Adv. Calc. Var. **1** (2008), 93–121 DOI 10.1515 / ACV.2008.003

Heisenberg isoperimetric problem. The axial case

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Communicated by N. Fusco

Abstract. We prove Pansu's conjecture about the Heisenberg isoperimetric problem in the class of axially symmetric sets. The result is based on a weighted rearrangement scheme in the half plane which is of independent interest.

Keywords. Isoperimetric problem, Heisenberg group.

AMS classification. 49Q20.

1 Introduction

We identify the Heisenberg group $\mathbb{H}^n$ with $\mathbb{C}^n \times \mathbb{R}$, $n \in \mathbb{N}$, endowed with the group law

$$(z,t)(z',t') = (z+z', t+t'+2\,\mathrm{Im}\, z\cdot \bar{z}'),$$

where $t,t' \in \mathbb{R}$, $z = x+iy$, $z' = x'+iy' \in \mathbb{C}^n$ with $x,y,x',y' \in \mathbb{R}^n$ and $z\cdot\bar{z}' = \sum_{j=1}^n z_j \bar{z}'_j$. The group is non-commutative and its center is $Z = \{(z,t) \in \mathbb{H}^n \mid z = 0\}$. The Lie algebra of left-invariant vector fields is spanned by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t} \quad \text{and} \quad T = \frac{\partial}{\partial t},$$

with $j = 1,\dots,n$. The distribution spanned by the vector fields X_j and Y_j, called horizontal distribution, generates the Lie algebra by brackets. The maps $\delta_\lambda : \mathbb{H}^n \to \mathbb{H}^n$, $\lambda > 0$,

$$\delta_\lambda(z,t) = (\lambda z, \lambda^2 t), \tag{1.1}$$

form a group of automorphisms of $\mathbb{H}^n$ called dilations.

The natural volume in $\mathbb{H}^n$ is the Haar measure, which, up to a positive factor, coincides with Lebesgue measure $\mathcal{L}^{2n+1}$ in $\mathbb{C}^n \times \mathbb{R}$. Let $\Omega \subset \mathbb{H}^n$ be an open set. The Heisenberg (horizontal) divergence of a vector field $\varphi \in C^1(\Omega;\mathbb{R}^{2n})$ is

$$\mathrm{div}_{\mathbb{H}}\varphi = \sum_{j=1}^n \big(X_j\varphi_{2j-1} + Y_j\varphi_{2j}\big).$$

The Heisenberg perimeter in Ω of a $\mathcal{L}^{2n+1}$-measurable set $E \subset \mathbb{H}^n$ is

$$\mathcal{P}(E;\Omega) = \sup\left\{ \int_E \mathrm{div}_{\mathbb{H}}\varphi(z,t)\,dz\,dt \;\middle|\; \varphi \in C^1_0(\Omega;\mathbb{R}^{2n}),\ \|\varphi\|_\infty \le 1 \right\}.$$

If $\mathcal{P}(E;\Omega) < +\infty$, we say that E has finite Heisenberg perimeter in Ω. In the case $\Omega = \mathbb{H}^n$, we let $\mathcal{P}(E) = \mathcal{P}(E;\mathbb{H}^n)$. The structure of sets with finite Heisenberg perimeter is studied by Franchi, Serapioni and Serra Cassano in [10]. In this article, among many other results, it is proved that for sets with boundary of class C^1 perimeter equals the $(2n+1)$-dimensional spherical Hausdorff measure of the boundary constructed by means of the Carnot–Carathéodory metric. For sets with boundary of class C^2, perimeter also coincides with Minkowski content of the boundary [20].

Volume and perimeter are related through the Heisenberg isoperimetric inequality: there exists a constant $C_n > 0$ such that for any $\mathcal{L}^{2n+1}$-measurable set $E \subset \mathbb{H}^n$

$$\min\left\{\mathcal{L}^{2n+1}(E), \mathcal{L}^{2n+1}(\mathbb{H}^n \setminus E)\right\} \le C_n \mathcal{P}(E)^{\frac{2n+2}{2n+1}}. \tag{1.2}$$

This inequality was first proved by Pansu in [21] and [22] in the case $n = 1$ for bounded smooth sets (with 3-Hausdorff measure replacing perimeter). In the general form (1.2), the inequality is due to Garofalo and Nhieu [11] (see also [8]). As far as the exponents appearing in (1.2) is concerned, note that volume and perimeter scale homogeneously with respect to dilations (1.1), and precisely for any $\lambda > 0$

$$\mathcal{L}^{2n+1}(\delta_\lambda(E)) = \lambda^{2n+2}\mathcal{L}^{2n+1}(E) \quad \text{and} \quad \mathcal{P}(\delta_\lambda(E)) = \lambda^{2n+1}\mathcal{P}(E).$$

Volume and perimeter are left invariant.

We denote by $\mathcal{E}$ the family of all $\mathcal{L}^{2n+1}$-measurable subsets E of $\mathbb{H}^n$ such that $0 < \mathcal{L}^{2n+1}(E) < +\infty$ and consider the infimum

$$\operatorname{Isop}(\mathcal{E}) = \inf\left\{ \frac{\mathcal{P}(E)^{2n+2}}{\mathcal{L}^{2n+1}(E)^{2n+1}} \,\middle|\, E \in \mathcal{E} \right\}. \tag{1.3}$$

By (1.2), it is $\operatorname{Isop}(\mathcal{E}) > 0$. A set $E \in \mathcal{E}$ at which the infimum is attained is called (Heisenberg) isoperimetric set. The existence of isoperimetric sets is proved by Leonardi and Rigot in [15]. The Heisenberg isoperimetric problem consists in computing isoperimetric sets. In [22], Pansu notes that the boundary of a smooth isoperimetric set in $\mathbb{H}^1$ has "constant mean curvature" and that a smooth surface has "constant mean curvature" if and only if it is foliated by horizontal lifts of plane circles with constant radius. Then he conjectures that an isoperimetric set in $\mathbb{H}^1$, if smooth, is obtained by rotating around the center of the group a geodesic joining two points in the center. Recently, Pansu's conjecture appeared again in [14].

We prove the natural generalization of Pansu's conjecture in $\mathbb{H}^n$ for any $n \ge 1$ under an additional symmetry assumption on the sets.

Definition 1.1 (Axially symmetric set). We say that a set $E \subset \mathbb{H}^n$ is axially symmetric if $(z,t) \in E$ implies that $(\zeta,t) \in E$ for all $\zeta \in \mathbb{C}^n$ such that $|\zeta| = |z|$. The set $F \subset \mathbb{R}^2_+ = \mathbb{R}^+ \times \mathbb{R}$ such that

$$E \setminus Z = \left\{ (z,t) \in \mathbb{H}^n \,\middle|\, (|z|,t) \in F \right\}$$

is called generating set of E.

Denote by $\mathcal{A}$ the family of all sets $E \in \mathcal{E}$ which are axially symmetric and consider the infimum

$$\mathrm{Isop}(\mathcal{A}) = \inf \left\{ \frac{\mathcal{P}(E)^{2n+2}}{\mathcal{L}^{2n+1}(E)^{2n+1}} \;\middle|\; E \in \mathcal{A} \right\}. \tag{1.4}$$

Clearly, it is $\mathrm{Isop}(\mathcal{A}) \geq \mathrm{Isop}(\mathcal{E})$. A set $E \in \mathcal{A}$ for which the infimum in (1.4) is attained is called an axially symmetric isoperimetric set. Our main result is the following

Theorem 1.2. *The infimum* $\mathrm{Isop}(\mathcal{A})$ *is attained. Moreover, up to a dilation, a vertical translation and a* $\mathcal{L}^{2n+1}$*-negligible set, any axially symmetric isoperimetric set coincides with*

$$E_{\mathrm{isop}} = \left\{ (z,t) \in \mathbb{H}^n \;\middle|\; |t| < \arccos|z| + |z|\sqrt{1-|z|^2},\ |z| < 1 \right\}. \tag{1.5}$$

Here, by "vertical translation" we mean a left translation by some $(0, t_0) \in \mathbb{H}^n$. This is actually an Euclidean vertical translation.

The boundary of the set (1.5) is obtained, for $n = 1$, by rotating around the center of the group a Heisenberg geodesic for the Carnot–Carathéodory metric joining the antipodal points $(0, \pm\pi/2)$, i.e., the set (1.5) is the solution to problem (1.3) conjectured by Pansu. The natural generalization of Pansu's conjecture states that E_{isop} is the unique solution of (1.3), up to group operations and negligible sets. There is a wide evidence supporting this conjecture.

1. Resolution with axial symmetry and regularity ([25]). If the boundary of an isoperimetric set in $\mathbb{H}^n$ is of class C^2 then it has "constant mean curvature", away from the characteristic set of the boundary, where the curvature is not defined. Complete constant mean curvature hypersurfaces which are rotationally invariant are classified by Ritoré and Rosales in [25]. Among other results, the authors prove that the unique compact rotationally invariant hypersurface of class C^2 with constant mean curvature in $\mathbb{H}^n$ is the boundary of the set (1.5), up to dilation and vertical translation.

2. Resolution with C^2 regularity ([26]). If an isoperimetric set in $\mathbb{H}^1$ has boundary of class C^2, then it is the set (1.5) with $n = 1$, up to dilation and left translation. This theorem is due to Ritoré and Rosales [26]. The authors first describe the structure of the characteristic set of the boundary using some results of [5] and then they determine the isoperimetric set using the ruling property (foliation by geodesics) of constant mean curvature surfaces. The regularity issue, however, is a delicate one: for instance, solutions to the Plateau problem in the Heisenberg group are not in general of class C^2 (see e.g. [23]).

3. Resolution with one spherical section and regularity ([6]). Consider the family of sets $E \subset \mathbb{H}^n$ which satisfy the properties:

(i) $\mathcal{L}^{2n+1}(E \cap \{t > 0\}) = \mathcal{L}^{2n+1}(E \cap \{t < 0\})$;

(ii) $E = \{(z,t) \in \mathbb{H}^n : -v(z) < t < u(z), |z| < 1\}$, for functions u, v which are non negative and of class C^2 in $\{|z| < 1\}$ and continuous on $\{|z| \leq 1\}$ with $u(z) = v(z) = 0$ for $|z| = 1$;

(iii) $u(z) = v(z) = 0$ implies $|z| = 1$.

In [6], Danielli, Garofalo and Nhieu prove that the isoperimetric problem restricted to the class of sets satisfying (i)–(iii) has a unique solution which is the set E_{isop} in (1.5).

4. Resolution with convexity ([19]). In the case $n = 1$, an isoperimetric set which is convex has the form (1.5), up to dilation, left translation and an $\mathcal{L}^3$-negligible set. This result is due to Rickly and the author ([19]). Distributional solutions of the Euler equation for the variational formulation of the problem are proved to have Sobolev regularity. This enables to solve the equation along a regular Lagrangian flow, thus establishing the "foliation by geodesics property" conjectured by Pansu.

5. Calibration argument ([24]). For $r > 0$ let $B_r = \{(z,0) \in \mathbb{H}^n \mid |z| < r\}$ and $C_r = \{(z,t) \in \mathbb{H}^n \mid |z| < r\}$. Let $E \subset \mathbb{H}^n$ be a bounded open set with finite perimeter such that:

(i) $B_r \subset E \subset C_r$ for some $r > 0$;

(ii) $\mathcal{L}^{2n+1}(E) = \mathcal{L}^{2n+1}(E_{\mathrm{isop}})$, where E_{isop} is the set in (1.5).

Then, it is $\mathcal{P}(E_{\mathrm{isop}}) \leq \mathcal{P}(E)$. In [24], Ritoré also discusses the equality case. The proof of this result is based on a calibration argument. The calibration is constructed using the horizontal unit normal to the boundary of E_{isop}. I would like to seize this opportunity to thank M. Ritoré for sending me an early version of his paper.

6. Other contributions. Some observations on the Heisenberg isoperimetric problem can be found in [16] and [17]. A 2-dimensional version of the problem is formulated and solved by Morbidelli and the author in [18]: in the Grushin plane, isoperimetric sets coincide with the section of the set $E_{\mathrm{isop}} \subset \mathbb{H}^1$ with the $y = 0$ plane, properly scaled and translated. Existence and uniqueness of p-area minimizers are studied by Cheng, Hwang and Yang in [4]. The compact version of $\mathbb{H}^1$ is the complex sphere of $\mathbb{C}^2$: several results of [26] have been generalized to this setting by Hurtado and Rosales [12]. Finally, the monograph [3] is a detailed introduction to the isoperimetric problem in the Heisenberg group.

By a rearrangement argument, we reduce Theorem 1.2 to a one dimensional problem which can be solved by elementary methods. The first step is the reduction of the Heisenberg isoperimetric problem with axial symmetry to an isoperimetric problem in the half plane $\mathbb{R}^2_+ = \mathbb{R}^+ \times \mathbb{R}$.

Let $F \subset \mathbb{R}^2_+$ be a measurable set and let $D \subset \mathbb{R}^2_+$ be an open set. We define the weighted perimeter of F in D

$$\mathcal{Q}(F; D) = \sup \left\{ \int_F \left(\partial_r\left(r^{2n-1}\psi_1\right) + 2r^{2n}\partial_t\psi_2\right) dr\, dt \,\middle|\, \begin{array}{l} \psi \in C^1_0(D; \mathbb{R}^2), \\ \|\psi\|_\infty \leq 1 \end{array} \right\}. \quad (1.6)$$

In the case $D = \mathbb{R}^2_+$, we let $\mathcal{Q}(F;\mathbb{R}^2_+) = \mathcal{Q}(F)$.

Proposition 1.3 (Planar reduction). *Let $E \subset \mathbb{H}^n$ be a measurable axially symmetric set with generating set $F \subset \mathbb{R}^2_+$, and let $\Omega \subset \mathbb{H}^n$ be an axially symmetric open set with generating set $D \subset \mathbb{R}^2_+$. Then E has finite Heisenberg perimeter in Ω if and only if F has finite weighted perimeter in D, and moreover*

$$\mathcal{P}(E;\Omega) = \omega_{2n-1}\mathcal{Q}(F;D), \tag{1.7}$$

where $\omega_{2n-1} = \mathcal{H}^{2n-1}(\mathbb{S}^{2n-1})$ is the standard surface measure of the $(2n-1)$-dimensional unit sphere.

Proposition 1.3 is proved in Section 4. Using spherical coordinates in $\mathbb{C}^n$, we find the representation for the Lebesgue measure of an axially symmetric set $E \subset \mathbb{H}^n$

$$\mathcal{L}^{2n+1}(E) = \omega_{2n-1}\int_F r^{2n-1}dr\,dt = \omega_{2n-1}\mathcal{V}(F), \tag{1.8}$$

where the volume $\mathcal{V}(F)$ of the generating set F is defined by the last equality.

By the isoperimetric inequality (1.2), from (1.7) and (1.8) it follows that there exists a constant $C'_n > 0$ such that

$$\min\left\{\mathcal{V}(F), \mathcal{V}(\mathbb{R}^2_+ \setminus F)\right\} \le C'_n \mathcal{Q}(F)^{\frac{2n+2}{2n+1}} \tag{1.9}$$

for $\mathcal{L}^2$-measurable sets $F \subset \mathbb{R}^2_+$. We also have the relation

$$\frac{\mathcal{P}(E)^{2n+2}}{\mathcal{L}^{2n+1}(E)^{2n+1}} = \omega_{2n-1}\mathrm{Isop}(F),$$

where the isoperimetric ratio of F is defined by

$$\mathrm{Isop}(F) = \frac{\mathcal{Q}(F)^{2n+2}}{\mathcal{V}(F)^{2n+1}}. \tag{1.10}$$

The infimum $\mathrm{Isop}(\mathcal{R})$ in (1.4) is thus equal to

$$\mathrm{Isop}(\mathcal{R}) = \omega_{2n-1}\inf\left\{\mathrm{Isop}(F)\,\middle|\,F \subset \mathbb{R}^2_+\ \mathcal{L}^2\text{-measurable}, 0 < \mathcal{V}(F) < +\infty\right\}. \tag{1.11}$$

A set $F \subset \mathbb{R}^2_+$ at which the infimum in (1.11) is attained is called $\mathcal{Q}$-isoperimetric set.

We look for a rearrangement scheme which, starting from a set $F \subset \mathbb{R}^2_+$, produces a set $F^\sharp \subset \mathbb{R}^2_+$ such that:

$$\begin{aligned}
&\text{(i) the sections } F^\sharp_t = \left\{ r \in \mathbb{R}^+ \,\middle|\, (r,t) \in F^\sharp \right\},\ t \in \mathbb{R}, \text{ are intervals}\\
&\quad\ \text{of the form } (0, g(t)) \text{ for some } g(t) \ge 0;\\
&\text{(ii) } \mathcal{Q}(F^\sharp) \le \mathcal{Q}(F);\\
&\text{(iii) } \mathcal{V}(F^\sharp) \ge \mathcal{V}(F).
\end{aligned} \tag{1.12}$$

The coefficient r^{2n} in front of the partial derivative ∂_t in the variational definition of $\mathcal{Q}(F;\cdot)$ in (1.6) makes it difficult to manage the contribution of "vertical" perimeter under rearrangement. This is the reason why the standard (Steiner) decreasing rearrangement does not seem to work in similar situations (see, e.g. for Dirichlet-type integrals, Theorem 2.13 in [13] and Theorem 1 in [2]).

Let us set the problem in a more general framework. We consider the following perimeter of a $\mathcal{L}^2$-measurable set $F \subset \mathbb{R}^2_+$ in an open set $D \subset \mathbb{R}^2_+$:

$$\mathcal{R}(F;D) = \sup\left\{ \int_F \left\{\partial_r\big(\varrho\psi_1\big) + \partial_t\big(\tau\psi_2\big)\right\} dr\, dt \,\middle|\, \begin{array}{l} \psi \in C^1_0(D;\mathbb{R}^2), \\ \|\psi\|_\infty \le 1 \end{array} \right\}. \tag{1.13}$$

As usual, we let $\mathcal{R}(F;\mathbb{R}^2_+) = \mathcal{R}(F)$. The functions $\varrho, \tau : \mathbb{R}^2_+ \to \mathbb{R}$ are assumed to satisfy the following properties:

$$\begin{array}{l} 1)\ \ \varrho, \tau \in C(\mathbb{R}^2_+) \text{ and } \varrho, \tau > 0 \text{ on } \mathbb{R}^2_+; \\ 2)\ \ \text{for } \mathcal{L}^1\text{-a.e. } t \in \mathbb{R}, \text{ the function } r \mapsto \varrho(r,t) \text{ is in } \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^+) \\ \quad\ \ \text{and increasing, i.e. } \varrho(r_1,t) \le \varrho(r_2,t) \text{ for } 0 < r_1 \le r_2; \\ 3)\ \ \tau(r,t) = \tau_1(r)\tau_2(t) \text{ with } \tau_1 \in L^1(0,\delta), \text{ for all } \delta > 0, \text{ and } \tau_2 \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}). \end{array} \tag{1.14}$$

By the Lipschitz regularity in 2) and 3), the integral in (1.13) is well defined. By 3), the function $\Theta : \mathbb{R}^2_+ \to \mathbb{R}^+$

$$\Theta(r,t) = \int_0^r \tau(s,t)\, ds \tag{1.15}$$

is defined for all $(r,t) \in \mathbb{R}^2_+$ and moreover, for any fixed $t \in \mathbb{R}$ the function $r \mapsto \Theta(r,t)$ is strictly increasing, by 1). We denote by Θ_t^{-1} the inverse function of $\Theta_t = \Theta(\cdot,t)$.

Definition 1.4 (τ-rearrangement). We say that a measurable set $F \subset \mathbb{R}^2_+$ is τ-rearrangeable if the function $f : \mathbb{R} \to [0,+\infty]$

$$f(t) = \int_{F_t} \tau(r,t)\, dr \tag{1.16}$$

is in $L^1_{\mathrm{loc}}(\mathbb{R})$. In this case, we let $g(t) = \Theta_t^{-1}(f(t))$, and we call the set

$$F^\sharp = \big\{(r,t) \in \mathbb{R}^2_+ : 0 < r < g(t)\big\}$$

the τ-rearrangement of F.

Related to Definition 1.4, we prove the main rearrangement result.

Theorem 1.5. *Assume that ϱ and τ satisfy conditions* (1.14). *Let $F \subset \mathbb{R}^2_+$ be a $\mathcal{L}^2$-measurable set which is τ-rearrangeable and such that $\mathcal{R}(F) < +\infty$. Then its τ-rearrangement $F^\sharp$ satisfies*

$$\mathcal{R}(F^\sharp) \le \mathcal{R}(F). \tag{1.17}$$

Moreover, if $\mathcal{R}(F^\sharp) = \mathcal{R}(F)$ then $F = F^\sharp$ up to a $\mathcal{L}^2$-negligible set.

The proof of Theorem 1.5 is contained in Section 2. We explain how inequality (1.17) is related to the notion of τ-rearrangement. If $\mathcal{R}(F) < +\infty$, the open sets map $D \mapsto \mathcal{R}(F; D)$ extends to a finite Borel measure on $\mathbb{R}^2_+$. This measure is the total variation of the vector valued Borel measure $\big(\mathcal{R}_1(F;\cdot), \mathcal{R}_2(F;\cdot)\big)$, where, for open sets $D \subset \mathbb{R}^2_+$,

$$\mathcal{R}_1(F; D) = \sup\left\{ \int_F \{\partial_r\big(\varrho\psi\big)\, dr\, dt \,\middle|\, \psi \in C^1_0(D), \|\psi\|_\infty \le 1 \right\},$$

$$\mathcal{R}_2(F; D) = \sup\left\{ \int_F \{\partial_t\big(\tau\psi\big)\, dr\, dt \,\middle|\, \psi \in C^1_0(D), \|\psi\|_\infty \le 1 \right\}.$$

The search for a rearrangement $F^\sharp$ of F such that $\mathcal{R}_2(F^\sharp; \mathbb{R}^+ \times A) \le \mathcal{R}_2(F; \mathbb{R}^+ \times A)$ for any open set $A \subset \mathbb{R}$ lead us to Definition 1.4 (see *Step 2* in the proof of Theorem 1.5). On the other hand, the monotonicity of $r \mapsto \varrho(r,t)$ required in (1.14) guarantees that $\mathcal{R}_1(F^\sharp; \mathbb{R}^+ \times A) \le \mathcal{R}_1(F; \mathbb{R}^+ \times A)$ for any open set $A \subset \mathbb{R}$. As both partial perimeters do not increase under τ-rearrangement, the same holds for their total variation and we get (1.17). The study of partial perimeters under Steiner symmetric decreasing rearrangement is a key step in De Giorgi's proof [7] of the isoperimetric property of the Euclidean ball in $\mathbb{R}^n$ in the class of sets with finite perimeter (see also the modern version of the proof by Talenti [27]).

In order to complete the program sketched in (1.12), we need to control the change of volume. Let us introduce a general volume in the half plane. We take a function $u : \mathbb{R}^2_+ \to \mathbb{R}^+$ which satisfies the following properties:

$$\begin{aligned} &1)\ u \text{ is } \mathcal{L}^2\text{-measurable and } u > 0 \text{ on } \mathbb{R}^2_+;\\ &2)\ r \mapsto u(r,t) \text{ is in } L^1(0,\delta) \text{ for all } \delta > 0 \text{ and for } \mathcal{L}^1\text{-a.e. } t \in \mathbb{R}. \end{aligned} \tag{1.18}$$

The function $U : \mathbb{R}^2_+ \to \mathbb{R}$

$$U(r,t) = \int_0^r u(s,t)\, ds, \tag{1.19}$$

is positive and moreover $r \mapsto U(r,t)$ is strictly increasing, by 1). We denote by U_t^{-1} the inverse function of $U_t = U(\cdot, t)$.

We define the volume of a $\mathcal{L}^2$-measureable set $F \subset \mathbb{R}^2_+$ by

$$\mathcal{U}(F) = \int_F u(r,t)\, dr\, dt.$$

The volume of F is finite if and only if $u \in L^1(F)$.

Definition 1.6. We say that the volume $\mathcal{U}$ is non decreasing with respect to the τ-rearrangement if for any $\mathcal{L}^1$-measurable set $A \subset \mathbb{R}^+$ we have

$$U_t^{-1}\Big(\int_A u(r,t)\,dr\Big) \leq \Theta_t^{-1}\Big(\int_A \tau(r,t)\,dr\Big) \tag{1.20}$$

for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$.

Condition (1.20) ensures that $\mathcal{U}(F^\sharp) \geq \mathcal{U}(F)$ (see Proposition 2.4).

Then it remains to check that the compatibility condition is verified in the case we are dealing with. In our case, we have $\tau(r,t) = 2r^{2n}$ and $u(r,t) = r^{2n-1}$ (see (1.8)). For this pair of functions, the compatibility condition (1.20) is a consequence of the elementary inequality

$$\Big((\alpha+1)\int_A r^\alpha dr\Big)^{\frac{1}{\alpha+1}} \leq \Big((\beta+1)\int_A r^\beta dr\Big)^{\frac{1}{\beta+1}}$$

for $A \subset \mathbb{R}^+$ $\mathcal{L}^1$-measurable and $-1 < \alpha < \beta$ (Example 2.5).

If the coefficients ϱ and τ appearing in the divergence in (1.13) do not depend on t, the Steiner rearrangement of a set in direction t is standard (see Section 3). In Section 5, we prove the following

Theorem 1.7. *The infimum in* (1.11) *is attained and a* $\mathcal{Q}$*-isoperimetric set* F *satisfies:*

(i) $F = F^\sharp$*, up to a* $\mathcal{L}^2$*-negligible set*;

(ii) the sections $F_r = \{t \in \mathbb{R} : (r,t) \in F\}$ *are equivalent to intervals, for* $\mathcal{L}^1$*-a.e.* $r \in \mathbb{R}^+$;

(iii) F *is contained in a bounded rectangle, and precisely*

$$F \subset \left\{ (r,t) \in \mathbb{R}^2_+ \;\middle|\; \begin{array}{l} 0 < r \leq c_n \mathcal{Q}(F)^{\frac{1}{2n+1}}, \\ |t - t_0| \leq d_n \mathcal{Q}(F)^{2n}/\mathcal{V}(F)^{2n-1} \end{array} \right\} \tag{1.21}$$

for some $t_0 \in \mathbb{R}$ *and for dimensional constants* $c_n, d_n > 0$.

Thanks to Theorem 1.7, the axial isoperimetric problem in the Heisenberg group can be reduced to a one dimensional problem which can be solved by elementary methods, thus proving Theorem 1.2.

Notation. We denote by $|z|$ the usual norm of $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ with $z_j = x_j + iy_j \equiv (x_j, y_j)$. J is the standard complex structure $J(z) = iz$. $\mathbb{R}^+ = (0, +\infty)$ is the positive open half line and $\mathbb{R}^2_+ = \mathbb{R}^+ \times \mathbb{R}$ is the open half plane. $\mathcal{L}^k$ is Lebesgue measure in $\mathbb{R}^k$, $\mathcal{H}^{2n}$ is the $2n$-dimensional Hausdorff measure in $\mathbb{R}^{2n+1}$ and $\mathcal{H}^1$ is the Hausdorff measure in the plane. By $\langle \cdot, \cdot \rangle$ we mean the standard inner product in $\mathbb{R}^k$.

2 Weighted rearrangement in the half plane

2.1 Preliminaries on perimeters

Let $\Omega \subset \mathbb{R}^k$, $k \geq 2$, be an open set and let $V_1, \ldots, V_m \in \mathrm{Lip}_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$, $m \geq 2$, be vector fields in Ω. We identify these vector fields with the differential operators

$$V_i(x) = \sum_{j=1}^{k} \varrho_{ij}(x) \frac{\partial}{\partial x_j}, \qquad i = 1, \ldots, m,$$

where $\varrho_{ij} \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$. We denote by V_i^* the adjoint operator of V_i in $L^2(\Omega)$, i.e., the operator is defined by

$$\int_\Omega \varphi\, V_i \psi\, dx = \int_\Omega \psi\, V_i^* \varphi\, dx$$

for all $\varphi, \psi \in C_0^1(\Omega)$.

We introduce the family of test functions

$$\mathfrak{F}_m(\Omega) = \big\{ \varphi = (\varphi_1, \ldots, \varphi_m) \in C_0^1(\Omega; \mathbb{R}^m) \;\big|\; \varphi_1^2 + \ldots + \varphi_m^2 \leq 1 \text{ in } \Omega \big\}.$$

Let $F \subset \Omega$ be a $\mathfrak{L}^k$-measurable set and let $A \subset \Omega$ be an open set. For any $i = 1, \ldots, m$, we define the ith partial perimeter of F in A by

$$\mathfrak{P}_i(F; A) = \sup_{\varphi \in \mathfrak{F}_1(A)} \int_F V_i^* \varphi\, dx. \tag{2.1}$$

By Riesz representation theorem, if $\mathfrak{P}_i(F; \Omega) < +\infty$ then the open sets function $A \mapsto \mathfrak{P}_i(F; A)$ extends to a finite Radon measure μ_i in Ω and there exists a Borel function $\sigma_i : \Omega \to \mathbb{R}$ with $|\sigma_i| = 1$ μ_i-a.e. such that

$$\int_F V_i^* \varphi\, dx = \int_\Omega \varphi\, \sigma_i\, d\mu_i \tag{2.2}$$

for all $\varphi \in C_0^1(\Omega)$. We denote by $\mu = (\mu_1, \ldots, \mu_m)$ the vector valued Radon measure in Ω whose ith component is the measure μ_i.

We introduce the formal divergence of a given vector field $\varphi = (\varphi_1, \ldots, \varphi_m) \in C^1(\Omega; \mathbb{R}^m)$ with respect to the frame of vector fields $V = (V_1, \ldots, V_m)$:

$$\mathrm{div}_V \varphi = -\sum_{i=1}^{m} V_i^* \varphi_i.$$

The V-perimeter in an open set $A \subset \Omega$ of a $\mathfrak{L}^k$-measurable set $F \subset \Omega$ is

$$\mathfrak{P}(F; A) = \sup_{\varphi \in \mathfrak{F}_m(A)} \int_F \mathrm{div}_V \varphi\, dx. \tag{2.3}$$

By Riesz representation theorem, if $\mathcal{R}(F;\Omega) < +\infty$ then the open sets function $A \mapsto \mathcal{R}(F;A)$ extends to a finite Radon measure ν in Ω. Moreover, there exists a Borel map $n : \Omega \to \mathbb{R}^m$ such that $|n| = 1$ ν-a.e. and

$$\int_F \mathrm{div}_V \varphi \, dx = \int_\Omega \langle \varphi, n \rangle \, d\nu \tag{2.4}$$

for all $\varphi \in C_0^1(\Omega;\mathbb{R}^m)$. Note that $\nu(\Omega) < +\infty$ implies $\mu_i(\Omega) < +\infty$ for all $i = 1,\dots,m$.

The measures ν and μ_i, $i = 1,\dots,m$, are finite Radon measure. In particular, they are Borel regular and ν is the total variation of $\mu = (\mu_1,\dots,\mu_m)$. Before sketching the proof of this fact we recall the following definition.

Definition 2.1. The total variation of a vector valued Borel measure $\mu = (\mu_1,\dots,\mu_m)$ in Ω is the Borel measure $|\mu|$ defined, for Borel sets $E \subset \Omega$, by

$$\begin{aligned}|\mu|(E) = \sup\Bigg\{ \sum_{j=1}^{+\infty} |\mu(E_j)| \;\Bigg|\; & (E_j)_{j\in\mathbb{N}} \text{ disjoint sequence} \\ & \text{of Borel sets with } E = \bigcup_{j=1}^{+\infty} E_j \Bigg\}.\end{aligned} \tag{2.5}$$

By the Radon–Nikodym theorem it is $\mu = \eta|\mu|$, where $\eta : \Omega \to \mathbb{R}^m$ is a Borel map such that $|\eta| = 1$ $|\mu|$-a.e. in Ω. From (2.2) and (2.4), we have for any $\varphi \in C_0^1(\Omega)$ and $i = 1,\dots,m$,

$$\int_\Omega \varphi\, n_i \, d\nu = \int_F V_i^* \varphi \, dx = \int_\Omega \varphi\, \sigma_i \, d\mu_i = \int_\Omega \varphi\, \sigma_i \eta_i \, d|\mu|,$$

where $n = (n_1,\dots,n_m)$. By density, this identity holds for any characteristic function φ of any open set in Ω. Thus, we deduce that $n_i\nu = \sigma_i\mu_i = \sigma_i\eta_i|\mu|$. The set $E = \{x \in \Omega : |\sigma_i(x)| \neq 1\}$ has vanishing μ_i measure. It follows that $\eta_i(x) = 0$ for $|\mu|$-a.e. $x \in E$, and then the Borel map $\widehat{\eta} : \Omega \to \mathbb{R}^m$ defined by $\widehat{\eta}_i = \sigma_i\eta_i$ satisfies $|\widehat{\eta}| = 1$ $|\mu|$-a.e. in Ω. For any Borel set $B \subset \Omega$, we have

$$|\mu|(B) = \int_B \langle \widehat{\eta}, \widehat{\eta} \rangle d|\mu| = \int_B \langle \widehat{\eta}, n \rangle d\nu \le \nu(B).$$

The same argument provides $\nu(B) \le |\mu|(B)$. This shows that $\nu = |\mu|$.

2.2 τ-rearrangement

From now on, we work in the half plane $\mathbb{R}^2_+ = \mathbb{R}^+ \times \mathbb{R}$ and we denote by $(r,t) \in \mathbb{R}^+ \times \mathbb{R}$ a generic point. Let $\varrho, \tau : \mathbb{R}^2_+ \to \mathbb{R}$ be functions satisfying (1.14) and consider the vector fields

$$V_1 = \varrho(r,t)\frac{\partial}{\partial r}, \quad V_2 = \tau(r,t)\frac{\partial}{\partial t}. \tag{2.6}$$

According to (2.1) and (2.3), we define the following perimeters of a $\mathcal{L}^2$-measurable set $F \subset \mathbb{R}^2_+$ in an open set $D \subset \mathbb{R}^2_+$

$$\begin{aligned} \mathcal{P}_1(F;D) &= \sup_{\psi\in\mathcal{F}_1(D)} \int_F \partial_r(\varrho\psi)\,dr\,dt, \\ \mathcal{P}_2(F;D) &= \sup_{\psi\in\mathcal{F}_1(D)} \int_F \partial_t(\tau\psi)\,dr\,dt, \end{aligned} \tag{2.7}$$

and the V-perimeter

$$\mathcal{P}(F;D) = \sup_{\psi\in\mathcal{F}_2(D)} \int_F \{\partial_r(\varrho\psi_1) + \partial_t(\tau\psi_2)\}\,dr\,dt. \tag{2.8}$$

We recall some standard facts in the following two propositions.

Proposition 2.2. *Let $A \subset \mathbb{R}$ be an open set and let $F \subset \mathbb{R}^2_+$ be a $\mathcal{L}^2$-measurable set such that $\mathcal{P}_1(F;\mathbb{R}^+ \times A) < +\infty$. Then we have*

$$\mathcal{P}_1(F;\mathbb{R}^+ \times A) = \int_A \Big(\sup_{\psi\in\mathcal{F}_1(\mathbb{R}^+)} \int_{F_t} \partial_r\big(\varrho(r,t)\psi(r)\big)\,dr\Big)\,dt. \tag{2.9}$$

Here, $F_t = \{r \in \mathbb{R}^+ : (r,t) \in F\}$ is the section of F at level $t \in \mathbb{R}$.

Proof. The inequality $\le$ in (2.9) is elementary. We prove the converse inequality. By Theorem 2.2.2 in [9], there exists a sequence of functions $f_j \in C^\infty(\mathbb{R}^+ \times A)$, $j \in \mathbb{N}$, such that:

1) $f_j \to \chi_F$ in $L^1_{\mathrm{loc}}(\mathbb{R}^+ \times A)$ as $j \to +\infty$;
2) we have

$$\lim_{j\to+\infty} \int_{\mathbb{R}^+\times A} |\partial_r f_j(r,t)|\,\varrho(r,t)\,dr\,dt = \mathcal{P}_1(F;\mathbb{R}^+ \times A). \tag{2.10}$$

By 1), for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ it is $f_j(\cdot,t) \to \chi_F(\cdot,t)$ in $L^1_{\mathrm{loc}}(\mathbb{R}^+)$ as $j \to +\infty$. On the other hand, by Fatou's lemma and by the lower semicontinuity of the total variation with respect to the L^1_{loc}-convergence, we have

$$\begin{aligned} &\lim_{j\to+\infty} \int_A \int_{\mathbb{R}^+} |\partial_r f_j(r,t)|\,\varrho(r,t)\,dr\,dt \\ &\qquad \ge \int_A \liminf_{j\to+\infty} \int_{\mathbb{R}^+} |\partial_r f_j(r,t)|\,\varrho(r,t)\,dr\,dt \\ &\qquad = \int_A \liminf_{j\to+\infty} \sup_{\psi\in\mathcal{F}_1(\mathbb{R}^+)} \int_{\mathbb{R}^+} f_j(r,t)\partial_r\big(\varrho(r,t)\psi(r)\big)\,dr\,dt \\ &\qquad \ge \int_A \sup_{\psi\in\mathcal{F}_1(\mathbb{R}^+)} \int_{F_t} \partial_r\big(\varrho(r,t)\psi(r)\big)\,dr\,dt. \end{aligned} \tag{2.11}$$

From (2.10) and (2.11) we get the inequality $\ge$ in (2.9). □

Proposition 2.3. *Let $\varrho_1 \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^+)$ be a function such that $\varrho_1 > 0$ in $\mathbb{R}^+$, and let $E \subset \mathbb{R}^+$ be a $\mathfrak{L}^1$-measurable set such that*

$$\sup_{\psi \in \mathfrak{F}_1(\mathbb{R}^+)} \int_E \partial_r(\varrho_1 \psi)\, dr < +\infty. \tag{2.12}$$

Then, up to a $\mathfrak{L}^1$-negligible set we have $E = \bigcup_{i\in\mathfrak{I}} E_i$, where $\mathfrak{I} \subset \mathbb{Z}$, $E_i = (a_i, b_i)$ is an interval, $0 \le a_i < b_i \le +\infty$ for all $i \in \mathfrak{I}$ and $b_i < a_j$ for $i, j \in \mathfrak{I}$ with $i < j$. Moreover,

$$\sup_{\psi \in \mathfrak{F}_1(\mathbb{R}^+)} \int_E \partial_r(\varrho_1 \psi)\, dr = \sum_{i\in\mathfrak{I}} \big(\varrho_1(a_i) + \varrho_1(b_i)\big), \tag{2.13}$$

where we agree that $\varrho_1(0) = \varrho_1(+\infty) = 0$.

Proof. For any bounded open subset $A \subset \mathbb{R}^+$ there is a constant $C > 0$ such that $\varrho_1 \ge C$ on A. Thus, by (2.12) we have

$$\sup_{\psi \in \mathfrak{F}_1(A)} \int_E \partial_r \psi\, dr < +\infty,$$

i.e., E has locally finite perimeter (unweighted perimeter) in $\mathbb{R}^+$. It follows that $E = \bigcup_{i\in\mathfrak{I}} E_i$ as in the statement of the proposition (see e.g. [1, Section 3.2]). Moreover, for any $\psi \in C_0^1(\mathbb{R}^+)$ we have

$$\int_E \partial_r(\varrho_1 \psi)\, dr = \sum_{i\in\mathfrak{I}} \big(\varrho_1(b_i)\psi(b_i) - \varrho_1(a_i)\psi(a_i)\big). \tag{2.14}$$

Now, (2.13) follows from (2.14) on taking the supremum over $\psi \in \mathfrak{F}_1(\mathbb{R}^+)$. □

Before proving Theorem 1.5, we need a couple of observations.

Let $f \in BV_{\mathrm{loc}}(\mathbb{R})$ be a function of locally bounded variation in $\mathbb{R}$. Upon modifying f on a set of vanishing $\mathfrak{L}^1$-measure, f is the difference of two increasing functions, by Jordan theorem. Thus f has left and right limit at any point, and moreover they are equal in the complement of an at most countable set $N = \{t_k \in \mathbb{R} : k \in K\}$ for some $K \subset \mathbb{N}$. Let

$$g(t) = \Theta_t^{-1}(f(t)), \tag{2.15}$$

where Θ_t^{-1} is the inverse of the function $\Theta_t = \Theta(\cdot, t)$ defined in (1.15). The function g has the same continuity properties as f. For $k \in K$ denote by S_k the closed segment in $\mathbb{R}^2$ with end-points $\big(g(t_k)^-, t_k\big)$ and $\big(g(t_k)^+, t_k\big)$, where $g(t_k)^-$ and $g(t_k)^+$ are the left and right limits of g at t_k. The "graph" of g

$$\Gamma_g = \bigcup_{k\in\mathbb{N}} S_k \cup \big\{ (g(t), t) \in \mathbb{R}^2 \;\big|\; t \in \mathbb{R} \setminus N \big\} \tag{2.16}$$

is relatively closed in $\mathbb{R}^2_+$. The measure $\mathcal{R}(F^\sharp;\cdot)$ associated with the set $F^\sharp = \{(r,t) \in \mathbb{R}^2_+ \mid 0 < r < g(t)\}$ is supported in Γ_g and in particular

$$\mathcal{R}(F^\sharp;\mathbb{R}^2_+ \setminus \Gamma_g) = 0. \tag{2.17}$$

In fact, for any $\psi \in \mathfrak{F}_2(\mathbb{R}^2_+ \setminus \Gamma_g)$ by the divergence theorem it is

$$\int_{F^\sharp} \big(\partial_r(\varrho\psi_1) + \partial_t(\tau\psi_2)\big) dr\, dt = 0.$$

A second observation concerns the factorization property $\tau(r,t) = \tau_1(r)\tau_2(t)$ in item 3) in (1.14). For some $\mathcal{L}^2$-measurable set $F \subset \mathbb{R}^2_+$, let

$$f_1(t) = \int_{F_t} \tau_1(r)\, dr. \tag{2.18}$$

Then, the function f in (1.16) is of the form $f(t) = f_1(t)\tau_2(t)$ and definition (2.15) is equivalent with

$$\int_0^{g(t)} \tau_1(r)\, dr = f_1(t), \tag{2.19}$$

because $\tau_2 > 0$.

Proof of Theorem 1.5. We introduce the Borel measures μ_1 and μ_2 on $\mathbb{R}$

$$\mu_1(B) = \mathcal{R}_1(F;\mathbb{R}^+ \times B),$$
$$\mu_2(B) = \mathcal{R}_2(F;\mathbb{R}^+ \times B),$$

where $B \subset \mathbb{R}$ is a Borel set. Analogously, starting from $F^\sharp$ we define

$$\mu_1^\sharp(B) = \mathcal{R}_1(F^\sharp;\mathbb{R}^+ \times B),$$
$$\mu_2^\sharp(B) = \mathcal{R}_2(F^\sharp;\mathbb{R}^+ \times B).$$

Step 1. We claim that $\mu_1^\sharp(B) \le \mu_1(B)$ for any Borel set $B \subset \mathbb{R}$.

Since the measures are Borel regular, it is sufficient to prove the claim for an open set $B \subset \mathbb{R}$. By assumption $\mathcal{R}(F) < +\infty$ and then $\mathcal{R}_1(F;\mathbb{R}^2_+) < +\infty$. By Proposition 2.2, it follows that for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ we have

$$\sup_{\psi \in \mathfrak{F}_1(\mathbb{R}^+)} \int_{F_t} \partial_r\big(\varrho(r,t)\psi(r)\big)\, dr < +\infty,$$

and then, by Proposition 2.3, possibly modifying F in a $\mathcal{L}^2$-negligible set, it is

$$F_t = \bigcup_{i \in \mathfrak{I}(t)} \big(a_i(t), b_i(t)\big), \tag{2.20}$$

where $\mathfrak{I}(t) \subset \mathbb{Z}$, $0 \leq a_i(t) < b_i(t) \leq +\infty$ for $i \in \mathfrak{I}(t)$ and $b_i(t) < a_j(t)$ for $i < j$. Note that $b_i(t) < +\infty$ for all $i \in \mathfrak{I}(t)$ and

$$i^\sharp(t) = \sup \mathfrak{I}(t) < +\infty,$$

because F is τ-rearrangeable and $r \mapsto \tau(r,t)$ is increasing. Moreover, the sum in (2.13) is finite. Using (2.9) and (2.13), we get

$$\begin{aligned}\mu_1(B) &= \sup_{\psi \in \mathfrak{F}_1(\mathbb{R}^+ \times B)} \int_F \partial_r \big(\varrho(r,t)\psi(r,t)\big)\, dr\, dt \\ &= \int_B \Big(\sup_{\psi \in \mathfrak{F}_1(\mathbb{R}^+)} \int_{F_t} \partial_r (\varrho(r,t)\psi(r))\, dr \Big)\, dt \\ &= \int_B \sum_{i \in \mathfrak{I}(t)} \big(\varrho(a_i(t),t) + \varrho(b_i(t),t)\big)\, dt.\end{aligned} \tag{2.21}$$

In order to compute $\mu_1^\sharp(B)$, recall that $F^\sharp = \big\{ (r,t) \in \mathbb{R}^2_+ \,\big|\, 0 < r < g(t) \big\}$, where $g(t) = \Theta_t^{-1}(f(t))$ and, by (1.16) and (2.20),

$$f(t) = \int_{F_t} \tau(r,t)\, dr = \sum_{i \in \mathfrak{I}(t)} \big\{\Theta(b_i(t),t) - \Theta(a_i(t),t)\big\}.$$

Then, by (2.9) and (2.13), we have

$$\mu_1^\sharp(B) = \int_B \varrho(g(t),t)\, dt. \tag{2.22}$$

Because $r \mapsto \Theta(r,t)$ is increasing, we have

$$g(t) = \Theta_t^{-1}\Big(\sum_{i \in \mathfrak{I}(t)} \big(\Theta\big(b_i(t),t\big) - \Theta\big(a_i(t),t\big)\big)\Big) \leq b_{i^\sharp(t)}(t), \tag{2.23}$$

and we finally obtain the pointwise estimate

$$\varrho\big(g(t),t\big) \leq \varrho\big(b_{i^\sharp(t)}(t),t\big) \leq \sum_{i \in \mathfrak{I}(t)} \big(\varrho(a_i(t),t) + \varrho(b_i(t),t)\big), \tag{2.24}$$

which is a consequence of (2.23), because $r \mapsto \varrho(r,t)$ is increasing, as well. Now, the claim of *Step 1* follows from (2.21), (2.22) and (2.24).

Step 2. We claim that $\mu_2^\sharp(B) \leq \mu_2(B)$ for any Borel set $B \subset \mathbb{R}$.

It is sufficient to consider open sets $B \subset \mathbb{R}$. By property 3) in (1.14), restricting the family of test functions and recalling (2.18) we get

$$
\begin{aligned}
\mu_2(B) &= \sup_{\psi\in\mathfrak{F}_1(\mathbb{R}^+\times B)} \int_F \partial_t(\tau(r,t)\psi(r,t))\,dr\,dt \\
&\geq \sup_{\psi\in\mathfrak{F}_1(B)} \int_F \tau_1(r)\partial_t\big(\tau_2(t)\psi(t)\big)\,dr\,dt \\
&= \sup_{\psi\in\mathfrak{F}_1(B)} \int_{\mathbb{R}} \left(\int_{F_t} \tau_1(r)\,dr\right)\partial_t\big(\tau_2(t)\psi(t)\big)\,dt \\
&= \sup_{\psi\in\mathfrak{F}_1(B)} \int_{\mathbb{R}} f_1(t)\partial_t\big(\tau_2(t)\psi(t)\big)\,dt.
\end{aligned}
\tag{2.25}
$$

Recall that $f \in L^1_{\mathrm{loc}}(\mathbb{R})$, because F is τ-rearrangeable and thus $f_1 \in L^1_{\mathrm{loc}}(\mathbb{R})$, as well. Let us define the total variation of f_1 in B with respect to the vector field $\tau_2(t)\frac{\partial}{\partial t}$

$$
|D_{\tau_2} f_1|(B) = \sup_{\psi\in\mathfrak{F}_1(B)} \int_{\mathbb{R}} f_1(t)\partial_t\big(\tau_2(t)\psi(t)\big)\,dt.
$$

With this notation, (2.25) reads $\mu_2(B) \geq |D_{\tau_2} f_1|(B)$. By the Coarea formula (see e.g. Theorem 4.2 in [20]), it holds

$$
|D_{\tau_2} f_1|(B) = \int_0^{+\infty} \sup_{\psi\in\mathfrak{F}_1(B)} \int_{\{f_1>s\}} \partial_t\big(\tau_2(t)\psi(t)\big)\,dt\,ds. \tag{2.26}
$$

Now we perform the change of variable

$$
s = \int_0^{\sigma} \tau_1(r)\,dr, \quad \text{with} \quad ds = \tau_1(\sigma)d\sigma.
$$

By (2.18)–(2.19), $f_1(t) > s$ is equivalent with $g(t) > \sigma$. Thus, we get

$$
\begin{aligned}
|D_{\tau_2} f_1|(B) &= \int_0^{+\infty} \tau_1(\sigma) \sup_{\psi\in\mathfrak{F}_1(B)} \int_{\{g>\sigma\}} \partial_t\big(\tau_2(t)\psi(t)\big)\,dt\,d\sigma \\
&\geq \sup_{\psi\in\mathfrak{F}_1(\mathbb{R}^+\times B)} \int_0^{+\infty} \int_{\{g>\sigma\}} \partial_t\big(\tau(\sigma,t)\psi(\sigma,t)\big)\,dt\,d\sigma \\
&= \sup_{\psi\in\mathfrak{F}_1(\mathbb{R}^+\times B)} \int_{F^\sharp} \partial_t\big(\tau(\sigma,t)\psi(\sigma,t)\big)\,d\sigma\,dt \\
&= \mu_2^\sharp(B).
\end{aligned}
\tag{2.27}
$$

Now our claim follows from (2.25) and (2.27).

Denote by $|\mu|$ and $|\mu^\sharp|$ the total variation measures on $\mathbb{R}$ of the vector valued measures

$\mu = (\mu_1, \mu_2)$ and $\mu^\sharp = (\mu_1^\sharp, \mu_2^\sharp)$ respectively.

Step 3. We claim that $|\mu|(E) \le \mathfrak{R}(F; \mathbb{R}^+ \times E)$ and $|\mu^\sharp|(E) = \mathfrak{R}(F^\sharp; \mathbb{R}^+ \times E)$ for any Borel set $E \subset \mathbb{R}$.

It is sufficient to prove the claim for an open set $E \subset \mathbb{R}$. Denote by $|\nu^\sharp| = \mathfrak{R}(F^\sharp; \cdot)$ the total variation of the vector valued measure $\nu^\sharp = (\nu_1^\sharp, \nu_2^\sharp) = \big(\mathfrak{R}_1(F^\sharp; \cdot), \mathfrak{R}_2(F^\sharp; \cdot)\big)$ on $\mathbb{R}^2_+$. For any Borel partition $(E_j)_{j\in\mathbb{N}}$ of E there is a Borel partition of product type $(\mathbb{R}^+ \times E_j)_{j\in\mathbb{N}}$ of $\mathbb{R}^+ \times E$. It follows that $|\mu^\sharp|(E) \le |\nu^\sharp|(\mathbb{R}^+ \times E)$. The same argument proves the claim in *Step 3* concerning $|\mu|$.

In order to prove the converse inequality $|\nu^\sharp|(\mathbb{R}^+ \times E) \le |\mu^\sharp|(E)$, it is sufficient to show that for any Borel partition $(F_j)_{j\in\mathbb{N}}$ of $\mathbb{R}^+ \times E$ there exists a Borel partition of product type $(\mathbb{R}^+ \times E_h)_{h\in\mathbb{N}}$ of $\mathbb{R}^+ \times E$ such that

$$\sum_{j\in\mathbb{N}} |\nu^\sharp(F_j)| \le \sum_{h\in\mathbb{N}} |\mu^\sharp(E_h)|. \tag{2.28}$$

By (2.25), $f_1 \in BV_{\mathrm{loc}}(\mathbb{R})$ and the same holds for f, because $\tau_2 \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$ is positive. Let Γ_g be the graph of g defined in (2.16) with $N = \{t_k \in \mathbb{R} : k \in K\}$ as in the discussion before formula (2.15). Let $J \subset \mathbb{N}$ be the set of all $j \in \mathbb{N}$ such that $F_j \cap \Gamma_g \neq \varnothing$. By (2.17), we have

$$\nu^\sharp(F_j) = 0 \quad \text{for all} \quad j \in \mathbb{N} \setminus J. \tag{2.29}$$

Letting shrink the open set A in (2.9) to one point, we see that $\nu_1^\sharp(\mathbb{R}^+ \times \{t\}) = 0$ for all $t \in \mathbb{R}$. Thus we have the countable additivity

$$\Big|\nu^\sharp\Big(\bigcup_{j\in\mathbb{N}} B_j \times \{t\}\Big)\Big| = \sum_{j\in\mathbb{N}} |\nu^\sharp(B_j \times \{t\})| \tag{2.30}$$

for any sequence of pairwise disjoint Borel sets $B_j \subset \mathbb{R}^+$, $j \in \mathbb{N}$.

Denote by $\pi : \mathbb{R}^2_+ \to \mathbb{R}$, $\pi(r,t) = t$, the standard projection onto the second coordinate. Let $(E_h)_{h\in\mathbb{N}}$ be the Borel covering of E made up by the sets $\pi(F_j) \setminus N$ with $j \in J$ together with $\{t_k\}$, $k \in K$, such that $t_k \in E$. By (2.29) and (2.30), we have

$$\begin{aligned}
\sum_{j\in\mathbb{N}} |\nu^\sharp(F_j)| &= \sum_{j\in J} |\nu^\sharp(F_j)| \\
&\le \sum_{j\in J} \big(|\nu^\sharp(F_j \setminus \mathbb{R}^+ \times N)| + |\nu^\sharp\big(F_j \cap (\mathbb{R}^+ \times N)\big)|\big) \\
&= \sum_{j\in J} |\nu^\sharp(\mathbb{R}^+ \times (\pi(F_j) \setminus N)| + \sum_{k\in K, t_k\in E} |\nu^\sharp(\mathbb{R}^+ \times \{t_k\})| \\
&= \sum_{h\in\mathbb{N}} |\mu^\sharp(E_h)|.
\end{aligned}$$

This ends the proof of (2.28) and of *Step 3*.

From the *Steps 1* and *2* and recalling definition (2.5), it follows that

$$|\mu^\sharp|(B) \le |\mu|(B) \tag{2.31}$$

for any Borel set $B \subset \mathbb{R}$. Now, (1.17) follows from *Step 3*. In fact,

$$\mathcal{R}(F^\sharp) = |\mu^\sharp|(\mathbb{R}) \le |\mu|(\mathbb{R}) \le \mathcal{R}(F). \tag{2.32}$$

Step 4. We claim that if $\mathcal{R}(F^\sharp) = \mathcal{R}(F)$ then $F = F^\sharp$ up to a $\mathcal{L}^2$-negligible set.

We have $|\mu^\sharp|(\mathbb{R}) = |\mu|(\mathbb{R})$, by (2.32), and thus $|\mu^\sharp|(E) = |\mu|(E)$ for any Borel set $E \subset \mathbb{R}$, by (2.31). By Radon–Nikodym theorem it is $\mu^\sharp = \eta^\sharp|\mu|$ and $\mu = \eta|\mu|$ for Borel maps $\eta, \eta^\sharp : \mathbb{R} \to \mathbb{R}^2$ such that $|\eta| = 1$ and $|\eta^\sharp| = 1$ $|\mu|$-a.e. in $\mathbb{R}$. On the other hand, $\eta_1^\sharp \le \eta_1$ and $\eta_2^\sharp \le \eta_2$, by *Steps 1 and 2*. We deduce that $\eta^\sharp = \eta$ $|\mu|$-a.e. in $\mathbb{R}$, and in particular we have $\mu_1^\sharp(B) = \mu_1(B)$ for any Borel set $B \subset \mathbb{R}$. From (2.21) and (2.22), we deduce that we have equalities in (2.23) and (2.24). This implies that $F_t = (0, g(t))$ for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$, up to a $\mathcal{L}^1$-negligible set of $\mathbb{R}^+$. □

2.3 τ-rearrangement and volume

Proposition 2.4. *Assume that* $u : \mathbb{R}^2_+ \to \mathbb{R}$ *satisfies* (1.18) *and* (1.20). *Let* $F \subset \mathbb{R}^2_+$ *be a* $\mathcal{L}^2$*-measurable set which is* τ*-rearrangeable and such that* $\mathcal{U}(F) < +\infty$. *Then its* τ*-rearrangement* $F^\sharp$ *satisfies*

$$\mathcal{U}(F^\sharp) \ge \mathcal{U}(F). \tag{2.33}$$

Proof. We have, by the Fubini–Tonelli theorem,

$$\begin{aligned} \mathcal{U}(F) &= \int_{\mathbb{R}} \int_{F_t} u(r,t)\,dr\,dt, \\ \mathcal{U}(F^\sharp) &= \int_{\mathbb{R}} \int_0^{g(t)} u(r,t)\,dr\,dt = \int_{\mathbb{R}} U(g(t),t)\,dt, \end{aligned} \tag{2.34}$$

where U is the function in (1.19) and

$$g(t) = \Theta_t^{-1}\Big(\int_{F_t} \tau(r)\,dr\Big).$$

We have $g(t) < +\infty$ for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$, because F is τ-rearrangeable. For any such t, condition (1.20) yields

$$U(g(t),t) \ge \int_{F_t} u(r,t)\,dr. \tag{2.35}$$

Now our claim (2.33) follows from (2.34) and from the pointwise estimate (2.35). □

Example 2.5. Let $\alpha, \beta \in \mathbb{R}$ be a pair of numbers such that $-1 < \alpha < \beta$ and consider the functions $u(r,t) = r^\alpha$ and $\tau(r,t) = r^\beta$, $r > 0$. We show that the volume $\mathcal{U}$ in (1.19) does not decrease under τ-rearrangement. Precisely, we have to check condition (1.20), which in the present case reduces to

$$\Big((\alpha+1)\int_A r^\alpha dr\Big)^{\frac{1}{\alpha+1}} \le \Big((\beta+1)\int_A r^\beta dr\Big)^{\frac{1}{\beta+1}} \tag{2.36}$$

for $\mathcal{L}^1$-measurable $A \subset \mathbb{R}^+$. It is sufficient to prove (2.36) when

$$A = \bigcup_{i=1}^k A_i$$

is the finite union of $k \in \mathbb{N}$ disjoint intervals $A_i = (a_i, b_i)$ with $0 \le a_i < b_i < a_{i+1}$. The case when A is a countable union of intervals is obtained by monotone convergence. The general case follows upon approximating a $\mathcal{L}^1$-measurable set $A \subset \mathbb{R}^+$ by open sets.

For $s > 0$ let

$$\psi(s) = \log\Big(\sum_{i=1}^k s\int_{A_i} r^{s-1} dr\Big)^{\frac{1}{s}} = \frac{1}{s}\log\sum_{i=1}^k (b_i^s - a_i^s).$$

Proving (2.36) for $-1 < \alpha < \beta$ is equivalent to showing that ψ is strictly increasing for $s > 0$. The inequality $\psi'(s) > 0$ is equivalent to

$$\sum_{i=1}^k \big(b_i^s \log b_i^s - a_i^s \log a_i^s\big) - \sum_{i=1}^k \big(b_i^s - a_i^s\big)\log\sum_{i=1}^k \big(b_i^s - a_i^s\big) > 0. \tag{2.37}$$

We prove (2.37) by induction on $k \in \mathbb{N}$. For $k = 1$, letting $b_1^s = x$ and $a_1^s = y$, we have to check that

$$L_x(y) = x\log x - y\log y - (x-y)\log(x-y) > 0 \qquad \text{for} \quad 0 \le y < x.$$

This follows from $L_x(0) = L_x(x) = 0$ and

$$L_x''(y) = \frac{x}{y(y-x)} < 0 \qquad \text{for} \quad 0 < y < x.$$

Now assume that (2.37) holds for $k \in \mathbb{N}$. We prove it for $k+1$. Letting $b_i^s = x_i$ and $a_i^s = y_i$, we have to check that

$$L(y_{k+1}) = \sum_{i=1}^{k+1} \big(x_i \log x_i - y_i \log y_i\big) - \sum_{i=1}^{k+1}(x_i - y_i)\log\sum_{i=1}^{k+1}(x_i - y_i) > 0$$

for $x_k < y_{k+1} < x_{k+1}$. We consider L as a function of y_{k+1} alone, for fixed $x_1, \dots, x_{k+1}$ and $y_1, \dots, y_k$. By the induction assumption, we have $L(x_k) > 0$ and $L(x_{k+1}) > 0$. The claim follows from

$$L''(y_{k+1}) = -\frac{1}{y_{k+1}} - \Big(\sum_{i=1}^{k+1}(x_i - y_i)\Big)^{-1} < 0$$

for $x_k < y_{k+1} < x_{k+1}$.

3 Steiner rearrangement in the vertical direction

Let $\varrho, \tau \in C(\mathbb{R}^2_+)$ be two functions which satisfy the following properties:

$$\begin{aligned} &1)\ \ \varrho(r,t) = \varrho(r) \text{ is positive in } \mathbb{R}^+ \text{ and in } \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^+);\\ &2)\ \ \tau(r,t) = \tau(r) \text{ is positive.}\end{aligned} \tag{3.1}$$

The vector fields $V = (V_1, V_2)$ are as in (2.6) and the perimeter $\mathcal{R}(F; D)$ of a $\mathcal{L}^2$-measurable set $F \subset \mathbb{R}^2_+$ in an open set $D \subset \mathbb{R}^2_+$ is defined in (2.8). The partial perimeters $\mathcal{R}_i(F; D)$, $i = 1, 2$, are defined in (2.7).

Definition 3.1 (Steiner symmetric decreasing rearrangement). We say that a measurable set $F \subset \mathbb{R}^2_+$ is t-rearrangeable if the function $h : \mathbb{R}^+ \to [0, +\infty]$, given by

$$h(r) = \frac{1}{2}\mathcal{L}^1(F_r), \tag{3.2}$$

is in $L^1_{\mathrm{loc}}(\mathbb{R}^+)$. Here, $F_r = \{ t \in \mathbb{R} \,|\, (r,t) \in F \}$ is the section of F at level $r \in \mathbb{R}^+$. In this case, we call the set

$$F^* = \{ (r,t) \in \mathbb{R}^2_+ \,|\, |t| < h(r) \}$$

the Steiner symmetric decreasing rearrangement (simply, t-rearrangement) of F.

Theorem 3.2. *Assume that ϱ and τ satisfy conditions* (3.1). *Let $F \subset \mathbb{R}^2_+$ be a $\mathcal{L}^2$-measurable set which is t-rearrangeable and such that $\mathcal{R}(F) < +\infty$. Then its t-rearrangement F^* satisfies*

$$\mathcal{R}(F^*) \le \mathcal{R}(F). \tag{3.3}$$

Moreover, if $\mathcal{R}(F^*) = \mathcal{R}(F)$ then $F_r \subset \mathbb{R}$ is equivalent with a segment for $\mathcal{L}^1$-a.e. $r \in \mathbb{R}^+$.

Proof. We introduce the Borel measures μ_1 and μ_2 on $\mathbb{R}^+$

$$\begin{aligned} \mu_1(B) &= \mathcal{R}_1(F; B \times \mathbb{R}),\\ \mu_2(B) &= \mathcal{R}_2(F; B \times \mathbb{R}),\end{aligned}$$

where $B \subset \mathbb{R}^+$ is a Borel set. Analogously, starting from F^*, we define

$$\mu_1^*(B) = \mathcal{R}_1(F^*; B \times \mathbb{R}),$$
$$\mu_2^*(B) = \mathcal{R}_2(F^*; B \times \mathbb{R}).$$

For any open set $B \subset \mathbb{R}^+$, we have by the Fubini–Tonelli theorem and by the Coarea formula (see (2.25)–(2.27))

$$\begin{aligned} \mu_1(B) &\geq 2 \sup_{\psi \in \mathcal{F}_1(B)} \int_{\mathbb{R}^+} h(r)\partial_r\big(\varrho(r)\psi(r)\big)\,dr \\ &\geq 2 \int_0^{+\infty} \sup_{\psi \in \mathcal{F}_1(B)} \int_{\{h>s\}} \partial_r\big(\varrho(r)\psi(r)\big)dr\,ds \\ &\geq \mu_1^*(B). \end{aligned} \tag{3.4}$$

On the other hand, as in the proof of *Step 1* in Theorem 1.5 (see the argument starting from formula (2.21)), it is

$$\mu_2(B) = \int_B \tau(r)\Big(\sup_{\psi \in \mathcal{F}_1(\mathbb{R})} \int_{F_r} \partial_t\psi(t)\,dt\Big)dr$$

with

$$\sup_{\psi \in \mathcal{F}_1(\mathbb{R})} \int_{F_r} \partial_t\psi(t)\,dt \geq \sup_{\psi \in \mathcal{F}_1(\mathbb{R})} \int_{F_r^*} \partial_t\psi(t)\,dt, \tag{3.5}$$

because the right hand side can be either 0 (in which case the left hand side is also 0) or 2 (in which case the left hand side is equal or larger than 2). It follows that $\mu_2(B) \geq \mu_2^*(B)$, with equality if and only if F_r is equivalent to a segment for $\mathcal{L}^1$-a.e. $r \in B$.

Now, the claim (3.3) follows by the same argument as in (2.31)–(2.32). Moreover, if $\mathcal{R}(F^*) = \mathcal{R}(F)$ then, arguing as in *Step 4* of the proof Theorem 1.5, we deduce that $\mu_2(B) = \mu_2^*(B)$ for any Borel set $B \subset \mathbb{R}^+$. This implies that we have equality in (3.5) for $\mathcal{L}^1$-a.e. $r \in \mathbb{R}^+$. Thus F_r is equivalent to a segment for $\mathcal{L}^1$-a.e. $r \in \mathbb{R}^+$. $\square$

4 Proof of Proposition 1.3

We first note that, by an easy approximation argument, we have $\mathcal{P}(E;\Omega) = P(E;\Omega \setminus Z)$. Without loss of generality, we prove Proposition 1.3 in the case $\Omega = \mathbb{H}^n$ and $D = \mathbb{R}^2_+$. Let $E \subset \mathbb{H}^n$ be an axially symmetric $\mathcal{L}^{2n+1}$-measurable set with generating set $F \subset \mathbb{R}^2_+$.

Step 1. We claim that we have the inequality

$$\mathcal{P}(E) \geq \omega_{2n-1}\mathcal{Q}(F). \tag{4.1}$$

For any $\psi \in \mathfrak{F}_2(\mathbb{R}^2_+)$ we define $\varphi \in \mathfrak{F}_{2n}(\mathbb{H}^n)$ by letting

$$\varphi(z,t) = \frac{1}{|z|}\big\{\psi_1(|z|,t)z - \psi_2(|z|,t)J(z)\big\}. \tag{4.2}$$

Here, $z = (z_1,\dots,z_n) \in \mathbb{C}^n \equiv \mathbb{R}^{2n}$ with $z_j \equiv (x_j, y_j)$ and $J(z) = iz$ is the standard complex structure. For z and $J(z)$ are orthogonal, it is easy to check that $\varphi_1^2 + \ldots + \varphi_{2n}^2 \le 1$ is equivalent with $\psi_1^2 + \psi_2^2 \le 1$. Moreover, we have

$$\begin{aligned}
\partial_{x_j}\varphi_{2j-1} &= -\frac{x_j}{|z|^3}\big(x_j\psi_1 + y_j\psi_2\big) + \frac{1}{|z|}\left(\psi_1 + \frac{x_j^2}{|z|}\partial_r\psi_1 + \frac{x_j y_j}{|z|}\partial_r\psi_2\right),\\
\partial_{y_j}\varphi_{2j} &= -\frac{y_j}{|z|^3}\big(y_j\psi_1 - x_j\psi_2\big) + \frac{1}{|z|}\left(\psi_1 + \frac{y_j^2}{|z|}\partial_r\psi_1 - \frac{x_j y_j}{|z|}\partial_r\psi_2\right),\\
2y_j\partial_t\varphi_{2j-1} &= \frac{2}{|z|}\big(x_j y_j\partial_t\psi_1 + y_j^2\partial_t\psi_2\big),\\
-2x_j\partial_t\varphi_{2j} &= \frac{2}{|z|}\big(-x_j y_j\partial_t\psi_1 + x_j^2\partial_t\psi_2\big),
\end{aligned}$$

and, summing up, we get the following expression for the Heisenberg divergence of φ

$$\mathrm{div}_{\mathbb{H}}\varphi(z,t) = \partial_r\psi_1(|z|,t) + \frac{2n-1}{|z|}\psi_1(|z|,t) + 2|z|\partial_t\psi_2(|z|,t). \tag{4.3}$$

Letting $E_t = \{z \in \mathbb{C}^n : (z,t) \in E\}$ and $F_t = \{r \in \mathbb{R}^+ : (r,t) \in F\}$, using the Fubini–Tonelli theorem and spherical coordinates (Coarea formula) in $\mathbb{C}^n$, we obtain by (4.3)

$$\begin{aligned}
&\int_E \mathrm{div}_{\mathbb{H}}\varphi(z,t)\,dzdt = \int_{-\infty}^{+\infty}\int_{E_t}\mathrm{div}_{\mathbb{H}}\varphi(z,t)\,dz\,dt\\
&= \int_{-\infty}^{+\infty}\int_0^{+\infty}\int_{|z|=r}\chi_E(z,t)\mathrm{div}_{\mathbb{H}}\varphi(z,t)\,d\mathcal{H}^{2n-1}\,dr\,dt\\
&= \omega_{2n-1}\int_{-\infty}^{+\infty}\int_{F_t}\Big(\partial_r\psi_1(r,t) + \frac{2n-1}{r}\psi_1(r,t) + 2r\partial_t\psi_2(r,t)\Big)r^{2n-1}\,dr\,dt\\
&= \omega_{2n-1}\int_F\Big(\partial_r\big(r^{2n-1}\psi_1(r,t)\big) + 2r^{2n}\partial_t\psi_2(r,t)\Big)\,dr\,dt.
\end{aligned} \tag{4.4}$$

Taking the supremum over all $\psi \in \mathfrak{F}_2(\mathbb{R}^2_+)$ we obtain (4.1).

Step 2. We claim that if $E \subset \mathbb{H}^n$ is an axially symmetric bounded open set with finite perimeter and with boundary which is of class C^∞ in $\mathbb{H}^n \setminus Z$, then we have

$$\mathcal{P}(E) = \omega_{2n-1}\mathcal{Q}(F). \tag{4.5}$$

Denote by $\nu = (\nu_z, \nu_t) \in \mathbb{S}^{2n}$ the exterior unit normal to $\partial E \setminus Z$, and let $(\nu_r, \nu_t) \in \mathbb{S}^1$ be the exterior unit normal to ∂F (the boundary of F in $\mathbb{R}^2_+$). We have the relation $\nu_z(z,t) = \nu_r(|z|,t)z/|z|$ for $z \neq 0$. By the divergence theorem, we get from (4.4)

$$\int_{\partial E} \langle \varphi, \nu_z - 2\nu_t J(z)\rangle d\mathcal{H}^{2n} = \omega_{2n-1} \int_{\partial F} \big(\psi_1 \nu_r + 2r\psi_2 \nu_t\big) r^{2n-1} d\mathcal{H}^1. \tag{4.6}$$

Note that $|\nu_z - 2\nu_t J(z)| = 0$ implies $z = 0$. The vector fields ψ on ∂F and φ on $\partial E \setminus Z$ which make maximum the right respectively the left hand side of (4.6) with the L^∞ bound $\|\psi\|_\infty \le 1$ on ∂F and $\|\varphi\|_\infty \le 1$ on $\partial E \setminus Z$, are

$$\psi_1 = \frac{\nu_r}{(\nu_r^2 + 4r^2\nu_t^2)^{1/2}}, \quad \psi_2 = \frac{2r\nu_t}{(\nu_r^2 + 4r^2\nu_t^2)^{1/2}},$$

and

$$\varphi = \frac{\nu_z - 2\nu_t J(z)}{|\nu_z - 2\nu_t J(z)|} = \frac{1}{|z|}\big\{\psi_1 z - \psi_2 J(z)\big\}.$$

Note that φ is related to ψ according to (4.2). Indeed, we have $|\nu_z - 2\nu_t J(z)|^2 = \nu_r^2 + 4r^2\nu_t^2$ because ν_z and $J(z)$ are orthogonal. The vector fields φ and ψ can be smoothly extended in a neighborhood of $\partial E \setminus Z$ respectively of ∂F, with L^∞ norms bounded by 1. These extended vector fields can be uniformly approximated (near the boundaries) by compactly supported smooth vector fields with the same L^∞ bound. This finishes the proof of (4.5).

Step 3. We claim that we have the inequality

$$\mathcal{P}(E) \le \omega_{2n-1}\mathcal{Q}(F).$$

We can assume that $\mathcal{Q}(F) < +\infty$. By Theorem 2.2.2 and Corollary 2.3.6 in [9], there exists a sequence $(F_j)_{j\in\mathbb{N}}$ of open sets of $\mathbb{R}^2_+$ with boundary of class C^∞ in $\mathbb{R}^2_+$ and such that

(i) $\lim_{j\to+\infty} \mathcal{L}^2\big((F_j \Delta F) \cap K\big) = 0$ for any compact set $K \subset \mathbb{R}^2_+$;

(ii) $\lim_{j\to+\infty} \mathcal{Q}(F_j) = \mathcal{Q}(F)$.

For any $j \in \mathbb{N}$, denote by $E_j \subset \mathbb{H}^n$ the axially symmetric open set with boundary of class C^∞ in $\mathbb{H}^n \setminus Z$ which has F_j as generating set. From (i), it follows that for any compact set $K' \subset \mathbb{H}^n \setminus Z$ we have

$$\lim_{j\to+\infty} \mathcal{L}^{2n+1}\big((E_j \Delta E) \cap K'\big) = 0,$$

and thus, by the lower semicontinuity of perimeter with respect to the L^1_{loc} convergence, it follows

$$\mathcal{P}(E) = \mathcal{P}(E; \mathbb{H}^n \setminus Z) \le \liminf_{j\to+\infty} \mathcal{P}(E_j; \mathbb{H}^n \setminus Z).$$

On the other hand, by the *Step 2* and by (ii), we see that we actually have a limit and

$$\lim_{j\to+\infty} \mathcal{P}(E_j;\mathbb{H}^n\setminus Z) = \omega_{2n-1}\lim_{j\to+\infty}\mathcal{Q}(F_j) = \omega_{2n-1}\mathcal{Q}(F).$$

This ends the proof of the *Step 3* and of Proposition 1.3.

5 Proof of Theorems 1.7 and 1.2

Let $n \in \mathbb{N}$ and introduce the functions

$$\varrho(r) = r^{2n-1}, \quad \tau(r) = 2r^{2n}, \quad v(r) = r^{2n-1}. \tag{5.1}$$

The functions ϱ and τ satisfy both conditions (1.14) and (3.1), and the function v satisfies conditions (1.18). For a $\mathcal{L}^2$-measurable set $F \subset \mathbb{R}^2_+$, the volume

$$\mathcal{V}(F) = \int_F v(r)\,dr\,dt$$

is non decreasing with respect to the τ-rearrangement, i.e., the condition (1.20) holds. This follows from Example 2.5 with $\alpha = 2n-1$ and $\beta = 2n$.

Proof of Theorem 1.7. We divide the proof into three steps.

Step 1. Let $F \subset \mathbb{R}^2_+$ be a $\mathcal{L}^2$-measurable set such that $0 < \mathcal{V}(F) < +\infty$ and $\mathcal{Q}(F) < +\infty$. Let F^* be the Steiner symmetric decreasing rearrangment of F in direction t introduced in Definition 3.1. Note that the function h in (3.2) belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^+)$, because $\mathcal{V}(F) < +\infty$. By Theorem 3.2, we have $\mathcal{Q}(F^*) \le \mathcal{Q}(F)$. Moreover $\mathcal{V}(F^*) = \mathcal{V}(F)$, because the density of the volume $\mathcal{V}$, the function v in (5.1), does not depend on t. Then we have

$$\mathrm{Isop}(F^*) \le \mathrm{Isop}(F). \tag{5.2}$$

Assume that $F = F^*$ and $\mathcal{Q}(F) < +\infty$. The function $f : \mathbb{R} \to [0,+\infty]$ associated with F as in (1.16) is even and decreasing on $(0,+\infty)$, because $F = F^*$. As in (2.25) with $B = (0,+\infty)$ and $\tau_2 = 1$, we have

$$\sup_{\psi\in\mathcal{F}_1(0,+\infty)} \int_{\mathbb{R}} f(t)\partial_t\psi(t)\,dt \le \mathcal{Q}(F) < +\infty,$$

i.e. $f \in BV(0,+\infty)$, and in particular f is essentially bounded near $t = 0$. Then it is $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ and F is τ-rearrangeable, as required in Definition 1.4. Denote by $F^\sharp$ the τ-rearrangement of F. By Theorem 1.5 we have $\mathcal{Q}(F^\sharp) \le \mathcal{Q}(F)$, and by Proposition 2.4 along with Example 2.5 we have $\mathcal{V}(F^\sharp) \ge \mathcal{V}(F)$. Moreover, by (1.9) it is $\mathcal{V}(F^\sharp) < +\infty$, because $\mathcal{V}(\mathbb{R}^2_+ \setminus F^\sharp) = +\infty$. Then we have

$$\mathrm{Isop}(F^\sharp) \le \mathrm{Isop}(F). \tag{5.3}$$

Step 2. Assume that $F^\sharp = F$ and that the sections F_r are (equivalent to) intervals. We claim that, possibly modifying F in a $\mathcal{L}^2$-negligible set, we have

$$F \subset \big[0, c_n \mathcal{Q}(F)^{\frac{1}{2n+1}}\big] \times \big[t_0 - d_n \mathcal{Q}(F)^{2n}/\mathcal{V}(F)^{2n-1}, t_0 + d_n \mathcal{Q}(F)^{2n}/\mathcal{V}(F)^{2n-1}\big], \tag{5.4}$$

for some $t_0 \in \mathbb{R}$ and for dimensional constants $c_n, d_n > 0$.

Up to a $\mathcal{L}^2$-negligible set, the set F is of the form

$$F = \big\{ (r,t) \in \mathbb{R}^2_+ \,\big|\, 0 < r < g(t),\ t \in \mathbb{R} \big\} \tag{5.5}$$

for some function $g : \mathbb{R} \to [0, +\infty]$ which is decreasing on $(t_0, +\infty)$ and increasing on $(-\infty, t_0)$ for some $t_0 \in \mathbb{R}$. We let $M = \sup_{t\in\mathbb{R}} g(t)$. As in (2.25)–(2.27), we have

$$\begin{aligned}
\mathcal{Q}(F) &\geq \sup_{\psi\in\mathcal{F}_1(\mathbb{R}^2_+)} \int_F \tau(r)\partial_t \psi(r,t)\, dr\, dt \\
&\geq \int_0^{+\infty} \tau(r) \sup_{\psi\in\mathcal{F}_1(\mathbb{R})} \int_{\{g>r\}} \partial_t \psi(t)\, dt\, dr \\
&= 4\int_0^M r^{2n}\, dr = \frac{4M^{2n+1}}{2n+1}.
\end{aligned}$$

Then, we get the estimate

$$M \leq \big((2n+1)\mathcal{Q}(F)/4\big)^{\frac{1}{2n+1}}. \tag{5.6}$$

The set F in (5.5) is also of the form

$$F = \big\{ (r,t) \in \mathbb{R}^2_+ \,\big|\, k(r) < t < h(r),\ r \in \mathbb{R}^+ \big\}$$

for some functions $k, h : \mathbb{R}^+ \to [-\infty, +\infty]$ such that h and $-k$ are decreasing, thanks to $F = F^\sharp$. Moreover, we can assume that $h(r) = k(r) = t_0$ for all $r > M$. Thus, as in (3.4), we have

$$\begin{aligned}
\mathcal{Q}(F) &\geq \sup_{\psi\in\mathcal{F}_1(\mathbb{R}^2_+)} \int_F \partial_r\big(\varrho(r)\psi(r,t)\big)\, dr\, dt \\
&\geq \sup_{\psi\in\mathcal{F}_1(\mathbb{R}^+)} \int_{\mathbb{R}^+} \big(h(r)-k(r)\big)\partial_r\big(\varrho(r)\psi(r)\big)\, dr \\
&\geq M^{2n-1} \sup_{\psi\in\mathcal{F}_1(\mathbb{R}^+)} \int_{\mathbb{R}^+} \big(h(r)-k(r)\big)\partial_r \psi(r)\, dr \\
&\geq M^{2n-1} \lim_{r\to 0^+} \big(h(r)-k(r)\big).
\end{aligned} \tag{5.7}$$

From (5.7), we infer that

$$F \subset R := \big[0, M\big] \times \big[t_0 - \mathcal{Q}(F)/M^{2n-1}, t_0 + \mathcal{Q}(F)/M^{2n-1}\big], \tag{5.8}$$

and from (5.8) we get an estimate from below for M

$$\mathcal{V}(F) \leq \mathcal{V}(R) = \frac{\mathcal{Q}(F)}{n} M. \tag{5.9}$$

Finally, from (5.6), (5.8) and (5.9), we obtain the inclusion (5.4) with the dimensional constants $c_n = \big((2n+1)/4\big)^{\frac{1}{2n+1}}$ and $d_n = 1/n^{2n-1}$.

Step 3. The infimum in (1.11) is attained.

Let $(F_j)_{j\in\mathbb{N}}$ be a minimizing sequence for (1.11): $F_j \subset \mathbb{R}^2_+$ are $\mathcal{L}^2$-measurable sets such that $0 < \mathcal{V}(F_j) < +\infty$ for all $j \in \mathbb{N}$ and

$$\lim_{j\to+\infty} \operatorname{Isop}(F_j) = \frac{\operatorname{Isop}(\mathcal{R})}{\omega_{2n-1}} > 0. \tag{5.10}$$

By (5.2) and (5.3) in *Step 1*, we can without loss of generality assume that $F_j = F_j^* = F_j^\sharp$ for all $j \in \mathbb{N}$. We can also assume that $\mathcal{V}(F_j) = 1$ for all $j \in \mathbb{N}$. If this is not the case, we replace F_j with $\delta_\lambda(F_j)$ where $\delta_\lambda(r,t) = (\lambda r, \lambda^2 t)$ and $\lambda > 0$ is fixed in such a way that $\lambda^{2n+2}\mathcal{V}(F_j) = \mathcal{V}(\delta_\lambda(F_j)) = 1$.

We have $F_j = \big\{(r,t) \in \mathbb{R}^2_+ \,\big|\, |t| < h_j(t),\ r \in \mathbb{R}^+\big\}$ for functions $h_j : \mathbb{R}^+ \to [0,+\infty]$ which are decreasing on $(0,+\infty)$. By *Step 2*, the functions h_j are uniformly bounded and moreover, by (5.4), there exists $r_0 > 0$ such $h_j(r) = 0$ for all $r \geq r_0$ and for all $j \in \mathbb{N}$. By Helly's theorem, possibly taking a subsequence, the sequence $(h_j)_{j\in\mathbb{N}}$ converges pointwise to a decreasing function $h : \mathbb{R}^+ \to [0,+\infty)$. Let

$$F = \big\{(r,t) \in \mathbb{R}^2_+ \,\big|\, |t| < h(r),\ r \in \mathbb{R}^+\big\}.$$

By the dominated convergence theorem, we have

$$\mathcal{V}(F) = \lim_{j\to+\infty} \mathcal{V}(F_j) = 1. \tag{5.11}$$

Moreover, χ_{F_j} converges to χ_F in $L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$. By the lower semicontinuity of the perimeter

$$\mathcal{Q}(F) \leq \liminf_{j\to+\infty} \mathcal{Q}(F_j). \tag{5.12}$$

From (5.10), (5.11) and (5.12), it follows that $\operatorname{Isop}(F) = \operatorname{Isop}(\mathcal{R})/\omega_{2n-1}$.

Now, let F be any $\mathcal{Q}$-isoperimetric set. By Theorem 1.5, F satisfies $F^\sharp = F$. Moreover, by Theorem 3.2, the sections F_r are intervals for $\mathcal{L}^1$-a.e. $r \in \mathbb{R}^+$. By *Step 2,* the set F satisfies the inclusion (1.21). □

Proof of Theorem 1.2. Theorem 1.2 now follows from Theorem 1.7 and from the results of [24]. Here, we give a brief self contained proof leaving the details to the reader.

By Theorem 1.7 and Proposition 1.3, the infimum in (1.4) is attained at a set $E \in \mathfrak{A}$. The generating set of $F \subset \mathbb{R}^2_+$ of E is a minimum for (1.11). After a vertical translation, we can assume that $t_0 = 0$ in (1.21). Possibly modifying F in a $\mathcal{L}^2$-negligible set, we can assume that $F \subset \mathbb{R}^2_+$ is open. The boundary ∂F of F in $\mathbb{R}^2_+$ is rectifiable, and precisely it is the union of two 1-Lipschitz curves, by properties (i) and (ii) of F in Theorem 1.7. Then the perimeter of F is

$$\mathcal{Q}(F) = \int_{\partial F} \sqrt{\nu_1^2 + 4r^2\nu_2^2}\, r^{2n-1} d\mathcal{H}^1, \tag{5.13}$$

where $\nu = (\nu_1, \nu_2)$ is the exterior unit normal to ∂F, that is defined $\mathcal{H}^1$-a.e. on ∂F. Formula (5.13) is obtained from (1.6), first transforming integrals into boundary integrals by the divergence theorem, and then taking the supremum over test functions.

The set F minimizes $\mathcal{Q}(F)$ among sets with the same volume $\mathcal{V}(F)$. Equivalently, F minimizes the isoperimetric ratio $\mathrm{Isop}(F)$ in (1.10). Because the integrand

$$\sqrt{\nu_1^2 + 4r^2\nu_2^2}\, r^{2n-1}$$

is elliptic away from the set $r = 0$, by standard regularity theory we deduce that $\partial F \cap \mathbb{R}^2_+$ is a curve of class C^∞.

If $\nu_2(r,t) \neq 0$ there is a neighborhood of $(r,t) \in \partial F$ in $\mathbb{R}^2_+$ such that in this neighborhood ∂F is the graph of a function $t = \psi(r)$ with $\psi \in C^\infty(I)$, for some maximal open interval $I \subset \mathbb{R}^+$. By the variational principle, the function ψ satisfies the weak equation

$$\int_I \frac{\psi'(r)\varphi'(r)}{\sqrt{\psi'(r)^2 + 4r^2}}\, r^{2n-1}\, dr = K \int_I \varphi(r)\, r^{2n-1}\, dr, \quad K = \frac{(2n+1)\mathcal{Q}(F)}{(2n+2)\mathcal{V}(F)} \tag{5.14}$$

for any $\varphi \in C_0^\infty(I)$. This can be obtained modifying the piece of boundary of F given by the graph of ψ with the graph of $\psi + \varepsilon\varphi$, $\varepsilon \in \mathbb{R}$, and using the fact that at $\varepsilon = 0$ the isoperimetric ratio is a minimum. The weak equation (5.14) yields

$$-\frac{d}{dr}\left(\frac{r^{2n-1}\psi'(r)}{\sqrt{\psi'(r)^2 + 4r^2}}\right) = Kr^{2n-1}. \tag{5.15}$$

Possibly replacing F with a rescaled set, we can assume that $K = 2n$. It is either $\psi \leq 0$ or $\psi \geq 0$. Assume that we are in the latter case: then we have $\psi' \leq 0$. Integrating equation (5.15) we get

$$\psi'(r) = -\frac{2r(r^{2n} + H)}{\sqrt{r^{4n-2} - (r^{2n} + H)^2}} \tag{5.16}$$

for some constant $H \in \mathbb{R}$. The function ψ' is defined in the interval $I = \{r \in \mathbb{R}^+ : r^{2n-1} > |r^{2n} + H|\}$. If $H < 0$ the derivative ψ' changes sign at $r^{2n} = -H$. This is not possible because it is $\psi' \le 0$ in I. The case $H < 0$ gives rise to rotational symmetric constant mean curvature hypersurfaces (in the Heisenberg group) which are called nodoids in [25].

If $H > 0$ then it is $I = (r_0, r_1)$ for some $0 < r_0 < r_1$ and in particular $\psi'(r)$ tends to $-\infty$ as $r \to r_0^+$. Let $J = \psi(I) = (s_0, s_1)$ and let $\chi : J \to I$ be the inverse function of ψ. This function solves the differential equation

$$\chi' = -\frac{\sqrt{\chi^{4n-2} - (\chi^{2n} + H)^2}}{2\chi(\chi^{2n} + H)}.$$

Squaring both sides, taking a derivative and simplifying χ', we get a second order equation for χ of the form $\chi'' = f(\chi)$ for some smooth function f, away from $\chi = 0$. The graph of the solution to this equation for $s \in (s_0, s_1)$ with data $\chi(s_1) = r_0$ and $\chi'(s_1) = 0$ is contained in ∂F. If $\chi''(s_1) = 0$ then χ is constant, which is not possible. If $\chi''(s_1) \neq 0$, i.e. $\chi''(s_1) > 0$, then we contradict property (ii) of F in Theorem 1.7, because ∂F would be a strictly convex graph $r = \chi(s)$ in a neighborhood of $(s_1, \chi(s_1)) \in \partial F$. The case $H > 0$ gives rise to rotational symmetric constant mean curvature hypersurfaces which are called unduloids in [25]. Eventually, it must be $H = 0$, and from (5.16) we get the ordinary differential equation

$$\psi'(r) = -\frac{2r^2}{\sqrt{1 - r^2}}, \quad r \in (0, 1).$$

For $\partial F \cap \mathbb{R}^2_+$ is of class C^∞ it must be $\psi(1) = 0$. With this condition, the solution of the equation is $\psi(r) = \arccos r + r\sqrt{1 - r^2}$, which gives formula (1.5). □

References

[1] L. Ambrosio, N. Fusco and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Mathematical Monographs, Oxford Science Publications, Clarendon Press, Oxford, 2000.

[2] F. Brock, Weighted Dirichlet-type inequalities for Steiner symmetrization, *Calc. Var.* **8** (1999), no. 1, pp. 15–25.

[3] L. Capogna, D. Danielli, D. P. Scott and J. T. Tyson, *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem*, Progress in Mathematics 259, Birkhäuser, Basel, 2007.

[4] J.-H. Cheng, J.-F. Hwang and P. Yang, Existence and uniqueness for p-area minimizers in the Heisenberg group, *Math. Ann.* **337** (2007), no. 2, pp. 253–293.

[5] J.-H. Cheng, J.-F. Hwang, A. Malchiodi and P. Yang, Minimal surfaces in pseudohermitian geometry, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **4** (2005), pp. 129–177.

[6] D. Danielli, N. Garofalo and D.-M. Nhieu, A partial solution of the isoperimetric problem for the Heisenberg group, *Forum Math.*, to appear.

[7] E. De Giorgi, Sulla proprietà isoperimetrica dell'ipersfera nella classe degli insiemi aventi frontiera orientata di misura finita, *Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. Sez. I (8)* **5** 1958, pp. 33–44 [English translation in E. De Giorgi, *Selected papers*, Springer, Berlin, 2006].

[8] B. Franchi, S. Gallot and R. L. Wheeden, Sobolev and isoperimetric inequalities for degenerate metrics, *Math. Ann.* **300** (1994), pp. 557–571.

[9] B. Franchi, R. Serapioni and F. Serra Cassano, Meyers–Serrin type theorems and relaxation of variational integrals depending on vector fields, *Houston J. Math.* **22** (1996), no. 4, pp. 859–890.

[10] B. Franchi, R. Serapioni and F. Serra Cassano, Rectifiability and perimeter in the Heisenberg group, *Math. Ann.* **321** (2001), pp. 479–531.

[11] N. Garofalo and D.-M. Nhieu, Isoperimetric and Sobolev inequalities for Carnot–Carathéodory spaces and the existence of minimal surfaces, *Comm. Pure Appl. Math.* **49** (1996), pp. 1081–1144.

[12] A. Hurtado and C. Rosales, Area-stationary surfaces inside the sub-Riemannian three sphere, *Math. Ann.*, to appear in 2007.

[13] B. Kawohl, *Rearrangements and Convexity of Level Sets in PDE*, Lecture Notes in Mathematics 1150, Springer, Berlin, 1985.

[14] G. P. Leonardi and S. Masnou, On the isoperimetric problem in the Heisenberg group $\mathbb{H}^n$, *Ann. Mat. Pura Appl. (4)* **184** (2005), pp. 533–553.

[15] G. P. Leonardi and D. Rigot, Isoperimetric sets on Carnot groups, *Houston J. Math.* **29** (2003), pp. 609–637.

[16] R. Monti, Some properties of Carnot–Carathéodory balls in the Heisenberg group, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.* **11** (2001), no. 3, pp. 155–167.

[17] R. Monti, Brunn–Minkowski and isoperimetric inequality in the Heisenberg group, *Ann. Acad. Sci. Fenn. Math.* **28** (2003), pp. 99–109.

[18] R. Monti and D. Morbidelli, Isoperimetric inequality in the Grushin plane, *J. Geom. Anal.* **14** (2004), pp. 355–368.

[19] R. Monti and M. Rickly, Convex isoperimetric sets in the Heisenberg group, preprint (2007).

[20] R. Monti and F. Serra Cassano, Surface measures in Carnot–Carathéodory spaces, *Calc. Var. Partial Differ. Eq.* **13** (2001), pp. 339–376.

[21] P. Pansu, Une inégalité isopérimétrique sur le groupe de Heisenberg, *C. R. Acad. Sci. Paris Sér. I Math.* **295** (1982), pp. 127–130.

[22] P. Pansu, An isoperimetric inequality on the Heisenberg group, in: *Proceedings of the Conference on Differential Geometry on Homogeneous Spaces (Turin, 1983)*, pp. 159–174, Rend. Sem. Mat. Univ. Politec. Torino, 1983.

[23] S. D. Pauls, H-minimal graphs of low regularity in $\mathbb{H}^1$, *Comment. Math. Helv.* **81** (2006), pp. 337–381.

[24] M. Ritoré, A proof by calibration of an isoperimetric inequality in the Heisenberg group, work in progress.

[25] M. Ritoré and C. Rosales, Rotationally invariant hypersurfaces with constant mean curvature in the Heisenberg group $\mathbb{H}^n$, *J. Geom. Anal.* **16** (2006), no. 4, pp. 703–720.

[26] M. Ritoré and C. Rosales, Area-stationary surfaces in the Heisenberg group $\mathbb{H}^1$, preprint, 2005.

[27] G. Talenti, The standard isoperimetric theorem, in: *Handbook of Convex Geometry*, Vol. A, pp. 73–123, North-Holland, Amsterdam, 1993.

Received September 15, 2007; revised October 31, 2007.

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