses, among others, many of the ones studied by Horie and Ishii [15] and by Lions and Souganidis [19]; hence, for these cases, this paper provides the rate of convergence.

Let us recall that, in presence of the only vanishing viscosity term, the rate of convergence is typically of order $\varepsilon^{\gamma/2}$ (see [6]). For the homogenization of first order coercive equations, a rate of convergence of order $\varepsilon^{1/3}$ was obtained in [10], while for the homogenization of second order convex, uniformly elliptic equations, a rate of order ε^{α} was proved in [9] (with α depending on the regularity of the solution of the limit problem). In the present paper, we establish a rate of convergence which depends on γ and the regularity of the solution of (1.2). For $\gamma = 1$ we extend the result of [10] for the homogenization of first order equations. Indeed, we have

$$\|u_\varepsilon - u\|_\infty \leq C\varepsilon^{\frac{\alpha}{3}}$$

where α is the Hölder exponent of u. For $\gamma < 1$, we obtain the estimate

$$\|u_\varepsilon - u\|_\infty \leq C\varepsilon^{\alpha \min\{\frac{\gamma}{2},(1-\gamma)\}};$$

finally, for $\gamma > 1$, we show that

$$\|u_\varepsilon - u\|_\infty \leq C\varepsilon^{\min\{\frac{1}{3},\frac{\gamma-1}{2}\}}.$$

We also prove that these three estimates can be improved under appropriate structural conditions on the Hamiltonian. As an interesting byproduct we shall ascertain that u_ε converges uniformly to u in the whole space $\mathbb{R}^N$.

Let us now stress some features of our arguments. To prove an estimate on $\|u_\varepsilon - u\|_\infty$ it is not possible to invoke standard regular perturbation results in viscosity solution theory since they are based on a priori estimate of the quantity $\|H - \overline{H}\|$, not available in this case. We will follow the same approach as in [10] and we use the technique of doubling variables to compare u^ε with u perturbed with an approximated corrector, i.e. the solution of the ergodic approximation to the appropriate cell problem. For $\gamma \leq 1$, since the corresponding cell problem is of 2^{nd} order, to insure the appropriate regularity of the approximate corrector we need to assume the uniform ellipticity of H. For $\gamma > 1$, since the cell problem is of 1^{st} order, the regularity of the approximate corrector is insured by the coercitivity of H.

The paper is organized as follows. Section 2 contains the main assumptions and some preliminary properties. Sections 3, 4, 5 are devoted, respectively, to the case $\gamma = 1$, $\gamma < 1$ and $\gamma > 1$. Since some of the arguments are the same in the three cases, we will detail them only in Section 3. Section 6 contains some examples and model problems.

2 Standing assumptions

For every $\varepsilon > 0$ and $\gamma > 0$, we consider the problem

$$u_\varepsilon + H\left(x, \frac{x}{\varepsilon}, Du_\varepsilon, \varepsilon^\gamma D^2 u_\varepsilon\right) = 0 \qquad x \in \mathbb{R}^N \tag{2.1}$$

where u_ε is a real-valued function, Du_ε and $D^2 u_\varepsilon$ stand respectively for its gradient and its Hessian matrix.

The following hypotheses will be required in this paper:

(H1) H is 1-periodic in y for all $x, p, X \in \mathbb{R}^N \times \mathbb{R}^N \times S^N$, i.e.

$$H(x, y + z, p, X) = H(x, y, p, X) \qquad \text{for all } z \in \mathbb{Z}^N;$$

(H2) there exists $C > 0$ such that, for all $x, y, z, w, p, q \in \mathbb{R}^N$, $X, Z \in S^N$

$$|H(x, y, 0, 0)| \le C,$$
$$|H(x, y, p, X) - H(x, y, q, Z)| \le C\left(|p - q| + \|X - Z\|\right),$$
$$|H(x, y, p, X) - H(z, w, p, X)| \le C(|x - z| + |y - w|)(1 + |p| + \|X\|).$$

For $\gamma \le 1$, we shall assume that the operator H is convex and uniformly elliptic:

(H3) $H(x, y, p, \cdot)$ is convex for any $x, y \in \mathbb{R}^N$, $p \in \mathbb{R}^N$, and H is uniformly elliptic, i.e. there exists $\theta > 0$ such that

$$H(x, y, p, X) - H(x, y, p, X + Y) \ge \theta(Y)$$

for any $X, Y \in S^N$, $Y \ge 0$, $x, y, p \in \mathbb{R}^N$.

For $\gamma > 1$, we will replace (H3) with a coercivity assumption:

(H4) H is coercive in p, i.e. it fulfills

$$\lim_{r \to \infty} \inf\{H(x, y, p, 0) : (x, y) \in \mathbb{R}^N \times \mathbb{R}^N, |p| > r\} = +\infty.$$

In the next propositions we state regularity results for the solutions of the second order problem (1.1) (see [14], [22]) and of the corresponding limit problems (see [6], [17]).

Proposition 2.1 *Consider the elliptic problem*

$$u + F(x, Du, D^2u) = 0$$

where F satisfies (H2) and (H3). Then this problem admits a unique bounded continuous viscosity solution u. Moreover there exist $C > 0$ and $\bar{\alpha} \in (0, 1)$ such that

$$\|u\|_\infty,\ \|Du\|_\infty \le C \quad \text{and} \quad \|u\|_{C^{2,\bar{\alpha}}(B(x,1))} \le C.$$

Proposition 2.2 *Consider the Hamilton-Jacobi equation*

$$u + F(x, Du) = 0 \qquad x \in \mathbb{R}^N$$

where F satisfies

$$|F(x_1, p) - F(x_2, p)| \le C|x_1 - x_2|(1 + |p|), \quad |F(x, p_1) - F(x, p_2)| \le C|p_1 - p_2| \tag{2.2}$$

for any $x_i, p_i \in \mathbb{R}^N$ ($i = 1, 2$). Then this problem admits a unique bounded continuous viscosity solution u. Moreover

a) $u \in W^{1,\infty}(\mathbb{R}^N)$ *if* $C \le 1$*;*

b) $u \in C^{0,\alpha}(\mathbb{R}^N)$ *for any* $\alpha \in (0, 1)$ *if* $C = 1$*;*

c) $u \in C^{0,\alpha}(\mathbb{R}^N)$ *with* $\alpha = \frac{1}{C}$ *if* $C > 1$.

Finally if F is coercive in p, i.e. $\lim_{r \to \infty} \inf\{F(x, p) : x \in \mathbb{R}^N, |p| > r\} = +\infty$, *then* $u \in W^{1,\infty}(\mathbb{R}^N)$ *for every* $C > 0$.

3 Rate of convergence, case $\gamma = 1$

This section is devoted to the analysis of the most interesting case, i.e. $\gamma = 1$, in which vanishing viscosity term and fast oscillation one have a strong interaction. Throughout this section we will assume (H1), (H2) and (H3). The main result in this section is Theorem 3.1.

3.1 Properties of the effective Hamiltonian and of the approximate correctors

In this subsection, for the sake of completeness, we recall well known properties of the effective Hamiltonian and of the correctors.

For each (x, p) fixed, we introduce the following ergodic problem: find the unique value $\overline{H}_1(x, p)$ such that there exists a periodic solution to

$$H(x, y, p + D\chi(y), D^2\chi(y)) = \overline{H}_1(x, p), \qquad y \in \mathbb{R}^N. \tag{3.1}$$

We will denote by $\chi^1(\cdot; x, p)$ a viscosity solution to (3.1) in order to recall the dependence on the fixed parameters x, p. A solution to (3.1) can be obtained as the limit as $\lambda \to 0$ of the solution of the approximated cell problem

$$\lambda w_\lambda(y) + H(x, y, p + Dw_\lambda(y), D^2 w_\lambda(y)) = 0 \qquad y \in \mathbb{R}^N. \tag{3.2}$$

Lemma 3.1 *Let w_λ be a periodic solution of the approximated cell problem* (3.2)*. Then there exists $C > 0$, independent of λ, such that*

(i) $\|\lambda w_\lambda(\cdot; x, p)\|_\infty \le C(1 + |p|)$*;*

(ii) for some $\alpha \in (0, 1]$, $\|w_\lambda(\cdot; x, p) - w_\lambda(0; x, p)\|_{C^{2,\alpha}(\mathbb{R}^N)} \le C(1 + |p|)$*;*

(iii) $\lambda|D_p w_\lambda| \le C$ *and* $\lambda|D_x w_\lambda| \le C(1 + |p|)$ *(in the viscosity sense);*

(iv) $\|\lambda w_\lambda(\cdot; x, p) + \overline{H}_1(x, p)\|_\infty \le \lambda C(1 + |p|)$ *where $\overline{H}_1$ is the effective Hamiltonian given by the ergodic problem* (3.1)*.*

Proof. For the proof, we refer the reader to [22] and [4, Theorem 4.1 and following Remark] (see also [9, Lemma 2.1] and [5, Theorem II.2]).

Lemma 3.2 *(i) $w_\lambda(\cdot; x, p) - w_\lambda(0; x, p)$ converge uniformly, as $\lambda \to 0$, to a periodic solution $\chi^1(\cdot; x, p)$ of* (3.1)*. Moreover the solution of* (3.1) *is unique up to an additive constant.*

(ii) Let χ^1 be a solution of (3.1)*. Then for some $C > 0$ and $\alpha \in (0, 1]$,*

$$\|\chi^1(\cdot; x, p)\|_{C^{2,\alpha}(\mathbb{R}^N)} \le C(1 + |p|)$$

(iii) $\overline{H}_1$ satisfies (2.2)*. Moreover if H is coercive (see (H4)), then also $\overline{H}_1$ is coercive.*

Proof. The proof of (i), (ii) can be found in [5, Theorem II.2], (see also [4]), while for (iii) we refer to [9, Lemma 2.2].

3.2 The rate of convergence for $\gamma = 1$

Let u be the unique bounded solution to

$$u + \overline{H}_1(x, Du) = 0 \tag{3.3}$$

with $\overline{H}_1$ given by the cell problem (3.1). Then, by Lemma 3.2.(iii) and Proposition 2.2, u is Hölder continuous with exponent $\alpha \in (0, 1]$.

Theorem 3.1 *Let u_ε and u be the unique bounded solutions resp. to* (2.1) *with* $\gamma = 1$ *and to* (3.3).

i) If H is independent of x (namely, $H(x, y, p, M) = H(y, p, M)$), then there exists a constant $M > 0$ such that

$$\|u_\varepsilon - u\|_\infty \leq M\varepsilon. \tag{3.4}$$

ii) In the general case, there exists a constant $M > 0$ such that

$$\|u_\varepsilon - u\|_\infty \leq M\varepsilon^{\frac{\alpha}{3}} \tag{3.5}$$

where $\alpha \in (0, 1]$ is the Hölder exponent of u .

The proof of this theorem is postponed until the next two subsections.

Remark 3.1 In particular, we deduce that, as $\varepsilon \to 0$, u_ε converges uniformly to u in $\mathbb{R}^N$.

Remark 3.2 Observe that if H satisfies also (H4), then, by Proposition 2.2 and Lemma 3.2.(iii), u is Lipschitz continuous. Therefore the convergence rate is $1/3$, as for homogenization of first order equations (see [10]).

3.3 Proof of Theorem 3.1.i)

The function u is the unique bounded solution to $u + \overline{H}_1(Du) = 0$. Hence, obviously, $u \equiv -\overline{H}_1(0)$. Define

$$v_\varepsilon^\pm(x) := u + \varepsilon\chi^1\left(\frac{x}{\varepsilon}\right) \pm \varepsilon K$$

where $K > 0$ is a positive constant to be fixed later and χ^1 is the solution of (3.1) with $p = 0$ such that $\chi^1(0) = 0$. Then, for K sufficiently large, v_ε^+ and v_ε^- are respectively a supersolution and a subsolution to (1.1). Indeed, by (3.1), we infer

$$\begin{aligned} v_\varepsilon^+ + H\left(\frac{x}{\varepsilon}, Dv_\varepsilon^+, \varepsilon D^2 v_\varepsilon^+\right) &= -\overline{H}_1(0) + \varepsilon\chi^1 + \varepsilon K + H\left(\tfrac{x}{\varepsilon}, D\chi^1, D^2\chi^1\right) \\ &\geq -\varepsilon\|\chi^1\|_\infty + \varepsilon K. \end{aligned}$$

We get an analogous result for v_ε^-. Then by standard comparison principles between bounded sub and supersolutions of uniformly elliptic PDEs (see[11]) we get that

$$v_\varepsilon^-(x) \leq u_\varepsilon(x) \leq v_\varepsilon^+(x) \qquad \forall x \in \mathbb{R}^N$$

and in particular

$$-2K\varepsilon \leq \varepsilon\left(\chi^1\left(\frac{x}{\varepsilon}\right) - K\right) \leq u_\varepsilon(x) + \overline{H}_1(0) \leq \varepsilon\left(\chi^1\left(\frac{x}{\varepsilon}\right) + K\right) \leq 2K\varepsilon.$$

3.4 Proof of Theorem 3.1.ii)

We will use some arguments introduced in [10, Thm 3.1] (see also [20],[9]). Fix $\varepsilon, \delta, \beta, \tau \in (0,1)$, $\lambda = \varepsilon^\delta$ (δ and β will be chosen later). Define the auxiliary function

$$\Phi(x,\xi,z) := u_\varepsilon(x) - u(\xi) - \varepsilon w_\lambda\left(\frac{x}{\varepsilon}; z, \frac{z-\xi}{\varepsilon^\beta}\right) - \frac{|x-\xi|^2}{2\varepsilon^\beta} - \frac{|x-z|^2}{2\varepsilon^\beta} - \frac{\tau}{2}|\xi|^2, \tag{3.6}$$

where the functions u_ε, u and w_λ, are respectively solutions to (1.1) with $\gamma = 1$, (3.3) and (3.2). Recall that, by Proposition 2.2, u is bounded and Hölder continuous with exponent α. Moreover u_ε are C^2 functions by Proposition 2.1 and are uniformly bounded in ε by assumption (H2) and a standard comparison principle (see [11]), i.e. there exists $C > 0$ independent of ε such that $\|u_\varepsilon\|_\infty \le C$, $\forall \varepsilon$. Moreover Lemma 3.1 ensures that $w_\lambda \in C^2$ and $\|w_\lambda(\cdot;,x,p)\|_\infty \le K\lambda^{-1}(1+|p|)$. Hence, the function Φ attains its maximum in some point $(\hat{x}, \hat{\xi}, \hat{z})$. Observe that we may assume that this maximum is strict (if necessary, adding to Φ a smooth function vanishing together with its first and second derivatives at $(\hat{x}, \hat{\xi}, \hat{z})$).

In the following lemma, we collect some estimates concerning $(\hat{x}, \hat{\xi}, \hat{z})$.

Lemma 3.3 *Let $(\hat{x}, \hat{\xi}, \hat{z})$ be a maximum point of the function Φ defined in* (3.6) *with $u \in C^{0,\alpha}$. Assume $\delta + \beta < 1$ and $\tau < 2\varepsilon^\beta$. Then there exists a constant K, which does not depend on $\varepsilon, \beta, \tau, \lambda = \varepsilon^\delta, \alpha$ such that*

$$\frac{\tau}{2}|\hat{\xi}|^2 \le K \tag{3.7}$$

$$|\hat{x} - \hat{\xi}| \le K\varepsilon^{\beta/(2-\alpha)} \tag{3.8}$$

$$|\hat{x} - \hat{z}| \le K\varepsilon^{1-\delta}. \tag{3.9}$$

Moreover $|\hat{z} - \hat{\xi}| \le K\varepsilon^{\beta/(2-\alpha)}$ since $\beta/(2-\alpha) \le \beta \le 1-\delta$.

For clarity we proceed with the proof of Theorem 3.1 and postpone the proof of the Lemma to paragraph 3.4.1.

Since we shall need several constants, for simplicity the same letter (usually, C_1) may denote different constants from line to line; however, it will always denote constants depending only on the Hamiltonian H (i.e., independent of $\varepsilon, \delta, \beta, \tau$).

Observe that by the definition of Φ, the function

$$u_\varepsilon(x) - \left[\varepsilon w_\lambda\left(\frac{x}{\varepsilon}; \hat{z}, \frac{\hat{z}-\hat{\xi}}{\varepsilon^\beta}\right) + \frac{|x-\hat{\xi}|^2}{2\varepsilon^\beta} + \frac{|x-\hat{z}|^2}{2\varepsilon^\beta}\right]$$

has a maximum at $\hat{x}$. Then, using the fact that u_ε is a subsolution to (2.1) we get

$$u_\varepsilon(\hat{x}) + H\left(\hat{x}, \frac{\hat{x}}{\varepsilon}, Dw_\lambda + \frac{\hat{x}-\hat{\xi}}{\varepsilon^\beta} + \frac{\hat{x}-\hat{z}}{\varepsilon^\beta}, D^2 w_\lambda + 2\varepsilon^{1-\beta}\mathbf{I}\right) \le 0. \tag{3.10}$$

By assumption (H3), the estimates (3.9), (3.8) and the properties of w_λ stated in Lemma 3.1.(ii), there exist a constant C_1 such that

$$\begin{aligned}
H\left(\hat{x}, \frac{\hat{x}}{\varepsilon}, Dw_\lambda + \frac{\hat{x}-\hat{\xi}}{\varepsilon^\beta} + \frac{\hat{x}-\hat{z}}{\varepsilon^\beta}, D^2 w_\lambda + 2\varepsilon^{1-\beta}\mathbf{I}\right) \\
\geq H\left(\hat{z}, \frac{\hat{x}}{\varepsilon}, Dw_\lambda + \frac{\hat{z}-\hat{\xi}}{\varepsilon^\beta}, D^2 w_\lambda\right) - C_1\left(2\frac{|\hat{x}-\hat{z}|}{\varepsilon^\beta} + \varepsilon^{1-\beta}\right) \\
-C_1|\hat{x}-\hat{z}|\left(1 + \frac{|\hat{z}-\hat{\xi}|}{\varepsilon^\beta}\right) \\
\geq H\left(\hat{z}, \frac{\hat{x}}{\varepsilon}, Dw_\lambda + \frac{\hat{z}-\hat{\xi}}{\varepsilon^\beta}, D^2 w_\lambda\right) - C_1\varepsilon^{1-\beta-\delta}.
\end{aligned}$$

Substituting this estimate in (3.10), using the fact that w_λ solves equation (3.2) centered in $(\hat{z}, (\hat{z} - \hat{\xi})/\varepsilon^\beta)$ and Lemma 3.1, we deduce

$$u_\varepsilon(\hat{x}) + \overline{H}_1\left(\hat{z}, \frac{\hat{z}-\hat{\xi}}{\varepsilon^\beta}\right) \leq C_1(\varepsilon^{1-\delta-\beta} + \varepsilon^{\delta-\beta(1-\alpha)/(2-\alpha)}). \tag{3.11}$$

Observe that, by the definition of Φ, the function

$$\psi(\xi) := u(\xi) - \left[-\varepsilon w_\lambda\left(\frac{\hat{x}}{\varepsilon}; \hat{z}, \frac{\hat{z}-\xi}{\varepsilon^\beta}\right) - \frac{|\hat{x}-\xi|^2}{2\varepsilon^\beta} - \frac{\tau}{2}|\xi|^2\right] \tag{3.12}$$

attains a strict minimum at $\hat{\xi}$. Since the function between square brackets is not sufficiently regular to be used as a test function, we use again the standard argument of doubling variables. For $\sigma > 0$, we introduce the function

$$\Psi(\xi, y) := u(\xi) + \varepsilon w_\lambda\left(\frac{\hat{x}}{\varepsilon}; \hat{z}, \frac{\hat{z}-y}{\varepsilon^\beta}\right) + \frac{|\hat{x}-\xi|^2}{2\varepsilon^\beta} + \frac{\tau}{2}|\xi|^2 + \frac{|\xi-y|^2}{2\sigma} \tag{3.13}$$

and we denote by (ξ_σ, y_σ) a minimum point of Ψ in $B(\hat{\xi}, 1) \times B(\hat{\xi}, 1)$. The following lemma gives useful estimates.

Lemma 3.4 *Let* (ξ_σ, y_σ) *be a minimum point of* Ψ *as defined in* (3.13) *in* $B(\hat{\xi}, 1) \times B(\hat{\xi}, 1)$. *There exists* $K > 0$, *independent of all the parameters* $\varepsilon, \beta, \tau, \lambda = \varepsilon^\delta, \sigma$, *such that*

$$\frac{|\xi_\sigma - y_\sigma|}{\sigma} \leq K\varepsilon^{1-\beta-\delta}. \tag{3.14}$$

Moreover

$$|\xi_\sigma - \hat{\xi}| + |y_\sigma - \hat{\xi}| \to 0 \qquad \text{as } \sigma \to 0 \tag{3.15}$$

As above, we postpone the proof of this lemma to paragraph 3.4.2. By the definition of Ψ, the function

$$u(\xi) - \left[-\frac{|\hat{x}-\xi|^2}{2\varepsilon^\beta} - \frac{\tau}{2}|\xi|^2 - \frac{|\xi-y_\sigma|^2}{2\sigma}\right]$$

has a minimum at ξ_σ. Then, by equation (3.3), we get

$$u(\xi_\sigma) + \overline{H}_1\left(\xi_\sigma, \frac{\hat{x}-\xi_\sigma}{\varepsilon^\beta} - \tau\xi_\sigma + \frac{y_\sigma - \xi_\sigma}{\sigma}\right) \geq 0.$$

By the properties of $\overline{H}_1$ proved in Lemma 3.2 , we obtain

$$u(\xi_\sigma) + \overline{H}_1\left(\xi_\sigma, \frac{\hat{x}-\xi_\sigma}{\varepsilon^\beta} - \tau\xi_\sigma + \frac{y_\sigma - \xi_\sigma}{\sigma}\right) \le u(\xi_\sigma) + \overline{H}_1\left(\hat{\xi}_\sigma, \frac{\hat{z}-\xi_\sigma}{\varepsilon^\beta}\right) + C_1\left[\frac{|\hat{x}-\hat{z}|}{\varepsilon^\beta} + \tau|\xi_\sigma| + \frac{|y_\sigma - \xi_\sigma|}{\sigma}\right].$$

By the estimates (3.14), (3.7), (3.9) and letting $\sigma \to 0$ in the previous inequality, we get

$$u(\hat{\xi}) + \overline{H}_1\left(\hat{\xi}, \frac{\hat{z}-\hat{\xi}}{\varepsilon^\beta}\right) \ge -C_1\left(\sqrt{\tau} + \varepsilon^{1-\beta-\delta}\right).$$

Using (3.8), (3.9) and the properties of $\overline{H}_1$ in Lemma 3.2, we obtain

$$\begin{aligned} u(\hat{\xi}) + \overline{H}_1\left(\hat{z}, \frac{\hat{z}-\hat{\xi}}{\varepsilon^\beta}\right) &\ge -C_1\left(\varepsilon^{1-\beta-\delta} + \sqrt{\tau}\right) - C_1|\hat{z}-\hat{\xi}|\left(1 + \frac{|\hat{z}-\hat{\xi}|}{\varepsilon^\beta}\right) \\ &\ge -C_1\left(\varepsilon^{1-\beta-\delta} + \varepsilon^{\alpha\beta/(2-\alpha)} + \sqrt{\tau}\right). \end{aligned} \tag{3.16}$$

Comparing (3.11) with (3.16) we infer

$$u_\varepsilon(\hat{x}) - u(\hat{\xi}) \le C_1(\varepsilon^{1-\delta-\beta} + \varepsilon^{\delta-\beta(1-\alpha)/(2-\alpha)} + \varepsilon^{\alpha\beta/(2-\alpha)}) + C_1\sqrt{\tau}. \tag{3.17}$$

Finally, the relation $\Phi(x, x, x) \le \Phi(\hat{x}, \hat{\xi}, \hat{z})$ entails

$$u_\varepsilon(x) - u(x) \le u_\varepsilon(\hat{x}) - u(\hat{\xi}) + \varepsilon\left(w_\lambda\left(\frac{x}{\varepsilon}; x, 0\right) - w_\lambda\left(\frac{\hat{x}}{\varepsilon}; \hat{z}, \frac{\hat{z}-\hat{\xi}}{\varepsilon^\beta}\right)\right) + \frac{\tau}{2}|x|^2 \qquad \forall x \in \mathbb{R}^N.$$

Taking into account (3.17), Lemma 3.1.(i) and the estimates in Lemma 3.3, we obtain

$$\begin{aligned} u_\varepsilon(x) - u(x) \le\ & C_1(\varepsilon^{1-\delta-\beta} + \varepsilon^{\delta-\beta(1-\alpha)/(2-\alpha)} + \varepsilon^{\alpha\beta/(2-\alpha)} + \varepsilon^{1-\delta-\beta(1-\alpha)/(2-\alpha)}) \\ & + C_1\sqrt{\tau} + \frac{\tau}{2}|x|^2. \end{aligned}$$

Letting $\tau \to 0$ we obtain that for every $x \in \mathbb{R}^N$

$$u_\varepsilon(x) - u(x) \le C_1(\varepsilon^{1-\delta-\beta} + \varepsilon^{\delta-\beta(1-\alpha)/(2-\alpha)} + \varepsilon^{\alpha\beta/(2-\alpha)}) \le C_1\varepsilon^{\alpha/3},$$

provided that we choose $\beta = (2-\alpha)/3$ and $\delta = 1/3$. The other inequality in (3.5) can be obtained in an analogous way and we omit its proof.

3.4.1 Proof of Lemma 3.3

As in the proof of Theorem 3.1, the same letter K_1 may denote different constants from line to line; however, all these constants are independent of $\varepsilon, \beta, \delta, \tau$.

Let us prove inequality (3.7). The relation $\Phi(\hat{x}, \hat{\xi}, \hat{z}) \ge \Phi(0, 0, 0)$ gives

$$\frac{\tau}{2}|\hat{\xi}|^2 \le 2\|u_\varepsilon\|_\infty + 2\|u\|_\infty + \varepsilon\left(w_\lambda(0; 0, 0) - w_\lambda\left(\frac{\hat{x}}{\varepsilon}; \hat{z}, \frac{\hat{z}-\hat{\xi}}{\varepsilon^\beta}\right)\right) - \frac{|\hat{x}-\hat{\xi}|^2}{2\varepsilon^\beta} - \frac{|\hat{x}-\hat{z}|^2}{2\varepsilon^\beta}. \tag{3.18}$$

By Lemma 3.1.(i) and using Young inequality, we get

$$\varepsilon\left(w_\lambda(0;0,0) - w_\lambda\left(\frac{\hat{x}}{\varepsilon};\hat{z},\frac{\hat{z}-\hat{\xi}}{\varepsilon^\beta}\right)\right) \le 2\varepsilon^{1-\delta}C + C\varepsilon^{1-\delta}\frac{|\hat{x}-\hat{\xi}|}{\varepsilon^\beta} + C\varepsilon^{1-\delta}\frac{|\hat{x}-\hat{z}|}{\varepsilon^\beta}$$
$$\le 2\varepsilon^{1-\delta}C + C^2\varepsilon^{2(1-\delta-\beta/2)} + \frac{|\hat{x}-\hat{\xi}|^2}{2\varepsilon^\beta} + \frac{|\hat{x}-\hat{z}|^2}{2\varepsilon^\beta}.$$

Substituting this relation in (3.18), we get the desired estimate (3.7).

We now pass to prove estimate (3.8); inequality $\Phi(\hat{x},\hat{\xi},\hat{z}) \ge \Phi(\hat{x},\hat{x},\hat{z})$ gives

$$\frac{|\hat{x}-\hat{\xi}|^2}{2\varepsilon^\beta} \le u(\hat{x}) - u(\hat{\xi}) + \varepsilon\left(w_\lambda\left(\frac{\hat{x}}{\varepsilon};\hat{z},\frac{\hat{z}-\hat{x}}{\varepsilon^\beta}\right) - w_\lambda\left(\frac{\hat{x}}{\varepsilon};\hat{z},\frac{\hat{z}-\hat{\xi}}{\varepsilon^\beta}\right)\right) - \frac{\tau}{2}|\hat{\xi}|^2 + \frac{\tau}{2}|\hat{x}|^2. \tag{3.19}$$

Taking into account estimate (3.7), we infer

$$-\frac{\tau}{2}|\hat{\xi}|^2 + \frac{\tau}{2}|\hat{x}|^2 \le \frac{\tau}{2}|\hat{x}-\hat{\xi}|^2 + \tau|\hat{x}-\hat{\xi}||\hat{\xi}| \le \frac{\tau}{2}|\hat{x}-\hat{\xi}|^2 + \sqrt{2\tau K}|\hat{x}-\hat{\xi}|.$$

Using again Lemma 3.1.(iii), for some constant K_1 we have

$$\varepsilon\left(w_\lambda\left(\frac{\hat{x}}{\varepsilon};\hat{z},\frac{\hat{z}-\hat{x}}{\varepsilon^\beta}\right) - w_\lambda\left(\frac{\hat{x}}{\varepsilon};\hat{z},\frac{\hat{z}-\hat{\xi}}{\varepsilon^\beta}\right)\right) \le \varepsilon^{1-\delta-\beta}K_1|\hat{x}-\hat{\xi}|.$$

Substituting the last two inequalities in (3.19) with $\tau \le 2\varepsilon^\beta$ (recall: $u \in C^{0,\alpha}$), we obtain

$$\frac{|\hat{x}-\hat{\xi}|^{2-\alpha}}{4\varepsilon^\beta} \le K_1 + (\varepsilon^{1-\delta-\beta}K_1 + \sqrt{2\tau K})|\hat{x}-\hat{\xi}|^{1-\alpha}.$$

We apply now Young inequality with $p = \frac{2-\alpha}{1-\alpha}$ and $q = 2-\alpha$ to the last term of the previous formula and get

$$\frac{|\hat{x}-\hat{\xi}|^{2-\alpha}}{8\varepsilon^\beta} \le K_1 + \frac{1}{2-\alpha}\left(8\varepsilon^\beta\frac{1-\alpha}{2-\alpha}\right)^{1-\alpha}\left(\varepsilon^{1-\delta-\beta}K_1 + \sqrt{2\tau K}\right)^{2-\alpha} \le \frac{K_1}{8}.$$

Finally we prove (3.9); by Lemma 3.1.(iii), for some constant K_1, the inequality $\Phi(\hat{x},\hat{\xi},\hat{z}) \ge \Phi(\hat{x},\hat{\xi},\hat{x})$ ensures

$$\begin{aligned}\frac{|\hat{x}-\hat{z}|^2}{2\varepsilon^\beta} &\le \varepsilon\left(w_\lambda\left(\frac{\hat{x}}{\varepsilon};\hat{x},\frac{\hat{x}-\hat{\xi}}{\varepsilon^\beta}\right) - w_\lambda\left(\frac{\hat{x}}{\varepsilon};\hat{z},\frac{\hat{z}-\hat{\xi}}{\varepsilon^\beta}\right)\right)\\ &\le \varepsilon^{1-\delta-\beta}K_1|\hat{x}-\hat{z}| + \varepsilon^{1-\delta-\beta}K_1|\hat{x}-\hat{z}|(\varepsilon^\beta + |\hat{x}-\hat{\xi}|).\end{aligned}$$

Therefore, if necessary increasing the constant K_1, we obtain

$$|\hat{x}-\hat{z}| \le \varepsilon^{1-\delta}K_1 + K_1\varepsilon^{1-\delta+\beta/(2-\alpha)} \le K_1\varepsilon^{1-\delta}.$$

3.4.2 Proof of Lemma 3.4.

Let us recall that (ξ_σ, y_σ) is a maximum point of Ψ. Lemma 3.1.(iii) and inequality $\Psi(\xi_\sigma, y_\sigma) \le \Psi(\xi_\sigma, \xi_\sigma)$ entail

$$\frac{|\xi_\sigma - y_\sigma|^2}{2\sigma} \le \varepsilon\left[w_\lambda\left(\frac{\hat{x}}{\varepsilon};\hat{z},\frac{\hat{z}-y_\sigma}{\varepsilon^\beta}\right) - w_\lambda\left(\frac{\hat{x}}{\varepsilon};\hat{z},\frac{\hat{z}-\xi_\sigma}{\varepsilon^\beta}\right)\right] \le C\varepsilon^{1-\delta-\beta}|\xi_\sigma - y_\sigma|.$$

Whence, estimate (3.14) easily follows.

Let us now pass to prove formula (3.15). By Lemma 3.1.(iii) and (3.14), we get

$$\begin{aligned}\Psi(\xi_\sigma, y_\sigma) &= u(\xi_\sigma) + \varepsilon w_\lambda\left(\frac{\hat{x}}{\varepsilon}; \hat{z}, \frac{\hat{z}-\xi_\sigma}{\varepsilon^\beta}\right) + \frac{|\hat{x}-\xi_\sigma|^2}{2\varepsilon^\beta} + \frac{\tau}{2}|\xi_\sigma|^2 + \\ &+ \varepsilon\left(w_\lambda\left(\frac{\hat{x}}{\varepsilon}; \hat{z}, \frac{\hat{z}-y_\sigma}{\varepsilon^\beta}\right) - w_\lambda\left(\frac{\hat{x}}{\varepsilon}; \hat{z}, \frac{\hat{z}-\xi_\sigma}{\varepsilon^\beta}\right)\right) \geq \psi(\xi_\sigma) - C\sigma\varepsilon^{1-2\beta-\delta},\end{aligned}$$

where ψ is defined in (3.12). Therefore

$$\psi(\xi_\sigma) - C\varepsilon^{1-2\beta-\delta}\sigma \leq \Psi(\xi_\sigma, y_\sigma) \leq \Psi(\hat{\xi}, \hat{\xi}) = \psi(\hat{\xi}) = \min \psi(\xi).$$

We fix $\varepsilon, \beta, \lambda, \tau$ and we let $\sigma \to 0$ in the previous inequality. So, eventually passing to a subsequence and recalling that ψ attains a strict minimum at $\hat{\xi}$, we get $\xi_\sigma \to \hat{\xi}$, which gives the desired conclusion.

4 Rate of convergence, case $\gamma < 1$

This section is devoted to the case $\gamma < 1$. Throughout this section we will assume (H1), (H2) and (H3). The main result in this section is Theorem 4.1.

4.1 Properties of the effective Hamiltonian and of the correctors

For $(x, p) \in \mathbb{R}^N \times \mathbb{R}^N$ fixed, we introduce the following ergodic problem: find a periodic solution to

$$H(x, y, p, D^2\chi(y)) = \overline{H}_-(x, p), \qquad y \in \mathbb{R}^N. \tag{4.1}$$

We also consider the approximate cell problems

$$\lambda w_\lambda(y) + H(x, y, p, D^2 w_\lambda) = 0. \tag{4.2}$$

Remark 4.1 The cell problems (4.1) and their approximations (4.2) are independent of $\gamma \in (0, 1)$. In particular, the effective Hamiltonian will be the same for every $\gamma \in (0, 1)$ and we will denote it by $\overline{H}_-(x, p)$.

Remark 4.2 Lemma 3.1 and Lemma 3.2 still hold in this case. In particular the solution of (4.1) is unique up to an additive constant.

In the following lemma, we establish some properties of χ^- under a structural assumption on H.

Lemma 4.1 *Assume that the Hamiltonian H, besides the standing assumptions, satisfies the following structural assumption*

$$H(x, y, p, X) = \max_{\zeta \in Z}\left\{-tr[a^\zeta(y)X] + F^\zeta(x, y, p)\right\}, \tag{4.3}$$

where Z is a compact metric space while a^ζ, F^ζ are continuous functions, uniformly bounded with respect to ζ, and moreover there exists $C > 0$ such that, for every $x, z, y, w, p, q \in \mathbb{R}^N$ and $\zeta \in Z$,

$$|F^\zeta(x, y, p) - F^\zeta(z, w, q)| \leq C|p - q| + C(|x - z| + |y - w|)(1 + |p| \vee |q|). \tag{4.4}$$

Fix the solution χ^- to (4.1) *such that* $\chi^-(0) = 0$. *Then, there exists* $C > 0$ *independent of* γ *such that*

$$|D_p\chi^-| \le C \quad \textit{and} \quad |D_x\chi^-| \le C(1 + |p|) \qquad \textit{(in viscosity sense).}$$

Proof. By Remark 4.2, it suffices to prove that the estimates in Lemma 3.1.(iii) are independent of λ, that is there exists $C > 0$ such that $|D_p w_\lambda^-| \le C$ and $|D_x w_\lambda^-| \le C(1 + |p|)$ (in the viscosity sense).

We fix (x_1, p_1) and (x_2, p_2) and consider $w_1 = w_\lambda^-(\cdot; x_1, p_1)$ and $w_2 = w_\lambda^-(\cdot; x_2, p_2)$. Define $w_\lambda = w_1 - w_2$. To get the result it is sufficient to prove that

$$\|w_\lambda - w_\lambda(0)\|_\infty \le C(1 + |p_1| \vee |p_2|)|x_1 - x_2| + C|p_1 - p_2|. \tag{4.5}$$

In order to prove (4.5), let us first observe that, using assumption (4.4), the functions $v^\pm = w_1 \pm C\lambda^{-1}[(1 + |p_1| \vee |p_2|)|x_1 - x_2| + |p_1 - p_2|]$ are respectively a super and a subsolution to equation (4.2) centered at (x_2, p_2). Then standard comparison principle yields

$$\|\lambda w_\lambda\|_\infty \le C(1 + |p_1| \vee |p_2|)|x_1 - x_2| + C|p_1 - p_2|. \tag{4.6}$$

We now proceed by contradiction, assuming that there exists $\lambda_k \to 0$, $x_1^k, x_2^k, p_1^k, p_2^k \in \mathbb{R}^N$ such that

$$c_k := \|w_k - w_k(0)\|_\infty \ge k(1 + |p_1^k| \vee |p_2^k|)|x_1^k - x_2^k| + k|p_1^k - p_2^k| \tag{4.7}$$

where $w_k(\cdot) := w_{\lambda_k}^-(\cdot; x_1^k, p_1^k) - w_{\lambda_k}^-(\cdot; x_2^k, p_2^k)$. Define $v_k := (w_k - w_k(0))/c_k$. Then observe that $\|v_k\|_\infty = 1$ and $v_k(0) = 0$ for every k. Moreover v_k is a supersolution to

$$\lambda_k v_k + \max_{\zeta\in Z}\left\{-\mathrm{tr}\left[a^\zeta(y)D^2 v_k\right] + \frac{F^\zeta(x_1^k, y, p_1^k) - F^\zeta(x_2^k, y, p_2^k)}{c_k}\right\} + \frac{\lambda_k w_k(0)}{c_k} \ge 0,$$

and a subsolution to

$$\lambda_k v_k + \min_{\zeta\in Z}\left\{-\mathrm{tr}\left[a^\zeta(y)D^2 v_k\right] + \frac{F^\zeta(x_1^k, y, p_1^k) - F^\zeta(x_2^k, y, p_2^k)}{c_k}\right\} + \frac{\lambda_k w_k(0)}{c_k} \le 0.$$

Let us observe that relations (4.4), (4.6) and (4.7) entail: $|F^\zeta(x_1^k, y, p_1^k) - F^\zeta(x_2^k, y, p_2^k)|/c_k \le C/k$ and $|\lambda_k w_k(0)|/c_k \le C/k$ for every (y, ζ), k. By the same argument as in [2, Prop 12] (see also [4, Thm 4.1] and [23, Thm 5.1]) we obtain that v_k are equi-continuous; by Ascoli-Arzelá's theorem, if necessary passing to a subsequence, v_k converges to v locally uniformly and v is a bounded supersolution to

$$\max_{\zeta\in Z}\left\{-\mathrm{tr}\left[a^\zeta(y)D^2 v\right]\right\} \ge 0 \qquad \text{in } \mathbb{R}^N.$$

The Strong Minimum Principle ([23]) entails that v is constant, in contradiction with the fact that $\|v\|_\infty = 1$ and $v(0) = 0$.

4.2 The rate of convergence for $\gamma < 1$

Let u^- be the unique bounded solution to

$$u + \overline{H}_-(x, Du) = 0 \tag{4.8}$$

with $\overline{H}_-$ given by the cell problem (4.1). Then, by Remark 4.2 and Proposition 2.2, u is Hölder continuous with exponent $\alpha \in (0, 1]$.

Theorem 4.1 *Let u_ε and u^- be resp. the unique bounded solutions to* (2.1) *with $\gamma < 1$ and to* (4.8)*.*

i) If $H(x, y, p, X) = -tr[a(y)X] + F(y, p)$, there exists $M > 0$ independent of γ such that

$$\|u_\varepsilon - u^-\|_\infty \le M\varepsilon^{1-\gamma}; \tag{4.9}$$

ii) If H satisfies (4.3)*, there exists $M > 0$ independent of γ such that*

$$\|u_\varepsilon - u^-\|_\infty \le M\varepsilon^{\min(\frac{\alpha\gamma}{2},1-\gamma)}. \tag{4.10}$$

iii) In the general case, there exists $M > 0$ independent of γ such that

$$\|u_\varepsilon - u^-\|_\infty \le M\varepsilon^{\alpha\min(\frac{\gamma}{2},1-\gamma)}. \tag{4.11}$$

Remark 4.3 The previous theorem says that u_ε converge uniformly in $\mathbb{R}^N$ as $\varepsilon \to 0$ for every $\gamma < 1$ to the same function u^-. Moreover, the best rate of convergence that can be obtained is, when H satisfies (4.3), $\frac{\alpha}{\alpha+2}$ (for $\gamma = \frac{2}{2+\alpha}$) and, in the general case $\frac{\alpha}{3}$ (for $\gamma = \frac{2}{3}$).

4.3 Proof of Theorem 4.1.i)

We shall argue as in the proof of Theorem 3.1.(i). The solution u^- is given by $-\overline{H}_-(0)$. Define $v_\varepsilon^\pm(x) := u^- + \varepsilon^{2-\gamma}\chi^-(x/\varepsilon) \pm \varepsilon^{1-\gamma}K$, where χ^- is the solution of (4.1) with $p = 0$ such that $\chi^-(0) = 0$. We show that, for $K > 0$ sufficiently large, v_ε^+ and v_ε^- are respectively a supersolution and a subsolution to (1.1). In fact

$$\begin{aligned}
& v_\varepsilon^+ - \varepsilon^\gamma \mathrm{tr}\left[a\left(\tfrac{x}{\varepsilon}\right)D^2 v_\varepsilon^+\right] + F\left(\tfrac{x}{\varepsilon}, Dv_\varepsilon^+\right) \\
&= -\overline{H}_-(0) + \varepsilon^{2-\gamma}\chi^-\left(\tfrac{x}{\varepsilon}\right) + \varepsilon^{1-\gamma}K - \mathrm{tr}\left[a\left(\tfrac{x}{\varepsilon}\right)D^2\chi^-\left(\tfrac{x}{\varepsilon}\right)\right] + F\left(\tfrac{x}{\varepsilon}, \varepsilon^{1-\gamma}D\chi^-\left(\tfrac{x}{\varepsilon}\right)\right) \\
&\ge \varepsilon^{2-\gamma}\chi^-\left(\tfrac{x}{\varepsilon}\right) + \varepsilon^{1-\gamma}K + F\left(\tfrac{x}{\varepsilon}, \varepsilon^{1-\gamma}D\chi^-\left(\tfrac{x}{\varepsilon}\right)\right) - F\left(\tfrac{x}{\varepsilon}, 0\right) \\
&\ge \varepsilon^{2-\gamma}\|\chi^-\|_\infty + \varepsilon^{1-\gamma}K - \varepsilon^{1-\gamma}\|D\chi^-\| \ge 0.
\end{aligned}$$

The proof for v_ε^- is similar and we shall omit it. By the comparison principles (see[11]), we get that

$$v_\varepsilon^-(x) \le u_\varepsilon(x) \le v_\varepsilon^+(x) \qquad \forall x \in \mathbb{R}^N$$

and then in particular

$$-2K\varepsilon^{1-\gamma} \le u_\varepsilon(x) - u^- \le 2K\varepsilon^{1-\gamma}.$$

4.4 Proof of Theorem 4.1.ii)

The argument of the proof is similar to that in the proof of Theorem 3.1.ii) and therefore we just sketch it. For $\tau, \beta \in (0, 1)$ (to be fixed later), define the auxiliary function

$$\Phi(x, \xi, z) := u_\varepsilon(x) - u^-(\xi) - \varepsilon^{2-\gamma}\chi^-\left(\frac{x}{\varepsilon}; z, \frac{z-\xi}{\varepsilon^\beta}\right) - \frac{|x-\xi|^2}{2\varepsilon^\beta} - \frac{|x-z|^2}{2\varepsilon^\beta} - \frac{\tau}{2}|\xi|^2, \tag{4.12}$$

where χ^- is a solution to (4.1). By Lemma 3.2.(ii), the function Φ attains a strict maximum in some point $(\hat{x}, \hat{\xi}, \hat{z})$. The following lemma gives some useful estimates.

Lemma 4.2 *Let $(\hat{x}, \hat{\xi}, \hat{z})$ be a maximum point of Φ as defined in (4.12) and $\alpha \in (0, 1]$ the Hölder exponent of u^-. Assume $\tau < 2\varepsilon^\beta$. Then there exists a constant K, which does not depend on $\varepsilon, \beta, \tau, \alpha$ such that*

$$\frac{\tau}{2}|\hat{\xi}|^2 \leq K \tag{4.13}$$
$$|\hat{x} - \hat{\xi}| \leq K\varepsilon^{\beta/(2-\alpha)} \tag{4.14}$$
$$|\hat{x} - \hat{z}| \leq K\varepsilon^{2-\gamma}. \tag{4.15}$$

Moreover $|\hat{z} - \hat{\xi}| \leq K\varepsilon^{\beta/(2-\alpha)}$.

The proof of this lemma follows by repeating the same argument of the proof of Lemma 3.3 (Lemma 3.1 on the properties of w_λ is replaced by Lemma 4.1 on the properties of χ^-); hence, we shall omit it.

Let us come back to the proof of the Theorem. As before, the letter C_1 may denote different constants from line to line; however, it will always denotes constants independent of ε, β, τ. By the definition of Φ, the function

$$u_\varepsilon(x) - \left[\varepsilon^{2-\gamma}\chi^-\left(\frac{x}{\varepsilon}; \hat{z}, \frac{\hat{z} - \hat{\xi}}{\varepsilon^\beta}\right) + \frac{|x - \hat{\xi}|^2}{2\varepsilon^\beta} + \frac{|x - \hat{z}|^2}{2\varepsilon^\beta}\right]$$

has a maximum at $\hat{x}$. Then, since u_ε is a subsolution to (1.1), we deduce

$$u_\varepsilon(\hat{x}) + \max_{\zeta\in Z}\left\{-\mathrm{tr}\left[a^\zeta\left(\frac{\hat{x}}{\varepsilon}\right)(D^2\chi^- + 2\varepsilon^{\gamma-\beta}\mathbf{I})\right] + F^\zeta\left(\hat{x}, \frac{\hat{x}}{\varepsilon}, \varepsilon^{1-\gamma}D\chi^- + \frac{\hat{x} - \hat{\xi}}{\varepsilon^\beta} + \frac{\hat{x} - \hat{z}}{\varepsilon^\beta}\right)\right\} \leq 0. \tag{4.16}$$

By assumption (4.4), Remark 4.2 and the estimates (4.15) and (4.14) we get that there exists a constant C_1 such that, for every $\zeta \in Z$,

$$\begin{aligned}
F^\zeta&\left(\hat{x}, \frac{\hat{x}}{\varepsilon}, \varepsilon^{1-\gamma}D\chi^- + \frac{\hat{x} - \hat{\xi}}{\varepsilon^\beta} + \frac{\hat{x} - \hat{z}}{\varepsilon^\beta}\right)\\
&\geq F^\zeta\left(\hat{x}, \frac{\hat{x}}{\varepsilon}, \frac{\hat{z} - \hat{\xi}}{\varepsilon^\beta}\right) - C_1\left(\varepsilon^{1-\gamma} + \frac{|\hat{x} - \hat{z}|}{\varepsilon^\beta}\right)\\
&\geq F^\zeta\left(\hat{z}, \frac{\hat{x}}{\varepsilon}, \frac{\hat{z} - \hat{\xi}}{\varepsilon^\beta}\right) - C_1\varepsilon^{1-\gamma} - C_1|\hat{x} - \hat{z}|\left(1 + \frac{|\hat{z} - \hat{\xi}|}{\varepsilon^\beta}\right)\\
&\geq F^\zeta\left(\hat{z}, \frac{\hat{x}}{\varepsilon}, \frac{\hat{z} - \hat{\xi}}{\varepsilon^\beta}\right) - C_1\varepsilon^{1-\gamma}.
\end{aligned}$$

Substituting last inequality in (4.16) and using the fact that χ^- is a classical solution to (4.1) for $(x, p) = (\hat{z}, (\hat{z} - \hat{\xi})/\varepsilon^\beta)$ we obtain

$$u_\varepsilon(\hat{x}) + \overline{H}_-\left(\hat{z}, \frac{\hat{z} - \hat{\xi}}{\varepsilon^\beta}\right) \leq C_1(\varepsilon^{1-\gamma} + \varepsilon^{\gamma-\beta}). \tag{4.17}$$

Observe now that, by the definition of Φ, the function

$$\psi(\xi) := u^-(\xi) - \left[-\varepsilon^{2-\gamma}\chi^-\left(\frac{\hat{x}}{\varepsilon}; \hat{z}, \frac{\hat{z} - \xi}{\varepsilon^\beta}\right) - \frac{|\hat{x} - \xi|^2}{2\varepsilon^\beta} - \frac{\tau}{2}|\xi|^2\right]$$

has a strict minimum at $\hat{\xi}$. Since the function between square brackets is not sufficiently regular, we perform again the argument of doubling variables and we introduce the function

$$\Psi(\xi, y) := u^-(\xi) + \varepsilon^{2-\gamma}\chi^-\left(\frac{\hat{x}}{\varepsilon}; \hat{z}, \frac{\hat{z}-y}{\varepsilon^\beta}\right) + \frac{|\hat{x}-\xi|^2}{2\varepsilon^\beta} + \frac{\tau}{2}|\xi|^2 + \frac{|\xi-y|^2}{2\sigma}, \tag{4.18}$$

where $\sigma > 0$ is a given parameter. Let (ξ_σ, y_σ) be a minimum point of Ψ in $B(\hat{\xi}, 1) \times B(\hat{\xi}, 1)$.

Lemma 4.3 *Let (ξ_σ, y_σ) be a minimum point of Ψ as defined in (4.18) in $B(\hat{\xi}, 1) \times B(\hat{\xi}, 1)$. There exists $K > 0$, independent of all the parameters $\varepsilon, \beta, \tau, \gamma, \sigma$, such that*

$$\frac{|\xi_\sigma - y_\sigma|}{\sigma} \leq K\varepsilon^{2-\gamma-\beta}. \tag{4.19}$$

Moreover there holds: $|\xi_\sigma - \hat{\xi}| + |y_\sigma - \hat{\xi}| \to 0$ as $\sigma \to 0$.

Its proof is analogous to that of Lemma 3.4; actually, it suffices to substitute w_λ with χ^- and Lemma 3.1 with Lemma 4.1. Hence, we shall omit it.

Observe that the function $u^-(\xi) - \left[-\frac{|\hat{x}-\xi|^2}{2\varepsilon^\beta} - \frac{\tau}{2}|\xi|^2 - \frac{|\xi-y_\sigma|^2}{2\sigma}\right]$ has a minimum at ξ_σ. By equation (4.8) and the properties of $\overline{H}_-$, we infer

$$\begin{aligned} 0 &\leq u^-(\xi_\sigma) + \overline{H}_-\left(\xi_\sigma, \frac{\hat{x}-\xi_\sigma}{\varepsilon^\beta} + \tau\xi_\sigma + \frac{\xi_\sigma - y_\sigma}{\sigma}\right) \\ &\leq u^-(\xi_\sigma) + \overline{H}_-\left(\hat{z}, \frac{\hat{z}-\xi_\sigma}{\varepsilon^\beta}\right) + C\left[\tau|\xi_\sigma| + \frac{|\xi_\sigma - y_\sigma|}{\sigma} + |\hat{z}-\xi_\sigma|\left(1 + \frac{|\hat{z}-\xi_\sigma|}{\varepsilon^\beta}\right)\right]. \end{aligned}$$

Letting $\sigma \to 0$, owing to estimate (4.19), we infer that

$$u^-(\hat{\xi}) + \overline{H}_-\left(\hat{z}, \frac{\hat{z}-\hat{\xi}}{\varepsilon^\beta}\right) \geq -C_1(\varepsilon^{2-\gamma-\beta} + \sqrt{\tau} + \varepsilon^{\alpha\beta/(2-\alpha)}). \tag{4.20}$$

By estimates (4.17) and (4.20) and inequality $2-\gamma-\beta \geq \gamma-\beta$, we obtain

$$u_\varepsilon(\hat{x}) - u^-(\hat{\xi}) \leq C_1(\varepsilon^{\gamma-\beta} + \varepsilon^{1-\gamma} + \varepsilon^{\alpha\beta/(2-\alpha)}) + C_1\sqrt{\tau}. \tag{4.21}$$

Finally the relation $\Phi(x, x, x) \leq \Phi(\hat{x}, \hat{\xi}, \hat{z})$ entails

$$u_\varepsilon(x) - u^-(x) \leq u_\varepsilon(\hat{x}) - u^-(\hat{\xi}) + \varepsilon^{2-\gamma}\left(\chi^-\left(\frac{x}{\varepsilon}; x, 0\right) - \chi^-\left(\frac{\hat{x}}{\varepsilon}; \hat{z}, \frac{\hat{z}-\hat{\xi}}{\varepsilon^\beta}\right)\right) + \frac{\tau}{2}|x|^2.$$

Estimates (4.21) and Lemma 4.2 ensure

$$u_\varepsilon(x) - u^-(x) \leq C_1(\varepsilon^{\gamma-\beta} + 2\varepsilon^{\alpha\beta/(2-\alpha)} + \varepsilon^{1-\gamma}) + C_1\sqrt{\tau} + \frac{\tau}{2}|x|^2.$$

Letting $\tau \to 0$ we obtain that for every $x \in \mathbb{R}^N$

$$u_\varepsilon(x) - u(x) \leq C_1(\varepsilon^{\gamma-\beta} + \varepsilon^{\alpha\beta/(2-\alpha)} + \varepsilon^{1-\gamma}) \leq C_1\varepsilon^{\min(\frac{\alpha\gamma}{2}, 1-\gamma)},$$

choosing $\beta = \gamma(2-\alpha)/2$.

4.5 Proof of Theorem 4.1.iii)

We shall follow the same arguments of the proof of Theorem 3.1.ii); hence, we shall only emphasize the main differences. For each $\delta, \beta, \tau \in (0,1)$, $\lambda = \varepsilon^\delta$, introduce the function

$$\Phi(x,\xi,z) := u_\varepsilon(x) - u^-(\xi) - \varepsilon^{2-\gamma} w_\lambda\left(\frac{x}{\varepsilon}; z, \frac{z-\xi}{\varepsilon^\beta}\right) - \frac{|x-\xi|^2}{2\varepsilon^\beta} - \frac{|x-z|^2}{2\varepsilon^\beta} - \frac{\tau}{2}|\xi|^2$$

where the functions u_ε, u^- and w_λ are solutions respectively to (1.1), (4.8) and (4.2). We denote by $(\hat{x}, \hat{\xi}, \hat{z})$ the maximum point of Φ. Arguing as before, we have

$$u_\varepsilon(\hat{x}) + \overline{H}_-\left(\hat{z}, \frac{\hat{z}-\hat{\xi}}{\varepsilon^\beta}\right) \le C_1(\varepsilon^{1-\gamma-\beta(1-\alpha)/(2-\alpha)} + \varepsilon^{2-\gamma-\beta-\delta} + \varepsilon^{\gamma-\beta} + \varepsilon^{\delta-\beta(1-\alpha)/(2-\alpha)}). \tag{4.22}$$

In order to have an inequality similar to (3.16), for every $\sigma \in (0,1)$, we introduce the function

$$\Psi(\xi, y) := u^-(\xi) + \varepsilon^{2-\gamma} w_\lambda\left(\frac{\hat{x}}{\varepsilon}; \hat{z}, \frac{\hat{z}-y}{\varepsilon^\beta}\right) + \frac{|\hat{x}-\xi|^2}{2\varepsilon^\beta} + \frac{\tau}{2}|\xi|^2 + \frac{|\xi-y|^2}{2\sigma},$$

and we denote by (ξ_σ, y_σ) its minimum point in $B(\hat{\xi},1)\times B(\hat{\xi},1)$. Arguing as in the proof of Theorem 3.1.ii), we infer

$$u^-(\hat{\xi}) + \overline{H}_-\left(\hat{z}, \frac{\hat{z}-\hat{\xi}}{\varepsilon^\beta}\right) \ge C_1(\tau^{1/2} + \varepsilon^{2-\gamma-\beta-\delta} + \varepsilon^{\alpha\beta/(2-\alpha)}). \tag{4.23}$$

Finally, we compare (4.22) with (4.23), we use relation $\Phi(x,x,x) \le \Phi(\hat{x}, \hat{\xi}, \hat{z})$, Remark 4.2 and we let $\tau \to 0$. So we obtain that, for every $x \in \mathbb{R}^N$,

$$\begin{aligned} u_\varepsilon(x) - u(x) &\le C_1\left(\varepsilon^{1-\gamma-\beta(1-\alpha)/(2-\alpha)} + \varepsilon^{2-\gamma-\beta-\delta} + \varepsilon^{\gamma-\beta} + \varepsilon^{\delta-\beta(1-\alpha)/(2-\alpha)} + \varepsilon^{\alpha\beta/(2-\alpha)}\right) \\ &\le C_1 \varepsilon^{\alpha \min\{\frac{\gamma}{2}, 1-\gamma\}}, \end{aligned}$$

choosing , for $\gamma \le \frac{2}{3}$, $\beta = \frac{2-\alpha}{2}\gamma$ and $\delta = 1 - \frac{3}{4}\gamma$ and, for $\gamma > \frac{2}{3}$, $\beta = (2-\alpha)(1-\gamma)$ and $\delta = \frac{1}{2}$.

5 Rate of convergence for $\gamma > 1$

This section is devoted to the analysis of the case $\gamma > 1$. In this case the vanishing viscosity term is faster than the oscillation term, so the convergence behavior is similar to the one in homogenization of first order Hamilton-Jacobi equations.

Throughout this section we will assume assumptions (H1), (H2) and (H4). Let us emphasize that, in this case, the ellipticity and the convexity of the operator (i.e. assumptions (H3)) are not required and they are replaced by the coercivity assumption (H4). The main result in this section is Theorem 5.1.

5.1 Properties of the effective Hamiltonian, of the approximate correctors and of their inf and sup-convolutions

For each $(x,p) \in \mathbb{R}^N \times \mathbb{R}^N$, consider the following ergodic equation

$$H(x,y,p+D\chi,0) = \overline{H}_+(x,p), \qquad \chi \text{ periodic} \tag{5.1}$$

and its approximations

$$\lambda w_\lambda(y) + H(x, y, p + Dw_\lambda, 0) = 0. \tag{5.2}$$

We will denote by $\chi^+(\cdot; x, p)$ a periodic viscosity solution to (5.1) to recall the dependence on the fixed parameters x, p.

Remark 5.1 As for $\gamma < 1$, the cell problems, their ergodic approximations and the effective Hamiltonian $\overline{H}_-(x, p)$ are independent of $\gamma > 1$.

We collect in a lemma the main properties of w_λ, χ^+ and $\overline{H}_+$.

Lemma 5.1 *Assume that H satisfies (H1), (H2) and (H4). Let w_λ and χ^+ be respectively the solution to* (5.2) *and to* (5.1) *with* $\chi^+(0) = 0$. *Denote by $\overline{H}_+$ the effective Hamiltonian. Then*

(i) there exists $C > 0$ independent of γ, λ such that

$||\lambda w_\lambda(\cdot; x, p)||_\infty \le C(1 + |p|)$;

$|D_y w_\lambda(\cdot; x, p)| \le C(1 + |p|)$ *(in the viscosity sense);*

$\lambda|D_p w_\lambda| \le C$ *and* $\lambda|D_x w_\lambda| \le C(1 + |p|)$ *(in the viscosity sense);*

$||\lambda w_\lambda(\cdot; x, p) + \overline{H}_+(x, p)|| \le \lambda C(1 + |p|)$.

(ii) χ^+ is Lipschitz continuous.

(iii) $\overline{H}_+$ satisfies (H4) (the coercivity assumption) and (2.2) *where C is given in (i).*

Proof. The proof of this lemma can be found [10, Lemma 2.3].

Let w_λ be a solution to the approximate cell problem (5.2) centered at (x, p). We define its η- inf convolution as

$$(w_\lambda)_\eta(z; x, p) := \inf_y \left\{ w_\lambda(y; x, p) + \frac{|z - y|^2}{2\eta} \right\}. \tag{5.3}$$

We refer the reader for the definition of inf-convolution to [7, Lemma II.4.11, Lemma II.4.12]. In the following lemma we collect some useful properties of the function $(w_\lambda)_\eta$.

Lemma 5.2 *The function $(w_\lambda)_\eta$ fullfils the following properties, for some constant $C > 0$:*

i) $y \to (w_\lambda)_\eta(y; x, p) - |y|^2/\eta$ is a concave function. In particular $(w_\lambda)_\eta$ is semiconcave;

ii) $||\lambda(w_\lambda)_\eta(\cdot; x, p)||_\infty \le C(1 + |p|)$;

iii) $|D_y(w_\lambda)_\eta(\cdot; x, p)| \le C(1 + |p|)$ (in the viscosity sense);

iv) $\lambda|D_p(w_\lambda)_\eta| \le C$ and $\lambda|D_x(w_\lambda)_\eta| \le C(1 + |p|)$ (in the viscosity sense)

v) $(w_\lambda)_\eta(y; x, p)$ is a viscosity supersolution to

$$\lambda(w_\lambda)_\eta + H(x, y, p + D(w_\lambda)_\eta) \ge -C\eta(1 + |p|). \tag{5.4}$$

Proof. Property i) is well known. Items ii), iii), iv) come directly from the definition of $(w_\lambda)_\eta$ and from the properties of w_λ stated in Lemma 5.1. The proof of item (v) can be obtained as in [7, Proposition II.4.13], recalling that w_λ is Lipschitz.

We conclude recalling also a lemma on semiconcave functions.

Lemma 5.3 *Fix $u,\ \phi \in C^2(\mathbb{R}^N)$, a semiconcave function $v : \mathbb{R}^N \to \mathbb{R}$ and $L > 0$ such that $v(x)-L|x|^2$ is concave. Assume that $\hat{x}$ is a strict maximum point for $u-\phi-v$, then*

$$\hat{p} := Du(\hat{x}) - D\phi(\hat{x}) \in D^- v(\hat{x}).$$

Moreover, since v is semiconcave,

$$\hat{p} \in D^+ v(\hat{x}) \quad \text{and} \quad (\hat{p}, L\mathbf{I}) \in J^{2,+} v(\hat{x}).$$

Proof. The proof of the first part is in Lemma 2.4 in [10]. The second part is based on properties of semiconcave functions (see [7], [11]).

5.2 The convergence rate for $\gamma > 1$

Let u^+ be the unique bounded solution to

$$u + \overline{H}_+(x, Du) = 0. \tag{5.5}$$

with $\overline{H}_+$ given by the cell problem (5.1). Then, by Lemma 3.2.(iii) and Prop. 2.2, u is Lipschitz continuous.

Theorem 5.1 *Let u_ε and u be the unique bounded solutions resp. to* (2.1) *with $\gamma > 1$ and* (5.5)*.*

i) If $H(x, y, p, X) = H(y, p, X)$, there exists a constant $M > 0$ independent of γ such that

$$\|u_\varepsilon - u^+\|_\infty \le M\varepsilon^{\min(1, \frac{\gamma-1}{2})}. \tag{5.6}$$

ii) In general, there exists a constant $M > 0$ independent of γ such that

$$\|u_\varepsilon - u^+\|_\infty \le M\varepsilon^{\min\left(\frac{1}{3}, \frac{\gamma-1}{2}\right)}. \tag{5.7}$$

Remark 5.2 In particular u_ε converge uniformly in $\mathbb{R}^N$ as $\varepsilon \to 0$ for every $\gamma > 1$ to the same function u^+.

5.3 Proof of Theorem 5.1.i)

The solution u^+ is given by $-\overline{H}_+(0)$. Let χ be a solution of (5.1) with $p = 0$. Let χ^η and χ_η be the sup and the inf convolution of χ (see [7]). Set $\eta = \varepsilon^h$ with $h > 0$ to be fixed later. It is well known (e.g., see Lemma II.4.11 and Proposition II.4.3 in [7]) that χ^η satisfies

$$H(y, D\chi^\eta, \varepsilon^{\gamma-1}D^2\chi^\eta) - \overline{H}_+(0) \le C_1(\varepsilon^{\gamma-1-h} + \varepsilon^h) = C_1\varepsilon^{(\gamma-1)/2}$$

for C_1 sufficiently large, provided that we set $h = (\gamma-1)/2$. Similarly χ_η satisfies

$$H(y, D\chi_\eta, \varepsilon^{\gamma-1}D^2\chi_\eta) - \overline{H}_+(0) \ge C_1(\varepsilon^{\gamma-1-h} + \varepsilon^h) = C_1\varepsilon^{(\gamma-1)/2}.$$

We define

$$v_\varepsilon^\pm(x) := u^+ + \varepsilon\chi_\eta\left(\frac{x}{\varepsilon}\right) \pm K\varepsilon^{\min(1,(\gamma-1)/2)}$$

By the same arguments of proof of Theorem 4.1.i), one can easily check that, for K sufficiently large, v_ε^+ and v_ε^- are respectively a super and a subsolution to (1.1). By the comparison principle we obtain

$$-C_1\varepsilon^{\min(1,(\gamma-1)/2)} \le u_\varepsilon(x) - u^+ \le C_1\varepsilon^{\min(1,(\gamma-1)/2)}.$$

5.4 Proof of Theorem 5.1.ii)

The arguments of this proof are analogous to that in the proof of Theorem 3.1.ii). Then we will just sketch them. We define the auxiliary function

$$\Phi(x,\xi,z) := u_\varepsilon(x) - u^+(\xi) - \varepsilon(w_\lambda)_\eta\left(\frac{x}{\varepsilon}; z, \frac{z-\xi}{\varepsilon^\beta}\right) - \frac{|x-\xi|^2}{2\varepsilon^\beta} - \frac{|x-z|^2}{2\varepsilon^\beta} - \frac{\tau}{2}|\xi|^2, \tag{5.8}$$

where $0 < \tau < 1$, $0 < \beta < 1$ and $\eta > 0$ are parameters to be fixed later. We denote by $(\hat{x}, \hat{\xi}, \hat{z})$ the maximum point of Φ. Arguing as before, by Lemma 5.2, items (ii)-(iv), we get the same estimates of Lemma 3.3.

By Lemma 5.3, with $u(x) = u_\varepsilon(x)$, $\phi(x) = |x-\hat{\xi}|^2/(2\varepsilon^\beta)+|x-\hat{z}|^2/(2\varepsilon^\beta)$ and $v(\hat{x}) = \varepsilon(w_\lambda)_\eta(\hat{x}/\varepsilon; \hat{z}, (\hat{z}-\hat{\xi})/\varepsilon^\beta)$, we get that

$$\hat{p} := Du_\varepsilon(\hat{x}) - \frac{\hat{x}-\hat{\xi}}{\varepsilon^\beta} - \frac{\hat{x}-\hat{z}}{\varepsilon^\beta} \in D^-(w_\lambda)_\eta\left(\frac{\hat{x}}{\varepsilon}; \hat{z}, \frac{\hat{z}-\hat{\xi}}{\varepsilon^\beta}\right)$$

and moreover

$$\left(\hat{p} + \frac{\hat{x}-\hat{\xi}}{\varepsilon^\beta} + \frac{\hat{x}-\hat{z}}{\varepsilon^\beta}, \frac{1}{\eta\varepsilon}\mathbf{I} + \frac{2}{\varepsilon^\beta}\mathbf{I}\right) \in J^{2,+}u_\varepsilon(\hat{x}).$$

Then, since u_ε is a subsolution to (2.1), we infer

$$0 \geq u_\varepsilon(\hat{x}) + H\left(\hat{x}, \frac{\hat{x}}{\varepsilon}, D(w_\lambda)_\eta\left(\frac{\hat{x}}{\varepsilon}; \hat{z}, \frac{\hat{z}-\hat{\xi}}{\varepsilon^\beta}, \left(2\varepsilon^{\gamma-\beta} + \frac{\varepsilon^{\gamma-1}}{\eta}\right)\mathbf{I}\right) + \frac{\hat{x}-\hat{\xi}}{\varepsilon^\beta} + \frac{\hat{x}-\hat{z}}{\varepsilon^\beta}\right). \tag{5.9}$$

Moreover, by Lemma 3.3 and Lemma 5.2-(iii), let us observe that

$$H\left(\hat{x}, \frac{\hat{x}}{\varepsilon}, \hat{p} + \frac{\hat{x}-\hat{\xi}}{\varepsilon^\beta} + \frac{\hat{x}-\hat{z}}{\varepsilon^\beta}\right) \geq H\left(\hat{z}, \frac{\hat{x}}{\varepsilon}, \hat{p} + \frac{\hat{z}-\hat{\xi}}{\varepsilon^\beta}\right) - K\varepsilon^{1-\delta-\beta}.$$

Taking into account equation (5.4), we deduce

$$\begin{aligned} H\left(\hat{z}, \frac{\hat{x}}{\varepsilon}, \hat{p} + \frac{\hat{z}-\hat{\xi}}{\varepsilon^\beta}\right) &\geq -\lambda(w_\lambda)_\eta\left(\frac{\hat{x}}{\varepsilon}; \hat{z}, \frac{\hat{z}-\hat{\xi}}{\varepsilon^\beta}\right) - C\eta\left(1 + \frac{|\hat{z}-\hat{\xi}|}{\varepsilon^\beta}\right) \\ &\geq -\lambda w_\lambda\left(\frac{\hat{x}}{\varepsilon}; \hat{z}, \frac{\hat{z}-\hat{\xi}}{\varepsilon^\beta}\right) - C\eta\left(1 + \frac{|\hat{z}-\hat{\xi}|}{\varepsilon^\beta}\right) \\ &\geq \overline{H}\left(\hat{z}, \frac{\hat{z}-\hat{\xi}}{\varepsilon^\beta}\right) - C(\varepsilon^\delta + \eta) \end{aligned}$$

where in the last inequality Lemma 3.3 and Lemma 5.1.(iii) have been used.

Substituting the last two inequalities in relation (5.9) and recalling assumption (H2), we obtain

$$u_\varepsilon(\hat{x}) + \overline{H}\left(\hat{z}, \frac{\hat{z}-\hat{\xi}}{\varepsilon^\beta}\right) \leq C_1\left(\varepsilon^{\gamma-\beta} + \frac{\varepsilon^{\gamma-1}}{\eta} + \varepsilon^{1-\beta-\delta} + \varepsilon^\delta + \eta\right).$$

On the other hand, by the same argument of Theorem 3.1 (with $\alpha = 1$), we infer

$$u^+(\hat{\xi}) + \overline{H}\left(\hat{z}, \frac{\hat{z}-\hat{\xi}}{\varepsilon^\beta}\right) \geq -C_1(\varepsilon^{1-\beta-\delta} + \sqrt{\tau} + \varepsilon^\beta).$$

The last two inequalities entail that there holds

$$u_\varepsilon(\hat{x}) - u^+(\hat{\xi}) \le C_1 \left(\varepsilon^{1-\delta-\beta} + \varepsilon^\delta + \varepsilon^\beta + \varepsilon^{\gamma-\beta} + \frac{\varepsilon^{\gamma-1}}{\eta} + \eta \right) + C_1 \sqrt{\tau}. \tag{5.10}$$

Finally, by the definition of Φ in (5.8), for every $x \in \mathbb{R}^N$ the following relation holds

$$u_\varepsilon(x) - u^+(x) \le u_\varepsilon(\hat{x}) - u^+(\hat{\xi}) + \varepsilon \left((w_\lambda)_\eta \left(\frac{x}{\varepsilon}; x, 0 \right) - (w_\lambda)_\eta \left(\frac{\hat{x}}{\varepsilon}; \hat{z}, \frac{\hat{z} - \hat{\xi}}{\varepsilon^\beta} \right) \right) + \frac{\tau}{2}|x|^2.$$

Taking into account relation (5.10), Lemma 3.3 and Lemma 5.2.(ii), we deduce

$$u_\varepsilon(x) - u^+(x) \le C_1 \left(\varepsilon^{1-\delta-\beta} + \varepsilon^\delta + \varepsilon^\beta + \varepsilon^{\gamma-\beta} + \frac{\varepsilon^{\gamma-1}}{\eta} + \eta \right) + C_1 \sqrt{\tau} + \frac{\tau}{2}|x|^2.$$

Letting $\tau \to 0$ we obtain that for every $x \in \mathbb{R}^N$

$$u_\varepsilon(x) - u^+(x) \le C_1 \left(\varepsilon^{1-\delta-\beta} + \varepsilon^\delta + \varepsilon^\beta + \varepsilon^{\gamma-\beta} + \frac{\varepsilon^{\gamma-1}}{\eta} + \eta \right).$$

Choosing by symmetry, $\beta = \delta = 1/3$ and $\eta = \varepsilon^{(\gamma-1)/2}$, we accomplish the first inequality of the statement. Being similar, the proof of other inequality is omitted.

6 Examples

In this section we describe some examples of problems satisfying our structural assumptions (H1)-(H4). The typical example is an Hamiltonian of Isaacs type

$$H\left(x, \frac{x}{\varepsilon}, Du, \varepsilon^\gamma D^2 u\right) = \min_{\theta \in \Theta} \max_{\zeta \in Z} \left\{ L^{\theta,\zeta}\left(x, \frac{x}{\varepsilon}, Du, D^2 u\right) \right\} \tag{6.1}$$

$$H\left(x, \frac{x}{\varepsilon}, Du, \varepsilon^\gamma D^2 u\right) = \max_{\zeta \in Z} \min_{\theta \in \Theta} \left\{ L^{\theta,\zeta}\left(x, \frac{x}{\varepsilon}, Du, D^2 u\right) \right\} \tag{6.2}$$

where

$$L^{\theta,\zeta}\left(x, \frac{x}{\varepsilon}, p, X\right) = -\mathrm{tr}\left(\varepsilon^\gamma a^{\theta,\zeta}\left(x, \frac{x}{\varepsilon}\right) X\right) - f^{\theta,\zeta}\left(x, \frac{x}{\varepsilon}\right) \cdot p - l^{\theta,\zeta}\left(x, \frac{x}{\varepsilon}\right). \tag{6.3}$$

Note that $L^{\theta,\zeta}$ is the generator of a diffusion process satisfying

$$dX_s = f^{\theta_s,\zeta_s}\left(X_s, \frac{X_s}{\varepsilon}\right) ds + \varepsilon^{\gamma/2} \sigma^{\theta_s,\zeta_s}\left(X_s, \frac{X_s}{\varepsilon}\right) dW_s$$

with $a^{\theta,\zeta} = \sigma^{\theta,\zeta}(\sigma^{\theta,\zeta})^T/2$. Assumptions (H1), (H2) are in this case satisfied if

(A_1) $a^{\theta,\zeta}$, $f^{\theta,\zeta}$ and $l^{\theta,\zeta}$ are 1-periodic in y for any $x \in \mathbb{R}^N$;

(A_2) Θ, Z are compact metric space; $a^{\theta,\zeta}$, $f^{\theta,\zeta}$ and $l^{\theta,\zeta}$ are bounded continuous functions, Lipschitz in (x, y) uniformly in θ, ζ with Lipschitz constant C.

For assumption (H3) we need

(A_3) Θ reduces to a singleton; $a^{\theta,z}(x,y) \geq \nu I$, $\forall (x,y,\theta,z) \in \mathbb{R}^N \times \mathbb{R}^N \times \Theta \times Z$.

Finally the coercitivity (H4) for the Hamiltonian (6.1) is implied by

(A_4) There exists $\nu > 0$ such that $B(0,\nu) \subset \overline{\text{co}}\{f^{\theta,\zeta}(x,y) : z \in Z\}$ for any $\theta \in \Theta$, where $\overline{\text{co}}$ indicates the closed convex hull.

An interesting collateral problem to that of convergence of (1.1) is the characterization of the effective Hamiltonian, the Hamiltonian of the limit problem. In general explicit formulas are not available and the problem must be treated numerically (see [1], [9], [21]). Nevertheless in some special cases it is possible to give representation formulas of the effective Hamiltonian.

Example 6.1 We consider $\gamma < 1$ and F as in Lemma 4.1, i.e. the problem

$$u_\varepsilon - \varepsilon^\gamma \text{tr}\left(a\left(x, \frac{x}{\varepsilon}\right) D^2 u_\varepsilon\right) + F\left(x, \frac{x}{\varepsilon}, Du_\varepsilon\right) = 0. \tag{6.4}$$

We observe that the effective equation (4.1) coincides with the one arising in homogenization of second order equations (i.e. $\gamma = 0$ in (6.4)). By the results in [4, 9], $\overline{H}_-$ is given by

$$\overline{H}_-(x,p) := \int_{[0,1)^N} F(x,y,p)\, d\mu_x(y).$$

where for $\bar{x}$ fixed the measure $\mu_{\bar{x}}$ is the unique invariant measure for the diffusion associated to the matrix $a(\bar{x},y)$ and it is characterized by the adjoint equation

$$\sum_{i,j=1}^N \frac{\partial^2}{\partial y_i\, \partial y_j}\left(a_{ij}(\bar{x},y)\mu_{\bar{x}}\right) = 0, \qquad \mu_{\bar{x}} \text{ periodic}, \int_{(0,1)^N} d\mu_x(y) = 1.$$

In this case, by Theorem 4.1.(ii), we have the estimate

$$\|u_\varepsilon - u^-\|_\infty \leq M\varepsilon^{\min(\frac{\alpha\gamma}{2}, 1-\gamma)} \qquad \forall \gamma < 1.$$

If we assume that the diffusion coefficient $a(x,y)$ is also independent of x, then for every $\bar{x}$, the unique invariant measure $\mu_{\bar{x}}$ coincides with the Lebesgue one and therefore for any x

$$\overline{H}_-(x,p) := \int_{[0,1)^N} F(x,y,p)\, dy. \tag{6.5}$$

In this case by Theorem 4.1.(i) we have the estimate

$$\|u_\varepsilon - u^-\|_\infty \leq M\varepsilon^{1-\gamma} \qquad \forall \gamma < 1.$$

Example 6.2 For $\gamma = 1$ we consider the classical case of a linear problem, i.e.

$$u_\varepsilon - \varepsilon \text{tr}\left(a\left(x, \frac{x}{\varepsilon}\right) D^2 u_\varepsilon\right) + F\left(x, \frac{x}{\varepsilon}, Du_\varepsilon\right) = 0 \tag{6.6}$$

where $F(x, y, p) = -f(x, y) \cdot p - l(x, y)$ with f, l smooth. The invariant measure $\mu_x(y)$ associated to the process generated by $a(x, y)$ and $f(x, y)$ is the unique solution of the adjoint equation

$$\sum_{i,j=1}^{N} \frac{\partial^2}{\partial y_i \partial y_j}(a_{ij}(x, y)\mu_x) + \sum_{i=1}^{N} \frac{\partial}{\partial y_i}(f_i(x, y)\mu_x) = 0, \qquad \mu_{\bar{x}} \text{ periodic},$$

such that $\int_{(0,1)^N} d\mu_x(y) = 1$. Then the effective Hamiltonian is given by

$$\overline{H}_1(x, p) = \int_{(0,1)^N} F(x, y, p) d\mu_x(y) = -\int_{(0,1)^N} f(x, y) \cdot p \, d\mu_x(y) - \int_{(0,1)^N} l(x, y) \, d\mu_x(y).$$

Example 6.3 Consider for $\gamma > 1$ the semilinear problem

$$u_\varepsilon - \varepsilon^\gamma \mathrm{tr}\left(a\left(x, \frac{x}{\varepsilon}\right) D^2 u_\varepsilon\right) + F\left(x, \frac{x}{\varepsilon}, Du_\varepsilon\right) = 0 \tag{6.7}$$

with

$$F(x, y, p) = \max_{\zeta \in Z} \{-f(x, y, \zeta) \cdot p - l(x, y, \zeta)\}.$$

Following [2], we give an explicit representation formula for $\overline{H}_+$ in terms of relaxed controls. Let $\mathcal{M}$ be the set of the probability measures on $[0, 1]^N \times Z$ and define a relaxed control problem by introducing for $\phi = f, l$ the relaxed function

$$\phi^r : \mathbb{R}^N \times \mathcal{M} \to \mathbb{R}^N \qquad \phi^r(x, \mu) := \int_{[0,1]^N \times Z} \phi(x, y, \zeta) d\mu.$$

A measure $\mu \in \mathcal{M}$ is a *limiting relaxed control* if there exists a control law ζ_s such that

$$\mu_n := \frac{1}{t_n} \int_0^{t_n} \delta_{Y_s, \zeta_s} ds \longrightarrow \mu \qquad \text{weak-}*$$

where $\dot{Y}_s = f(x, Y_s, \zeta_s)$, $Y_0 = x$ and $\delta_{Y,\zeta}$ is the Dirac's mass at $(y, \zeta) \in [0, 1]^N \times Z$. We denote by $\mathcal{M}_l(x)$ the set of the limiting relaxed controls for the initial condition $Y_0 = x$. By [2, Thm 7], we have

$$\overline{H}_+(x, p) = \sup_{\mu \in \mathcal{M}_l(x)} \{-f^r(x, \mu) \cdot p - l^r(x, \mu)\}.$$

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