Modalities in Temporal Logic*

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ABSTRACT

In logics of branching-time, ‘possibility’ can be conceived as ‘existence of a suitable set of histories’ passing through the moment under consideration. A particular limit case of this is the Ockhamist notion of possibility, which is explained as truth at some history. The tree-like representation of time offers other ways of defining possibility as, for instance, truth at any history in some equivalence class modulo *undividedness*. In general, we can consider representations of time in which, at any moment t, the set of histories passing through t can be decomposed into indistinguishability classes. This yields to a new general notion of possibility including, as particular cases, other notions previously considered.

1. INTRODUCTION

The modal notions considered in this paper are closely related to the assumption that, according to Indeterminism, moments in Time have many different, incompatible, possible futures.[1] If a coin is tossed at moment t_0, we can think of two moments t and t', both in the future of t_0, in which it comes out tails and heads, respectively. This means in particular that Time is *branching*: it does not consist of a single linear sequence of moments; it is made of different *possible courses of events*. A further assumption of Indeterminism is that only the future (of a given moment) is manifold; the past is unique. Then, if two courses of events share a common moment, they also overlap in the past of that moment.

From a set-theoretical point of view, the above considerations lead to conceive Time as a *tree*. In the context of branching-time semantics, a *tree* is a pair **T** = $\langle T,<\rangle$ in which T is a set and $<$ is a binary relation on T with the following properties: *irreflexivity* ($t \not< t$ for all $t \in T$), *transitivity* (if $t < t'$ and $t' < t''$, then $t < t''$), and *left-linearity* (if $t' < t$ and $t'' < t$, then either $t' < t''$, or $t'' < t'$, or $t' = t''$). The elements of T represent (and are called) *moments*, and $<$ is the *earlier/later* relation between them. Thus, $t < t'$ can be read as ‘t is in the past of t' ’, or as t' is in the future of t'. By irreflexivity, no moment is in the past or in the future of itself. Left-linearity is the set-theoretical correspondent of the uniqueness of the past. Figure 1 below represents a tree in which $t' < t$ whenever t can be reached from t' moving upward along a line. Then t_0 is in the past of both t_1 and t_2, but these two moments are not temporally comparable.

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[1] I will not consider, then, notions of possibility defined in terms of temporal notions like, e.g., the *Diodorean possibility* which is defined as “truth now, or in the future”. For such notions, see (Denyer 2009) and (Ciuni, 2009), in this volume.

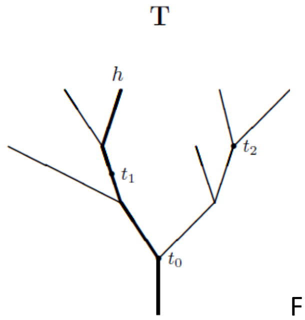

Figure 1

A *linear order* < on the set X is an irreflexive and transitive relation on X such that, for all $x \neq y$ in X, either $x < y$ or $y < x$. A history in a tree **T** is a subset h of T, which is linearly ordered by < and is maximal for inclusion: for every $X \subseteq T$, if $h \subseteq X$ and < linearly orders X, then $X = h$.[2] The marked line h in Figure 1 represents a history. Histories correspond to (complete) courses of events and play a crucial role when possibilities are involved in branching-time contexts. Sentences like "it is possible that it will come up tails" allow representations in terms of (first-order) quantification over moments: "there is a future moment in which it comes up tails". But sentences like "it is possible that it will *never* come up heads" involve a (second order) quantification over courses of events.

Quantification over histories is a peculiar aspect of Prior's Ockhamist and Peircean semantics for branching-time (Prior, 1967), which are defined in Section 2. In both these semantics, possibility is viewed as existence of a history: 'possible at moment t_0' in Figure 1, means 'true in (at least) one of the histories passing through t_0'. This agrees with the above example of the toss of a coin.

There are other ways, though, in which possibility can be conceived. Considering Figure 1 again, we can observe that, at t_0, Time branches out in *only two ways*, despite the fact that there are six histories passing through that moment. Possibility at t_0 can be viewed as openness with respect to take one path or the other. From the set-theoretical point of view, this notion of possibility is based on the *undividedness* relation between histories (Section 3), which is particularly relevant in Belnap's *s.t.i.t.* logic of agency (Belnap et al. 2001).

The notion of undividedness can be generalized by considering other ways of 'grouping histories together'. In (Zanardo, 1998) I have considered the notion of *indistinguishability* (possibly for a given agent) between histories. Intuitively, we can assume that, at any given moment in Time, some histories passing through that moment cannot be distinguished from one-another.

Consider for instance a game G with two players, P_1 and P_2, and let M be the set of all possible moves. Then a match is a sequence $m_0^1, m_0^2, m_1^1, m_1^2 ...$ of moves, where the superscripts denote the player and m_n^i belongs to a set $M_n^i \subseteq M$. The set M_n^i is determined by the situation reached at that step of the match. The rules of G might establish, for instance, that the match ends (and P_i loses) when M_n^i is empty. The set of all possible matches of the game can be

[2] Histories are sometimes *chronicles*, for instance in (Øhrstrøm and Hasle, 1995) and (Øhrstrøm, 2009)

viewed as the set of histories in a tree structure $\mathbf{T}_G$.[3] At any stage t_0 of a match, the choice of the player in turn depends, among other things, on the investigation of the possible evolutions of the match after that stage. In this context, it is quite reasonable to assume that the player is unable to distinguish different evolutions if they agree (overlap) on a sufficiently large number of moves. That number depends of course on the complexity of the game as well as on the computing ability of the player.

Differently from undividedness, indistinguishability is a primitive notion. Thus, in Section 4, I will consider temporal structures consisting of a tree endowed with a family of binary (indistinguishability) relations, indexed on the set of moments. This will allow us to define other kinds of possibilities. In particular, in Section 5, I will consider the notion of possibility related to the notion of choice. It will be shown that indistinguishability provides a general framework for dealing with all these modal notions.

2. OCKHAMIST AND PEIRCEAN SEMANTICS

In this paper only propositional languages are considered: starting with a denumerable set $\{p_0,p_1,...,p_n,...\}$ of propositional variables, complex formulas are built by means of the usual Boolean operators.[4] Temporal languages have in general two further operators, *P* and *F*, which are read as 'at least once in the past' and as 'at least once in the future' (of the moment under consideration). The interpretation of the past operator is rather obvious: if α is true at a moment t, then $P\alpha$ is true at any t' in the future of t. When Time is given a tree-like structure, the interpretation of the operator *F* is more controversial. This issue is widely discussed in (Prior, 1967), where Ockhamist and Peircean semantics are proposed as solutions to the problem of interpreting formulas of the forma $F\alpha$ in branching-time contexts.[5]

The peculiar aspect of the Ockhamist reading of the operator *F* is that, in general, it makes no sense to ask whether formulas of the form $F\alpha$ are true or false at a given moment. Ockhamist truth is relative to pairs $\langle t, h\rangle$, where the moment t belongs to the history h. This means that the truth value of $F\alpha$ at $\langle t, h\rangle$ is established on the basis of the truth value of α at pairs $\langle t', h\rangle$, where t' is in the future of t and belongs to h.[6]

Since histories are linear orders, Ockhamist logic of the operators *P* and *F* is linear-time logic. In order to deal with the branching aspect of Time, Ockhamist language has a modal operator $\Diamond$ which is read "at some history passing through the moment under consideration".

[3] The technical details of the definition of T_G are a bit complex since a single move m can occur in different matches. Thus, we have to consider as elements of T_G the finite sequences $\sigma = \langle m_0, m_1,...,m_n\rangle$ that are compatible with the rules of G, and we set $\sigma < \sigma'$ whenever σ is an initial segment of σ'.

[4] The restriction to the propositional case agrees with most of the works on the subject of this paper. The extension to first-order languages are technically very complex and the quantification within or without tense operators raises difficult, but extremely intriguing, philosophical problem

[5] It is interesting to observe that Ockhamist and Peircean logics are quite similar to the logics CTL* and CTL (*Computation Tree Logic*), which were independently defined as application of temporal logic to Theoretical Computer Science (Clark et al., 1986; Emerson and Halpern, 1986).

[6] Sometimes, truth at $\langle t, h\rangle$ is explained as "truth at t, under the assumption that h is the history that will *actually* take place". But, as shown in (Belnap and Green, 1994) and (Belnap et al., 2001), the notion of *actual future* is rather debatable. This matter is widely discussed in (Øhrstrøm, 2009) in this volume.

Then, as observed in the introduction, the first notion of possibility that we find in a branching-time context is "truth, at some course of events".

Given any tree **T**, the set of all histories in it will be written as **H**(**T**). We write $\mathbf{H}_t(\mathbf{T})$, or $\mathbf{H}_t$ when **T** is given by the context, to denote the set of histories passing through the moment *t*. An Ockhamist evaluation of the propositional variables in **T** is a function *V* assigning each propositional variable a set pairs $\langle t', h\rangle$ in which $t \in h \in \mathbf{H}(\mathbf{T})$. We read $\langle t', h\rangle \in V(p_n)$ as "p_n is true at $\langle t', h\rangle$".[7]

Ockhamist truth relation will be written $\vDash_{Ock}$ and is recursively defined by rules $\mathbf{O}_0$ to $\mathbf{O}_5$ below. $\mathbf{T},V \vDash_{Ock} \alpha\,[t, h]$ means that "α is true at $\langle t', h\rangle$ in **T** with the evaluation *V*".

$\mathbf{O}_0 : \mathbf{T},V \vDash_{Ock} p_n\,[t, h]$	iff	$\langle t',h\rangle \in V(p_n)$
$\mathbf{O}_1 : \mathbf{T},V \vDash_{Ock} \neg\alpha\,[t, h]$	iff	$\mathbf{T},V \nvDash_{Ock} \alpha\,[t, h]$
$\mathbf{O}_2 : \mathbf{T},V \vDash_{Ock} \alpha \wedge \beta\,[t, h]$	iff	$\mathbf{T},V \vDash_{Ock} \alpha\,[t, h]$ and $\mathbf{T},V \vDash_{Ock} \beta\,[t, h]$
$\mathbf{O}_3 : \mathbf{T},V \vDash_{Ock} F\alpha\,[t, h]$	iff	$\exists\, t' \in h : t < t'$ and $\mathbf{T},V \vDash_{Ock} \alpha\,[t', h]$
$\mathbf{O}_4 : \mathbf{T},V \vDash_{Ock} P\alpha\,[t, h]$	iff	$\exists\, t' < t : \mathbf{T},V \vDash_{Ock} \alpha\,[t', h]$
$\mathbf{O}_5 : \mathbf{T},V \vDash_{Ock} \Diamond\alpha\,[t, h]$	iff	$\exists\, h' \in \mathbf{H}_t : \mathbf{T},V \vDash_{Ock} \alpha\,[t, h']$

Universal closures of $\mathbf{T},V \vDash_{Ock} \alpha\,[t, h]$ (with respect to *V*, or $\langle t, h\rangle$, etc.) are written in the usual way. For instance $\mathbf{T},V \vDash_{Ock} \alpha$ means that $\mathbf{T},V \vDash_{Ock} \alpha\,[t, h]$ holds for all *t* and $h \ni t$. If $\vDash_{Ock} \alpha$, then we say that α is an Ockhamist *validity*.

The dual operators *H*, *G*, and $\Box$ are defined in the usual way as $\neg P\neg$, $\neg F\neg$, and $\neg\Diamond\neg$, respectively, and their meaning is given by the obvious universal quantification over moments or over histories.

Ockhamist truth of formulas of the form $\Diamond\alpha$ or $\Box\alpha$ is history independent, in the sense that $\mathbf{T},V \vDash_{Ock} \Diamond\alpha\,[t, h]$ implies $\mathbf{T},V \vDash_{Ock} \Diamond\alpha\,[t, h']$ for all $h' \ni t$, and similarly for $\Box\alpha$.[8] Also formulas of the form $P\alpha$ enjoy a sort of history independency: the truth of $P\alpha$ at $\langle t, h\rangle$ does not depend on *h* whenever α is built from formulas of the form $\Box\alpha_1,\ldots, \Box\alpha_n$ without any use of the operator *F*. If history independent evaluations are adopted (see Footnote 7), then we have only to assume that α is *F*-free. There is a substantial difference, though, between the independence from histories of formulas of the form $\Diamond\alpha$ and that of formulas $P\alpha$. In the first case, the property is due to the quantification over histories in the semantics of the operator $\Diamond$, while, in the second case, the property is due to the tree-like structure of time: all histories passing through the

[7] Some authors consider *history independent* evaluations, that is, evaluations assigning sets of *moments* to propositional variables. With these evaluations, the truth condition $\mathbf{O}_0$ below would be simply $t \in V(p_n)$ and $p_n \equiv \Box p_n$ would be a validity. This corresponds to the idea that propositional variables represent atomic facts (like "it is raining") and hence their truth depends only on the moment under consideration. Prior himself discusses this issue in (Prior, 1967). A brief discussion can also be found in (Zanardo, 2006b).

[8] The logic CTL* distinguishes between *state formulas* and *path formulas*. The former are those equivalent to formulas of the form $\Box\alpha$, whose truth depends only on the moment (state) and does not depend on the particular history (path) we are considering.

moment at hand agree in the past of that moment.

Peircean truth is relative to moments and the quantification over histories is implicit in the operator **F** which is interpreted as "at some future moment, *on each history*". Peircean language has also a 'weak' future operator, **f**, whose meaning is "at some future moment, *on some history*".

Peircean evaluations on a tree $\langle T,<\rangle$ are functions V assigning a set of moments to each propositional variable. Peircean truth relation $\vDash_{\mathrm{Peir}}$ is defined by rules $\mathbf{P}_0$ to $\mathbf{P}_5$ below. We read $\mathbf{T},V \vDash_{\mathrm{Peir}} \alpha\ [t]$ as "α is true at t in $\mathbf{T}$ with the evaluation V".

$\mathbf{P}_0$: $\mathbf{T},V \vDash_{\mathrm{Peir}} \mathrm{p}_n\ [\mathrm{t}]$ iff $t \in V(p_n)$

$\mathbf{P}_{1,2}$: the usual rules for $\neg$ and $\wedge$

$\mathbf{P}_3$: $\mathbf{T},V \vDash_{\mathrm{Peir}} \mathbf{F}\alpha\ [t]$ iff $\forall h \in \mathbf{H}_t, \exists t' \in h :$ $t < t'$ and $\mathbf{T},V \vDash_{\mathrm{Peir}} \alpha\ [t']$

$\mathbf{P}_4$: $\mathbf{T},V \vDash_{\mathrm{Peir}} \mathbf{f}\alpha\ [t]$ iff $\exists h \in \mathbf{H}_t, \exists t' \in h :$ $t < t'$ and $\mathbf{T},V \vDash_{\mathrm{Peir}} \alpha\ [t']$

$\mathbf{P}_5$: $\mathbf{T},V \vDash_{\mathrm{Peir}} P\alpha\ [t]$ iff $\exists t' < t : \mathbf{T},V \vDash_{\mathrm{Peir}} \alpha\ [t']$

As observed above, in Peircean semantics the second-order quantification over histories is implicit in the truth rules for the operators **F** and **f**. Thus, we don't have modal operators in the usual sense.[9] On the technical side, it must be observed that in rule $\mathbf{P}_4$ the expression $\exists h \in \mathbf{H}_t$, $\exists t_h \in h : t < t'$ ... is equivalent to $\exists t' > t$... and hence this rule is expressible by a first-order quantification over moments.

The dual operator $H = \neg P \neg$ has the obvious meaning also in Peircean logic, and this holds similarly for future universal operators $\mathbf{G} = \neg\mathbf{f}\neg$ which can be read "always in the future". The operator $\mathbf{g} = \neg\mathbf{F}\neg$ is more interesting: by $\mathbf{P}_3$, its meaning is "always in the future, *on some possible history*". If α means "it comes up heads", then the sentence "it is possible that it will never come up heads" considered above is expressed by $\mathbf{g}\neg\alpha$.

Peircean language can be viewed as a fragment of the Ockhamist one because the operators **f** and **F** can be expressed as $\Diamond F$ and $\Box F$.[10] Despite this technical relation, it is evident that the two approaches correspond to deeply different conceptions of the meaning of tensed assertions in branching-time contexts.

2.1 BUNDLED TREE SEMANTICS

The Ockhamist operator $\Diamond$ and the Peircean operator **F** quantify over the set $\mathbf{H}_t$ of *all* the histories passing through the moment t at hand. Various works in the literature have

[9] In the *neighborhood semantics* of (Segerberg, 1971), or in the *minimal models* of (Chellas, 1980) we can actually see a semantics for modal logic similar to Peircean semantics for the operator **F**.

[10] Since, in Peircean semantics, propositional variables are evaluated at sets of moments, this embedding of Peircean language into Ockhamist one preserves truth on if history independent (Ockhamist) evaluations are considered -see Footnote 7.

considered branching time semantics in which these operators quantify over a fixed set of histories passing through t, possibly different from $\mathbf{H}_t$. Formally, a bundled tree is a pair ⟨T, β⟩ in which β is a set of histories such that $\cup \beta = T$, that is, every moment in **T** belongs to some element of β. In Figure 2, for instance, every moment belongs to some h_i and hence we can consider a bundle β consisting of those histories. We have $\beta \neq$ H(**T**) because $h_\omega \notin \beta$.

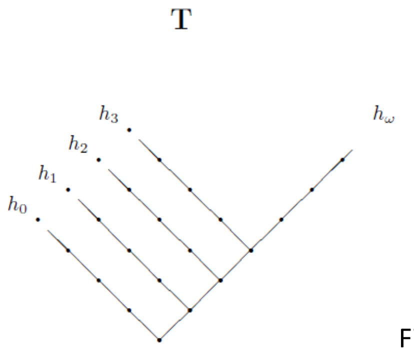

Figure 2

Ockhamist and Peircean semantics can be based on bundled trees: the only difference is that the quantification over $\mathbf{H}_t$ in $\mathbf{O}_5$ and $\mathbf{P}_3$ is replaced by quantification over $\mathbf{H}_t \cap \beta$.

From the mathematical logic point of view, moving from trees to bundled trees allows to turn a second-order quantification into a first-order one and the matter offers many technical problems and results (Burgess, 1979; Burgess, 1980; Zanardo, 2006b; Zanardo et al. 1999). On the philosophical side, quantifying over histories in a bundle amounts to hold that, at any moment in Time we can consider a set of *admissible histories* and that there might be maximal linear sequences of moments which are not admissible. This matter is discussed, for instance, in (Belnap et al., 2001) and in (Thomason, 1984), where this point of view is criticized: roughly speaking, excluding the history h_ω in the structure of Figure 2, seems to lead to counterintuitive consequences of some plausible premises -see (Belnap et al., 2001, pp.199-203). In (van Benthem, 1986), instead, the admissible history approach is defended. In private correspondence, van Benthem writes: *putting in a set of runs [*i.e. histories*] explicitly at least invites us to state interesting conditions on them, that explain the temporal reasoning practice we want to analyze.*

In the sequel of this paper I will still consider only *standard structures* in which history quantifiers act on the whole sets $\mathbf{H}_t$, but for all those structures the bundled tree semantics can be adopted as well.

3. UNDIVIDED HISTORIES - IMMEDIATE POSSIBILITIES

Definition 3.1 *The histories h_1 and h_2 are* undivided *at the moment t in the tree ⟨T,<⟩ (in symbols,* $\mathrm{U}_t(h_1, h_2)$*) whenever there exists a moment $t' > t$ such that $t' \in h_1 \cap h_2$.*

The relation U_t is called undividedness at t and is an equivalence relation on $\mathbf{H}_t$. The equivalence class of h modulo U_t will be written $[h]_t^{\mathrm{U}}$. As observed in the introduction, equivalence classes modulo U_t represent the ways in which Time branches out at t and it is natural to refer to them as *immediate possibilities* at t (Belnap, 1992). For instance, in Figure 1 there are two immediate possibilities at t_0 while at any $t < t_0$ there is only one immediate possibility. In Belnap's *s.t.i.t.* logic of agency (Belnap et al., 2001), undividedness is deeply involved in connection with the notion of *choice* (see also the *Theory of Causation* in (von Kutschera, 1993)). These issues will be discussed below in Section 5.

In (Zanardo, 1998) I have considered an extension of Ockhamist language obtained by adding an operator $\Diamond^{\mathrm{U}}$ quantifying within equivalence classes modulo undividedness. The semantics of this operator is given by the following rule

$$\mathbf{T},V \vDash_{\mathrm{Ock}} \Diamond^{\mathrm{U}} \alpha\ [t, h] \ \text{ iff } \ \exists h' : \mathrm{U}_t\,(h, h') \text{ and } \mathbf{T},V \vDash_{\mathrm{Ock}} \alpha\ [t, h']$$

Some combinations of $\Diamond^{\mathrm{U}}$ with Ockhamist operators are equivalent to Ockhamist expressions. For instance, it easy to verify that

$$\vDash_{\mathrm{Ock}} \Diamond^{\mathrm{U}} \Diamond\alpha \leftrightarrow \Diamond\alpha \ \text{ and } \vDash_{\mathrm{Ock}} \Diamond^{\mathrm{U}} \Box\alpha \leftrightarrow \Box\alpha$$

but in general the operator $\Diamond^{\mathrm{U}}$ is not expressible in Ockhamist language: for instance, the formula $\Diamond^{\mathrm{U}} G p_0$ is not equivalent to any Ockhamist formula (Zanardo, 1998, Prop. 3.1). Beyond the technical details, the non-equivalence between the standard Ockhamist language and the present enriched one reflects the fact that, in general, quantifications over a given equivalence class modulo U_t cannot be simulated by quantifications over the whole $\mathbf{H}_t$.

In this enriched version of Ockhamist language, the possibility operators, $\Diamond$ and $\Diamond^{\mathrm{U}}$, still quantify over histories, but a quantification over immediate possibilities can be simulated by combining those operators. For instance, the formula $\Diamond\Box^{\mathrm{U}}\alpha$ expresses the fact that α holds at (every history of) some immediate possibility.

In the above-mentioned paper I have also considered Peircean-like operators quantifying over the set of immediate possibilities at a given moment of a branching-time structure. For instance, I have considered an operator therein written as $\mathbf{f}^{\mathrm{U}}$ defined by

$$\vDash_{\mathrm{Peir}} \mathbf{f}^{\mathrm{U}} \alpha\ [t] \ \text{ iff } \ \textit{there is an immediate possibility } \pi \textit{ at } t \textit{ such that}$$
$$\forall h \in \pi, \exists t' \in h : t < t' \text{ and } \vDash_{\mathrm{Peir}} \alpha\ [t']$$

Also this operator is not definable in the usual Peircean language (Zanardo, 1998, Prop. 3.5).

The results sketched above show that, in order to deal with reasonable notions of possibility related to undividedness, new operators are needed. In Section 5 I will show that these notions of possibility can be viewed as a particular case of possibility related to indistinguishability.

3.1 RELATIVE CLOSENESS -TOPOLOGY

Undividedness at t, as a partition of $\mathbf{H}_t$, determines a sort of relative closeness relation in this set: if $U_t(h, h_1)$ holds, but $U_t(h, h_2)$ does not, then it makes sense to say that h is closer, or more similar, to h_1 than to h_2. In general, we can set

$$Cl(h, h_1,h_2) \Leftrightarrow h \cap h_2 \subset h \cap h_1 \tag{3.1}$$

where $\subset$ is proper inclusion. The ternary relation Cl is called *relative closeness* and $Cl(h, h_1, h_2)$ is read as *h is closer to h_1 than to h_2*.

Differently from undividedness, relative closeness does not depend on a particular moment. It is also worth observing that this relation is not always definable in terms of undividedness. Consider for instance the tree of Figure 3 and assume that the intersection $h \cap h_1 \cap h_2$ has no maximum, while $h \cap h_1$ has just one moment which does not belong to h_2. In this case, for any moment t, $U_t(h, h_1) \Leftrightarrow U_t(h, h_2)$, but we have also $Cl(h, h_1, h_2)$.

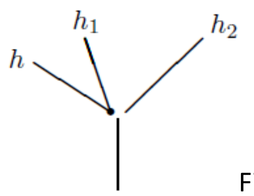

Figure 2

The natural environment for relative closeness issues is topology. Given any tree **T**, we can consider the topology $\tau_\mathbf{T}$ on H(**T**) generated by the set $\{\mathbf{H}_t : t \in T\}$ (Sabbadin and Zanardo, 2003). It is straightforward to verify that $Cl(h, h_1,h_2)$ holds if and only if there exists an open set X in $\tau_\mathbf{T}$ such that $h, h_1 \in X$, but $h_2 \notin X$.

The topological approach is not only a different way of describing the usual set-theoretical relations between moments and histories in a tree. This approach provides also a different ontological perspective under which branching-time semantics can be viewed. In the papers (Zanardo, 2004; Zanardo, 2006a) I have inverted (dualized) the usual perspective which describes histories as set of moments. It turns out that we can start from a primitive notion of history with a suitable topology on the set of all histories, or with a (primitive) notion of relative closeness having two suitable, quite natural, properties: (1) *every history is closer to itself than to any other history*, and (2) *if h is closer to h′ than to h″, then it is not the case that h is closer to h″ than to h′*. Representation results can be proved and, as far as Ockhamist validity is concerned, the dual approach turns out to be equivalent to the original one.

4. INDISTINGUISHABLE HISTORIES - RECOGNIZED POSSIBILITIES

In the Introduction I considered an example of indistinguishable histories in the framework of a given game. In that example two histories cannot be distinguished at the moment t if, in the future of t, they overlap on a segment Δt whose length depends on the complexity of the game and on the computing ability of the player. If $\Delta t > 0$, then indistinguishability can be viewed as a strengthening of undividedness: any two indistinguishable histories are also undivided.

Examples of the opposite situation can be considered as well. Assume for instance that the agent *a* is at a cross-roads at the present moment t_0, and that he can decide either to turn left or to turn right. Then the set $\mathbf{H}_{t_0}$ can be decomposed into two sets $\mathbf{H}_{\mathrm{left}}$ and $\mathbf{H}_{\mathrm{right}}$ according to the choice of the agent a. Given any $h_l \in \mathbf{H}_{\mathrm{left}}$ and $h_r \in \mathbf{H}_{\mathrm{right}}$ we have that h_l and h_r are divided at t_0 because, in the near future of t_0, they differ, at least, in *a*'s choice. If the notion of indistinguishability we are considering is relative to the knowledge of some other agent *b* who has no access to *a*'s activity, then h_l and h_r might be indistinguishable at t_0. This happens, for instance, when the only difference between h_l and h_r in the near future of t_0 is just the choice of the agent *a*.

In any case, any reasonable notion of indistinguishability seems to have a temporal dimension: if two histories are distinguishable now, they cannot become indistinguishable at some future moment. This justifies the following formal definition.

Definition 4.1 *An* **I**-tree *is a pair* $\langle \mathbf{T},\mathbf{I}\rangle$ *in which* **T** *is a tree and* **I** *is an* indistinguishability function *on* **T***: the domain of* **I** *is T and, for all* $t \in T$, $\mathbf{I}_t$ *in an equivalence relation on* $\mathbf{H}_t$ *such that,* $\mathbf{I}_t(h, h')$ *&* $t' < t \Rightarrow \mathbf{I}_{t'}(h, h')$*.*

At any moment *t* in Time, we can consider the partition of $\mathbf{H}_t$ into equivalence classes modulo $\mathbf{I}_t$. The class of the history *h*, that is $\{h' : \mathbf{I}_t(h, h')\}$, will be denoted by $[h]^{\mathbf{I}}_t$. According to the intended meaning of the relations $\mathbf{I}_t$, single histories cannot be recognized at that moment: the only recognizable entities at *t* are classes $[h]^{\mathbf{I}}_t$. Each of these classes represents a recognized way in which Time branches out at *t*, and hence we will refer to them as *recognized possibilities at t*. Sometimes we will write $i \in \mathbf{I}_t$ to mean that *i* is an equivalence class modulo $\mathbf{I}_t$; this will make the notation lighter.[11]

The set of all recognized possibilities in an **I**-tree $\langle \mathbf{T},\mathbf{I}\rangle$ will be written as $T^{\mathbf{I}}$. In the technical definition of this set we have to take into account that there might be recognized possibilities $[h]^{\mathbf{I}}_t = [h]^{\mathbf{I}}_{t'}$ with $t \neq t'$. No confusion can arise if we consider pairs $\langle t, i\rangle$:

$$T^{\mathbf{I}} = \{\langle t, i\rangle : t \in T \text{ and } i \in \mathbf{I}_t\} \qquad (4.2)$$

Quantifying over $\mathbf{I}_t$ or within some element of It give rise to different new notions of possibility at *t*.[12] In (Zanardo, 1998) I have considered a language containing Ockhamist-like operators, as well as Peircean-like ones. The starting point is an Ockhamist notion of truth, but,

[11] In mathematical terms, this means that the equivalence $\mathbf{I}_t$ and its quotient set are identified.

[12] A more exhaustive treatment of indistinguishability would have required a notion of agent-indexed indistinguishability functions $\mathbf{I}_a$, where *a* ranges over a set of agents: two histories may be distinguishable for an agent, but indistinguishable for another one. Accordingly, we would have different, simultaneous, notions of possibility and the language would have agent-indexed possibility operators. Such a distinction, though, goes beyond the aim of this work, where we consider only the basic properties of possibility related to indistinguishability.

since some histories may be indistinguishable from one-another, we consider truth at pairs $\langle t, i\rangle$: *at the moment t*, truth at $\langle t, h\rangle$ and truth at $\langle t, h'\rangle$ cannot be distinguished if $\mathbf{I}_t(h, h')$. On the other hand, differently from $\mathbf{O}_3$, we interpret the operator for the future in a Peircean way, because i is generally constituted by many histories.

An evaluation on an **I**-tree $\langle \mathbf{T},\mathbf{I}\rangle$ is a function V assigning a set of pairs $\langle t, i\rangle$, with $i \in \mathbf{I}_t$, to each propositional variable. Then truth is relative to recognized possibilities. We write $\vDash_{\text{Ind}}$ for truth in **I**-trees. The following rules provide a semantics for a language with an Ockhamist operator $\Diamond$, and the Peircean operators **F** and **f** (in addition to the past operator P).

$\mathbf{I}_0$: $\langle \mathbf{T},\mathbf{I}\rangle, V \vDash_{\text{Ind}} p_n\ [t, i]$	iff	$\langle t, i\rangle \in V(p_n)$
$\mathbf{I}_{1,2}$: the usual rules for $\neg$ and $\wedge$		
$\mathbf{I}_3$: $\langle \mathbf{T},\mathbf{I}\rangle, V \vDash_{\text{Ind}} \mathbf{F}\alpha\ [t, i]$	iff	$\forall h \in i, \exists t' \in h$: $t < t'$ and $\langle \mathbf{T},\mathbf{I}\rangle, V \vDash_{\text{Ind}} \alpha\ [t', [h]^{\mathbf{I}}_{t'}]$
$\mathbf{I}_4$: $\langle \mathbf{T},\mathbf{I}\rangle, V \vDash_{\text{Ind}} \mathbf{f}\alpha$ [t, i]	iff	$\exists h \in i, \exists t' \in h$: $t < t'$ and $\langle \mathbf{T},\mathbf{I}\rangle, V \vDash_{\text{Ind}} \alpha\ [t', [h]^{\mathbf{I}}_{t'}]$
$\mathbf{I}_5$: $\langle \mathbf{T},\mathbf{I}\rangle, V \vDash_{\text{Ind}} P\alpha\ [t, i]$	iff	$\exists t' < t : \langle \mathbf{T},\mathbf{I}\rangle, V \vDash_{\text{Ind}} \alpha\ [t', [h]^{\mathbf{I}}_{t'}]$
$\mathbf{I}_6$: $\langle \mathbf{T},\mathbf{I}\rangle, V \vDash_{\text{Ind}} \Diamond\alpha\ [t, i]$	iff	$\exists i' \in \mathbf{I}_t : \langle \mathbf{T},\mathbf{I}\rangle, V \vDash_{\text{Ind}} \alpha\ [t, i']$

We observed above that the second-order quantification in Rule $\mathbf{P}_4$ is equivalent to a first-order quantification moments. This does not happen for the quantification over histories in $\mathbf{I}_4$ because this quantification is restricted to the elements of the recognized possibility i. The quantification over $\mathbf{I}_t$ in $\mathbf{I}_6$, instead, can be replaced by a quantification over $\mathbf{H}_t$: the right side of this rule is equivalent to $\exists h \in \mathbf{H}_t : \mathbf{T}, V \vDash_{\text{Ock}} \alpha\ [t, [h]^{\mathbf{I}}_t]$. Like in the case of Ockhamist semantics, the truth of $\Diamond\alpha$, or of $\Box\alpha$ at $\langle t, i\rangle$ does not depend on i.

The set $T^{\mathbf{I}}$ of recognized possibilities in an **I**-tree can be endowed with an order relation and with an equivalence relation in a natural way. We set

$$\langle t, i\rangle \prec \langle \tau, j\rangle \overset{\text{Def}}{\equiv} t < \tau \text{ and } j \subseteq i$$
$$\langle t, i\rangle \sim \langle \tau, j\rangle \overset{\text{Def}}{\equiv} t = \tau \tag{4.3}$$

The following proposition is a straightforward consequence of Definition 4.1.

Proposition 4.2 For every **I**-tree $\langle \mathbf{T},\mathbf{I}\rangle$,
(1) $\prec$ *is a tree relation on* $T^{\mathbf{I}}$; *and*
(2) if $\langle t, i\rangle \sim \langle \tau, j\rangle$, *then the restriction of* $\sim$ *to* $\{\langle t', i'\rangle : \langle t', i'\rangle \prec \langle t, i\rangle\} \times \{\langle \tau', j'\rangle : \langle \tau', j'\rangle \prec \langle \tau, j\rangle\}$ *is an order isomorphism*.

The tree $\langle T^{\mathbf{I}}, \prec\rangle$ will be denoted by $\mathbf{T}^{\mathbf{I}}$. On the basis of Proposition 4.2, the evaluation rules $\mathbf{I}_0$ to $\mathbf{I}_6$ can be rewritten as evaluation rules in structures $\langle T^{\mathbf{I}}, \prec, \sim\rangle$. It turns out that the semantics for the operators **F**, **f** and *P* in these structures is just Peircean semantics in the tree $\mathbf{T}^{\mathbf{I}}$, while ◊ is interpreted as the modal S5-operator with accessibility relation ∼.

An interesting peculiarity of the **I**-tree semantics is that it can be viewed as a unified framework having Ockhamist and Peircean semantics as limit cases (Zanardo, 1998). If in fact it is the total relation for any *t*, that is $\mathbf{I}_t(h, h')$ for all $h, h' \in \mathbf{H}_t$, then every It contains only one equivalence class which is $\mathbf{H}_t$. In this case the map $\varphi : \langle t, i\rangle \to t$ is an isomorphism from $\mathbf{T}^{\mathbf{I}}$ onto **T**, and the equivalence ∼ is equality. This means in particular that **F** and **f** have the same meaning as in Peircean logic and that the possibility operator is vacuous, i.e. $\langle \mathbf{T},\mathbf{I}\rangle \vDash_{\mathrm{Ind}} \alpha \equiv \Diamond\alpha$ for all formulas α.

If conversely It is the diagonal relation for all *t*, that is $\mathbf{I}_t(h, h')$ iff $h = h'$, then every class in It contains exactly one history. In particular, the operators **F** and **f** coincide - i.e. $\langle \mathbf{T},\mathbf{I}\rangle \vDash_{\mathrm{Ind}} \mathbf{f}\alpha \equiv \mathbf{F}\alpha$ for all α - and ◊ has the same meaning as in Ockhamist logic. In this case $\mathbf{T}^{\mathbf{I}}$ is the union of disjoint linear orders, and the relation ∼ renders the structure an *Ockhamist frame* (Zanardo, 1985; Zanardo, 1996) or, with some minor differences, a *Leibnizian structure*, in the terminology of (Øhrstrøm and Hasle, 1995).

The properties of **I**-trees have of course a topological counterpart. In particular, the topology $\tau_{\mathbf{TI}}$ is a refinement of $\tau_{\mathbf{T}}$ (see Subsection 3.1). In the two limit cases considered above we have that $\tau_{\mathbf{TI}}$ is not a proper refinement when **T** and $\mathbf{T}^{\mathbf{I}}$ are isomorphic, while, in the other case, $\tau_{\mathrm{T}}{}^{\mathbf{I}}$ is the discrete topology.

5. CHOICES

The general framework provided by the **I**-tree semantics allows us to deal with the particular case in which possibility is meant as possibility (for a given agent, at a given moment) of choosing among different alternatives. The notion of choice we consider here is the one involved in Belnap’s logic of agency, as well as von Kutschera’s logic of causation.

A *choice function* for an agent a in a tree-like representation of Time is a function C_a assigning a partition $C_{a,t}$ of $\mathbf{H}_t$ to each moment *t*. In (Belnap et al., 2001, p. 34) we read (using the notation of the present paper)

> *...the idea is that, by acting at t, the agent a is able to determine a particular one of the equivalence classes from* $C_{a,t}$ *within which the future course of history must then lie, but this is the extent of his influence.*

The elements of $C_{a,t}$ can be thought of as the ways in which the world goes on, depending on *a*’s actions. Thus, if *a* decides to spend the week-end at home, and *this is really a choice allowed to him*, then he constraints the course of events to lie in an element of $C_{a,t}$ which contains only histories in which *a* is at home in the week-end. One of the requests on the partitions $C_{a,t}$ corresponds to the idea that no choice of *a* at *t* can distinguish two histories that are undivided at *t*:

$$\text{if } \mathrm{U}_t(h, h') \text{ and } h \in X \in C_{a,t} \text{ then } h' \in X \qquad (5.4)$$

The other property that choice functions must have in the context of logic of agency is significant when multiple agents are considered. For any set A of agents and any $X_a \in C_{a,t}$ ($a \in A$), $\cap_{a \in A} X_a \neq \emptyset$. This property is discussed in (Belnap et al., 2001, Sect. 7C.4) where it is called *Independence of Agents*.[13] As observed in Footnote 12, considering more than one agent is beyond the goals of the present paper; thus, in the sequel, we denote choice functions by C and the index a is suppressed.

Choice functions are particular indistinguishability functions. If in fact $C_t(h, h')$ and $t' < t$, then h and h' are undivided at t' and hence, by (5.4), $C_{t'}(h, h')$ holds as well. On the other hand, if the indistinguishability function I contains undividedness, then it is a trivially a choice function, and hence choice functions are precisely the indistinguishability functions that contain undividedness. The following proposition provides a characterization of the **I**-trees with this property.

Proposition 5.1 (Proposition 4.2 in (Zanardo, 1998)) *For every* **I***-tree* $\langle \mathbf{T}, \mathbf{I} \rangle$

$$\langle \mathbf{T}, \mathbf{I} \rangle \vDash_{\mathrm{Ind}} Pp \to \Box Pp \Leftrightarrow$$
$$\Leftrightarrow \forall t \in \mathbf{T}, \forall h, h' \in \mathbf{H}_t, U_t(h, h') \to \mathbf{I}_t(h, h')$$

It is interesting to observe that this characterization of choice functions involves the formula $Pp \to \Box Pp$ which expresses the unpreventability of the past: if something happened, then it is necessary (now unpreventable) that it happened.

In Ockhamist logic, the formula $P\alpha \to \Box P\alpha$ is valid when α is constructed from formulas of the form $\Box\alpha_1, \ldots, \Box\alpha_n$ without any use of the operator F. If propositional variables are evaluated at sets of moments (see Footnote 7), then the equivalences $p_i \equiv \Box p_i$ are Ockhamist validities, so that the formula $Pp_n \to \Box Pp_n$ is valid as well. Also in this case, though, it is not difficult to find counterexamples to the Ockhamist validity of, e.g., $PFp \to \Box PFp$.

Adopting the **I**-tree semantics, instead, the assumption that **I** is a choice function (that is $U_t \subseteq I_t$ for all t) guarantees that $P_\alpha \to \Box P\alpha$ is true for any formula α, possibly containing future operators. I think that this unexpected relation between the notion of choice in branching-time and unrestricted formulation of unpreventability of the past is rather intriguing and deserves further investigations.

The following proposition shows that the case in which indistinguishability is contained in undividedness is definable as well. Then, we can characterize the particular case in which indistinguishability is exactly undividedness.

Proposition 5.2 *For every* **I***-tree* $\langle \mathbf{T}, \mathbf{I} \rangle$,

$$\langle \mathbf{T}, \mathbf{I} \rangle \vDash_{\mathrm{Ind}} \mathbf{f}p \wedge \mathbf{g}\neg p \to \mathbf{F}(\Diamond \mathbf{g}\neg p \wedge \Diamond(p \vee \mathbf{f}p)) \Leftrightarrow$$

[13] In many works on Belnap's theory of agency this property is expressively described as "something happens".

$$\Leftrightarrow \forall t \in \mathbf{T}, \forall h, h' \in \mathbf{H}_t, \mathbf{I}_t(h, h') \to \mathrm{U}_t(h, h')$$

Proof. Assume the right side of ⇔ and $\langle \mathbf{T},\mathbf{I}\rangle, V \vDash_{\mathrm{Ind}} \mathbf{f}p \wedge \mathbf{g}\neg p\ [t, i]$ for some evaluation V, moment t, and $i \in \mathbf{I}_t$. This means that there are two histories $h_1, h_2 \in i$ such that, for some $t_1 > t$ in h_1, and for all $t' > t$ in h_2,

$$(*)\ \langle \mathbf{T}, \mathbf{I}\rangle, V \vDash_{\mathrm{Ind}} p\ [t_1, [h_1]^{\mathbf{I}}_{t'}] \text{ and } (**)\ \langle \mathbf{T}, \mathbf{I}\rangle, V \vDash_{\mathrm{Ind}} \neg p\ [t, [h_2]^{\mathbf{I}}_{t'}]$$

Let h be any history in the recognized possibility i; then $\mathbf{I}_t(h, h_1)$ and $\mathbf{I}_t(h, h_2)$. Since we are assuming that indistinguishability is contained in undividedness, there is a moment $t_0 \in h \cap h_1 \cap h_2$ such that $t < t_0$.

Two cases can be considered: either t has an immediate successor in $h \cap h_1 \cap h_2$, or t has no immediate successor in $h \cap h_1 \cap h_2$. In both cases we can assume that $t_0 \leq t_1$. By (∗), $\langle \mathbf{T},\mathbf{I}\rangle, V \vDash_{\mathrm{Ind}} p \vee \mathbf{f}p\ [t_0, [h_1]^{\mathbf{I}}_{t0}]$, and, by (∗∗), we have also $\langle \mathbf{T}, \mathbf{I}\rangle, V \vDash_{\mathrm{Ind}} \mathbf{g}\neg p\ [t_0, [h_2]^{\mathbf{I}}_{t0}]$. Then $\langle \mathbf{T}, \mathbf{I}\rangle, V \vDash_{\mathrm{Ind}} \Diamond \mathbf{g}\neg p \wedge \Diamond (p \vee \mathbf{f}p)[t_0, [h]^{\mathbf{I}}_{t0}]$. Since h is an arbitrary element of i, this implies that $\langle \mathbf{T},\mathbf{I}\rangle, V \vDash_{\mathrm{Ind}} \mathbf{F}(\Diamond \mathbf{g}\neg p \wedge \Diamond (p \vee \mathbf{f}p))\ [t, i]$. This concludes the first part of the proof. Conversely, assume that there exist t_0, h_1, h_2 in $\langle \mathbf{T},\mathbf{I}\rangle$ such that $\mathbf{I}_{t0}(h_1, h_2)$, but not-$\mathrm{U}_{t0}(h_1, h_2)$. Consider any evaluation V such that

$$V(p) = \{(t, [h]^{\mathbf{I}}) : h \in [h_1]^{\mathbf{I}}_{t0} \cap [h]^{\mathbf{U}}_{t0} \text{ and } t_0 < t\}$$

This implies that $\langle \mathbf{T}, \mathbf{I}\rangle, V \vDash_{\mathrm{Ind}} \mathbf{f}p \wedge \mathbf{g}\neg p\ [t_0, [h_2]^{\mathbf{I}}_{t0}]$. Consider any moment t such that $t \geq t'$ for some $t' > t_0$ in h_2. Since h_2 is not $\mathbf{U}_{t0}$-related to h_1, we have that, for every h passing through t, $\langle \mathbf{T}, \mathbf{I}\rangle, V \vDash_{\mathrm{Ind}} \neg p\ [t, [h]^{\mathbf{I}}_{t}]$. This implies in particular that $\langle \mathbf{T}, \mathbf{I}\rangle, V \vDash_{\mathrm{Ind}} \neg\Diamond(p \wedge \mathbf{f}p)[t', [h_2]^{\mathbf{I}}_{t'}]$ for every $t' > t_0$ in h_2. Then $\langle \mathbf{T}, \mathbf{I}\rangle, V \vDash_{\mathrm{Ind}} \mathbf{F}(\Diamond \mathrm{g}\neg \mathrm{p} \wedge \Diamond(p \vee \mathbf{f}p))\ [t_0, [h_2]^{\mathbf{I}}_{t0}]$. ■

6. CONCLUSIONS

We have considered various modal notions in branching-time contexts and in all cases possibility is viewed as existence of a suitable set of courses of events. In Ockhamist logic possibility is 'existence of (at least) one history', while, if undividedness is taken into account, possibility can be conceived as 'existence of an equivalence class modulo undividedness'. In these two perspectives, possibility can be defined on the basis of the set-theoretical structure of Time.

Some examples show that it makes sense to assume that, at any moment in Time, some histories cannot be distinguished from others. We can have various notions of indistinguishability depending on the context to which branching time logic is applied. This yields to consider tree-like structures endowed with (moment relative) indistinguishability relations, and to conceive possibility as existence of indistinguishability classes.

These enriched structures provide a unified semantics for branching-time logics as well as a general framework for dealing with choices. It is shown that the particular indistinguishability relations corresponding to choices and undividedness are definable relations.

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