Deformation Quantization of Complex Involutive Submanifolds

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Introduction

Let M be a complex manifold, and T^*M its cotangent bundle endowed with the canonical symplectic structure. The sheaf of rings $\mathcal{W}_M$ of WKB microdifferential operators (WKB operators, for short) provides a deformation quantization of T^*M (we refer to [15, 1, 16] for the definition of $\mathcal{W}_M$ via the sheaf of microdifferential operators). On a complex symplectic manifold X there may not exist a sheaf of rings locally isomorphic to $i^{-1}\mathcal{W}_M$, for any symplectic local chart $i\colon X \supset U \to T^*M$. The idea is then to consider the whole family of locally defined sheaves of WKB operators as the deformation quantization of X. To state it precisely, one needs the notion of algebroid stack, introduced by Kontsevich [12]. In particular, the stack of WKB modules over X defined in Polesello-Schapira [16] (see also Kashiwara [9] for the contact case) is better understood as the stack of $\mathfrak{W}_X$-modules, where $\mathfrak{W}_X$ denotes the algebroid stack of deformation quantization of X.

Let $V \subset X$ be an involutive (*i.e.* coisotropic) submanifold. Assume for simplicity that the quotient of V by its bicharacteristic leaves is isomorphic to a complex symplectic manifold Z, and denote by $q\colon V \to Z$ the quotient map. If $\mathcal{L}$ is a simple WKB module along V, then the algebra of its endomorphisms is locally isomorphic to $q^{-1}i^{-1}\mathcal{W}_N^{\mathrm{op}}$, for $i\colon Z \supset U \to T^*N$ a symplectic local chart. Hence one may say that $\mathcal{L}$ provides a deformation quantization of V. Again, since in general there do not exist globally defined simple WKB modules, the idea is to consider the algebroid stack of locally defined simple WKB modules as the deformation quantization of V. We then establish a relation between this algebroid stack and that of deformation quantization of Z. This generalizes a result of D'Agnolo-Schapira [6] for the Lagrangian case (see also [9] for the contact case), which turns out to be an essential ingredient of our proof.

In this paper we start by defining what an algebroid stack is, and how it is locally described. We then discuss the algebroid stack of WKB operators on a complex symplectic manifold X and define the deformation quantization of an involutive submanifold $V \subset X$ by means of simple WKB modules along V. Finally, we relate this deformation quantization to that given by WKB operators on the quotient of V by its bicharacteristic leaves.

1 Algebroid stacks

We start here by recalling the categorical realization of an algebra as in [13], and we then sheafify that construction. We assume that the reader is familiar with the basic notions from the theory of stacks which are, roughly speaking, sheaves of categories. (The classical reference is [7], and a short presentation is given *e.g.* in [9, 4].)

Let R be a commutative ring. An R-linear category (R-category for short) is a category whose sets of morphisms are endowed with an R-module structure, so that composition is bilinear. An R-functor is a functor between R-categories which is linear at the level of morphisms.

If A is an R-algebra, we denote by A^+ the R-category with a single object, and with A as set of morphisms. This gives a fully faithful functor from R-algebras to R-categories. If $f, g\colon A \to B$ are R-algebra morphisms, transformations $f^+ \Rightarrow g^+$ are in one-to-one correspondence with the set

$$\{b \in B\colon bf(a) = g(a)b,\ \forall a \in A\}, \tag{1.1}$$

with vertical composition of transformations corresponding to multiplication in B. Note that the category $\mathsf{Mod}(A)$ of left A-modules is R-equivalent to the category $\mathsf{Hom}_R(A^+, \mathsf{Mod}(R))$ of R-functors from A^+ to $\mathsf{Mod}(R)$ and that the Yoneda embedding

$$A^+ \to \mathsf{Hom}_R((A^+)^{\mathrm{op}}, \mathsf{Mod}(R)) \approx_R \mathsf{Mod}(A^{\mathrm{op}})$$

identifies A^+ with the full subcategory of right A-modules which are free of rank one. (Here $\approx_R$ denotes R-equivalence.)

Let X be a topological space, and $\mathcal{R}$ a (sheaf of) commutative algebra(s). As for categories, there are natural notions of $\mathcal{R}$-linear stacks ($\mathcal{R}$-stacks for short), and of $\mathcal{R}$-functor between $\mathcal{R}$-stacks.

If $\mathcal{A}$ is an $\mathcal{R}$-algebra, we denote by $\mathcal{A}^+$ the $\mathcal{R}$-stack associated with the separated prestack $X \supset U \mapsto \mathcal{A}(U)^+$. This gives a functor from $\mathcal{R}$-algebras to $\mathcal{R}$-categories which is faithful and locally full. If $f, g\colon \mathcal{A} \to \mathcal{B}$ are $\mathcal{R}$-algebra morphisms, transformations $f^+ \Rightarrow g^+$ are described as in (1.1). As above, the stack $\mathfrak{Mod}(\mathcal{A})$ of left $\mathcal{A}$-modules is $\mathcal{R}$-equivalent to the stack of $\mathcal{R}$-functors $\mathfrak{Hom}_{\mathcal{R}}(\mathcal{A}^+, \mathfrak{Mod}(\mathcal{R}))$, and the Yoneda embedding gives a fully faithful functor

$$\mathcal{A}^+ \to \mathfrak{Hom}_{\mathcal{R}}((\mathcal{A}^+)^{\mathrm{op}}, \mathfrak{Mod}(\mathcal{R})) \approx_{\mathcal{R}} \mathfrak{Mod}(\mathcal{A}^{\mathrm{op}}) \tag{1.2}$$

into the stack of right $\mathcal{A}$-modules. This identifies $\mathcal{A}^+$ with the full substack of locally free right $\mathcal{A}$-modules of rank one.

Recall that one says a stack $\mathfrak{A}$ is non-empty if $\mathfrak{A}(X)$ has at least one object; it is locally non-empty if there exists an open covering $X = \bigcup_i U_i$ such that $\mathfrak{A}|_{U_i}$ is non-empty; and it is locally connected by isomorphisms if for any open subset $U \subset X$ and any $F, G \in \mathfrak{A}(U)$ there exists an open covering $U = \bigcup_i U_i$ such that $F|_{U_i} \simeq G|_{U_i}$ in $\mathfrak{A}(U_i)$.

Lemma 1.1. *Let $\mathfrak{A}$ be an $\mathcal{R}$-stack. The following are equivalent*

(i) $\mathfrak{A} \approx_{\mathcal{R}} \mathcal{A}^+$ *for an $\mathcal{R}$-algebra $\mathcal{A}$,*

(ii) $\mathfrak{A}$ *is non-empty and locally connected by isomorphisms.*

Proof. By (1.2), $\mathcal{A}^+$ is equivalent to the stack of locally free right $\mathcal{A}$-modules of rank one. Since $\mathcal{A}$ has a structure of right module over itself, (i) implies (ii). Conversely, let $\mathfrak{A}$ be an $\mathcal{R}$-stack as in (ii) and $\mathcal{P}$ an object of $\mathfrak{A}(X)$. Then $\mathcal{A} = \mathcal{E}nd_{\mathfrak{A}}(\mathcal{P})$ is an $\mathcal{R}$-algebra and the assignment $\mathcal{Q} \mapsto \mathcal{H}om_{\mathfrak{A}}(\mathcal{P}, \mathcal{Q})$ gives an $\mathcal{R}$-equivalence between $\mathfrak{A}$ and $\mathcal{A}^+$. □

We are now ready to give a definition of algebroid stack, equivalent to that in Kontsevich [12]. It is the linear analogue of the notion of gerbe (groupoid stack) from algebraic geometry [7].

Definition 1.2. An $\mathcal{R}$-algebroid stack is an $\mathcal{R}$-stack $\mathfrak{A}$ which is locally non-empty and locally connected by isomorphisms.

The $\mathcal{R}$-stack of $\mathfrak{A}$-modules is $\mathfrak{Mod}(\mathfrak{A}) = \mathfrak{Hom}_{\mathcal{R}}(\mathfrak{A}, \mathfrak{Mod}(\mathcal{R}))$.

Note that $\mathfrak{Mod}(\mathfrak{A})$ is an example of stack of twisted modules over not necessarily commutative rings (see [10, 4]). As above, the Yoneda embedding identifies $\mathfrak{A}$ with the full substack of $\mathfrak{Mod}(\mathfrak{A}^{\mathrm{op}})$ consisting of locally free objects of rank one.

2 Cocycle description of algebroid stacks

We will explain here how to recover an algebroid stack from local data. The parallel discussion for the case of gerbes can be found for example in [2, 3].

Let $\mathfrak{A}$ be an $\mathcal{R}$-algebroid stack. By definition, there exists an open covering $X = \bigcup_i U_i$ such that $\mathfrak{A}|_{U_i}$ is non-empty. By Lemma 1.1 there are $\mathcal{R}$-algebras $\mathcal{A}_i$ on U_i such that $\mathfrak{A}|_{U_i} \approx_{\mathcal{R}} \mathcal{A}_i^+$. Let $\Phi_i \colon \mathfrak{A}|_{U_i} \to \mathcal{A}_i^+$ and $\Psi_i \colon \mathcal{A}_i^+ \to \mathfrak{A}|_{U_i}$ be quasi-inverse to each other. On double intersections $U_{ij} = U_i \cap U_j$ there are equivalences $\Phi_{ij} = \Phi_i \Psi_j \colon \mathcal{A}_j^+|_{U_{ij}} \to \mathcal{A}_i^+|_{U_{ij}}$. On triple intersections U_{ijk} there are invertible transformations $\alpha_{ijk} \colon \Phi_{ij}\Phi_{jk} \Rightarrow \Phi_{ik}$ induced by $\Psi_j \Phi_j \Rightarrow \mathrm{id}$. On quadruple intersections U_{ijkl} the following diagram commutes

$$\begin{array}{ccc} \Phi_{ij}\Phi_{jk}\Phi_{kl} & \xRightarrow{\alpha_{ijk}\,\mathrm{id}_{\Phi_{kl}}} & \Phi_{ik}\Phi_{kl} \\ \Big\Downarrow{\scriptstyle \mathrm{id}_{\Phi_{ij}}\,\alpha_{jkl}} & & \Big\Downarrow{\scriptstyle \alpha_{ikl}} \\ \Phi_{ij}\Phi_{jl} & \xRightarrow{\alpha_{ijl}} & \Phi_{il}. \end{array} \tag{2.1}$$

These data are enough to reconstruct $\mathfrak{A}$ (up to equivalence), and we will now describe them more explicitly.

On double intersections U_{ij}, the $\mathcal{R}$-functor $\Phi_{ij}\colon \mathcal{A}_j^+ \to \mathcal{A}_i^+$ is locally induced by $\mathcal{R}$-algebra isomorphisms. There thus exist an open covering $U_{ij} = \bigcup_\alpha U_{ij}^\alpha$ and isomorphisms of $\mathcal{R}$-algebras $f_{ij}^\alpha\colon \mathcal{A}_j \to \mathcal{A}_i$ on U_{ij}^α such that $(f_{ij}^\alpha)^+ = \Phi_{ij}|_{U_{ij}^\alpha}$.

On triple intersections $U_{ijk}^{\alpha\beta\gamma} = U_{ij}^\alpha \cap U_{ik}^\beta \cap U_{jk}^\gamma$, we have invertible transformations $\alpha_{ijk}|_{U_{ijk}^{\alpha\beta\gamma}}\colon (f_{ij}^\alpha)^+(f_{jk}^\gamma)^+ \Rightarrow (f_{ik}^\beta)^+$. There thus exist invertible sections $a_{ijk}^{\alpha\beta\gamma} \in \mathcal{A}_i^\times(U_{ijk}^{\alpha\beta\gamma})$ such that

$$f_{ij}^\alpha f_{jk}^\gamma = \mathrm{Ad}(a_{ijk}^{\alpha\beta\gamma}) f_{ik}^\beta.$$

On quadruple intersections $U_{ijkl}^{\alpha\beta\gamma\delta\epsilon\varphi} = U_{ijk}^{\alpha\beta\gamma} \cap U_{ijl}^{\alpha\delta\epsilon} \cap U_{ikl}^{\beta\delta\varphi} \cap U_{jkl}^{\gamma\epsilon\varphi}$, the diagram (2.1) corresponds to the equalities

$$a_{ijk}^{\alpha\beta\gamma} a_{ikl}^{\beta\delta\varphi} = f_{ij}^\alpha(a_{jkl}^{\gamma\epsilon\varphi}) a_{ijl}^{\alpha\delta\epsilon}.$$

Indices of hypercoverings are quite cumbersome, and we will not write them explicitly anymore[1].

Let us summarize what we just obtained.

Proposition 2.1. *Up to equivalence, an $\mathcal{R}$-algebroid stack is given by the following data:*

(a) an open covering $X = \bigcup_i U_i$,

(b) $\mathcal{R}$-algebras $\mathcal{A}_i$ on U_i,

(c) isomorphisms of $\mathcal{R}$-algebras $f_{ij}\colon \mathcal{A}_j \to \mathcal{A}_i$ on U_{ij},

(d) invertible sections $a_{ijk} \in \mathcal{A}_i^\times(U_{ijk})$,

such that

$$\begin{cases} f_{ij} f_{jk} = \mathrm{Ad}(a_{ijk}) f_{ik}, & \textit{as morphisms } \mathcal{A}_k \to \mathcal{A}_i \textit{ on } U_{ijk}, \\ a_{ijk} a_{ikl} = f_{ij}(a_{jkl}) a_{ijl} & \textit{in } \mathcal{A}_i(U_{ijkl}). \end{cases}$$

Example 2.2. If the $\mathcal{R}$-algebras $\mathcal{A}_i$ are commutative, then the 1-cocycle $f_{ij} f_{jk} = f_{ik}$ defines an $\mathcal{R}$-algebra $\mathcal{A}$ on X and $\{a_{ijk}\}$ induces a 2-cocycle with values in $\mathcal{A}^\times$. In particular, if $\mathcal{A}_i = \mathcal{R}|_{U_i}$, then $f_{ij} = \mathrm{id}$ and $\mathcal{A} = \mathcal{R}$. Hence, $\mathcal{R}$-algebroid stacks locally $\mathcal{R}$-equivalent to $\mathcal{R}^+$ are determined by the 2-cocycle $a_{ijk} \in \mathcal{R}^\times(U_{ijk})$. One checks that two such stacks are (globally) $\mathcal{R}$-equivalent if and only if the corresponding cocycles give the same cohomology class in $H^2(X; \mathcal{R}^\times)$.

[1] Recall that, on a paracompact space, usual coverings are cofinal among hypercoverings.

Example 2.3. Let X be a complex manifold, and $\mathcal{O}_X$ its structural sheaf. A line bundle $\mathcal{L}$ on X is determined (up to isomorphism) by its transition functions $f_{ij} \in \mathcal{O}_X^\times(U_{ij})$, where $X = \bigcup_i U_i$ is an open covering such that $\mathcal{L}|_{U_i} \simeq \mathcal{O}_{U_j}$. Let $\lambda \in \mathbb{C}$, and choose determinations g_{ij} of the multivalued functions f_{ij}^λ. Since $g_{ij}g_{jk}$ and g_{ik} are both determinations of f_{ik}^λ, one has $g_{ij}g_{jk} = c_{ijk}g_{ik}$ for $c_{ijk} \in \mathbb{C}_X^\times(U_{ijk})$.

Let us denote by $\mathbb{C}_{\mathcal{L}^\lambda}$ the $\mathbb{C}$-algebroid stack associated with the cocycle $\{c_{ijk}\}$ as in the previous example. For $\lambda = m \in \mathbb{Z}$ we have $\mathbb{C}_{\mathcal{L}^m} \approx_{\mathbb{C}} \mathbb{C}_X^+$, but in general $\mathbb{C}_{\mathcal{L}^\lambda}$ is non trivial. On the other hand, $\mathcal{L}^\lambda$ defines a global object of the algebroid stack[2] $\mathcal{O}_X^+ \otimes_{\mathbb{C}} \mathbb{C}_{\mathcal{L}^\lambda}$, so that $\mathcal{O}_X^+ \otimes_{\mathbb{C}} \mathbb{C}_{\mathcal{L}^\lambda} \approx_{\mathbb{C}} \mathcal{O}_X^+$ is always trivial. Forgetting the $\mathcal{O}$-linear structure, the Yoneda embedding identifies $\mathcal{L}^\lambda$ with a twisted sheaf in (*i.e.* a global object of) $\mathfrak{Mod}(\mathbb{C}_{\mathcal{L}^{-\lambda}})$. (Here we used the equivalence $(\mathbb{C}_{\mathcal{L}^\lambda})^{\mathrm{op}} \approx_{\mathbb{C}} \mathbb{C}_{\mathcal{L}^{-\lambda}}$.)

3 Quantization of complex symplectic manifolds

The relation between microdifferential operators (see [17]) and WKB operators[3] is classical, and is discussed *e.g.* in [15, 1] for the cotangent bundle of the complex line. We follow here the presentation in [16].

Let M be a complex manifold, and denote by $\rho\colon J^1M \to T^*M$ the projection from the 1-jet bundle to the cotangent bundle. Let $(t;\tau)$ be the system of homogeneous symplectic coordinates on $T^*\mathbb{C}$, and recall that J^1M is identified with the affine chart of the projective cotangent bundle $P^*(M \times \mathbb{C})$ given by $\tau \neq 0$. Denote by $\mathcal{E}_{M\times\mathbb{C}}$ the sheaf of microdifferential operators on $P^*(M \times \mathbb{C})$. Its twist by half-forms $\mathcal{E}^{\sqrt{v}}_{M\times\mathbb{C}} = \pi^{-1}\Omega^{1/2}_{M\times\mathbb{C}} \otimes_{\pi^{-1}\mathcal{O}} \mathcal{E}_{M\times\mathbb{C}} \otimes_{\pi^{-1}\mathcal{O}} \pi^{-1}\Omega^{-1/2}_{M\times\mathbb{C}}$ is endowed with a canonical anti-involution. (Here $\pi\colon P^*(M\times\mathbb{C}) \to M \times \mathbb{C}$ denotes the natural projection.)

In a local coordinate system (x,t) on $M \times \mathbb{C}$, consider the subring $\mathcal{E}^{\sqrt{v}}_{M\times\mathbb{C},\hat{t}}$ of operators commuting with ∂_t. The ring of WKB operators (twisted by half-forms) is defined by

$$\mathcal{W}^{\sqrt{v}}_M = \rho_*(\mathcal{E}^{\sqrt{v}}_{M\times\mathbb{C},\hat{t}}|_{J^1M}).$$

It is endowed with a canonical anti-involution $*$, and its center is the constant sheaf k_{T^*M} with stalk the subfield $k = \mathcal{W}_{\mathrm{pt}} \subset \mathbb{C}[\![\tau^{-1},\tau]$ of WKB operators over a point.

In a local coordinate system (x) on M, with associated symplectic local coordinates $(x;u)$ on T^*M, a WKB operator P of order m defined on a open subset U of T^*M has a total symbol

$$\sigma_{\mathrm{tot}}(P) = \sum_{j=-\infty}^{m} p_j(x;u)\tau^j,$$

[2] We denote here by $\otimes_{\mathcal{R}}$ the tensor product of $\mathcal{R}$-linear stacks. In particular, $(\mathcal{A}_1 \otimes_{\mathcal{R}} \mathcal{A}_2)^+ \approx \mathcal{A}_1^+ \otimes_{\mathcal{R}} \mathcal{A}_2^+$ for $\mathcal{R}$-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$.

[3] WKB stands for Wentzel-Kramer-Brillouin.

where the p_j's are holomorphic functions on U subject to the estimates

$$\left\{\begin{array}{l}\text{for any compact subset } K \text{ of } U \text{ there exists a constant} \\ C_K > 0 \text{ such that for all } j < 0, \sup\limits_K |p_j| \le C_K^{-j}(-j)!.\end{array}\right. \tag{3.1}$$

If Q is another WKB operator defined on U, of total symbol $\sigma_{\mathrm{tot}}(Q)$, then

$$\sigma_{\mathrm{tot}}(P \circ Q) = \sum_{\alpha \in \mathbb{N}^n} \frac{\tau^{-|\alpha|}}{\alpha!} \partial_u^\alpha \sigma_{\mathrm{tot}}(P) \partial_x^\alpha \sigma_{\mathrm{tot}}(Q).$$

Remark 3.1. The ring $\mathcal{W}_M^{\sqrt{v}}$ is a deformation quantization of T^*M in the following sense. Setting $\hbar = \tau^{-1}$, the sheaf of formal WKB operators (obtained by dropping the estimates (3.1)) of degree less than or equal to 0 is locally isomorphic to $\mathcal{O}_{T^*M}[\![\hbar]\!]$ as $\mathbb{C}_{T^*M}$-modules (via the total symbol), and it is equipped with an unitary associative product which induces a star-product on $\mathcal{O}_{T^*M}[\![\hbar]\!]$.

Let X be a complex symplectic manifold of dimension $2n$. Then there are an open covering $X = \bigcup_i U_i$ and symplectic embeddings $\Phi_i \colon U_i \to T^*M$, for $M = \mathbb{C}^n$. Let $\mathcal{A}_i = \Phi_i^{-1}\mathcal{W}_M^{\sqrt{v}}$. Adapting Kashiwara's construction (cf [9]), Polesello-Schapira [16] proved that there exist isomorphisms of k-algebras f_{ij} and invertible sections a_{ijk} as in Proposition 2.1. Their result may thus be restated as

Theorem 3.2. *(cf [16]) On any symplectic complex manifold X there exists a canonical k-algebroid stack $\mathfrak{W}_X$ locally equivalent to $(i^{-1}\mathcal{W}_M^{\sqrt{v}})^+$ for any symplectic local chart $i \colon X \supset U \to T^*M$.*

Note that the canonical anti-involution $*$ on $\mathcal{W}_M^{\sqrt{v}}$ extends to an equivalence of k-stacks $\mathfrak{W}_X \approx_k \mathfrak{W}_X^{\mathrm{op}}$. Note also that, by Lemma 1.1, there exists a deformation quantization algebra on X if the k-algebroid stack $\mathfrak{W}_X$ has a global object, or equivalently if the stack $\mathfrak{Mod}(\mathfrak{W}_X)$ has a global object locally isomorphic to $i^{-1}\mathcal{W}_M^{\sqrt{v}}$ for any symplectic local chart $i \colon X \supset U \to T^*M$.

4 Quantization of involutive submanifolds

Let M be a complex manifold and $V \subset T^*M$ an involutive[4] submanifold. Similarly to the case of microdifferential operators (for which we refer to [11] and [8]), one introduces the sub-sheaf of rings $\mathcal{W}_V^{\sqrt{v}}$ of $\mathcal{W}_M^{\sqrt{v}}$ generated over $\mathcal{W}_M^{\sqrt{v}}(0)$ by the WKB operators $P \in \mathcal{W}_M^{\sqrt{v}}(1)$ such that $\sigma_1(P)$ vanishes on V. Here $\mathcal{W}_M^{\sqrt{v}}(m)$ denotes the sheaf of operators of order less or equal to m, and $\sigma_m(\cdot) \colon \mathcal{W}_M^{\sqrt{v}}(m) \to \mathcal{W}_M^{\sqrt{v}}(m)/\mathcal{W}_M^{\sqrt{v}}(m-1) \simeq \mathcal{O}_{T^*M} \cdot \tau^m$ is the symbol map of order m (which does not depend on the local coordinate system on M).

[4]Recall that V is involutive if for any pair f, g of holomorphic functions vanishing on V, their Poisson bracket $\{f, g\}$ vanishes on V.

Definition 4.1. Let $\mathcal{M}$ be a coherent $\mathcal{W}_M^{\sqrt{v}}$-module.

(i) $\mathcal{M}$ is a regular WKB module along V if locally there exists a coherent sub-$\mathcal{W}_M^{\sqrt{v}}(0)$-module $\mathcal{M}_0$ of $\mathcal{M}$ which generates it over $\mathcal{W}_M^{\sqrt{v}}$, and such that $\mathcal{W}_V^{\sqrt{v}} \cdot \mathcal{M}_0 \subset \mathcal{M}_0$.

(ii) $\mathcal{M}$ is a simple WKB module along V if locally there exists a $\mathcal{W}_M^{\sqrt{v}}(0)$-module $\mathcal{M}_0$ as above such that $\mathcal{M}_0/\mathcal{W}_M^{\sqrt{v}}(-1) \cdot \mathcal{M}_0 \simeq \mathcal{O}_V$.

Example 4.2. Recall that locally, any involutive submanifold $V \subset T^*M$ of codimension d may be written as:

$$V = \{(x;u); u_1 = \cdots = u_d = 0\},$$

in a local symplectic coordinate system $(x;u) = (x_1,\ldots,x_n;u_1,\ldots,u_n)$ on T^*M. In this case, any simple WKB module along V is locally isomorphic to

$$\mathcal{W}_M^{\sqrt{v}}/\mathcal{W}_M^{\sqrt{v}} \cdot (\partial_{x_1},\ldots,\partial_{x_d}).$$

Let X be a complex symplectic manifold of dimension $2n$, and $V \subset X$ an involutive submanifold. The notions of regular and simple module along V being local, they still make sense in the stack $\mathfrak{Mod}_{coh}(\mathfrak{W}_X)$ of coherent WKB modules on X.

Definition 4.3. Denote by $\mathfrak{Mod}_{V\text{-}reg}(\mathfrak{W}_X)$ the full substack of regular objects along V in $\mathfrak{Mod}_{coh}(\mathfrak{W}_X)$, and by $\mathfrak{S}_V$ its full substack of simple objects along V.

By Example 4.2, $\mathfrak{S}_V$ is locally non-empty and locally connected by isomorphisms. Hence it is a k-algebroid stack on X. Since it is supported by V, we consider $\mathfrak{S}_V$ as a stack on V.

The first equivalence in the following theorem asserts that the deformation quantization of V by means of simple WKB modules is equivalent, up to a twist, to that given by WKB operators on the quotient of V by its bicharacteristic leaves.

Theorem 4.4. *Let X be a complex symplectic manifold, and $V \subset X$ an involutive submanifold. Assume that there exist a complex symplectic manifold Z and a map $q\colon V \to Z$ whose fibers are the bicharacteristic leaves of V. Then there are an equivalence of k-algebroid stacks*[5] *on V*

$$\mathfrak{S}_V \approx_k q^{-1}\mathfrak{W}_Z \otimes_{\mathbb{C}} \mathbb{C}_{\Omega_V^{1/2}}, \tag{4.1}$$

and a k-equivalence

$$\mathfrak{Mod}_{V\text{-}reg}(\mathfrak{W}_X) \approx_k \mathfrak{Mod}_{coh}(q^{-1}\mathfrak{W}_Z^{\mathrm{op}} \otimes_{\mathbb{C}} \mathbb{C}_{\Omega_V^{-1/2}}). \tag{4.2}$$

[5]We denote here by q^{-1} the stack-theoretical inverse image. In particular, $q^{-1}(\mathcal{A}^+) \approx_k (q^{-1}\mathcal{A})^+$, for a k-algebra $\mathcal{A}$ on Z.

Note that the statement still holds for a general involutive submanifold $V \subset X$, replacing $q^{-1}\mathfrak{W}_Z$ with the algebroid stack obtained by adapting [16, Proposition 7.3] for the symplectic case.

If $V = X$, then $\mathfrak{S}_X \approx_k \mathfrak{W}_X^{\mathrm{op}}$ is the stack of locally free left WKB modules of rank one, and $\mathfrak{Mod}_{X\text{-}reg}(\mathfrak{W}_X) \approx_k \mathfrak{Mod}_{coh}(\mathfrak{W}_X)$. Since $\Omega_X \simeq \mathcal{O}_X$ by the nth power of the symplectic form, one has $\mathbb{C}_{\Omega_X^{1/2}} \approx_{\mathbb{C}} \mathbb{C}_X^+$. As $q = \mathrm{id}$, the theorem thus reduces to the equivalence $\mathfrak{W}_X^{\mathrm{op}} \approx_k \mathfrak{W}_X$ given by the involution $*$.

If $V = \Lambda$ is Lagrangian, then $Z = \mathrm{pt}$. Hence $q^{-1}\mathfrak{W}_{\mathrm{pt}} \approx_k k_\Lambda^+$, and (4.1) asserts that

$$\mathfrak{S}_\Lambda \otimes_{\mathbb{C}} \mathbb{C}_{\Omega_\Lambda^{-1/2}} \approx_k k_\Lambda^+.$$

In other words, it asserts that $\mathfrak{S}_\Lambda \otimes_{\mathbb{C}} \mathbb{C}_{\Omega_\Lambda^{-1/2}} \subset \mathfrak{Mod}(\mathfrak{W}_X \otimes_{\mathbb{C}} \mathbb{C}_{\Omega_\Lambda^{1/2}})$ has a global object. This is a result of D'Agnolo-Schapira [6], obtained by adapting a similar theorem of Kashiwara [9] for microdifferential operators, along the techniques of [16] (see also [14] for the smooth case). As for (4.2), we recover the WKB analogue of [9, Proposition 4], also stated in [6],

$$\mathfrak{Mod}_{\Lambda\text{-}reg}(\mathfrak{W}_X) \approx_k \mathfrak{Mod}_{loc\text{-}sys}(k_\Lambda^+ \otimes_{\mathbb{C}} \mathbb{C}_{\Omega_\Lambda^{-1/2}}),$$

where the right-hand side denotes the stack of twisted locally constant k-modules of finite rank.

Proof of Theorem 4.4. Consider the two projections

$$X \xleftarrow[p_1]{} X \times Z \xrightarrow[p_2]{} Z.$$

By the graph embedding, V is identified with a Lagrangian submanifold V' of $X \times Z$. By [6] there exists a simple module $\mathcal{L}$ along V' in $\mathfrak{Mod}(\mathfrak{W}_{X\times Z} \otimes_{\mathbb{C}} \mathbb{C}_{\Omega_{V'}^{1/2}})$. Using [4], we get an integral transform k-functor with kernel $\mathcal{L}$

$$\mathfrak{Mod}(p_1^{-1}\mathfrak{W}_X) \longrightarrow \mathfrak{Mod}(p_2^{-1}\mathfrak{W}_Z^{\mathrm{op}} \otimes_{\mathbb{C}} \mathbb{C}_{\Omega_{V'}^{-1/2}}).$$

Since q is identified with $p_2|_{V'}$ and the inclusion $V \subset X$ with $p_1|_{V'}$, we get an induced functor

$$\mathfrak{Mod}(\mathfrak{W}_X|_V) \longrightarrow \mathfrak{Mod}(q^{-1}\mathfrak{W}_Z^{\mathrm{op}} \otimes_{\mathbb{C}} \mathbb{C}_{\Omega_V^{-1/2}}).$$

This restricts to functors

$$\begin{array}{ccc} \mathfrak{Mod}_{V\text{-}reg}(\mathfrak{W}_X) & \longrightarrow & \mathfrak{Mod}_{coh}(q^{-1}\mathfrak{W}_Z^{\mathrm{op}} \otimes_{\mathbb{C}} \mathbb{C}_{\Omega_V^{-1/2}}) \\ \cup & & \cup \\ \mathfrak{S}_V & \longrightarrow & q^{-1}\mathfrak{W}_Z \otimes_{\mathbb{C}} \mathbb{C}_{\Omega_V^{1/2}}, \end{array}$$

which are local, and hence global, equivalences by the WKB analogue of [5, Proposition 4.6] and by a direct computation, respectively. □

As a corollary, we get a sufficient condition for the existence of a globally defined twisted simple WKB module along V.

Corollary 4.5. *In the situation of the above theorem, assume that there exists a deformation quantization algebra $\mathcal{A}$ on Z such that $\mathcal{A}^+ \approx_k \mathfrak{W}_Z$. Then there exists a globally defined simple module along V in $\mathfrak{Mod}(\mathfrak{W}_X \otimes_{\mathbb{C}} \mathbb{C}_{\Omega_V^{1/2}})$.*

Proof. By Lemma 1.1, the k-algebroid $\mathfrak{W}_Z$ has a global object. Then its image by the adjunction functor $\mathfrak{W}_Z \to q_* q^{-1}\mathfrak{W}_Z$ gives a globally defined twisted simple WKB module along V. □

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