

UNIQUENESS FOR A STOCHASTIC INVISCID DYADIC MODEL

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ABSTRACT. For the deterministic dyadic model of turbulence, there are examples of initial conditions in l^2 which have more than one solution. The aim of this paper is to prove that uniqueness, for all l^2 -initial conditions, is restored when a suitable multiplicative noise is introduced. The noise is formally energy preserving. Uniqueness is understood in the weak probabilistic sense.

1. INTRODUCTION

The infinite system of nonlinear differential equations

$$(1.1) \quad \begin{aligned} \frac{dX_n(t)}{dt} &= k_{n-1}X_{n-1}^2(t) - k_nX_n(t)X_{n+1}(t), \quad t \geq 0, \\ X_n(0) &= x_n \end{aligned}$$

for $n \geq 1$, with coefficients $k_n > 0$ for each $n \geq 1$, $X_0(t) = 0$ and $k_0 = 0$, is one of the simplest models which presumably reflect some of the properties of 3D Euler equations. At least, it is infinite dimensional, formally conservative (the energy $\sum_{n=1}^{\infty} X_n^2(t)$ is formally constant), and quadratic. One of its ‘pathologies’ is the lack of uniqueness of solutions, in the space l^2 of square summable sequences: when, for instance, $k_n = \lambda^n$ with $\lambda > 1$, there are examples of initial conditions $x = (x_n)_{n \geq 1} \in l^2$ such that there exist at least two solutions in l^2 on some interval $[0, T]$, with continuous components. This has been proved in [3]: on one side, given any $x \in l^2$, there exists a solution such that $\sum_{n=1}^{\infty} X_n^2(t) \leq \sum_{n=1}^{\infty} x_n^2$ for all $t \geq 0$; on the other side, for special elements $a = (a_n)_{n \geq 1} \in l^2$, the strictly increasing sequence

$$\frac{a_n}{t_0 - t}, \quad t \in [0, t_0)$$

is a (self-similar) solution. Other counterexamples can be done by time-reversing any solution that dissipates energy (this happens for all solutions having positive components). System (1.1) and variants of it have other special features, such as energy dissipation and loss of regularity; see [5], [6], [7], [14], [17], [18], [26], [3].

In this paper we prove that uniqueness is restored under a suitable random perturbation. On a filtered probability space (Ω, F_t, P) , let $(W_n)_{n \geq 1}$ be a sequence

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of independent Brownian motions. We consider the infinite system of stochastic differential equations in Stratonovich form:

$$(1.2) \quad dX_n = (k_{n-1}X_{n-1}^2 - k_n X_n X_{n+1}) dt + \sigma k_{n-1} X_{n-1} \circ dW_{n-1} - \sigma k_n X_{n+1} \circ dW_n$$

for $n \geq 1$, with $X_0(t) = 0$ and $\sigma \neq 0$. The concept of an exponentially integrable solution, used in the following theorem, is defined in the next section. By classical arguments, we shall prove weak existence in the class of exponentially integrable solutions. Our main theorem is:

Theorem 1.1. *Given $x \in l^2$, in the class of exponentially integrable solutions on an interval $[0, T]$ there is weak uniqueness for equation (1.2).*

The proof is given in section 4. Weak uniqueness here means uniqueness of the law of the process on the space $C([0, T]; \mathbb{R})^{\mathbb{N}}$. Our approach is based on the Girsanov transformation, so this is the natural result one expects. We do not know about strong uniqueness. Our use of the Girsanov transformation is not the most classical one and is inspired by [1], [8].

The multiplicative noise in equation (1.2) preserves the formal energy conservation. By applying the rules of Stratonovich calculus (see the same computation at the Itô level in the proof of Theorem 3.2) we have

$$\begin{aligned} dX_n^2 &= 2(k_{n-1}X_{n-1}^2 X_n - k_n X_n^2 X_{n+1}) dt \\ &\quad + \sigma k_{n-1} X_{n-1} X_n \circ dW_{n-1} - \sigma k_n X_n X_{n+1} \circ dW_n \end{aligned}$$

so that, *formally*, $d\sum_{n=1}^{\infty} X_n^2(t) = 0$. Only a multiplicative noise of a special form has this property, which is one of the key properties formally verified by the Euler equations. Notice that the Itô formulation (2.1) below (see also the linear analog (3.1)) contains a dissipative term, which however is exactly balanced by the correction term when the Itô formula is applied. Thus equation (2.1) below is not formally dissipative as it may appear at first glance.

Having in mind the lack of uniqueness, or at least the open problems about uniqueness, typical of various deterministic models in fluid dynamics, we think it is relevant to know that suitable stochastic perturbations may restore uniqueness. An example in this direction is known for the linear transport equation with poor regularity of coefficients; see [12]. The model of the present paper seems to be the first nonlinear example of this regularization phenomenon (in the area of equations of fluid dynamic type; otherwise see [15], [16] and related works, based on completely different methods). Partial results in the direction of improvements of well posedness, by means of additive noise, have been obtained for the 3D Navier-Stokes equations and other models by [9], [13].

Let us remark that, although equation (1.2) is not a PDE, it has a vague correspondence with the stochastic Euler equation

$$du + [u \cdot \nabla u + \nabla p] dt + \sigma \sum_j \nabla u \circ dW^j(t) = 0, \quad \operatorname{div} u = 0.$$

The energy is formally conserved also in this equation. The noise of this equation is multiplicative as in [12] and linearly dependent on the first derivatives of the solution. For a Lagrangian motivation of such a noise, in the case of stochastic Navier-Stokes equations, see [22].

In equation (1.2), we have inserted the parameter $\sigma \neq 0$ just to emphasize the basic open problem of understanding the zero-noise limit, $\sigma \rightarrow 0$. For simple examples of linear transport equations this is possible and yields a nontrivial selection principle among different solutions of the deterministic limit equation; see [2]. In the nonlinear case of the present paper the small coefficient σ appears in the form of a singular perturbation in the Girsanov density; thus the analysis of $\sigma \rightarrow 0$ is nontrivial.

2. ITÔ FORMULATION

The Itô form of equation (1.2) is

$$(2.1) \quad dX_n = (k_{n-1}X_{n-1}^2 - k_nX_nX_{n+1}) dt + \sigma k_{n-1}X_{n-1}dW_{n-1} - \sigma k_nX_{n+1}dW_n \\ - \frac{\sigma^2}{2} (k_n^2 + k_{n-1}^2) X_n dt$$

for all $n \geq 1$, with $k_0 = 0$ and $X_0 = 0$, as explained at the end of this section. All our rigorous analyses are based on the Itô form, the Stratonovich one serving mainly as an heuristic guideline.

Let us introduce the concept of a weak solution (equivalent to the concept of a solution of the martingale problem). Since our main emphasis is on uniqueness, we shall always restrict ourselves to a finite time horizon $[0, T]$.

By a filtered probability space (Ω, F_t, P) , on a finite time horizon $[0, T]$, we mean a probability space (Ω, F_T, P) and a right-continuous filtration $(F_t)_{t \in [0, T]}$.

Definition 2.1. Given $x \in l^2$, a weak solution of equation (1.2) in l^2 is a filtered probability space (Ω, F_t, P) , a sequence of independent Brownian motions $(W_n)_{n \geq 1}$ on (Ω, F_t, P) , and an l^2 -valued stochastic process $(X_n)_{n \geq 1}$ on (Ω, F_t, P) , with continuous adapted components X_n , such that

$$X_n(t) = x_n + \int_0^t (k_{n-1}X_{n-1}^2(s) - k_nX_n(s)X_{n+1}(s)) ds \\ + \int_0^t \sigma k_{n-1}X_{n-1}(s) dW_{n-1}(s) - \int_0^t \sigma k_nX_{n+1}(s) dW_n(s) \\ - \int_0^t \frac{\sigma^2}{2} (k_n^2 + k_{n-1}^2) X_n(s) ds$$

for each $n \geq 1$, with $k_0 = 0$ and $X_0 = 0$. We denote this solution by (Ω, F_t, P, W, X) , or simply by X .

To prove uniqueness we need the following technical condition, which we call exponential integrability, for brevity.

Definition 2.2. We say that a weak solution (Ω, F_t, P, W, X) is exponentially integrable if

$$E^P \left[e^{\frac{1}{\sigma^2} \int_0^T \sum_{n=1}^{\infty} X_n^2(t) dt} \left(1 + \int_0^T X_i^4(t) dt \right)^2 \right] < \infty$$

for all $i \in \mathbb{N}$.

We say that a weak solution is of class L^∞ if there is a constant $C > 0$ such that $\sum_{n=1}^{\infty} X_n^2(t) \leq C$ for a.e. $(\omega, t) \in \Omega \times [0, T]$.

L^∞ -solutions are exponentially integrable. Our main result, Theorem 1.1, states the weak uniqueness in the class of exponentially integrable solutions. In addition, we have

Theorem 2.3. *Given $(x_n) \in l^2$, there exists a weak L^∞ -solution to equation (1.2).*

The proof is given in section 4 and is based again on the Girsanov transform. However, we remark that existence can be proved also by the compactness method, similarly to the case of stochastic Euler or Navier-Stokes equations; see for instance [4] and [11]. In both cases, notice that it is a weak existence result: the solution is not necessarily adapted to the completed filtration of the Brownian motions.

The following proposition clarifies that a process satisfying (2.1) rigorously satisfies also (1.2).

Proposition 2.4. *If X is a weak solution of equation (1.2), then for every $n \geq 1$, the process $(X_n(t))_{t \geq 0}$ is a continuous semimartingale; hence the two Stratonovich integrals*

$$\begin{aligned} & \int_0^t k_{n-1} X_{n-1}(s) \circ dW_{n-1}(s) \quad \text{for } n \geq 2, \\ & - \int_0^t k_n X_{n+1}(s) \circ dW_n(s) \quad \text{for } n \geq 1 \end{aligned}$$

are well defined and equal, respectively, to

$$\begin{aligned} & \int_0^t k_{n-1} X_{n-1}(s) dW_{n-1}(s) - \frac{\sigma}{2} \int_0^t k_{n-1}^2 X_n(s) ds, \\ & - \int_0^t k_n X_{n+1}(s) dW_n(s) - \frac{\sigma}{2} \int_0^t k_n^2 X_n(s) ds. \end{aligned}$$

Hence X satisfies the Stratonovich equations (1.2).

Proof. We use a number of concepts and rules of stochastic calculus that can be found for instance in [21]. We have

$$\int_0^t X_{n-1}(s) \circ dW_{n-1}(s) = \int_0^t X_{n-1}(s) dW_{n-1}(s) + \frac{1}{2} [X_{n-1}, W_{n-1}]_t,$$

where $[X_{n-1}, W_{n-1}]_t$ is the joint quadratic variation of X_{n-1} and W_{n-1} . From the equation for $X_{n-1}(t)$, using the independence of the Brownian motions, we can compute $[X_{n-1}, W_{n-1}]_t = - \int_0^t \sigma k_{n-1} X_n(s) ds$. Similarly

$$\int_0^t X_{n+1}(s) \circ dW_n(s) = \int_0^t X_{n+1}(s) dW_n(s) + \frac{1}{2} [X_{n+1}, W_n]_t$$

and $[X_{n+1}, W_n]_t = \int_0^t \sigma k_n X_n(s) ds$. The proof is complete. □

3. AUXILIARY LINEAR EQUATION

Up to a Girsanov transform (section 4), our results are based on the following infinite system of *linear* stochastic differential equations:

$$\begin{aligned} dX_n &= \sigma k_{n-1} X_{n-1} \circ dB_{n-1} - \sigma k_n X_{n+1} \circ dB_n, \\ X_n(0) &= x_n \end{aligned}$$

for $n \geq 1$, with $X_0(t) = 0$ and $\sigma \neq 0$, where $(B_n)_{n \geq 0}$ is a sequence of independent Brownian motions. The Itô formulation is

$$(3.1) \quad \begin{aligned} dX_n &= \sigma k_{n-1} X_{n-1} dB_{n-1} - \sigma k_n X_{n+1} dB_n - \frac{\sigma^2}{2} (k_n^2 + k_{n-1}^2) X_n dt, \\ X_n(0) &= x_n. \end{aligned}$$

Definition 3.1. Let (Ω, F_t, Q) be a filtered probability space and let $(B_n)_{n \geq 0}$ be a sequence of independent Brownian motions on (Ω, F_t, Q) . Given $x \in l^2$, a solution of equation (3.1) on $[0, T]$ in the space l^2 is an l^2 -valued stochastic process $(X(t))_{t \in [0, T]}$, with continuous adapted components X_n , such that Q -a.s.

$$\begin{aligned} X_n(t) &= x_n + \int_0^t \sigma k_{n-1} X_{n-1}(s) dB_{n-1}(s) - \int_0^t \sigma k_n X_{n+1}(s) dB_n(s) \\ &\quad - \int_0^t \frac{\sigma^2}{2} (k_n^2 + k_{n-1}^2) X_n(s) ds \end{aligned}$$

for each $n \geq 1$ and $t \in [0, T]$, with $k_0 = 0$ and $X_0 = 0$.

Our main technical result is the following theorem.

Theorem 3.2. *Given $x \in l^2$, in the class of solutions of equation (3.1) on $[0, T]$ such that*

$$(3.2) \quad \int_0^T E^Q [X_n^4(t)] dt < \infty$$

for each $n \geq 1$ and

$$(3.3) \quad \lim_{n \rightarrow \infty} \int_0^T E^Q [X_n^2(t)] dt = 0$$

there is at most one element.

Proof. By linearity, it is sufficient to prove that a solution $(X_n)_{n \geq 1}$, with properties (3.2) and (3.3), with null initial condition is the zero solution. Assume thus that $x = 0$. We have

$$\begin{aligned} X_n(t) &= x_n + \int_0^t \sigma k_{n-1} X_{n-1}(s) dB_{n-1}(s) - \int_0^t \sigma k_n X_{n+1}(s) dB_n(s) \\ &\quad - \int_0^t \frac{\sigma^2}{2} (k_n^2 + k_{n-1}^2) X_n(s) ds; \end{aligned}$$

hence, from the Itô formula, we have

$$\begin{aligned} \frac{1}{2} dX_n^2 &= X_n dX_n + \frac{1}{2} d[X_n]_t \\ &= -\frac{\sigma^2}{2} (k_n^2 + k_{n-1}^2) X_n^2 dt + dM_n + \frac{\sigma^2}{2} (k_{n-1}^2 X_{n-1}^2 + k_n^2 X_{n+1}^2) dt, \end{aligned}$$

where

$$M_n(t) = \int_0^t \sigma k_{n-1} X_{n-1}(s) X_n(s) dB_{n-1}(s) - \int_0^t \sigma k_n X_n(s) X_{n+1}(s) dB_n(s).$$

From (3.2), $M_n(t)$ is a martingale, for each $n \geq 1$; hence $E^Q [M_n(t)] = 0$. Moreover, for each $n \geq 1$, $E^Q [X_n^2(t)]$ is finite and continuous in t : it follows easily

from condition (3.2) and equation (3.1) itself. From the previous equation (and the property $E^Q [X_n^2(0)] = 0$) we deduce that $E^Q [X_n^2(t)]$ satisfies

$$E^Q [X_n^2(t)] = -\sigma^2 (k_n^2 + k_{n-1}^2) \int_0^t E^Q [X_n^2(s)] ds + \sigma^2 k_{n-1}^2 \int_0^t E^Q [X_{n-1}^2(s)] ds + \sigma^2 k_n^2 \int_0^t E^Q [X_{n+1}^2(s)] ds$$

for $n \geq 1$, with $u_0(t) = 0$ for $t \geq 0$. It follows that

$$\int_0^t E^Q [(X_{n+1}^2(s) - X_n^2(s))] ds \geq \frac{k_{n-1}^2}{k_n^2} \int_0^t E^Q [(X_n^2(s) - X_{n-1}^2(s))] ds.$$

Since $X_0 \equiv 0$, we have $\int_0^t E^Q [(X_1^2(s) - X_0^2(s))] ds \geq 0$ and thus

$$\int_0^t E^Q [(X_{n+1}^2(s) - X_n^2(s))] ds \geq 0$$

for every $n \geq 1$, by induction. This implies that

$$\int_0^T E^Q [X_n^2(s)] ds \leq \int_0^T E^Q [X_{n+1}^2(s)] ds$$

for all $n \geq 1$. Therefore, by assumption (3.3), for every $n \geq 1$ we have

$$\int_0^T E^Q [X_n^2(s)] ds = 0.$$

This implies that $X_n^2(s) = 0$ a.s. in (ω, s) ; hence X is the null process. The proof is complete. \square

We complete this section with an existence result. The class $L^\infty(\Omega \times [0, T]; l^2)$ is included in the class described by the uniqueness theorem.

Notice that this is a result of strong existence and strong (or pathwise) uniqueness.

Theorem 3.3. *Given $x \in l^2$, there exists a unique solution in $L^\infty(\Omega \times [0, T]; l^2)$, with continuous components.*

Proof. We have only to prove existence. For every positive integer N , consider the finite dimensional stochastic system

$$dX_n^{(N)} = \sigma k_{n-1} X_{n-1}^{(N)} dB_{n-1} - \sigma k_n X_{n+1}^{(N)} dB_n - \frac{\sigma^2}{2} (k_n^2 + k_{n-1}^2) X_n^{(N)} dt, \\ X_n^{(N)}(0) = x_n$$

for $n = 1, \dots, N$, with $k_0 = k_N = 0$, $X_0^{(N)}(t) = X_{N+1}^{(N)}(t) = 0$. This linear finite dimensional equation has a unique global strong solution. By the Itô formula,

$$(3.4) \quad \frac{1}{2} d(X_n^{(N)})^2 = X_n^{(N)} dX_n^{(N)} + \frac{1}{2} d[X_n^{(N)}, X_n^{(N)}]_t \\ = \sigma k_{n-1} X_n^{(N)} X_{n-1}^{(N)} dB_{n-1} - \sigma k_n X_n^{(N)} X_{n+1}^{(N)} dB_n - \frac{\sigma^2}{2} (k_n^2 + k_{n-1}^2) (X_n^{(N)})^2 dt \\ + \frac{\sigma^2}{2} \left(k_{n-1}^2 (X_{n-1}^{(N)})^2 + k_n^2 (X_{n+1}^{(N)})^2 \right);$$

hence

$$\begin{aligned} \frac{1}{2}d \sum_{n=1}^N \left(X_n^{(N)}\right)^2 &= \sum_{n=1}^N \sigma k_{n-1} X_n^{(N)} X_{n-1}^{(N)} dB_{n-1} - \sum_{n=1}^N \sigma k_n X_n^{(N)} X_{n+1}^{(N)} dB_n \\ &\quad - \frac{\sigma^2}{2} \sum_{n=1}^N k_n^2 \left(X_n^{(N)}\right)^2 dt + \frac{\sigma^2}{2} \sum_{n=1}^N k_{n-1}^2 \left(X_{n-1}^{(N)}\right)^2 \\ &\quad - \frac{\sigma^2}{2} \sum_{n=1}^N k_{n-1}^2 \left(X_n^{(N)}\right)^2 dt + \frac{\sigma^2}{2} \sum_{n=1}^N k_n^2 \left(X_{n+1}^{(N)}\right)^2. \end{aligned}$$

This is equal to zero. Thus

$$\sum_{n=1}^N \left(X_n^{(N)}\right)^2 (t) = \sum_{n=1}^N x_n^2, \quad Q\text{-a.s.}$$

In particular, this very strong bound implies that there exists a subsequence $N_k \rightarrow \infty$ such that $\left(X_n^{(N_k)}\right)_{n \geq 1}$ converges weakly to some $(X_n)_{n \geq 1}$ in $L^p(\Omega \times [0, T]; l^2)$ for every $p > 1$ and also weak star in $L^\infty(\Omega \times [0, T]; l^2)$. Hence in particular $(X_n)_{n \geq 1}$ belongs to $L^\infty(\Omega \times [0, T]; l^2)$. Now the proof proceeds by standard arguments typical of equations with monotone operators (which thus apply to linear equations), presented in [23], [20]. The subspace of $L^p(\Omega \times [0, T]; l^2)$ of progressively measurable processes is strongly closed, hence weakly closed, and hence $(X_n)_{n \geq 1}$ is progressively measurable. The one-dimensional stochastic integrals which appear in each equation of system (3.1) are (strongly) continuous linear operators from the subspace of $L^2(\Omega \times [0, T]; l^2)$ of progressively measurable processes to $L^2(\Omega)$; hence they are weakly continuous, a fact that allows us to pass to the limit in each one of the linear equations of system (3.1). A posteriori, from these integral equations, it follows that there is a modification such that all components are continuous. The proof of existence is complete. \square

4. GIRSANOV TRANSFORM

The idea is that equation (2.1) written in the form

$$\begin{aligned} dX_n &= \sigma k_{n-1} X_{n-1} \left(\frac{1}{\sigma} X_{n-1} dt + dW_{n-1} \right) - \sigma k_n X_{n+1} \left(\frac{1}{\sigma} X_n dt + dW_n \right) \\ &\quad - \frac{\sigma^2}{2} (k_n^2 + k_{n-1}^2) X_n dt \end{aligned}$$

becomes equation (3.1) because the processes $B_n(t) := \frac{1}{\sigma} \int_0^t X_n(s) ds + W_n(t)$ are Brownian motions with respect to a new measure Q on (Ω, F_T) ; conversely, both weak existence and weak uniqueness statements transfer from equation (3.1) to equation (2.1). Equation (3.1) was also proved to be *strongly* well posed, but the same problem for the nonlinear model (2.1) is open.

Let us give the details. We use results about the Girsanov theorem that can be found in [25], Chapter VIII, and an infinite dimensional version proved in [19], [10].

4.1. **Proof of Theorem 1.1.** Let us prepare the proof with a few remarks. Assume that $(X_n)_{n \geq 1}$ is an exponentially integrable solution. Since in particular $E \left[\int_0^T \sum_{n=1}^\infty X_n^2(s) ds \right] < \infty$, the process $L_t := -\frac{1}{\sigma} \sum_{n=1}^\infty \int_0^t X_n(s) dW_n(s)$ is well defined, is a martingale and its quadratic variation $[L, L]_t$ is $\frac{1}{\sigma^2} \int_0^t \sum_{n=1}^\infty X_n^2(s) ds$. Since $E \left[\exp \left(\frac{1}{2\sigma^2} \int_0^T \sum_{n=1}^\infty X_n^2(t) dt \right) \right] < \infty$, the Novikov criterium applies, so $\exp \left(L_t - \frac{1}{2} [L, L]_t \right)$ is a strictly positive martingale. Define the probability measure Q on F_T by setting

$$(4.1) \quad \frac{dQ}{dP} = \exp \left(L_T - \frac{1}{2} [L, L]_T \right).$$

Notice also that Q and P are equivalent on F_T , by the strict positivity and

$$(4.2) \quad \frac{dP}{dQ} = \exp \left(Z_T - \frac{1}{2} [Z, Z]_T \right),$$

where

$$Z_t = \sum_{n=1}^\infty \int_0^t \frac{1}{\sigma} X_n(s) dB_n(s),$$

$$B_n(t) = W_n(t) + \int_0^t \frac{1}{\sigma} X_n(s) ds.$$

Indeed $\frac{dP}{dQ} = \exp \left(-L_T + \frac{1}{2} [L, L]_T \right)$, and one can check that $-L_T + \frac{1}{2} [L, L]_T = Z_T - \frac{1}{2} [Z, Z]_T$.

Under Q , $(B_n(t))_{n \geq 1, t \in [0, T]}$ is a sequence of independent Brownian motions. Since

$$\int_0^t k_{n-1} X_{n-1}(s) dB_{n-1}(s) = \int_0^t k_{n-1} X_{n-1}(s) dW_{n-1}(s) + \int_0^t k_{n-1} X_{n-1}(s) X_{n-1}(s) ds$$

and similarly for $\int_0^t k_n X_{n+1}(s) dB_n(s)$, we see that

$$X_n(t) = X_n(0) + \int_0^t k_{n-1} X_{n-1}(s) dB_{n-1}(s) - \int_0^t k_n X_{n+1}(s) dB_n(s) - \int_0^t \frac{1}{2} (k_n^2 + k_{n-1}^2) X_n(s) ds.$$

This is equation (3.1). We have proved the first half of the following lemma:

Lemma 4.1. *If (Ω, F_t, P, W, X) is an exponentially integrable solution of the non-linear equation (1.2), then it is a solution of the linear equation (3.1), where the processes*

$$B_n(t) = W_n(t) + \int_0^t \frac{1}{\sigma} X_n(s) ds$$

are a sequence of independent Brownian motions on (Ω, F_T, Q) , Q defined by (4.1). In addition, the process X on (Ω, F_T, Q) satisfies the assumptions of Theorem 3.2.

Proof. It remains to prove that conditions (3.2) and (3.3) hold true. We have

$$\begin{aligned} E^Q \left[\int_0^T X_n^4(t) dt \right] &= E^P \left[\mathcal{E}(L)_T \int_0^T X_n^4(t) dt \right] \\ &= E^P \left[\exp \left(L_T - [L, L]_T + \frac{1}{2} [L, L]_T \right) \int_0^T X_n^4(t) dt \right] \\ &\leq E^P [\exp(2L_T - 2[L, L]_T)]^{1/2} E^P \left[\left(\int_0^T X_n^4(t) dt \right)^2 \exp[L, L]_T \right]^{1/2}. \end{aligned}$$

The second factor is finite by the condition of exponential integrability of X . The term $E^P [\exp(2L_T - 2[L, L]_T)]$ is equal to one, by the Girsanov theorem applied to the martingale $2L_t$. The proof of condition (3.2) is complete. As to condition (3.3), it follows from the fact that $E \left[\int_0^T \sum_{n=1}^{\infty} X_n^2(s) ds \right] < \infty$, a consequence of the exponential integrability of X . The proof is complete. \square

One may also check that

$$dX_n = \sigma k_{n-1} X_{n-1} \circ dB_{n-1} - \sigma k_n X_{n+1} \circ dB_n,$$

so the previous computations could be described at the level of Stratonovich calculus.

Let us now prove weak uniqueness (the proof is now classical). Assume that $(\Omega^{(i)}, F_t^{(i)}, P^{(i)}, W^{(i)}, X^{(i)})$, $i = 1, 2$, are two exponentially integrable solutions of equation (1.2) with the same initial condition $x \in l^2$. Then

$$(4.3) \quad dX_n^{(i)} = \sigma k_{n-1} X_{n-1}^{(i)} dB_{n-1}^{(i)} - \sigma k_n X_{n+1}^{(i)} dB_n^{(i)} - \frac{\sigma^2}{2} (k_n^2 + k_{n-1}^2) X_n^{(i)} dt,$$

where, for each $i = 1, 2$,

$$B_n^{(i)}(t) = W_n^{(i)}(t) + \int_0^t \frac{1}{\sigma} X_n^{(i)}(s) ds$$

is a sequence of independent Brownian motions on $(\Omega^{(i)}, F_T^{(i)}, Q^{(i)})$, $Q^{(i)}$ defined by (4.1) with respect to $(P^{(i)}, W^{(i)}, X^{(i)})$.

We have proved in Theorem 3.2 that equation (3.1) has a unique strong solution. Thus it has uniqueness in law on $C([0, T]; \mathbb{R})^{\mathbb{N}}$, by the Yamada-Watanabe theorem (see [25], [24]); namely, the laws of $X^{(i)}$ under $Q^{(i)}$ are the same. The proof of the Yamada-Watanabe theorem in this infinite dimensional context, with the laws on $C([0, T]; \mathbb{R})^{\mathbb{N}}$, is step by step identical to the finite dimensional proof, for instance of [25], Chapter 9, Lemma 1.6 and Theorem 1.7. We do not repeat it here.

Given $n \in \mathbb{N}$, $t_1, \dots, t_n \in [0, T]$ and a measurable bounded function $f : (l^2)^n \rightarrow \mathbb{R}$, from (4.2) we have

$$\begin{aligned} &E^{P^{(i)}} \left[f \left(X^{(i)}(t_1), \dots, X^{(i)}(t_n) \right) \right] \\ &= E^{Q^{(i)}} \left[\exp \left(Z_t^{(i)} - \frac{1}{2} [Z^{(i)}, Z^{(i)}]_t \right) f \left(X^{(i)}(t_1), \dots, X^{(i)}(t_n) \right) \right], \end{aligned}$$

where $Z_t^{(i)} := \sum_{n=1}^{\infty} \int_0^t \frac{1}{\sigma} X_n^{(i)}(s) dB_n^{(i)}(s)$. Under $Q^{(i)}$, the law of $(Z^{(i)}, X^{(i)})$ on $C([0, T]; \mathbb{R})^{\mathbb{N}} \times C([0, T]; \mathbb{R})^{\mathbb{N}}$ is independent of $i = 1, 2$. A way to explain this fact

is to consider the enlarged system of stochastic equations composed of equation (4.3) and equation

$$dZ^{(i)} = \sum_{n=1}^{\infty} \frac{1}{\sigma} X_n^{(i)} dB_n^{(i)}.$$

This enlarged system has strong uniqueness, for trivial reasons, and thus also weak uniqueness by the Yamada-Watanabe theorem.

Hence

$$E^{P^{(1)}} \left[f \left(X^{(1)}(t_1), \dots, X^{(1)}(t_n) \right) \right] = E^{P^{(2)}} \left[f \left(X^{(2)}(t_1), \dots, X^{(2)}(t_n) \right) \right].$$

Thus we have uniqueness of the laws of $X^{(i)}$ on $C([0, T]; \mathbb{R})^{\mathbb{N}}$. The proof of uniqueness is complete.

4.2. Proof of Theorem 2.3. Let (Ω, F_t, Q, B, X) be a solution in $L^\infty(\Omega \times [0, T]; l^2)$ of the linear equation (3.1), provided by Theorem 3.3. Let us argue as in the previous subsection but from Q to P , namely by introducing the new measure P on (Ω, F_T) defined as $\frac{dP}{dQ} = \exp\left(Z_T - \frac{1}{2}[Z, Z]_T\right)$, where

$$Z_t := \sum_{n=1}^{\infty} \int_0^t \frac{1}{\sigma} X_n(s) dB_n(s).$$

Under P , the processes

$$W_n(t) := B_n(t) - \int_0^t \frac{1}{\sigma} X_n(s) ds$$

are a sequence of independent Brownian motions. We obtain that (Ω, F_t, P, W, X) is an L^∞ -solution of the nonlinear equation (2.1). The L^∞ -property is preserved since P and Q are equivalent. The proof of existence is complete.

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