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# Monodromy of logarithmic Barsotti-Tate groups attached to 1-motives

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**Abstract.** Let $K$ be a complete discrete valuation field with residue field of positive characteristic. We study the Barsotti-Tate group of a $K$-1-motive and we give a condition to extend it to a logarithmic BT-group over the valuation ring. We compare two notions of monodromy appearing in the literature.

## Introduction

Let $R$ be a complete discrete valuation ring with residue field $k$ of positive characteristic $p$ and field of fractions $K$. In this paper we consider a $K$-1-motive $M_K$ as in [15] and its associated Barsotti-Tate group. This last does not in general extend to a Barsotti-Tate group over $R$. However, with some assumptions, it extends to a logarithmic Barsotti-Tate group over $R$. This follows from [15] and Kato's results on finite logarithmic group schemes. Once chosen a uniformizing parameter $\pi$ of $R$, any logarithmic Barsotti-Tate group over $R$ is described by two data $(G, N)$ where $G$ is a classical Barsotti-Tate group over $R$ and $N$ is a homomorphism of classical Barsotti-Tate groups. Moreover, if $k$ is perfect and $R = W(k)$, $N$ induces a $W(k)$-homomorphism $\mathcal{N} : \mathbb{M}(G_k) \to \mathbb{M}(G_k)$ on Dieudonné modules such that $F\mathcal{N}V = \mathcal{N}$ and $\mathcal{N}^2 = 0$. In the first part of the paper we recall these constructions and we show how to relate $N$ with the "geometric monodromy" introduced by Raynaud. In the second part of the paper we give an explicit description of $\mathcal{N}$ in terms of additive extensions and integrals. In the last part of the paper we describe how to recover the logarithmic Barsotti-Tate group attached to a 1-motive from concrete schemes endowed with a suitable logarithmic structure.

## 1. 1-motives

**Definition 1.** Let $S$ be a scheme. An $S$-1-*motive* $M = [u : Y \to G]$ is a two term complex (in degree $-1$, 0) of commutative group schemes over $S$ such that:

1) Partially supported by Progetto Giovani Ricercatori CPDG021784 Università di Padova.

1. $Y$ is an $S$-group scheme that locally for the étale topology on $S$ is isomorphic to a constant group of type $\mathbb{Z}^r$.

2. $G$ is an $S$-group scheme extension of an abelian scheme $A$ over $S$ by a torus $T$.

3. $u$ is an $S$-homomorphism $Y \to G$.

Morphisms of $S$-1-motives are usual morphisms of complexes.

**Definition 2.** Let $M_K$ be a $K$-1-motive. One says that $M_K$

1. has *good reduction* if $M_K$ extends to a 1-motive over $R$, i.e. if

- $Y_K$ is not ramified over $R$,
- $T_K$ has good reduction over $R$,
- $A_K$ has good reduction over $R$,
- $u_K$ extends to a homomorphism $u : Y \to G$.

(Hence $G_K$ extends to a semi-abelian $R$-group scheme $G$.)

2. has *semistable reduction* if

- $Y_K$ is not ramified over $R$,
- $T_K$ has good reduction over $R$,
- $A_K$ has semistable reduction over $R$.

(Hence $G_K$ extends to a smooth $R$-group scheme with semi-abelian special fibre.[2)])

3. has *potentially semistable* (resp. *good*) *reduction* if it acquires semistable (resp. good) reduction after a finite extension of $K$.

4. is *strict* if $G_K$ has potentially good reduction.

Observe that any $K$-1-motive has potentially semistable reduction. However, even if we allow base change, the morphism $u_K$ does not in general extend over $R$. A simple example is the Tate curve $u_K : \mathbb{Z} \to \mathbb{G}_{m,K}$ with $u_K(1) = \pi$ a uniformizing element. It has semistable reduction but no good reduction.

In the following we will consider only $K$-1-motives or $R$-1-motives. For more details see Raynaud's paper [15]. We recall now a definition from [11], 4.6.1.

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2) Cf. [15], §4.

**Definition 3.** A *log* 1-*motive* over $R$ is a triple $(Y, G, u_K)$ where $Y, G$ are commutative group schemes over $R$ with $Y$ (resp. $G$) satisfying condition 1 (resp. 2) in Definition 1 for $S = \mathrm{Spec}(R)$ and $u_K : Y_K \to G_K$ is a homomorphism on generic fibres.

Observe that if $(Y, G, u_K)$ is a log 1-motive then $[u_K : Y_K \to G_K]$ is a strict $K$-1-motive.

**1.1. The Barsotti-Tate group attached to a $K$-1-motive.** Let $n$ be any positive integer and denote by ${}_nH$ the kernel of $n$-multiplication on a group $H$. For any $K$-1-motive $M_K = [u_K : Y_K \to G_K]$ one can construct an exact sequence of finite $n$-torsion group schemes over $K$:

$$\eta(n, u_K) : 0 \to {}_nG_K \to {}_nM_K \to Y_K/nY_K \to 0 \tag{1}$$

where ${}_nM_K$ is the cokernel of the homomorphism

$$Y_K \xrightarrow{(-n,-u_K)} Y_K \times_{G_K} G_K;$$

here the fibre product is taken with respect to $u_K$ on $Y_K$ and the $n$-multiplication on $G_K$. As explained in [15], 3.1, ${}_nM_K$ is the $\mathrm{H}^{-1}(C(M,n))$ with $C(M,n)$ the cone of the $n$-multiplication on the 1-motive $M_K$, i.e.

$$\begin{array}{rcccc} C(M,n) : Y_K & \to & Y_K \oplus G_K & \to & G_K, \\ y & \mapsto & \big(-nx, -u_K(y)\big), & & \\ & & (y,g) & \mapsto & u_K(y) - ng, \end{array}$$

in degree $-2$, $-1$, 0.

**Definition 4.** The $p$-divisible group or Barsotti-Tate group of the $K$-1-motive $M_K$ is $\varinjlim({}_{p^m}M_K)$.

In the previous notations we have then an exact sequence of BT-groups:

$$0 \to \varinjlim({}_{p^m}G_K) \to \varinjlim({}_{p^m}M_K) \to \varinjlim(Y_K/p^mY_K) \to 0.$$

It is clear that if $M_K$ has good reduction then $\varinjlim({}_{p^r}M_K)$ extends to a BT-group over $R$. We want to understand what happens in the general case. We state now a result that we will need later.

**Lemma 5.** *Let notations be as above.*

1. *Consider the following diagram obtained via push-out by* $u_K$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y_K & \xrightarrow{-n} & Y_K & \longrightarrow & Y_K/nY_K & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle u_K} & & \Big\downarrow & & \Big\downarrow & & \\ 0 & \longrightarrow & G_K & \longrightarrow & Y_K \amalg_{Y_K} G_K & \longrightarrow & Y_K/nY_K & \longrightarrow & 0. \end{array}$$

*The short exact sequence $\eta(n, u_K)$ in (1) is isomorphic to the sequence of kernels for the n-multiplication of the lower sequence.*

2. *Consider the following diagram obtained via pull-back by $u_K$:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & {}_nG_K & \longrightarrow & Y_K \times_{G_K} G_K & \longrightarrow & Y_K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle u_K} & & \\ 0 & \longrightarrow & {}_nG_K & \longrightarrow & G_K & \xrightarrow{n} & G_K & \longrightarrow & 0. \end{array}$$

*The short exact sequence $\eta(n, u_K)$ in (1) is isomorphic to the sequence of cokernels for the n-multiplication of the upper sequence.*

Raynaud shows in [15] that to any $K$-1-motive $M_K$ it is possible to associate in a canonical way a strict $K$-1-motive $M_K'$ having the same BT-group of $M_K$. His construction makes use of rigid analytic methods. As a consequence, working with BT-groups attached to $K$-1-motives, *one can always assume the $K$-1-motives to be strict*. We will do so in the sequel.

**1.2. Geometric monodromy.** Given a strict $K$-1-motive, the failure of good reduction is controlled by a pairing, the so-called geometric monodromy. To define it we need to recall some facts on the Poincaré bundle.

**Remark 6.** Let $M_K = [u_K : Y_K \to G_K]$ be a $K$-1-motive and $Y_K^*$ be the group of characters of the torus part $T_K$ of $G_K$ and $A_K$ the abelian variety $G_K/T_K$. It is known[3] that to give a 1-motive as above is equivalent to giving morphisms $h_K : Y_K \to A_K$, $h_K^* : Y_K^* \to A_K^*$ (with $A_K^*$ the dual variety of $A_K$) and a trivialization $s_K : Y_K \times Y_K^* \to \mathscr{P}_K^Y$ with $\mathscr{P}_K^Y$ the pull-back via $h_K \times h_K^*$ of the biextension $\mathscr{P}_K$. Suppose that $G_K$ has good reduction. Then both $A_K$ and the dual abelian variety $A_K^*$ have good reduction and the Poincaré bundle $\mathscr{P}_K$ extends to a biextension $\mathscr{P}$ in $\mathrm{Biext}^1(A, A^*; \mathbb{G}_{m,R})$ on Néron models. Also $h_K, h_K^*$ extend to morphisms $h, h^*$ over $R$ and the pull-back of $\mathscr{P}$ via $h \times h^*$ provides a biextension $\mathscr{P}^Y$ in $\mathrm{Biext}^1(Y, Y^*; \mathbb{G}_{m,R})$, whose generic fibre is $\mathscr{P}_K^Y$.

**Definition 7** ([15], §4.3). Let $M_K = [u_K : Y_K \to G_K]$ be a strict $K$-1-motive and $Y_K^*$ the group of characters of $T_K$. The *geometric monodromy* of $M_K$ is a morphism

$$\mu : Y_K \otimes Y_K^* \to \mathbb{Q} \tag{2}$$

defined as follows:

1. Suppose that $G_K$ has good reduction. Then there exists a trivialization $s_K$ of $\mathscr{P}^Y \in \mathrm{Biext}^1(Y, Y^*; \mathbb{G}_{m,R})$ on generic fibres (see Remark 6) and hence a trivialization $s$ of the image of $\mathscr{P}_K^Y$ in $\mathrm{Biext}^1(Y, Y^*; \mathscr{G})$.[4] Therefore the biextension $\mathscr{P}^Y$ is the pull-back of

$$0 \to \mathbb{G}_{m,R} \to \mathscr{G} \to i_*\mathbb{Z} \to 0$$

[3] See for example [3], 10.2.14 and [1], II, 2.3.3.

[4] Notations are those in [6], VIII; we use that $\mathrm{Biext}^1(Y, Y^*; \mathscr{G}) \cong \mathrm{Biext}^1(Y_K, Y_K^*; \mathbb{G}_{m,K})$.

via a unique $\mu_0 \in \mathrm{Hom}(Y_K \otimes Y_K^*, \mathbb{Z}) = \mathrm{Hom}(Y \otimes Y^*, i_*\mathbb{Z}) = \mathrm{Biext}^0(Y, Y^*; i_*\mathbb{Z})$ that factors through $s$. One sets $\mu = \mu_0$.

2. In the general situation, $G_K$ reaches good reduction after a Galois extension $K'$ of $K$. Now the monodromy on $K'$ is compatible with Galois action and can be descended to a $\mu$ as in (2).

Observe that $\mathbb{Q}$ has to be thought of as the group of values of the valuation of the algebraic closure of $K$ with $\mathbb{Z}$ the group of values assumed on $K$.

Let $K^{\mathrm{un}}$ be the maximal unramified extension of $K, v : (K^{\mathrm{un}})^* \to \mathbb{Z}$ the valuation and $R^{\mathrm{un}}$ its valuation ring. Observe that in the hypothesis of Definition 7/1 there is a valuation $v_{\mathscr{P}}$ on $\mathscr{P}_K(K^{\mathrm{un}})$ and that $\mu_0 = v_{\mathscr{P}} \circ s_K$ holds. Moreover if the abelian part is trivial, then $\mathscr{P}_K = \mathbb{G}_{m,K}$, $h_K$ and $h_K^*$ are the structure morphisms and $s_K : Y_K \otimes Y_K^* \to \mathbb{G}_{m,K}$ is the usual pairing $(y, y^*) \mapsto y^*(y)$. Hence $\mu_0(y, y^*) = v\big(y^*(y)\big)$. These results can be generalized. See also 4.6/6 in [15].

**Lemma 8.** *Let notations be as above. Suppose that $G_K$ has good reduction over a finite field extension $L$ of $K$. Then the geometric monodromy pairing $\mu$ coincides with the pairing*

$$Y_K \otimes Y_K^* \to \mathbb{Q}, \quad (y, y^*) \mapsto \frac{1}{e} v_L\big(y^*(t)\big)$$

*where $e$ is the index of ramification of $L/K$, $v_L$ is the extension of the valuation of $L$ to $L^{\mathrm{un}}$, $t \in T_K(L^{\mathrm{un}})$ is any point whose image in the component group $\mathbb{Z}^d$ of $T_{L^{\mathrm{un}}}$ (as well as of $G_{L^{\mathrm{un}}}$) coincides with the image of $u_K(y)$ and $y^*(t) \in (L^{\mathrm{un}})^*$ for any $y^* : T_{L^{\mathrm{un}}} \to \mathbb{G}_{m,L^{\mathrm{un}}}$.*

*Proof.* We may reduce to the case $L = K$. As explained above $\mu_0(y, y^*)$ is obtained via the valuation on $\mathscr{P}_K(K^{\mathrm{un}})$. Consider the push-out

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_K & \longrightarrow & G_K & \longrightarrow & A_K & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle y^*} & & \downarrow{\scriptstyle g_{y^*}} & & \| & & \\ 0 & \longrightarrow & \mathbb{G}_{m,K} & \longrightarrow & G_{y^*} & \longrightarrow & A_K & \longrightarrow & 0 \end{array} \tag{3}$$

where $G_{y^*} \cong \mathscr{P}_{K|A_K \times h^*(y^*)}$. The valuation $v_{\mathscr{P}}$ on $\mathscr{P}_K(K^{\mathrm{un}})$ restricts to a $v_{y^*}$ on $G_{y^*}(K^{\mathrm{un}})$ and $\mu_0(y, y^*) = v_{y^*}\big(g_{y^*} \circ u_K(y)\big)$ holds. Let now $t \in T_K(K^{\mathrm{un}})$ be a point having the same image as $u_K(y)$ in the component group $\mathbb{Z}^d$. Then one has $v_{y^*}\big(g_{y^*} \circ u_K(y)\big) = v\big(y^*(t)\big)$ for any character $y^*$ and $u_K(y) - t \in G(R^{\mathrm{un}})$. To conclude it is now sufficient to observe that $v_{y^*} \circ g_{y^*}$ is zero on $G(R^{\mathrm{un}})$ and that $v\big(y^*(t)\big) = v_{y^*}(g_{y^*}t)$. $\square$

**1.3. Devissage.** Once having realized that the defect of good reduction is controlled by the geometric monodromy, Raynaud explains, under the hypothesis that the geometric monodromy takes integer values, how to decompose a strict 1-motive into the sum of two 1-motives, the first having potentially good reduction and the second codifying the monodromy.

**Theorem 9.** *Let $M_K = [u_K : Y_K \to G_K]$ be a strict $K$-1-motive such that the geometric* *monodromy $\mu$ factors through $\mathbb{Z}$. Then for any choice of a uniformizing parameter $\pi$ of $R$ there is a canonical decomposition*

$$u_K = u^1_{K,\pi} + u^2_{K,\pi} \tag{4}$$

*where $u^2_{K,\pi}$ factors through the torus part $T_K$ and is given by the formula*

$$\begin{aligned} u^2_{K,\pi} : Y_K \to T_K = \underline{\mathrm{Hom}}(Y^*_K, \mathbb{G}_{m,K}) \xrightarrow{\iota} G_K, \\ y \mapsto (y^* \mapsto \pi^{\mu(y,y^*)}) \end{aligned} \tag{5}$$

*while $u^1_{K,\pi}$ has potentially good reduction.*

*Proof.* [15], 4.5.1. □

**Remark 10.** If both $G_K$ and $Y_K$ have good reduction (i.e. if $M_K$ comes from a log 1-motive), the geometric monodromy factors through $\mathbb{Z}$ and the 1-motive $u^1_{K,\pi}$ in the previous decomposition has good reduction.

**Remark 11.** Let notations be as above and $\rho : G_K \to A_K$ be the morphism of $G_K$ in its abelian quotient. It is clear that $\rho \circ u_K = \rho \circ u^1_{K,\pi}$ has potentially good reduction. Moreover the push-out of $\eta(n, u_K)$ with respect to $\rho_n : {}_nG_K \to {}_nA_K$ (the restriction of $\rho$ to the kernels of $n$-multiplication) is $\eta(n, \rho \circ u_K)$. More precisely we have:

$$\begin{tikzcd} & & {}_nT_K \arrow[r, equal] \arrow[d, "w"] & {}_nT_K \arrow[d, "\tau"] & & \\ \eta(n,u_K): & 0 \arrow[r] & {}_nG_K \arrow[r] \arrow[d, "\rho_n"] & {}_nM_K \arrow[r, "h"] \arrow[d, "g"] & Y_K/nY_K \arrow[r] \arrow[d, equal] & 0 \\ \eta(n,\rho\circ u_K): & 0 \arrow[r] & {}_nA_K \arrow[r] & {}_nM^A_K \arrow[r, "f"] & Y_K/nY_K \arrow[r] & 0. \end{tikzcd} \tag{6}$$

Suppose that the hypothesis of Theorem 9 holds. We wish to compare the sequence of finite $n$-torsion group schemes $\eta(n, u_K)$ in (1) associated to $u_K$ with the sequences associated to $u^1_{K,\pi}$ and $u^2_{K,\pi}$.

**Lemma 12.** *Let $M_K : [u_K : Y_K \to G_K]$ be a strict $K$-1-motive such that the geometric monodromy factors through $\mathbb{Z}$. Let $u_K = u^1_{K,\pi} + u^2_{K,\pi}$ be the decomposition of Theorem 9. Then $\eta(n, u_K)$ is isomorphic to $\eta(n, u^1_{K,\pi}) + \eta(n, u^2_{K,\pi})$ where $+$ denotes Baer's sum.*

*Proof.* Consider the homomorphism

$$\mathrm{Hom}(Y_K, G_K) \xrightarrow{\partial} \mathrm{Ext}^1(Y_K, {}_nG_K) \cong \mathrm{Ext}^1_{\mathbb{Z}/n\mathbb{Z}}(Y_K/nY_K, {}_nG_K)$$

that associates to a 1-motive $u_K$ the pull-back of $0 \to {}_nG_K \to G_K \xrightarrow{n} G_K \to 0$ by $u_K$, resp. the sequence of cokernels of such pull-back. Here the subscript $\mathbb{Z}/n\mathbb{Z}$ stands for extensions in the category of $\mathbb{Z}/n\mathbb{Z}$-modules. We have already seen in Lemma 5/2 that the isomorphism class of $\eta(n, u_K)$ is $\partial(u_K)$. The result follows from the fact that $\partial$ is a homomorphism. □

**Lemma 13.** *Given a $K$-1-motive $u_K : \mathbb{Z}^r \to \mathbb{G}^d_{m,K}$, (the isomorphism class of) the sequence $\eta(n, u_K)$ extends over $R$ if and only if $u^2_{K,\pi}$ is divisible by $n$, i.e. if and only if $\eta(n, u^2_{K,\pi})$ is isomorphic to the trivial sequence.*

*Proof.* Recall the following diagram:

$$\begin{array}{ccccccc}
\mathrm{Hom}(\mathbb{Z}^r, \mathbb{G}^d_{m,R}) & \xrightarrow{n} & \mathrm{Hom}(\mathbb{Z}^r, \mathbb{G}^d_{m,R}) & \xrightarrow{\partial} & \mathrm{Ext}^1(\mathbb{Z}^r, \boldsymbol{\mu}^d_{n,R}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Hom}(\mathbb{Z}^r, \mathbb{G}^d_{m,K}) & \xrightarrow{n} & \mathrm{Hom}(\mathbb{Z}^r, \mathbb{G}^d_{m,K}) & \xrightarrow{\partial} & \mathrm{Ext}^1(\mathbb{Z}^r, \boldsymbol{\mu}^d_{n,K}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \\
\mathbb{Z}^{rd} & \xrightarrow{n} & \mathbb{Z}^{rd} & & & &
\end{array}$$

where as above $\partial$ is obtained by pull-back. Suppose now that $\eta(n, u_K) = \partial(u_K)$ extends over $R$. The same is true for $\eta(n, u^1_{K,\pi})$ because $u^1_{K,\pi}$ has good reduction and hence also $\eta(n, u^2_{K,\pi})$ extends over $R$. Let $w_K$ be a 1-motive with good reduction such that $\eta(n, w_K) = \eta(n, u^2_{K,\pi})$. Define $w'_K := w_K - u^2_{K,\pi}$; observe that this is also the Raynaud decomposition of $w'_K$ and that $\partial(w'_K) = 0$. Hence there exists a 1-motive $w''_K$ such that $n \cdot w''_K = w'_K$. Let $w''_K = w^1_{K,\pi} + w^2_{K,\pi}$ be Raynaud's decomposition. It is clear that the $n$-multiplication preserves Raynaud's decompositions and hence $n \cdot w^1_{K,\pi} = w_K$ while $n \cdot w^2_{K,\pi} = -u^2_{K,\pi}$. Hence $u^2_{K,\pi}$ is divisible by $n$. $\square$

**1.3.1. Geometric monodromy à la Kato.** In the following we will work with group schemes as sheaves of $\mathbb{Z}$-modules on the flat site. We wish to understand better the $K$-1-motive $u^2_{K,\pi}$ in (4) in order to compare Kato's monodromy and Raynaud's monodromy.

Given an étale group scheme $N_K$ isomorphic to some $\mathbb{Z}^r$ over an algebraic closure of $K$, denote by $N_K^\vee$ the étale group scheme $\underline{\mathrm{Hom}}(N_K, \mathbb{Z})$ and by $N_K^{\vee D}$ its Cartier dual. We have $\mathbb{Z}^{\vee D} = \mathbb{G}_{m,K}$ and $N_K^{\vee D} = N_K \otimes_{\mathbb{Z}} \mathbb{G}_{m,K}$. The geometric monodromy $\mu : Y_K \otimes_{\mathbb{Z}} Y_K^* \to \mathbb{Z}$ of $u_K$, provides a morphism

$$(7) \qquad \begin{aligned} v : Y_K &\to \underline{\mathrm{Hom}}(Y_K^*, \mathbb{Z}) =: (Y_K^*)^\vee, \\ y &\mapsto \mu(y, -), \end{aligned}$$

and hence a morphism of tori

$$(8) \qquad v \otimes \mathrm{id} : Y_K \otimes_{\mathbb{Z}} \mathbb{G}_{m,K} \to (Y_K^*)^\vee \otimes_{\mathbb{Z}} \mathbb{G}_{m,K} = (Y_K^*)^D = T_K.$$

Let $H_K(1)$ denote the Cartier dual of the Pontrjagin dual[5)] for any finite étale $K$-group scheme $H_K$. Then $\boldsymbol{\mu}_n = \mathbb{Z}/n\mathbb{Z}(1)$ and if $n$ kills $H_K$ one has $H_K(1) = H_K \otimes_{\mathbb{Z}/n\mathbb{Z}} \boldsymbol{\mu}_n$. Hence we can introduce a "monodromy" homomorphism of level $n$

$$(9) \qquad v_n : Y_K/nY_K(1) = Y_K \otimes_{\mathbb{Z}} \boldsymbol{\mu}_n \to (Y_K^*)^\vee \otimes_{\mathbb{Z}} \boldsymbol{\mu}_n = {}_nT_K \; (\hookrightarrow {}_nG_K)$$

as the restriction of $v \otimes \mathrm{id}$ to the $n$-torsion subgroups. It was defined in [15], 4.6.

5) The Pontrjagin dual of $H_K$ is $\underline{\mathrm{Hom}}(H_K, \mathbb{Q}/\mathbb{Z})$.

Consider now the 1-motive (Tate's curve)

$$\boldsymbol{\pi} : \mathbb{Z} \to \mathbb{G}_{m,K}, \quad 1 \mapsto \pi.$$

It is clear from (5) that $u_{K,\pi}^2$ has the following factorization:

$$(10)\quad \begin{array}{ccccccc} Y_K & \xrightarrow{\boldsymbol{\pi}_Y := \mathrm{id}_{Y_K} \otimes \boldsymbol{\pi}} & Y_K \otimes_{\mathbb{Z}} \mathbb{G}_{m,K} & \xrightarrow{v \otimes \mathrm{id}} & (Y_K^*)^\vee \otimes_{\mathbb{Z}} \mathbb{G}_{m,K} = T_K & \xrightarrow{\iota} & G_K, \\ y & \longmapsto & y \otimes \pi & \longmapsto & \mu(y, -) \otimes \pi = \pi^{\mu(y,-)}, & & \end{array}$$

where $v$ was defined in (7).

**Lemma 14.** *Let $(Y, G, u_K)$ be a log 1-motive and denote by $\iota : T_K \to G_K$ the torus part. Then the homomorphism*

$$(11)\qquad \mathrm{Hom}(Y, G) \times \mathrm{Hom}(Y_K \otimes \mathbb{G}_{m,K}, T_K) \to \mathrm{Hom}(Y_K, G_K),$$
$$(u^1, w_K) \mapsto u_K^1 + \iota \circ w_K \circ \boldsymbol{\pi}_Y$$

*is an isomorphism.*

The homomorphism $\boldsymbol{\pi}_Y$ was defined in (10).

*Proof.* The surjectivity is clear because any homomorphism $u_K$ on the right hand side represents a $K$-1-motive and we have just seen how to decompose $u_K$ as sum of a 1-motive $u_{K,\pi}^1$ and a 1-motive $u_{K,\pi}^2 = \iota \circ (v \otimes \mathrm{id}) \circ \boldsymbol{\pi}_{Y_K}$ with $v$ the monodromy homomorphism. In our hypothesis $u_{K,\pi}^1$ extends to an $R$-1-motive $u_\pi^1$. Hence $u_K$ is the image of the pair $(u_\pi^1, v \otimes \mathrm{id})$. For the injectivity: the first group on the left injects into $\mathrm{Hom}(Y_K, G_K)$, so we are reduced to showing that given a $w_K \in \mathrm{Hom}(Y_K \otimes \mathbb{G}_{m,K}, T_K)$ the $K$-1-motive $\iota \circ w_K \circ \boldsymbol{\pi}_Y$ has good reduction if and only if $w_K$ is trivial. Denote by $\mu(w_K)$ the pairing corresponding to $w_K$ via the canonical isomorphisms

$$\mathrm{Hom}(Y_K \otimes \mathbb{G}_{m,K}, T_K) = \mathrm{Hom}(Y_K^*, Y_K^\vee) = \mathrm{Hom}(Y_K \otimes Y_K^*, \mathbb{Z}).$$

The image of $(0, w_K)$ via the map in (11) is a $K$-1-motive; let $\mu(w_K)$ denote its geometric monodromy. Such $K$-1-motive has good reduction if and only if $\mu(w_K) = 0$ and this last occurs if and only if $w_K = 0$. $\square$

We restrict again to the consideration of the 1-motive $\boldsymbol{\pi} : \mathbb{Z} \to \mathbb{G}_{m,K}$, $1 \mapsto \pi$. Observe that it satisfies the hypothesis of Remark 10 and that in this case $u_{K,\pi}^1$ is trivial. Let us denote by

$$(12)\qquad \theta_{n,K}^\pi : 0 \to \boldsymbol{\mu}_n \to {}_nE_K \to \mathbb{Z}/n\mathbb{Z} \to 0$$

the short exact sequence $\eta(n, \boldsymbol{\pi})$. The following results will be used in Theorem 19 to compare Raynaud's monodromy and Kato's monodromy.

**Theorem 15.** *Let $n$ be a positive integer and $\theta_{n,K}^\pi$ the short exact sequence just defined. Suppose that $u_K$ is a strict $K$-1-motive as in Theorem 9 with $u_K = u_{K,\pi}^1 + u_{K,\pi}^2$ its Raynaud decomposition.*

1. *The geometric monodromy of* $u_K$ *and the one of* $u^2_{K,\pi}$ *coincide.*

2. *The short exact sequence* $\eta(n, u^2_{K,\pi})$ *associated to* $M^2_K = [u^2_{K,\pi} : Y_K \to T_K]$ *is isomorphic to the push-out via* $v_n$ *of the sequence*

$$\theta^{\pi}_{n,K} \otimes_{\mathbb{Z}/n\mathbb{Z}} Y_K/nY_K : 0 \to \boldsymbol{\mu}_n \otimes_{\mathbb{Z}/n\mathbb{Z}} Y_K/nY_K \to {}_nE_K \otimes_{\mathbb{Z}/n\mathbb{Z}} Y_K/nY_K \to Y_K/nY_K \to 0.$$

*Proof.* The first fact follows immediately from the definition of $u^2_{K,\pi}$. For the second assertion, consider the factorization of $u^2_{K,\pi}$ described in (10). It says that there is a commutative diagram

$$\begin{array}{lccccccccc}
\eta(n, \boldsymbol{\pi}_Y): & 0 & \longrightarrow & {}_n(Y_K^\vee)^D & \longrightarrow & {}_nM^\pi_K & \longrightarrow & Y_K/nY_K & \longrightarrow & 0 \\
 & & & \Big\downarrow{\scriptstyle v_n} & & \Big\downarrow & & \Big\| & & \\
\eta(n, u^2_{K,\pi}): & 0 & \longrightarrow & {}_nT_K & \longrightarrow & {}_nM^2_K & \longrightarrow & Y_K/nY_K & \longrightarrow & 0
\end{array}$$

where $M^\pi_K = [\boldsymbol{\pi}_Y : Y_K \to Y_K \otimes \mathbb{G}_{m,K}]$. It is clear that $\eta(n, \boldsymbol{\pi}_Y) = \theta^\pi_{n,K} \otimes Y_K/nY_K$ by definition of $\boldsymbol{\pi}_Y$. Hence $\eta(n, u^2_{K,\pi})$ is isomorphic to $v_{n*}(\theta^\pi_{n,K} \otimes Y_K/nY_K)$, i.e. to the push-out via $v_n$ of the sequence $\theta^\pi_{n,K} \otimes_{\mathbb{Z}/n\mathbb{Z}} Y_K/nY_K$. $\square$

**Corollary 16.** *Suppose furthermore that* $Y_K$ *and* $G_K$ *have good reduction and consider the following homomorphism*:

$$(13) \qquad \Psi : \mathrm{Ext}^1_{\mathbb{Z}/n\mathbb{Z}}\left(\frac{Y}{nY}, {}_nG\right) \times \mathrm{Hom}\left(\frac{Y_K}{nY_K}(1), {}_nG_K\right) \to \mathrm{Ext}^1_{\mathbb{Z}/n\mathbb{Z}}\left(\frac{Y_K}{nY_K}, {}_nG_K\right),$$

$$(\eta^1, h) \mapsto \eta^1_K + h_*\left(\theta^\pi_{n,K} \otimes_{\mathbb{Z}/n\mathbb{Z}} \frac{Y_K}{nY_K}\right)$$

*where* $\eta^1_K$ *means the restriction of* $\eta^1$ *to generic fibres.*

1. *If* $u_K : Y_K \to G_K$ *is a* $K$-1-*motive with Raynaud's decomposition* $u_K = u^1_{K,\pi} + u^2_{K,\pi}$, *then the class of* $\eta(n, u_K)$ *lies in the image of* $\Psi$. *More precisely it corresponds to the pair* $(\eta(n, u^1_\pi), v_n)$ *where* $v_n$ *is the "monodromy" homomorphism of level n as in* (9) *and* $u^1_\pi$ *is the* $R$-1-*motive that extends* $u^1_{K,\pi}$.

2. *If* $Y_K \cong \mathbb{Z}^r$ *and* $G_K \cong \mathbb{G}^d_{m,K}$, *then* $\Psi$ *is an isomorphism.*

*Proof.* The first assertion is an immediate consequence of the previous theorem, part 1. We restrict then to the case ${}_nG_K = \boldsymbol{\mu}^d_n$ and $Y_K/nY_K = \mathbb{Z}^r/n\mathbb{Z}^r$. For the surjectivity it is sufficient to remark that any extension class on the right is represented by an $\eta(n, u_K)$ for a strict $K$-1-motive $u_K$ because of the vanishing of $\mathrm{H}^1(K, \mathbb{G}^d_{m,K}) = \mathrm{Ext}^1(\mathbb{Z}, \mathbb{G}^d_{m,K})$. For the injectivity: the group of extensions on the left injects in the group of extensions on the right. It remains to check that $h_*(\theta^\pi_{n,K} \otimes \mathbb{Z}^r/n\mathbb{Z}^r)$ extends over $R$ if and only if $h = 0$. Now, $h : \boldsymbol{\mu}^r_n \to \boldsymbol{\mu}^d_n$ extends to many homomorphisms $\tilde{h}_K : \mathbb{G}^r_{m,K} \to \mathbb{G}^d_{m,K}$. Choose one of them and let $u_K : \mathbb{Z}^r \to \mathbb{G}^d_{m,K}$ be $\tilde{h}_K \circ \boldsymbol{\pi}_{\mathbb{Z}^r}$ with $\boldsymbol{\pi}_{\mathbb{Z}^r}$ as in (10). It is clear that $h$ coincides with the monodromy homomorphism of level $n$ of the $K$-1-motive $u_K$. By hypothesis $\eta(n, u_K) \cong h_*(\theta^\pi_{n,K} \otimes \mathbb{Z}^r/n\mathbb{Z}^r)$ extends over $R$. Hence $u_K = u^1_{K,\pi} + u^2_{K,\pi}$ with $u^2_{K,\pi}$ divisible by $n$ (see Lemma 13). This implies that the monodromy of $u_K$ that equals the monodromy of $u^2_{K,\pi}$ is a multiple of $n$ and hence its monodromy homomorphism of level $n$ is trivial, i.e. $h = 0$. □

Theorem 15/2 and its corollary are the only original results of the first part of this paper. They become interesting once one realizes that Kato proves an analogous result for extensions of finite logarithmic group schemes (cf. Theorem 17). The comparison of these two results makes it possible to extend ${}_nM_K$ to a finite logarithmic group scheme over $R$.

We close this section by giving an example that should clarify all the previous constructions.

**1.3.2. Tate's curve.** Let $n$ be a positive integer and

$$u_K : \mathbb{Z} \to \mathbb{G}_{m,K}; \quad 1 \mapsto q = \varepsilon\pi^{nr+s}, \quad 0 \leqq s \leqq n-1,\ 0 \leqq r,\ \varepsilon \in R^*$$

an elliptic curve with split multiplicative reduction. The canonical decomposition of Theorem 9 provides $u^2_{K,\pi} : \mathbb{Z} \to \mathbb{G}_{m,K}$, $1 \mapsto \pi^{nr+s}$ and $u^1_{K,\pi} : \mathbb{Z} \to \mathbb{G}_{m,K}$, $1 \mapsto \varepsilon$. The geometric monodromy $\mu : \mathbb{Z} \otimes \mathbb{Z} \to \mathbb{Z}$ depends only on $u^2_{K,\pi}$ and it sends $1 \otimes 1$ to $rn + s$. The "monodromy" homomorphism of level $n$, $v_n : \mathbb{Z}/n\mathbb{Z} \otimes \boldsymbol{\mu}_n = \boldsymbol{\mu}_n \to \boldsymbol{\mu}_n$, is the $s$-multiplication. It is also clear that $\eta(n, u^2_{K,\pi})$ is isomorphic to $s \cdot \theta^{\pi}_{n,K}$.

## 2. Finite logarithmic group objects

For the theory of logarithmic spaces we refer to [7] and [10]. We need also some definitions and results in [8], [9].

Let $\pi$ be a fixed uniformizing element of $R$ and $\underline{T}$ the spectrum of $R$ with the standard log structure given by the chart $\mathbb{N} \to R$, $1 \mapsto \pi$. Denote by $T^{\log}_{\text{fl}}$ the logarithmic flat site over $\underline{T}$. A finite (representable) logarithmic group $G$ over $R$ is a sheaf of abelian groups over $T^{\log}_{\text{fl}}$ that is represented by a fine saturated log-scheme over $R$, log flat and of Kummer type over $R$ so that its underlying scheme is finite over $R$. For an example, consider Tate's elliptic curve $E_K$ defined via $\boldsymbol{\pi} : \mathbb{Z} \to \mathbb{G}_{m,K}$, $1 \mapsto \pi$. Kato shows how to extend $E_K$ to a group object $\underline{E}^{\pi}$ in the category of valuative logarithmic spaces over $R$. This is explained by Illusie in [7], 3.1. The kernel of $n$-multiplication on $\underline{E}^{\pi}$, denoted by ${}_n(\underline{E}^{\pi})$, is obtained via log blow-ups from a logarithmic space having

$$(14) \qquad {}_nE = \operatorname{Spec}\left( \bigoplus_{i=0}^{n-1} \frac{R[x_i]}{(x_i^n - \pi^i)} \right)$$

as underlying scheme. Moreover there is a short exact sequence of finite logarithmic groups given by

$$(15) \qquad \theta^{\pi}_n : 0 \to \mathbb{Z}/n\mathbb{Z}(1) \to {}_n(\underline{E}^{\pi}) \to \mathbb{Z}/n\mathbb{Z} \to 0$$

(cf. [7], 3.2.1.4) whose restriction to generic fibres is the short exact sequence $\theta^{\pi}_{n,K}$ that we used in Theorem 15 (cf. [7], 3.2.1.4).

Let now $F$ (resp. $H$) be an $n$-torsion finite (resp. a finite étale) group scheme over $R$

endowed with the inverse image log structure. A result of Kato (cf. [8], p. 84) says that extensions (of sheaves in $T_{\mathrm{fl}}^{\log}$)

$$\eta^{\log} : 0 \to F \to G^{\log} \to H \to 0$$

correspond bijectively (up to isomorphisms) to pairs $(G^{\mathrm{cl}}, N)$ where

$$\eta^{\mathrm{cl}} : 0 \to F \to G^{\mathrm{cl}} \to H \to 0$$

is a classical extension of group schemes over $R$ and $N : H(1) \to F$ is a morphism of $R$-group schemes where $H(1) = H \otimes_{\mathbb{Z}/n\mathbb{Z}} \boldsymbol{\mu}_n$. Moreover $\eta^{\log}$ is the Baer sum of $\eta^{\mathrm{cl}}$ and the push-out by $N$ of the extension $\theta_n^{\pi} \otimes_{\mathbb{Z}/n\mathbb{Z}} H$.

**Theorem 17** (Kato). *Let notations be as above. There is an isomorphism*

$$\mathrm{Ext}^1_{R^{\mathrm{fl}}}(H,F) \times \mathrm{Hom}\big(H(1),F\big) \xrightarrow{\sim} \mathrm{Ext}_{T^{\log}}(H,F),$$
$$(\eta^{\mathrm{cl}}, N) \mapsto \eta^{\mathrm{cl}} + N_*(\theta_n^{\pi} \otimes H).$$

*Proof.* Cf. [8]. □

Observe that the statement of this theorem is similar to that of Corollary 16. We will explain in Theorem 19 that they are deeply related. Before proceeding we need the following result.

**Lemma 18.** *Let $M_K = [u_K : Y_K \to G_K]$ be a $K$-1-motive and suppose that it comes from a log 1-motive $(Y, G, u_K)$. Having fixed a uniformizing parameter $\pi$ of $R$, let $u_K = u^1_{K,\pi} + u^2_{K,\pi}$ be Raynaud's decomposition of Theorem 9. Then*:

1. *$u^1_{K,\pi}$ extends to an $R$-1-motive $u^1_{\pi}$.*

2. *The monodromy homomorphism of level $n$ of $u_K$, i.e. $v_n : Y_K/nY_K(1) \to {}_nG_K$, extends to a homomorphism $v_{n,R} : Y/nY(1) \to {}_nG$.*

*Proof.* Both assertions are evident because it follows from the hypothesis that the torus part $T_K$ of $G_K$ extends to a torus over $R$; hence also its group of characters $Y_K^*$ extends to an étale group over $R$, say $Y^*$. This implies that the geometric monodromy takes values in $\mathbb{Z}$ and it extends to a biadditive map $Y \otimes Y^* \to \mathbb{Z}$ over $R$. This last provides the homomorphism $v_{n,R}$ we are looking for. □

We can now state the relation between Raynaud's geometric monodromy of a $K$-1-motive and Kato's monodromy of its logarithmic BT-group.

**Theorem 19.** *Let $M_K = [u_K : Y_K \to G_K]$ be a $K$-1-motive coming from a log 1-motive $(Y, G, u_K)$. Let $v_{n,R}$ be the homomorphism in Lemma 18 with $u^1_{\pi} : Y \to G$ the $R$-1-motive that extends the $u^1_{K,\pi}$ of Raynaud's decomposition.*

*The sequence $\eta(n, u_K)$ in (1) extends (up to isomorphisms) to a sequence of finite logarithmic group schemes and precisely, in the notations of Theorem 17, to the one associated to $\big(\eta(n, u^1_{\pi}), v_{n,R}\big)$.*

*In particular, $v_n$ (i.e. Raynaud's monodromy homomorphism of level n associated to $u_K$) is Kato's monodromy homomorphism N restricted to generic fibres.*

*Proof.* We know from Lemma 12 and Theorem 15 that $\eta(n, u_K)$ is isomorphic to

$$\eta(n, u^1_{K,\pi}) + (v_n)_*(\theta^\pi_{n,K} \otimes Y_K/nY_K).$$

Moreover, as $u^1_{K,\pi}$ extends to an $R$-1-motive $u^1_\pi$ (cf. Lemma 18) also $\eta(n, u^1_{K,\pi})$ extends to a sequence of classical group schemes $\eta(n, u^1_\pi)$; on the other hand the sequence $\theta^\pi_{n,K}$ in (12) extends to the sequence of logarithmic groups that we denoted by $\theta^\pi_n$ in (15). Hence the sequence of logarithmic groups $\eta(n, u^1_\pi) + (v_{n,R})_*(\theta^\pi_n \otimes Y/nY)$ restricted to generic fibres is (up to isomorphisms) $\eta(n, u_K)$. □

As all constructions above behave well with respect to inclusion homomorphisms ${}_{p^m}M_K \to {}_{p^{m+1}}M_K$ we can conclude that:

**Corollary 20.** *With hypothesis as above, let $M^1_\pi$ denote the $R$-1-motive $[u^1_\pi : Y \to G]$. The* BT*-group of $u_K$, $\varinjlim({}_{p^m}M_K)$, extends to a logarithmic* BT*-group $\varinjlim({}_{p^m}M^1_\pi, v_{p^m,R})$ where $\varinjlim {}_{p^m}M^1_\pi$ is the* BT*-group of $M^1_\pi$.*

**Remark 21.** As suggested in [11], 4.7 in the case of equal characteristic $p$ one could use the previous corollary for giving an alternative construction of the functor $D_{\log}$ in loc. cit. that associate to a log 1-motive over $R$ a Dieudonné crystal.

**Remark 22.** Let $F = (\mathbb{Z}/n\mathbb{Z})^r$ and $H = \boldsymbol{\mu}^d_n$. The decomposition in Theorem 17 restricted to generic fibres coincides with the isomorphism of Corollary 16. In particular this is true working with the $K$-1-motive $E_K = [u_K : \mathbb{Z} \to \mathbb{G}_{m,K}]$ given by $1 \mapsto q = \varepsilon\pi^{nr+s}$, $0 \leqq s \leqq n-1$, $\varepsilon \in R^*$. The kernel of $n$-multiplication ${}_nE_K$ extends (up to isomorphism) to a finite logarithmic group $({}_nE^{\mathrm{cl}}, N)$ where $N$ is the $s$-multiplication on $\boldsymbol{\mu}_n$ and ${}_nE^{\mathrm{cl}}$ is the finite group scheme that lies in the middle of $\eta(n, u^1_\pi)$ for $u^1_\pi : \mathbb{Z} \to \mathbb{G}_{m,R}$, $1 \mapsto \varepsilon$.

## 3. Monodromy on Dieudonné modules

Throughout this section $R = W(k)$, with $k$ perfect, and $M_K = [u_K : Y_K \to G_K]$ will be a fixed $K$-1-motive with $Y_K \cong \mathbb{Z}^r$. We will suppose that $G_K$ has good reduction and its extension over $R$, $G$, has split torus part of rank $d > 0$. We have seen in Corollary 20 that under these hypotheses the BT-group $\boldsymbol{M}_K(p) := \varinjlim({}_{p^m}M_K)$ associated to $u_K$ extends to a logarithmic BT-group, say $\boldsymbol{M}(p)^{\log} = (\boldsymbol{M}(p), N)$ where $\boldsymbol{M}(p) = \varinjlim({}_{p^m}M^1_\pi)$ is a classical BT-group over $R$ and

$$N : Y \otimes_{\mathbb{Z}} \boldsymbol{\mu}_{p^\infty} \to Y^{*\vee} \otimes \boldsymbol{\mu}_{p^\infty} \to \varinjlim({}_{p^m}G)$$

is the monodromy homomorphism: the first morphism is $v_R \otimes \mathrm{id}$ where $v_R$ is the extension of $v : Y_K \to Y^{*\vee}_K$ in (7) over $R$, which exists since $Y_K$, $Y^*_K$ are unramified.

**3.1. The identification of Dieudonné modules of $\mathbb{Q}_p/\mathbb{Z}_p$ and $\boldsymbol{\mu}_{p^\infty}$.** For the theory of Dieudonné modules we refer to [5]: If $G$ is a BT-group over $k$, its Dieudonné module is $\mathbb{M}(G) = \mathrm{Hom}(G, \widehat{CW}_k)$, where Hom means homomorphisms of $k$-formal groups. In particular, $\mathbb{M}\big((\mathbb{Q}_p/\mathbb{Z}_p)_k\big) = \zeta W(k)$ is a free $W(k)$-module of rank 1 whose canonical generator $\zeta$ is the natural embedding of $(\mathbb{Q}_p/\mathbb{Z}_p)_k$ in $\widehat{CW}_k$ once $\mathbb{Q}_p/\mathbb{Z}_p$ is identified with $CW(\mathbb{F}_p)$; more precisely, $\zeta$ corresponds to the covector $y = (\dots, y_{-2}, y_{-1}) \in \widehat{CW}_k(k^{\mathbb{Q}_p/\mathbb{Z}_p})$ defined by $y_{-i} = \sum\limits_{a \in \mathbb{Q}_p/\mathbb{Z}_p} a_{-i} f_a$, where $f_a(b) = \delta_{ab}$ is the Kronecker delta.

Let us recall that elements of $\mathbb{M}(G)$ can also be described as isomorphism classes of rigidified extensions of $G$ by $\mathbb{G}_{a,k}$ (as fppf sheaves, cf. [12], §5.2 or [13], §15). Given a $\varphi \in \mathbb{M}(G)$, the corresponding additive extension is obtained as the pull-back via $\varphi$ of the extension

$$0 \to \mathbb{G}_{a,k} \to \widehat{CW}_k \xrightarrow{V} \widehat{CW}_k \to 0$$

where $V$ is the Verschiebung of $\widehat{CW}_k$; it will be denoted by $\varphi_{p^\infty}^{\mathrm{add}}$. In particular, one can prove that the extension

$$\zeta_{p^\infty}^{\mathrm{add}} : 0 \to \mathbb{G}_{a,k} \to F \to (\mathbb{Q}_p/\mathbb{Z}_p)_k \to 0,$$

is isomorphic to the push-out of

$$\zeta_{p^\infty} : 0 \to \mathbb{Z} \to \mathbb{Z}[1/p] \xrightarrow{f} \mathbb{Q}_p/\mathbb{Z}_p \to 0 \tag{16}$$

via the canonical homomorphism $\mathbb{Z} \to \mathbb{G}_{a,k}$. Let $\sigma : \mathbb{Q}_p/\mathbb{Z}_p \to \mathbb{Z}[1/p]$ be the section of $f$ such that $0 \leqq \sigma(a) < 1$, for $a \in \mathbb{Q}_p/\mathbb{Z}_p$. The factor set of $\zeta_{p^\infty}$ (and hence of $\zeta_{p^\infty}^{\mathrm{add}}$) corresponding to the section $\sigma$ is then

$$\gamma : (\mathbb{Q}_p/\mathbb{Z}_p) \times (\mathbb{Q}_p/\mathbb{Z}_p) \to \mathbb{Z}\ (\to \mathbb{G}_{a,R}), \quad (a,b) \mapsto [\sigma(a) + \sigma(b)] \tag{17}$$

where square brackets mean integral part. The extension $\zeta_{p^\infty}^{\mathrm{add}}$ has a canonical lifting $(\zeta_{p^\infty}^{\mathrm{add}})_R$ to $R$ obtained as the push-out of $\zeta_{p^\infty}$ in (16) (now as a sequence over $R$) via the morphism $\mathbb{Z} \to \mathbb{G}_{a,R}$. The restriction of $(\zeta_{p^\infty}^{\mathrm{add}})_R$ on generic fibres splits, and the map

$$h : \varinjlim \mathbb{Z}/p^i\mathbb{Z} = \mathbb{Q}_p/\mathbb{Z}_p \to \mathbb{G}_{a,K}, \quad a \mapsto \sigma(a)$$

is the trivialisation. Let $K[X]$ be the affine algebra of $\mathbb{G}_{a,K}$ and let $h_i^* : K[X] \to K^{\mathbb{Z}/p^i\mathbb{Z}}$, $i > 0$, be the corresponding $K$-homomorphisms; one gains an element

$$h^*(X) = \sum_{a \in \mathbb{Q}_p/\mathbb{Z}_p} \sigma(a) f_a \in K^{\mathbb{Q}_p/\mathbb{Z}_p} = \varprojlim K^{\mathbb{Z}/p^i\mathbb{Z}}. \tag{18}$$

If $\mathbb{P}$ denotes the coproduct of $K^{\mathbb{Q}_p/\mathbb{Z}_p}$, then

$$\mathbb{P}h^*(X) - 1 \mathbin{\hat{\otimes}} h^*(X) - h^*(X) \mathbin{\hat{\otimes}} 1 = \sum_{a,b \in \mathbb{Q}_p/\mathbb{Z}_p} [\sigma(a) + \sigma(b)] f_a \mathbin{\hat{\otimes}} f_b \in R^{\mathbb{Q}_p/\mathbb{Z}_p} \mathbin{\hat{\otimes}} R^{\mathbb{Q}_p/\mathbb{Z}_p};$$

this tells us that $h^*(X)$ is an integral of second kind[6)] of $R^{\mathbb{Q}_p/\mathbb{Z}_p}$.

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[6)] We recall that given the algebra $\mathscr{A}$ of a formal group over $R$, an *integral of the second kind* of $\mathscr{A}$ is an element $f \in \mathscr{A} \mathbin{\hat{\otimes}}_R K$ such that $df \in \Omega_R(\mathscr{A})$ and $\mathbb{P}f - f \mathbin{\hat{\otimes}} 1 - 1 \mathbin{\hat{\otimes}} f \in \mathscr{A} \mathbin{\hat{\otimes}} \mathscr{A}$, where $\mathbb{P}$ denotes the coproduct in $\mathscr{A}$; we will denote by $I_2(\mathscr{A})$ the $R$-module of integrals of the second kind. Moreover if $\mathbb{P}f - f \mathbin{\hat{\otimes}} 1 - 1 \mathbin{\hat{\otimes}} f = 0$, $f$ is called an *integral of the first kind*.

Moreover, the covector $y$, and hence $\zeta$, can be recovered from $h^*(X)$. In fact by a direct computation one can check that $h^*(X) = \sum_{i=0}^{\infty} p^{-i}\hat{y}_{-i}^{p^i}$, where $\hat{y}_{-i} \in R^{\mathbb{Q}_p/\mathbb{Z}_p}$ is a lifting of $y_{-i} \in k^{\mathbb{Q}_p/\mathbb{Z}_p}$.

Also the Dieudonné module of $\boldsymbol{\mu}_{p^\infty}$ is a free $W(k)$-module of rank 1. Let $R[\![Y]\!]$ be the affine algebra of $\boldsymbol{\mu}_{p^\infty,R}$, where $Y$ is the canonical parameter and let $l(Y) \in \mathbb{Z}_{(p)}[\![Y]\!]$ be the Artin-Hasse logarithm of $1+Y$, i.e.

$$\exp\big(-l(Y) - p^{-1}l(Y)^p - p^{-2}l(Y)^{p^2} - \cdots\big) = 1 + Y,$$

then, $\mathbb{M}\big((\boldsymbol{\mu}_{p^\infty})_k\big) = \delta W(k)$, where $\delta = \big(\ldots, l_0(Y_0), l_0(Y_0)\big)$, and $l_0(Y_0)$ is the image of $l(Y)$ in the affine algebra of $\boldsymbol{\mu}_{p^\infty,k}$.

Let us remark that $-\log(1+Y) = l(Y) + p^{-1}l(Y)^p + p^{-2}l(Y)^{p^2} + \cdots$ is the integral of first kind of $(\boldsymbol{\mu}_{p^\infty})_R$ obtained by lifting $\delta$.

Let $(\mathbb{Q}_p/\mathbb{Z}_p)_k^\vee = \varinjlim(\mathbb{Z}/p^n\mathbb{Z})_k^\vee$ be the Pontrjagin dual of $(\mathbb{Q}_p/\mathbb{Z}_p)_k$; then $\boldsymbol{\mu}_{p^\infty,k}$ is the Cartier dual of $(\mathbb{Q}_p/\mathbb{Z}_p)_k^\vee$; as a consequence there exist two perfect pairings of $W(k)$-modules:

$$\langle -,- \rangle_C : \mathbb{M}(\boldsymbol{\mu}_{p^\infty,k}) \times \mathbb{M}\big((\mathbb{Q}_p/\mathbb{Z}_p)_k^\vee\big) \to W(k),$$
$$\langle -,- \rangle_P : \mathbb{M}\big((\mathbb{Q}_p/\mathbb{Z}_p)_k\big) \times \mathbb{M}\big((\mathbb{Q}_p/\mathbb{Z}_p)_k^\vee\big) \to W(k).$$

We will denote by

$$\mathrm{id}(1) : \mathbb{M}(\boldsymbol{\mu}_{p^\infty,k}) \to \mathbb{M}\big((\mathbb{Q}_p/\mathbb{Z}_p)_k\big) \tag{19}$$

the $W(k)$-isomorphism such that

$$\langle m,n \rangle_C = \langle \mathrm{id}(1)(m), n \rangle_P$$

for every $m \in \mathbb{M}(\boldsymbol{\mu}_{p^\infty,k})$ and $n \in \mathbb{M}\big((\mathbb{Q}_p/\mathbb{Z}_p)_k^\vee\big)$. One can check that $\mathrm{id}(1)(\delta) = \zeta$, so that

$$F \circ \mathrm{id}(1) \circ V = \mathrm{id}(1).$$

**3.2. Monodromy and additive extensions.** We want to define a $W(k)$-endomorphism of the Dieudonné module $\mathbb{M}\big(\boldsymbol{M}(p)_k\big)$ depending on the monodromy $N$. (See also [8], 5.2.2.) We proceed as follows:

$$\begin{tikzcd}
\mathbb{M}(Y^{*\vee} \otimes \boldsymbol{\mu}_{p^\infty,k}) \arrow[r, "\mathbb{M}(N_k)"] & \mathbb{M}(Y \otimes_{\mathbb{Z}} \boldsymbol{\mu}_{p^\infty,k}) \arrow[r, "\mathrm{id}_Y(1)"] & \mathbb{M}(Y \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p)_k \arrow[d] \\
\mathbb{M}\big(\boldsymbol{M}(p)_k\big) \arrow[u] \arrow[rr, dashed, "\mathcal{N}"] & & \mathbb{M}\big(\boldsymbol{M}(p)_k\big)
\end{tikzcd}$$

where the vertical map on the left comes from the obvious inclusion

$$Y^{*\vee} \otimes \boldsymbol{\mu}_{p^n,k} = \varinjlim({}_{p^m}T_k) \to \varinjlim({}_{p^m}M^1_{\pi,k})$$

and the vertical map on the right is obtained from the projection

$$\varinjlim({}_{p^m}M^1_{\pi,k}) \to Y \otimes_{\mathbb{Z}} (\mathbb{Q}_p/\mathbb{Z}_p)_k.$$

Recall now that both $Y$ and $Y^*$ are constant groups and hence

$$(20) \qquad \mathbb{M}(Y^{*\vee} \otimes \boldsymbol{\mu}_{p^\infty,k}) = Y^* \otimes \mathbb{M}(\boldsymbol{\mu}_{p^\infty,k}), \quad \mathbb{M}(Y \otimes \boldsymbol{\mu}_{p^\infty,k}) = Y^\vee \otimes_{\mathbb{Z}} \mathbb{M}(\boldsymbol{\mu}_{p^\infty,k}).$$

The morphism $\mathbb{M}(N_k)$ is then

$$\mathbb{M}(N_k) = v^\vee \otimes \mathrm{id},$$

where $v^\vee : Y^* \to Y^\vee$ comes from the geometric monodromy $\mu$ and is the transpose of $v$ in (7) (on special fibres). Using decompositions as in (20) we can define

$$(21) \qquad \mathrm{id}_Y(1) := \mathrm{id} \otimes \mathrm{id}(1) : Y^\vee \otimes \mathbb{M}(\boldsymbol{\mu}_{p^\infty,k}) \to Y^\vee \otimes \mathbb{M}\big((\mathbb{Q}_p/\mathbb{Z}_p)_k\big)$$

where $\mathrm{id}(1)$ is the canonical identification explained in (19). The map $\mathrm{id}_Y(1)$ is an isomorphism of $W(k)$-modules such that $F \circ \mathrm{id}_Y(1) \circ V = \mathrm{id}_Y(1)$ and the dotted arrow $\mathcal{N}$ turns out to be a $W(k)$-homomorphism such that $\mathcal{N}^2 = 0$ and $F\mathcal{N}V = \mathcal{N}$.

We want now to describe how the composition $\mathrm{id}_Y(1) \circ \mathbb{M}(N_k)$ works: Given an element $\chi \otimes \delta \in Y^* \otimes \mathbb{M}(\boldsymbol{\mu}_{p^\infty,k})$, with $\delta$ the canonical generator of $\mathbb{M}(\boldsymbol{\mu}_{p^\infty,k})$,

$$\big(\mathrm{id}_Y(1) \circ \mathbb{M}(N_k)\big)(\chi \otimes \delta) = \mathrm{id}_Y(1)\big(v^\vee(\chi) \otimes \delta\big) = v^\vee(\chi) \otimes \zeta.$$

**Remark 23.** The construction above could also be done restricting to kernels of $p^n$-multiplication ${}_{p^n}M$ and hence working with the monodromy homomorphism of level $p^n$,

$$v_{p^n} : Y \otimes \mathbb{Z}/p^n\mathbb{Z} \to Y^* \otimes \boldsymbol{\mu}_{p^n}.$$

This is what Kato does in [8], 5.2.2. Hence the construction above is simply a way to summarize Kato's construction for all $p^n$.

The element $v^\vee(\chi) \otimes \zeta \in Y^\vee \otimes \mathbb{M}\big((\mathbb{Q}_p/\mathbb{Z}_p)_k\big) = \mathbb{M}\big(Y \otimes (\mathbb{Q}_p/\mathbb{Z}_p)_k\big)$ can be described in different ways; we give here some examples without proofs:

1. If we interpret it as extension of $Y \otimes (\mathbb{Q}_p/\mathbb{Z}_p)_k$ by the additive group $\mathbb{G}_{a,k}$, $v^\vee(\chi) \otimes \zeta$ is represented by the pull-back of the sequence $\zeta^{\mathrm{add}}_{p^\infty}$ with respect to

$$\big(v^\vee(\chi), \mathrm{id}\big) : Y \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p \to \mathbb{Q}_p/\mathbb{Z}_p.$$

2. Given an extension of $\mathbb{Q}_p/\mathbb{Z}_p$ by $\mathbb{G}_{a,k}$, the push-out with respect to the $m$-multiplication and the pull-back with respect to the $m$-multiplication provide isomorphic sequences. In a similar way one proves that the sequence $\big(v^\vee(\chi), \mathrm{id}\big)^* \zeta^{\mathrm{add}}_{p^\infty}$ is also isomorphic to the push-out with respect to $\big(v^\vee(\chi), \mathrm{id}\big)$ of the sequence $Y \otimes \zeta^{\mathrm{add}}_{p^\infty}$.

3. Starting with the 1-motive $\boldsymbol{\pi} : \mathbb{Z} \to \mathbb{G}_{m,K}$, consider the sequence

$$0 \to \mathbb{G}_{m,K} \to F_K \to \mathbb{Q}_p/\mathbb{Z}_p \to 0$$

obtained by first applying push-out $\boldsymbol{\mu}_{p^n} \to \mathbb{G}_{m,K}$ to the sequence $\eta(p^n, \boldsymbol{\pi})$ as in (1) and then passing to limit on $\mathbb{Z}/p^n\mathbb{Z}$. The sequence extends over $R$ (passing to Néron models) and then provides a sequence on component groups

$$0 \to \mathbb{Z} \to \phi_F \to \mathbb{Q}_p/\mathbb{Z}_p \to 0$$

over $k$ that coincides with the opposite of the sequence $\zeta_{p^\infty}$ in (16). The minus sign depends on Lemma 5.

More generally, given the 1-motive $u^2_{\pi,K} : Y_K \to T_K$ and a character $\chi \in Y^*$, the sequence $v^\vee(\chi) \otimes \zeta$ is obtained as follows: first consider the push-out ${}_{p^n}T_K \to T_K$ in $\eta(p^n, u^2_{\pi,K})$ and then pass to limit on $Y_K/p^n Y_K$. At this point we have a sequence

$$0 \to T_K \to F_K \to Y_K \otimes \mathbb{Q}_p/\mathbb{Z}_p \to 0. \tag{22}$$

Passing to Néron models and taking the induced sequence on component groups we get a sequence

$$0 \to Y^{*\vee} \to \phi_F \to Y \otimes \mathbb{Q}_p/\mathbb{Z}_p \to 0. \tag{23}$$

This sequence is nothing else than the opposite of push-out with respect to $v : Y \to Y^{*\vee}$ of the sequence $Y \otimes \zeta_{p^\infty}$. Once fixed a character $\chi \in Y^*$, we can consider the induced homomorphism $\chi^\vee : Y^{*\vee} \to \mathbb{Z}$ (evaluation at $\chi$). Now the additive extension turns out to be opposite of the push-out of (23) via the composition of $\chi^\vee$ with the canonical homomorphism $\mathbb{Z} \to \mathbb{G}_{a,k}$.

4. We can also describe $v^\vee(\chi) \otimes \zeta$ in terms of integrals of the second kind generalizing what was done in (17) and (18).

Once fixed an isomorphism $Y \cong \bigoplus_i \mathbb{Z}e_i$, the factor set $\gamma$ in (17) provides a factor set of $Y \otimes \zeta^{\mathrm{add}}_{p^\infty}$

$$Y \otimes \gamma : Y \otimes (\mathbb{Q}_p/\mathbb{Z}_p) \times Y \otimes (\mathbb{Q}_p/\mathbb{Z}_p) \to Y \otimes_{\mathbb{Z}} \mathbb{Z},$$
$$\Big(\sum_i e_i \otimes a_i, \sum_i e_i \otimes b_i\Big) \mapsto \sum_i e_i \otimes [\sigma(a_i) + \sigma(b_i)]$$

and hence a factor set

$$Y \otimes (\mathbb{Q}_p/\mathbb{Z}_p) \times Y \otimes (\mathbb{Q}_p/\mathbb{Z}_p) \to \mathbb{G}_{a,R}, \tag{24}$$
$$\Big(\sum_i e_i \otimes a_i, \sum_i e_i \otimes b_i\Big) \mapsto \sum_i [\sigma(a_i) + \sigma(b_i)]\mu(e_i, \chi)$$

of $\big(v^\vee(\chi), \mathrm{id}\big)_*(Y \otimes \zeta^{\mathrm{add}}_{p^\infty})$. This factor set becomes trivial on generic fibres and a trivialization is given by

$$h : Y_K \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \mathbb{G}_{a,K}, \quad \sum_i e_i \otimes a_i \mapsto \mu(e_i, \chi)\sigma(a_i).$$

If we read this trivialization in terms of formal groups and then pass to the affine algebras, $h$ corresponds to a $K$-homomorphism

$$h^* : K[X] \to K^{\bigoplus_i \mathbb{Q}_p/\mathbb{Z}_p e_i}, \quad X \mapsto \sum_{a \in \bigoplus_i \mathbb{Q}_p/\mathbb{Z}_p e_i} \sum_i \mu(e_i, \chi)\sigma(a_i) f_a$$

where $f_a : \bigoplus_i \mathbb{Q}_p/\mathbb{Z}_p e_i \to K$ is 1 in $a$ and 0 otherwise. Also in this case $h^*(X)$ is an integral of the second kind in $K^{\bigoplus_i \mathbb{Q}_p/\mathbb{Z}_p e_i}$ that is sent to the class of $\big(v^\vee(\chi), \mathrm{id}\big)_*(Y \otimes \zeta_{p^\infty}^{\mathrm{add}})$ via the map

$$I_2(Y \otimes \mathbb{Q}_p/\mathbb{Z}_p) \to \mathbb{M}(Y \otimes \mathbb{Q}_p/\mathbb{Z}_p).$$

**Remark 24.** The above constructions involve the part $u_{K,\pi}^2$ in Raynaud's decomposition of the 1-motive $u_K$. In particular, the multiplicative factor set

$$\text{(25)} \qquad Y_K \otimes (\mathbb{Q}_p/\mathbb{Z}_p) \times Y_K \otimes (\mathbb{Q}_p/\mathbb{Z}_p) \to \mathbb{G}_{m,K},$$
$$\Big(\sum_i e_i \otimes a_i, \sum_i e_i \otimes b_i\Big) \mapsto \pi^{-\sum_i [\sigma(a_i)+\sigma(b_i)]\mu(e_i,\chi)},$$

whose valuation is the opposite of the additive factor set in (24), gives the extension obtained by taking the push-out with respect to $\chi$ of the sequence (22). If we think of the valuation as a sort of logarithm killing elements in $R^*$, something similar in form, even if quite different in nature, happens when working with 1-motives of the type $u_{K,\pi}^1 : Y_K \to T_K$, and $u_{K,\pi}^2 = 0$: Also in the present situation we have a multiplicative factor set, it may be chosen as follows:

$$\text{(26)} \qquad \Big(\sum_i e_i \otimes a_i, \sum_i e_i \otimes b_i\Big) \mapsto \prod_i u(e_i, \chi)^{-[\sigma(a_i)+\sigma(b_i)]},$$

where the $u(e_i, \chi)$ are principal units in $R$, so that the corresponding sequence extends to $R$. Now, we can obtain from this an additive factor set by just taking the $p$-adic logarithm. As before, the additive factor set we get,

$$\Big(\sum_i e_i \otimes a_i, \sum_i e_i \otimes b_i\Big) \mapsto \sum_i -[\sigma(a_i) + \sigma(b_i)] \log(u_i)$$

is trivial on generic fibres and its trivialisation provides an integral of the second kind $h(\chi) \in K^{\mathbb{Q}_p/\mathbb{Z}_p}$ and the BT-group of $u_{K,\pi}^1$ is completely determined by the $W(k)$-module generated by the $h(\chi)$, as $\chi$ varies in the group of the characters of $T_R$ (cf. [5], IV). Finally, let us observe that if one is just interested in computing the monodromy, then the use of the valuation is quite appropriate, but if one needs to consider the integrals of $u_{K,\pi} = u_{K,\pi}^1 + u_{K,\pi}^2$, i.e. one needs to integrate logarithmic differentials (in the style of [2]), one is forced to extend the $p$-adic logarithm defining $\log \pi$ in a way allowing one to distinguish the integrals of $u_{K,\pi}^1$ from those of $u_{K,\pi}^2$.

## 4. The finite logarithmic group that extends ${}_nM_K$

Throughout this section we work with a strict $K$-1-motive $u_K : \mathbb{Z}^r \to G_K$ where $G_K$ is a semiabelian scheme with split torus part $\mathbb{G}_{m,K}^d$ and with abelian quotient $A_K$ having good reduction. Given the $n$-torsion ${}_nM_K$ of such a 1-motive, we know from Theorem 19 that it extends to a finite logarithmic group over $R$, ${}_nM^{\log}$, but we still know little concerning the scheme ${}_nM$ underlying ${}_nM^{\log}$. In this section we are going to describe the logarithmic group scheme ${}_nM^{\log}$ for any $n$ and precisely we will show that ${}_nM^{\log}$ is the valuative space associated to $(\mathrm{Spec}({}_nA), \mathscr{M}_A)$ where ${}_nA$ is an algebra over $R$, constructed in a somewhat canonical way, and $\mathscr{M}_A$ is a logarithmic structure on $\mathrm{Spec}({}_nA)$ such that the structure morphism over $R$ induces a morphism of log schemes $(\mathrm{Spec}({}_nA), \mathscr{M}_A) \to \underline{T}$.

We have seen that ${}_nM_K$ is an extension of $Y_K/nY_K$ by ${}_nG$, hence a ${}_nG_K$-torsor over $Y_K/nY_K$. If ${}_nG_K = \boldsymbol{\mu}_n^d$, it is easy to describe the algebra of ${}_nM_K$; cf. [14], [4]. For example for the Tate curve $\boldsymbol{\pi} : \mathbb{Z} \to \mathbb{G}_{m,K}$ we denoted ${}_nM_K$ by

$$(27) \qquad {}_nE_K = \mathrm{Spec}\big(K^{\mathbb{Z}/n\mathbb{Z}}[x]/(x^n - b_{\pi,n})\big)$$

with $b_{\pi,n} := \sum_{j=0}^{n-1} \pi^j v_j$, where $\{v_0, \dots, v_{n-1}\}$ is the canonical basis of $K^{\mathbb{Z}/n\mathbb{Z}}$. See also (14).

For the general case, we read from (6) that ${}_nM_K$ is indeed a $\boldsymbol{\mu}_n^d$-torsor over the finite $K$-group scheme ${}_nM_K^A$. Recalling that $\mathrm{Pic}({}_nM_K^A) = 0$, ${}_nM_K$ can easily be described via [14], III, §4.

**Lemma 25.** *Let $\mathscr{B}_K$ be the algebra of ${}_nM_K^A$. Then*

$$ {}_nM_K = \mathrm{Spec}\big(\mathscr{B}_K[T_1, \dots, T_d]/(T_1^n - b_1, \dots, T_d^n - b_d)\big)$$

*for suitable $b_i \in \mathscr{B}_K^*$, $i = 1, \dots, d$.*

We need now to describe how the $b_i$ above depend on Raynaud's decomposition of the 1-motive $u_K$.

**4.1. The 1-motive $\boldsymbol{\pi}^{-1}$.** Before proceeding with the description of ${}_nM_K$ we need to know what happens when working with the 1-motive $\boldsymbol{\pi}^{-1} : \mathbb{Z} \to \mathbb{G}_{m,K}$, $1 \mapsto \pi^{-1}$. Its $n$-torsion group scheme is

$$(28) \qquad {}_nE_K^{\frac{1}{\pi}} = \mathrm{Spec}\left(\frac{K^{\mathbb{Z}/n\mathbb{Z}}[x]}{\left(x^n - \sum_{i=0}^{n-1} \pi^{-i} v_i\right)}\right)$$

with $\{v_0, \dots, v_{n-1}\}$ the canonical base of $K^{\mathbb{Z}/n\mathbb{Z}}$ over $K$. It is clear that we can not extend its algebra over $R$ via the equation $x^n - \sum_{i=0}^{n-1} \pi^{-i} v_i$. However, ${}_nE^{\frac{1}{\pi}}$ is isomorphic to the group scheme obtained from (15) by push-out with respect to $-1$ or by pull-back with respect to $-1$. Hence it is also

$$
{}_nE_K^{\frac{1}{\pi}} \cong \operatorname{Spec}\left(\frac{K^{\mathbb{Z}/n\mathbb{Z}}[y]}{\left(y^n - v_0 - \sum_{i=1}^{n-1} \pi^{n-i} v_i\right)}\right) \tag{29}
$$

because $-1 : \mathbb{Z} \to \mathbb{Z}$ sends $v_0 \mapsto v_0$, $v_i \mapsto v_{n-i}$ for $i > 0$. Moreover this scheme extends over $R$ to

$$
{}_nE^{\frac{1}{\pi}} = \operatorname{Spec}\left(\frac{R^{\mathbb{Z}/n\mathbb{Z}}[y]}{\left(y^n - v_0 - \sum_{i=1}^{n-1} \pi^{n-i} v_i\right)}\right). \tag{30}
$$

We can endow ${}_nE^{\frac{1}{\pi}}$ with the logarithmic structure coming from the special fibre and it becomes a logarithmic group scheme over $R$. Denote by ${}_n(\underline{E}^{\frac{1}{\pi}})$ its valuative logarithmic space; this lies in the middle of $-\theta_n^\pi$ for $\theta_n^\pi$ as in (15).

Another decomposition of a 1-motive that will be useful later is the following:

**Lemma 26.** *Let $u_K$ be a 1-motive as in Theorem 9. Suppose furthermore that $Y_K$ is split and also the torus part $T_K$ of $G_K$ is split. Once fixed a uniformizing parameter $\pi \in R$, a basis $(e_j)_j$ of $Y_K \cong \mathbb{Z}^r$ and a basis $(e_i^*)_i$ of $Y_K^*$ there are decompositions*

$$
u_K = u_{K,\pi}^1 + u_{K,\pi}^2 = u_{K,\pi}^1 + u_{K,\pi}^+ + u_{K,\pi}^-
$$

*where the first one is the decomposition in Theorem 9. The second decomposition is uniquely determined by the following conditions*:

- *$u_{K,\pi}^\pm : Y_K \to G_K$ factor through the torus part.*

- *If $\mu^+$ (resp. $\mu^-$) denotes the geometric monodromy of $u_{K,\pi}^+$ (resp. of $u_{K,\pi}^-$), then one has $\mu^+(e_j, e_i^*) \geqq 0$ (resp. $\mu^-(e_j, e_i^*) \leqq 0$).*

- *$u_{K,\pi}^2 = u_{K,\pi}^+ + u_{K,\pi}^-$ and $\mu = \mu^+ + \mu^-$.*

*Proof.* We are reduced to working with $u_{K,\pi}^2$. We define

$$
\mu^+ : Y_K \otimes Y_K^* \to \mathbb{Z}, \quad (e_j, e_i^*) \to \begin{cases} \mu(e_j, e_i^*) & \text{if this is positive,} \\ 0 & \text{otherwise.} \end{cases}
$$

Similar for $\mu^-$. Moreover $u_{K,\pi}^\pm$ is defined as done for $u_{K,\pi}^2$ in (10) with $\mu^\pm$ in place of $\mu$. □

**Remark 27.** Recall the decomposition in Lemma 26 and the factorization $u_{K,\pi}^2 = (v \otimes \mathrm{id})\boldsymbol{\pi}_Y$. Let us construct $v^\pm : Y_K \to (Y_K^*)^\vee$ from the geometric monodromy $\mu^\pm$ as in (7). Then we have factorizations $u_{K,\pi}^+ = (v^+ \otimes \mathrm{id})\boldsymbol{\pi}_Y$ (resp. $u_{K,\pi}^- = (v^- \otimes \mathrm{id})\boldsymbol{\pi}_Y^{-1}$) with

$$
\boldsymbol{\pi}_Y^{-1} : Y_K \to Y_K \otimes_{\mathbb{Z}} \mathbb{G}_{m,K}, \quad y \mapsto y \otimes \pi^{-1}.
$$

**4.2. A general result on short exact sequences.** Looking at the diagram (6) where ${}_nT_K = \boldsymbol{\mu}_n^d$ we realize that we should move our attention from the horizontal sequence in the middle $\eta(n, u_K)$ to the vertical sequence in the middle, because, as we have already observed $\boldsymbol{\mu}_n$-torsors can easily be described. Moreover we need to know how such a "vertical sequence" depends on the analogous sequences for $u^1_{\pi,K}$ and $u^2_{\pi,K}$. For this we need a general result.

Let $\bar{\psi} : 0 \to I \xrightarrow{w} L \xrightarrow{k} P \to 0$ be an exact sequence of group schemes over a base scheme $S$. Let $N$ be another group scheme over $S$ and consider two extensions $\eta^i : 0 \to L \to M^i \to N \to 0$, $i = 1, 2$. Let $\eta : 0 \to L \to M \to N \to 0$ be a sequence isomorphic to the Baer sum $\eta^1 + \eta^2$. Consider then the following diagrams:

$$
\begin{tikzcd}
 & & I \arrow[r, equal] \arrow[d, hook, "w"] & I \arrow[d, hook, "\tau^i"] & & \\
\eta^i : & 0 \arrow[r] & L \arrow[r] \arrow[d, two heads, "k"] & M^i \arrow[r, "h^i"] \arrow[d, two heads, "g^i"] & N \arrow[r] \arrow[d, equal] & 0, \\
 & 0 \arrow[r] & P \arrow[r] & Q^i \arrow[r, "f^i"] & N \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & I \arrow[r, equal] \arrow[d, hook, "w"] & I \arrow[d, hook, "\tau"] & & \\
\eta : & 0 \arrow[r] & L \arrow[r] \arrow[d, two heads, "k"] & M \arrow[r, "h"] \arrow[d, two heads, "g"] & N \arrow[r] \arrow[d, equal] & 0 \\
 & 0 \arrow[r] & P \arrow[r] & Q \arrow[r, "f"] & N \arrow[r] & 0
\end{tikzcd}
$$

where the vertical sequence on the left is $\bar{\psi}$ and the upper horizontal sequence is $\eta^i$ for $i = 1, 2$ (resp. $\eta$). Call $\psi^i$ for $i = 1, 2$ (resp. $\psi$) the vertical sequence in the middle. Suppose now that there is a sequence

$$\tilde{\eta}^2 : 0 \to I \to \tilde{M}^2 \to N \to 0$$

such that $\eta^2 = w_* \tilde{\eta}^2$. Summarizing we have

$$\eta \cong \eta^1 + w_* \tilde{\eta}^2.$$

We are going to see that a similar relation holds also for the vertical sequences $\psi, \psi^1$.

Now $k_* w_* \tilde{\eta}^2$ is isomorphic to the trivial extension and we choose a section $\sigma$ of $f^2$. Moreover $k_* \eta \cong k_* \eta^1$ and there is then an isomorphism $\imath^\sigma : Q \to Q^1$ depending on $\sigma$. It is also not difficult to check that

$$\psi \cong (\imath^\sigma)^* \psi^1 + (\sigma f)^* \psi^2. \tag{31}$$

Consider also the following push-out diagram:

$$
\begin{tikzcd}
\tilde{\eta}^2 : & 0 \arrow[r] & I \arrow[r] \arrow[d, "w"] \arrow[dr, dashed, "\tau^2"] & \tilde{M}^2 \arrow[r, "\tilde{h}^2"] \arrow[d, "\delta"] & N \arrow[r] \arrow[d, equal] & 0 \\
w_* \tilde{\eta}^2 = \eta^2 : & 0 \arrow[r] & L \arrow[r] \arrow[d, "k"] & M^2 \arrow[r, "h^2"] \arrow[d, "g^2"] & N \arrow[r] \arrow[d, equal] & 0 \\
k_* \eta^2 : & 0 \arrow[r] & P \arrow[r] & Q^2 \arrow[r, "f^2"] & N \arrow[r] & 0
\end{tikzcd}
$$

where the lower sequence is isomorphic to the trivial one. We denoted by $\psi^2$ the exact sequence involving $g^2$ and $\tau^2$. There exists then a canonical homomorphism $\sigma^c : N \to Q^2$ such that

$$\sigma^c \tilde{h}^2 = g^2 \delta;$$

it is not difficult to check that $\sigma^c$ is a section of $f^2$. Moreover by construction it satisfies

$$(\sigma^c)^* \psi^2 \cong \tilde{\eta}^2.$$

Taking now $\sigma = \sigma^c$ in (31) and setting $\imath := \imath^{\sigma^c}$ we get that

$$\psi \cong \imath^* \psi^1 + f^* \tilde{\eta}^2. \tag{32}$$

**4.3. ${}_nM_K$ as torsor under $\boldsymbol{\mu}_n^d$.** Let $u_K$ be a 1-motive as in Theorem 9. We have then a decomposition $u_K = u_{K,\pi}^1 + u_{K,\pi}^2$ and an isomorphism of sequences $\eta(n, u_K) \cong \eta(n, u_{K,\pi}^1) + \eta(n, u_{K,\pi}^2)$ for $\eta(n, -)$ the sequence introduced in (1).

The $n$-torsion ${}_{p^n}M_K$ of $u_K$ lies in the middle of $\eta(n, u_K)$ and we have already seen what it looks like in Lemma 25. Moreover $\eta(n, u_{K,\pi}^2) = w_* \eta(n, \tilde{u}_K^2)$, where the 1-motive $\tilde{u}_K^2 : Y_K \to T_K$ is obtained from $u_{K,\pi}^2$ by forgetting the inclusion $T_K \to G_K$ and $w$ is this inclusion restricted to kernels of $n$-multiplication. Hence we are in the situation of the subsection 4.2.

Considering diagram (6) for $\eta(n, u_{K,\pi}^1)$ in place of $\eta(n, u_{K,\pi})$ we get

$$\tag{33}
\begin{array}{ccccccccc}
 & & & & {}_nT_K & = & {}_nT_K & & & & \\
 & & & & \big\downarrow{\scriptstyle w} & & \big\downarrow{\scriptstyle \tau^1} & & & & \\
\eta(n, u_{K,\pi}^1): & 0 & \longrightarrow & & {}_nG_K & \longrightarrow & {}_nM_K^1 & \xrightarrow{h^1} & Y_K/nY_K & \longrightarrow & 0 \\
 & & & & \big\downarrow{\scriptstyle \rho_n} & & \big\downarrow{\scriptstyle g^1} & & \big\| & & \\
\eta(n, \rho u_{K,\pi}^1): & 0 & \longrightarrow & & {}_nA_K & \longrightarrow & Q_K^1 & \xrightarrow{f^1} & Y_K/nY_K & \longrightarrow & 0.
\end{array}$$

Let $\psi(n, u_{K,\pi}^1)$ be the vertical sequence in the middle of this diagram and $\psi(n, u_{K,\pi})$ the corresponding vertical sequence in the middle of (6). By (32) we have

$$\psi(n, u_K) \cong \imath^* \psi(n, u_{K,\pi}^1) + f^* \eta(n, \tilde{u}_K^2) \tag{34}$$

for $\imath$ the isomorphism ${}_nM_K^A \to Q_K^1$ constructed as in §4.2. By abuse of notations, we denote by ${}_nM_K^1$ also the group scheme in the middle of $\imath^* \psi(n, u_{K,\pi}^1)$; indeed it is isomorphic to the one in the middle of $\eta(n, u_{K,\pi}^1)$. The scheme ${}_nM_K^1$ is a $\boldsymbol{\mu}_n^d$-torsor over ${}_nM_K^A$ and hence

$${}_nM_K^1 \cong \mathrm{Spec}\big(\mathscr{B}_K[T_1, \ldots, T_d]/(T_1^n - b_1^{(1)}, \ldots, T_d^n - b_d^{(1)})\big).$$

On the other hand, call ${}_n\tilde{M}_K^2$ the group in the middle of $\eta(n, \tilde{u}_K^2)$. It is a $\boldsymbol{\mu}_n^d$-torsor over $\mathbb{Z}^r/n\mathbb{Z}^r$ and hence it has the form

$$ {}_n\tilde{M}_K^2 \cong \mathrm{Spec}\big(K^{\mathbb{Z}^r/n\mathbb{Z}^r}[T_1,\dots,T_d]/(T_1^n - b_1^{(2)},\dots,T_d^n - b_d^{(2)})\big). $$

Hence the group in the middle of $f^*\eta(n,\tilde{u}^2_{K,\pi})$ will be

$$ \mathrm{Spec}\big(\mathscr{B}_K[T_1,\dots,T_d]/(T_1^n - b_1^{(2)},\dots,T_d^n - b_d^{(2)})\big). $$

Recall now that $\tilde{u}^2_{K,\pi} = u^+_{K,\pi} + u^-_{K,\pi}$ by Lemma 26 and hence

$$ \eta(n,u^2_{K,\pi}) \cong \eta(n,u^+_{K,\pi}) + \eta(n,u^-_{K,\pi}). $$

Let

$$ \mathrm{Spec}\big(K^{\mathbb{Z}^r/n\mathbb{Z}^r}[T_1,\dots,T_d]/(T_1^n - b_1^+,\dots,T_d^n - b_d^+)\big) $$

be the group scheme in the middle of the sequence $\eta(n,u^+_{K,\pi})$ and analogously for $u^-_{K,\pi}$ with elements $b_i^- \in K^{\mathbb{Z}^r/n\mathbb{Z}^r}$ in place of $b_i^+$. Using the sequence

$$ (35) \qquad 0 \to \boldsymbol{\mu}_n^d(X) \to \Gamma(X,\mathscr{O}_X)^{d*} \xrightarrow{n} \Gamma(X,\mathscr{O}_X)^{d*} \to \mathrm{H}(X,\boldsymbol{\mu}_n^d) \to 0 $$

we may assume that $b_i^{(2)} = b_i^+ b_i^-$ and

$$ b_i = b_i^{(1)} b_i^+ b_i^- \quad \text{for all } 1 \leqq i \leqq d. $$

Recall now that the vertical sequence on the left in (6) or (33) extends over $R$ because $G_K$ has good reduction and that also ${}_nM_K^A$ extends to a finite group scheme over $R$, say ${}_nM^A = \mathrm{Spec}(\mathscr{B})$, because the $K$-1-motive

$$ \rho u_K : \mathbb{Z}^r \to G_K \to A_K $$

has good reduction. Finally also $\eta(n,u^1_{K,\pi})$ extends over $R$ and hence the same is true for the vertical sequence $\psi(n,u^1_{K,\pi})$ and so we may assume $b_i^{(1)} \in \mathscr{B}^*$. This implies that if we want to extend ${}_nM_K$ over $R$ we have to understand better ${}_n\tilde{M}_K^2$ and hence $b_i^{(2)} = b_i^+ b_i^-$. Recall the description of $u^2_{K,\pi}$ in (10) where now $Y_K = \mathbb{Z}^r$ and $T_K = \mathbb{G}^d_{m,K}$. It is an easy exercise to check that

$$ b_i^+ = (b_{\pi,n})^{\sum_{j=1}^r \mu^+(e_j,e_i^*)} \qquad \text{with } b_{\pi,n} := \sum_{j=0}^{n-1} \pi^j v_j \in R^{\mathbb{Z}^r/n\mathbb{Z}^r}, $$

with $v_0,\dots,v_{n-1}$ the standard basis of $K^{\mathbb{Z}/n\mathbb{Z}}$ (resp. of $R^{\mathbb{Z}/n\mathbb{Z}}$), $e_1,\dots,e_r$ the usual basis of $\mathbb{Z}^r$ and $e_i^*$ the character in $Y_K^*$ such that $e_i^*(T_h) = \delta_{ih}$, $\mu^+$ the geometric monodromy of $u^+_{K,\pi}$. Moreover $\mu^+(e_j,e_i^*) \geqq 0$ by definition of $u^+_{K,\pi}$; hence $b_i^+ \in \mathscr{B}$ for all $i$. In a similar way

$$ b_i^- = (b_{\pi^{-1},n})^{\sum_{j=1}^r -\mu^-(e_j,e_i^*)} \qquad \text{with } b_{\pi^{-1},n} := v_0 + \sum_{i=1}^{n-1} \pi^{n-i} v_i \in R^{\mathbb{Z}/n\mathbb{Z}}. $$

Hence also $b_i^- \in \mathscr{B}$ for all $i$ because $\mu^-(e_j,e_i^*) \leqq 0$ by definition of $u^-_{K,\pi}$. Summarizing, $b_i = b_i^{(1)} b_i^{(2)} = b_i^{(1)} b_i^+ b_i^- \in \mathscr{B}$ for all $i$. Hence ${}_nM_K$ extends to a finite scheme over $\mathrm{Spec}(\mathscr{B})$

$$(36) \qquad {}_nM = \operatorname{Spec}\big(\mathscr{B}[T_1,\dots,T_d]/(T_1^n - b_1,\dots,T_d^n - b_d)\big).$$

The following lemma says that ${}_nM$ is indeed a "nice" model.

**Lemma 28.** *Let notations be as above and endow ${}_nM^A = \operatorname{Spec}(\mathscr{B})$ with the inverse image log structure of the base $\underline{T}$. Let $({}_nM, \mathscr{M}_{{}_nM})$ be the scheme* (36) *with the logarithmic structure induced by the special fibre. Then the canonical morphism ${}_nM_K \to {}_nM_K^A$ extends to a unique morphism of logarithmic groups over $R$. Moreover the following universal property holds: For any fine saturated logarithmic scheme $(S, \mathscr{M}_S)$ over $({}_nM^A, \mathscr{M}_{{}_nM^A})$ there is a bijection between the scheme theoretic morphisms $S_K \to {}_nM_K$ over ${}_nM_K^A$ and the logarithmic morphisms $(S, \mathscr{M}_S) \to ({}_nM, \mathscr{M}_{{}_nM})$ over $({}_nM^A, \mathscr{M}_{{}_nM^A})$.*

*Proof.* We may assume $S = \operatorname{Spec}(C)$ affine. Any ${}_nM_K^A$-morphism $S_K \to {}_nM_K$ is described as a homomorphism of $\mathscr{B}_K$-algebras $\varphi_K : \mathscr{B}_K[T_1,\dots,T_n]/(T_i^n - b_i) \to C_K$. The only problem for the extension is to prove that the images of all $T_i$ lie in $C$ and more precisely in $\Gamma(S, \mathscr{M}_S)$. However $b_i = b_i^{(1)} b_i^{(2)}$ with $b_i^{(1)} \in \mathscr{B}^*$ and $b_i^{(2)} \in \Gamma(Y, \mathscr{M}_Y)$ for any logarithmic space over the group $\mathbb{Z}/n\mathbb{Z}$ endowed with the inverse image log structure because of the description of $b_i^{(2)}$ in terms of $b_{\pi,n}$ and $b_{\pi^{-1},n}$. Moreover $\varphi_K(T_i^n) = \varphi_K(b_i) \in \Gamma(S, \mathscr{M}_S)$. It is now sufficient to recall that $S$ is saturated to conclude that also $\varphi_K(T_i) \in \Gamma(S, \mathscr{M}_S)$. □

**Proposition 29.** *Let notations be as above. The finite group scheme ${}_nM_K$ over ${}_nM_K^A$ extends* (*up to isomorphisms*) *to a finite logarithmic group ${}_nM^{\log}$ that is the valuative logarithmic space associated to a logarithmic scheme whose underlying scheme is*

$$(37) \qquad {}_nM = \operatorname{Spec}\big(\mathscr{B}[T_1,\dots,T_d]/(T_1^n - b_1,\dots,T_d^n - b_d)\big).$$

*Moreover the diagram* (6) *extends to a diagram of finite logarithmic groups with ${}_nM^{\log}$ in the middle.*

*Proof.* It remains only to prove that ${}_nM$ with the logarithmic structure induced by the special fibre induces a group functor on the category of fine saturated logarithmic schemes over $R$, i.e. it is in $T_{\mathrm{fl}}^{\log}$. However this is immediate consequence of the previous lemma.

Also the assertion on the diagram follows applying the previous lemma. □

Observe that we had already proved in Theorem 18 that ${}_nM_K$ extends to a logarithmic group over $\underline{T}$, however, in the hypothesis of this section, it is possible to describe it in terms of algebras and not only as a "sum" of two extensions.

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