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Liouville type results for periodic and almost periodic linear operators

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Abstract

This paper is concerned with some extensions of the classical Liouville theorem for bounded harmonic functions to solutions of more general equations. We deal with entire solutions of periodic and almost periodic parabolic equations including the elliptic framework as a particular case. We derive a Liouville type result for periodic operators as a consequence of a result for operators periodic in just one variable, which is new even in the elliptic case. More precisely, we show that if $c \leqslant 0$ and a_{ij}, b_i, c, f are periodic in the same space direction or in time, with the same period, then any bounded solution u of

$$\partial_t u - a_{ij}(x,t)\partial_{ij}u - b_i(x,t)\partial_i u - c(x,t)u = f(x,t), \quad x \in \mathbb{R}^N,\ t \in \mathbb{R},$$

is periodic in that direction or in time. We then derive the following Liouville type result: if $c \leqslant 0$, $f \equiv 0$ and a_{ij}, b_i, c are periodic in all the space/time variables, with the same periods, then the space of bounded solutions of the above equation has at most dimension one. In the case of the equation $\partial_t u - Lu = f(x,t)$, with L periodic elliptic operator independent of t, the hypothesis $c \leqslant 0$ can be weakened by requiring that the periodic principal eigenvalue λ_p of $-L$ is nonnegative. Instead, the periodicity assumption cannot be relaxed, because we explicitly exhibit an almost periodic function b such that the space of bounded solutions of $u'' + b(x)u' = 0$ in $\mathbb{R}$ has dimension 2, and it is generated by the constant solution and a non-almost periodic solution.

The above counterexample leads us to consider the following problem: under which conditions are bounded solutions necessarily almost periodic? We show that a sufficient condition in the case of the equation $\partial_t u - Lu = f(x,t)$ is: f is almost periodic and L is periodic with $\lambda_p \geqslant 0$.

Finally, we consider problems in general periodic domains under either Dirichlet or Robin boundary conditions. We prove analogous properties as in the whole space, together with some existence and uniqueness results for entire solutions.
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1. Introduction

1.1. Statement of the main results

We study the properties of bounded **entire solutions** – that is, solutions for all times – of the parabolic equation

$$Pu = 0, \quad x \in \mathbb{R}^N,\ t \in \mathbb{R}, \tag{1}$$

with

$$Pu = \partial_t u - a_{ij}(x,t)\partial_{ij} u - b_i(x,t)\partial_i u - c(x,t)u$$

(the convention is adopted for summation from 1 to N on repeated indices, and ∂_i, ∂_{ij} denote the space-directional derivatives). We want to find in particular conditions under which the Liouville property (LP) holds. In analogy with the classical result for harmonic functions, we say that the LP holds if the space of bounded solutions has at most dimension one.

In some statements, we will restrict ourselves to time-independent operators, that we write as $P = \partial_t - L$, with L a general elliptic operator in non-divergence form:

$$Lu = a_{ij}(x)\partial_{ij} u + b_i(x)\partial_i u + c(x)u.$$

The associated stationary solutions satisfy the elliptic equation $-Lu = 0$ in $\mathbb{R}^N$.

Our assumptions on the coefficients are: $a_{ij}, b_i \in L^\infty(\mathbb{R}^N \times \mathbb{R}) \cap UC(\mathbb{R}^N \times \mathbb{R})$ (where UC stands for uniformly continuous), $c \in L^\infty(\mathbb{R}^N \times \mathbb{R})$ and the matrix field $(a_{ij})_{i,j}$ is symmetric and uniformly elliptic, that is,

$$\forall t \in \mathbb{R},\ x, \xi \in \mathbb{R}^N, \quad \underline{a}|\xi|^2 \leqslant a_{ij}(x,t)\xi_i\xi_j \leqslant \bar{a}|\xi|^2,$$

for some constants $0 < \underline{a} \leqslant \bar{a}$. Let us mention that, in the case of elliptic equations, the uniform continuity of the b_i can be dropped. We will sometimes denote the generic space/time point $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ by $X \in \mathbb{R}^{N+1}$.

We consider in particular operators with *periodic* and *almost periodic* coefficients. We say that a function $\phi : \mathbb{R}^{N+1} \to \mathbb{R}$ is periodic in the m-th variable, $m \in \{1, \dots, N+1\}$, with period $l_m > 0$, if $\phi(X + l_m e_m) = \phi(X)$ for $X \in \mathbb{R}^{N+1}$, where $(e_1, \dots, e_{N+1})$ denotes the canonical basis of $\mathbb{R}^{N+1}$. If ϕ is periodic in all the variables we simply say that it is periodic, with period $(l_1, \dots, l_{N+1})$. A linear operator is said to be periodic (resp. periodic in the m-th variable) if all its coefficients are periodic (resp. periodic in the m-th variable) **with the same period**.

The crucial step to prove the LP consists in showing that the periodicity of the operator P and of the function f is inherited by bounded solutions[1] of

$$Pu = f(x,t), \quad x \in \mathbb{R}^N,\ t \in \mathbb{R}. \tag{2}$$

Unless otherwise specified, the function f is only assumed to be measurable.

Theorem 1.1. *Let u be a bounded solution of* (2)*, with P, f periodic in the m-th variable, with the same period l_m, and with $c \leqslant 0$. Then, u is periodic in the m-th variable, with period l_m.*

From the above result it follows in particular that if P and f do not depend on t and $c \leqslant 0$, then all bounded solutions of (2) are stationary, that is, constant in time. Another consequence of Theorem 1.1 is that if P and f are periodic (in all the variables) with the same period, then all bounded solutions are periodic. In particular, they admit global maximum and minimum and then the strong maximum principle implies the LP.

[1] For us, a solution of a parabolic equation such as (2) is a function $u \in L^p_{loc}(\mathbb{R}^{N+1})$, for all $1 < p < \infty$, such that $\partial_t u, \partial_i u, \partial_{ij} u \in L^p_{loc}(\mathbb{R}^{N+1})$ and the equation holds a.e. We use an analogous definition for elliptic equations. In the sequel, we will omit to write a.e. for properties concerning measurable functions, and we will simply denote by inf and sup the ess inf and ess sup.

Corollary 1.2. *Let u be a bounded solution of* (1) *with P periodic and $c \leqslant 0$. Then, two possibilities occur*:

(1) $c \equiv 0$ *and u is constant*;
(2) $c \not\equiv 0$ *and* $u \equiv 0$.

Clearly, without the assumption $c \leqslant 0$ the LP no longer holds in general, even in the case of constant coefficients. As an example, the space of solutions of $-u'' + u = 0$ in $\mathbb{R}$ is generated by $u_1 = \sin x$ and $u_2 = \cos x$. However, if $P = \partial_t - L$, condition $c \leqslant 0$ in Corollary 1.2 is not necessary and can be relaxed by requiring that the *periodic principal eigenvalue* of $-L$ in $\mathbb{R}^N$ be nonnegative (cf. Theorem 1.3 below). Henceforth, $\lambda_p(-L)$ will always stand for the periodic principal eigenvalue of $-L$ in $\mathbb{R}^N$ and φ_p for the associated principal eigenfunction (see Section 2 for the definitions).

Theorem 1.3. *Let $P = \partial_t - L$, with L periodic with period $(l_1, \dots, l_N)$, and let f be periodic with period $(l_1, \dots, l_{N+1})$. If u is a bounded solution of* (2) *we have that:*

(i) *if $\lambda_p(-L) \geqslant 0$ then u is periodic, with period $(l_1, \dots, l_{N+1})$*;
(ii) *if $\lambda_p(-L) = 0$ and either $f \leqslant 0$ or $f \geqslant 0$ then $u \equiv k\varphi_p$, for some $k \in \mathbb{R}$, and $f \equiv 0$*;
(iii) *if $\lambda_p(-L) > 0$ and $f \equiv 0$ then $u \equiv 0$.*

In the particular case of stationary solutions, that is, solutions of the elliptic equation $Lu = 0$, statements (ii) and (iii) of Theorem 1.3 are contained in [21] (see the next section for further details). Theorem 1.3 part (iii) immediately implies the uniqueness of bounded solutions to (2). The existence result is also derived (cf. Corollary 2.2 below).

We next consider the problem of the validity of the LP if we relax the periodicity assumptions on a_{ij}, b_i, c and f. A natural generalization of periodic functions of a single real variable are *almost periodic* functions, introduced by Bohr [7]. This notion can be readily extended to functions of several variables through a characterization of **continuous** almost periodic functions due to Bochner [6].

Definition 1.4. We say that a function $\phi \in C(\mathbb{R}^{N+1})$ is almost periodic (a.p.) if from any arbitrary sequence $(X_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^{N+1}$ can be extracted a subsequence $(X_{n_k})_{k\in\mathbb{N}}$ such that $(\phi(X + X_{n_k}))_{k\in\mathbb{N}}$ converges uniformly in $X \in \mathbb{R}^{N+1}$.

It is straightforward to check that continuous periodic functions are a.p. (this is no longer true if we drop the continuity assumption). We say that a linear operator is a.p. if its coefficients are a.p.

By explicitly constructing a counterexample, we show that the Liouville type result of Corollary 1.2 does not hold in general – even for elliptic equations – if we require the operator to be only a.p.

Counterexample 1. There exists an a.p. function $b : \mathbb{R} \to \mathbb{R}$ such that the space of bounded solutions to

$$u'' + b(x)u' = 0 \quad \text{in } \mathbb{R} \tag{3}$$

has dimension 2, and it is generated by the function $u_1 \equiv 1$ and a function u_2 which is not a.p.

This also shows that bounded solutions of a.p. equations with nonpositive zero order term may not be a.p., in contrast with what happens for the periodicity (cf. Theorem 1.1). Actually, the function b in Counterexample 1 is *limit periodic*, that is, it is the uniform limit of a sequence of continuous periodic functions (see Definition 3.1 below). Limit periodic functions are a subset of a.p. functions because, as it is easily seen from Definition 1.4, the space of a.p. functions is closed with respect to the L^∞ norm (see e.g. [2,14]).

Next, we look for sufficient conditions under which all bounded solutions of (2) are necessarily a.p. Under the additional assumption $c \in C(\mathbb{R}^N)$, we derive the following.

Theorem 1.5. *Let $P = \partial_t - L$ with L periodic and let f be a.p. If u is a bounded solution of* (2) *we have that:*

(i) *if $\lambda_p(-L) \geqslant 0$ then u is a.p.*;
(ii) *if $\lambda_p(-L) = 0$ and either $f \leqslant 0$ or $f \geqslant 0$ then $u \equiv k\varphi_p$, for some $k \in \mathbb{R}$, and $f \equiv 0$.*

In the above statement, we require c to be continuous because, in the proof, we will make use of the fact that it is in particular a.p. Actually, using some weak compactness arguments, one can check that the continuity assumption on c could be removed.

Lastly, we prove analogous results to Theorems 1.1, 1.3 and Corollary 1.2 for equations in general periodic domains, under either Dirichlet or Robin boundary conditions. The analogue of Theorem 1.1 holds, in the case of Dirichlet boundary conditions, for domains periodic just in the direction e_m, whereas under Robin conditions we are able to prove the result only for domains periodic in all the directions. The Liouville type result in the Dirichlet case is stronger than in the whole space (Corollary 1.2) and it is actually a uniqueness result. An existence result is also obtained using a sub and supersolution method. In some of the statements for general domains, we require that the coefficients of the operator are Hölder continuous because we need some gradient estimates near the boundary.

1.2. A brief survey of the related literature

Starting from the end of the 50s, the classical Liouville theorem has been improved to the self-adjoint elliptic equation

$$\partial_i\big(a_{ij}(x)\partial_j u\big)+c(x)u=0 \quad \text{in } \mathbb{R}^N. \tag{4}$$

In the case $c\equiv 0$ (without any periodicity assumption on a_{ij}), the LP follows directly from the estimate on the oscillation of weak solutions proved by De Giorgi in the celebrated paper [13]. Another classical way to derive the LP is by applying the Harnack inequality in the balls B_r, provided that one can bound the constants uniformly with respect to r. This has been done by Gilbarg and Serrin [16] for the equation $a_{ij}(x)\partial_{ij}u=0$, with $a_{ij}(x)$ converging to constants as $|x|\to\infty$. Analogous Liouville type results can be derived in the parabolic case using the same type of estimates (see e.g. [22]). The case $c\not\equiv 0$ has been treated in many papers, using different techniques, such as probabilistic methods, semigroup or potential theory. With a purely pde approach, it is proved by Brezis, Chipot and Xie in the recent work [8], that the LP holds for (4) in the following cases: $N\leqslant 2$ and $c\leqslant 0$; $N>2$, $c\leqslant 0$ and $c(x)\leqslant -c_0(x)$ for $|x|$ large, with either $c_0(x)=C|x|^{-\beta}$, $C>0$, $\beta>2$, or c_0 nonnegative nontrivial periodic function. General nonvariational operators such as L defined in Section 1.1 are also considered in [8] and the LP is derived for the equation

$$Lu=0 \quad \text{in } \mathbb{R}^N, \tag{5}$$

when $c\leqslant 0$ and $c(x)\leqslant -C|x|^{-2}$, $\sum_{i=1}^N b_i(x)x_i\leqslant C$, for some $C>0$ and $|x|$ large. For the parabolic case, Hile and Mawata proved in [18] that the LP holds for a class of quasilinear equations satisfying some conditions at infinity. Their result applies in particular to Eq. (1) when $c\leqslant 0$, $(a_{ij}(X))_{i,j}\to$ identity and $b_i(X), c(X)\to$ constant, with a suitable rate, as $|X|\to\infty$.

Some authors treated the problem of the existence and uniqueness of nonnegative bounded solutions to linear elliptic equations in divergence form from the point of view of the criticality property of the operator. Starting from the ideas of Agmon [1] and Pinchover, and combining analytic and probabilistic techniques (such as the Martin representation theorem) Pinsky showed in [30] that if L is a periodic operator satisfying $\lambda_p(-L)=0$, then the unique (up to positive multiples) positive bounded solution of (5) is φ_p (see [29] for an extensive treatment of the subject). This result is a particular case of Theorem 1.3 part (ii) above and, since when $c\equiv 0$ there is no difference between studying bounded solutions and positive bounded solutions, it contains the case (1) of Corollary 1.2 when one restricts it to elliptic equations (except for the fact that some stronger regularity assumptions are required in [30]).

Another related topic is that of the characterization of polynomially growing solutions of periodic equations in the whole space. We stress out that the LP – as intended in the present paper – is obtained as a particular case considering polynomials of degree zero. In that framework, using some homogenization techniques, Avellaneda and Lin [3], and later Moser and Struwe [27], proved that the LP holds for (4) if $c\equiv 0$ and the a_{ij} are periodic (with the same period). The results of [3] and [27] have been improved by Kuchment and Pinchover [21] to the general non-self-adjoint elliptic equation (5), with L periodic and a_{ij}, b, c smooth (see also Li and Wang [23] for the case a_{ij} measurable and b_i, $c\equiv 0$). The restriction to bounded solutions of Theorem 28 part 3 in [21] is equivalent to the restriction to stationary solutions of statements (ii) and (iii) of Theorem 1.3 here. However, the method used in [21] – based on the Floquet theory – is quite involved and it is not clear to us whether or not it adapts to operators with non-smooth coefficients.

To our knowledge, no results about operators periodic in just one variable, such as Theorem 1.1, have been previously obtained.

For nonlinear operators, the LP simply refers to uniqueness of bounded (sometimes nonnegative) solutions in unbounded domains. The following works – amongst many others – deal with this subject in the elliptic case: [15,4,24] (semilinear operators, see also [5] for the parabolic case), [12] (quasilinear operators), [9,11,10,32] (fully nonlinear operators).

There is a vast literature on the problem of almost periodicity of bounded solutions of linear equations with a.p. coefficients (see e.g. [2,14,28,20]). Usually ordinary differential equations or systems are considered, often of the first order. As emphasized in [26], some authors made use in proofs of the claim that any bounded solution in $\mathbb{R}$ of a second order linear elliptic equation with a.p. coefficients has to be a.p. This claim is false, as shown by Counterexample 1 and also by a counterexample in [26]. There, the authors constructed an a.p. function $c(x)$ such that the equation

$$u'' + c(x)u = 0 \quad \text{in } \mathbb{R},$$

admits bounded solutions which are not a.p. In their case, the space of bounded solutions has dimension one and then the LP holds. They also addressed the following open question: if every solution of a linear equation in $\mathbb{R}$ with a.p. coefficients is bounded are all solutions necessarily a.p.? Counterexample 1 shows that the answer is no. A negative answer was also given in [19], where the authors exhibit a class of linear ordinary differential equations of order $n \geqslant 2$ for which all solutions are bounded in $\mathbb{R}$, yet no nontrivial solution is a.p. Thus, this also provides an example where the LP does not hold, but it is not interesting in this sense because the zero order term considered there is not nonpositive.

1.3. Organization of the paper

In Section 2, we consider the case when P and f are periodic and we prove Theorem 1.1, Corollary 1.2 and Theorem 1.3. In order to prove the periodicity of any bounded solution u, we show that the difference between u and its translation by one period is identically equal to 0. This is achieved by passing to a limit equation and making use of a supersolution v with positive infimum. We take $v \equiv 1$ in the case of Theorem 1.1 and $v \equiv \varphi_p$ in the case of Theorem 1.3. We further derive the existence and uniqueness of bounded entire solutions to (2) when $P = \partial_t - L$ and L is periodic and satisfies $\lambda_p(-L) > 0$.

Section 3 is devoted to the construction of the function b of Counterexample 1, which will be defined by an explicit recursive formula.

Theorem 1.5 is proved in Section 4. The basic idea to prove statement (i) is that, up to subsequences, all subsequences of a given sequence of translations of u converge to a solution of the same equation. Also, one can come back to the original equation by translating in the opposite direction. Then, the result follows from Theorem 1.3 parts (ii) and (iii).

In Sections 5, we derive results analogous to Theorems 1.1 and 1.3 for the Dirichlet and the Robin problems in periodic domains. There, the periodic principal eigenvalue $\lambda_p(-L)$ is replaced respectively by $\lambda_{p,D}(-L)$ (see Section 5.1) and $\lambda_{p,\mathcal{N}}(-L)$ (see Section 5.2) which take into account the boundary conditions. Existence and uniqueness results are presented as well.

2. The LP for periodic operators

Let us preliminarily recall the notion of periodic principal eigenvalue and eigenfunction. If L is periodic then the Krein Rutman theory yields the existence of a unique real number λ, called periodic principal eigenvalue of $-L$ (in $\mathbb{R}^N$), such that the eigenvalue problem

$$\begin{cases} -L\varphi = \lambda\varphi & \text{in } \mathbb{R}^N, \\ \varphi \text{ is periodic, with the same period as } L \end{cases}$$

admits positive solutions. Furthermore, the positive solution φ is unique up to a multiplicative constant, and it is called periodic principal eigenfunction. We denote by $\lambda_p(-L)$ and φ_p respectively the periodic principal eigenvalue and eigenfunction of $-L$.

The next lemma is the key tool to prove our results for periodic operators.

Lemma 2.1. *Assume that the operator P and the function f are periodic in the m-th variable, with the same period l_m. If there exists a bounded function v satisfying*

$$\inf_{\mathbb{R}^{N+1}} v > 0, \qquad Pv = \phi \quad \text{for } x \in \mathbb{R}^N,\ t \in \mathbb{R},$$

for some nonnegative function $\phi \in L^\infty(\mathbb{R}^{N+1})$, then any bounded solution u of (2) *is periodic in the m-th variable, with period l_m.*

Proof. Let u be a bounded solution of (2). Define the functions

$$\psi(X) := u(X + l_m e_m) - u(X), \qquad w(X) := \frac{\psi(X)}{v(X)}.$$

We want to show that they are nonpositive. Assume by way of contradiction that $k := \sup_{\mathbb{R}^{N+1}} w > 0$, and consider a sequence $(X_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^{N+1}$ such that $w(X_n) \to k$. Define the sequence of functions $(\psi_n)_{n\in\mathbb{N}}$ by $\psi_n(X) := \psi(X + X_n)$. Since the ψ_n are uniformly bounded and satisfy

$$\partial_t \psi_n - a_{ij}(X + X_n)\partial_{ij}\psi_n - b_i(X + X_n)\partial_i \psi_n - c(X + X_n)\psi_n = 0 \quad \text{in } \mathbb{R}^{N+1}, \tag{6}$$

interior parabolic estimates together with the Rellich–Kondrachov compactness theorem (see e.g. [17, Chapter 7]) imply that (a subsequence of) the sequence $(\psi_n)_{n\in\mathbb{N}}$ converges locally uniformly in $\mathbb{R}^{N+1}$ to a bounded function ψ_∞ and that $\partial_t \psi_n \to \partial_t \psi_\infty$, $\partial_i \psi_n \to \partial_i \psi_\infty$, $\partial_{ij}\psi_n \to \partial_{ij}\psi_\infty$ weakly in $L^p_{loc}(\mathbb{R}^{N+1})$, for any $1 < p < \infty$. Let $\tilde{a}_{ij}, \tilde{b}_i$ be the locally uniform limits and $\tilde{c}$ be the weak limit in $L^p_{loc}(\mathbb{R}^{N+1})$ of a converging subsequence respectively of $a_{ij}(X + X_n)$, $b_i(X + X_n)$ and $c(X + X_n)$. Thus, passing to the weak limit in (6) we derive

$$\tilde{P}\psi_\infty = 0 \quad \text{in } \mathbb{R}^{N+1},$$

where

$$\tilde{P} := \partial_t - \tilde{a}_{ij}(X)\partial_{ij} - \tilde{b}_i(X)\partial_i - \tilde{c}(X).$$

Analogously, the functions $v(X + X_n)$ converge (up to subsequences) locally uniformly in $\mathbb{R}^{N+1}$ to a function v_∞ satisfying

$$\inf_{\mathbb{R}^{N+1}} v_\infty > 0, \qquad \tilde{P}v_\infty \geqslant 0 \quad \text{in } \mathbb{R}^{N+1}.$$

The function $w_\infty := \psi_\infty / v_\infty$ reaches its maximum value k at 0. Moreover,

$$0 = \frac{\tilde{P}\psi_\infty}{v_\infty} = \partial_t w_\infty - \tilde{M} w_\infty + \frac{\tilde{P}v_\infty}{v_\infty} w_\infty \quad \text{in } \mathbb{R}^{N+1},$$

where the operator $\tilde{M}$ is defined by

$$\tilde{M} := \tilde{a}_{ij}\partial_{ij} + \big(2v_\infty^{-1}\tilde{a}_{ij}\partial_j v_\infty + \tilde{b}_i\big)\partial_i.$$

Since the term $(\tilde{P}v_\infty)/v_\infty$ is nonnegative, we can apply the parabolic strong maximum principle to the function w_∞ (see [31] for the smooth case and [22,25] for the case of strong solutions) and derive

$$\forall x \in \mathbb{R}^N,\ t \leqslant 0, \quad w_\infty(x,t) = k.$$

Using a diagonal method, we can find a subsequence of $(X_n)_{n\in\mathbb{N}}$ (that we still call $(X_n)_{n\in\mathbb{N}}$) and a sequence $(\zeta_h)_{h\in\mathbb{N}}$ in $[0, \sup u]$ such that

$$\forall h \in \mathbb{N}, \quad \lim_{n\to\infty} u(-hl_m e_m + X_n) = \zeta_h.$$

As a consequence,

$$\forall h \in \mathbb{N}, \quad \zeta_h - \zeta_{h+1} = \psi_\infty\big(-(h+1)l_m e_m\big) = k v_\infty\big(-(h+1)l_m e_m\big),$$

and then $\lim_{h\to\infty}\zeta_h=-\infty$: contradiction. We have shown that $w\leqslant 0$, that is, $u(X+l_m e_m)\leqslant u(X)$ for $X\in\mathbb{R}^N$. The opposite inequality can be obtained following the same arguments, with l_m replaced by $-l_m$. This time, the contradiction reached is that the sequence $(\zeta_h)_{h\in\mathbb{N}}$ as defined above tends to $+\infty$ as $h\to\infty$. $\square$

Proof of Theorem 1.1. Apply Lemma 2.1 with $v\equiv 1$. $\square$

Proof of Corollary 1.2. If u is a bounded solution to (1) then Theorem 1.1 implies that u is periodic (in all the variables). In particular, it attains its maximum M and minimum m in $\mathbb{R}^{N+1}$ at some points (x_M,t_M) and (x_m,t_m) respectively. Hence, the strong maximum principle implies that if $M\geqslant 0$ then $u(x,t)=M$ for $t\leqslant t_M$ and $x\in\mathbb{R}^N$, otherwise $u(x,t)=m$ for $t\leqslant t_m$ and $x\in\mathbb{R}^N$. Therefore, u is constant because it is periodic in t. The statement then follows. $\square$

Proof of Theorem 1.3. (i) The function $v(x,t):=\varphi_p(x)$ is bounded and satisfies

$$\inf_{\mathbb{R}^{N+1}} v>0,\qquad Pv=\phi\quad\text{for } x\in\mathbb{R}^N,\ t\in\mathbb{R},$$

with $\phi=\lambda_p(-L)\varphi_p\geqslant 0$. Hence, the statement is a consequence of Lemma 2.1.

(ii) Up to replacing u with $-u$, it is not restrictive to assume that $f\leqslant 0$. Set

$$k:=\sup_{\substack{x\in\mathbb{R}^N\\ t\in\mathbb{R}}}\frac{u(x,t)}{\varphi_p(x)}.$$

Since u is periodic by (i) – with the same space period $(l_1,\dots,l_N)$ as φ_p – it follows that there exists $X_0\in[0,l_1)\times\cdots\times[0,l_{N+1})$ where the nonnegative function $w(x,t):=k\varphi_p(x)-u(x,t)$ vanishes. Furthermore,

$$Pw=k\lambda_p(-L)\varphi_p-f\geqslant 0\quad\text{in }\mathbb{R}^{N+1}.$$

Therefore, the strong maximum principle and the time-periodicity of w yield $w\equiv 0$, that is, $u\equiv k\varphi_p$ and $f\equiv 0$.

(iii) Suppose that $\sup u\geqslant 0$ (otherwise replace u with $-u$). Proceeding as in (ii), one can find a constant $k\geqslant 0$ such that the periodic function $w(x,t):=k\varphi_p(x)-u(x,t)$ is nonnegative, vanishes at some point $X_0\in\mathbb{R}^{N+1}$ and satisfies $Pw=k\lambda_p(-L)\varphi_p\geqslant 0$. Once again, the strong maximum principle implies $w\equiv 0$. Hence, $k\lambda_p(-L)\varphi_p\equiv 0$, that is, $k=0$ and then $u\equiv 0$. $\square$

Remark 1. If L is periodic and $c\equiv 0$ then $\lambda_p(-L)=0$, with $\varphi_p\equiv 1$. If $c\leqslant 0$, $c\not\equiv 0$, then $\lambda_p(-L)>0$, as it is easily seen by applying the strong maximum principle to φ_p. Hence, in the case $P=\partial_t-L$, Corollary 1.2 is contained in Theorem 1.3 parts (ii) and (iii). Furthermore, the existence and uniqueness result of Corollary 2.2 below apply when $c\leqslant 0$, $c\not\equiv 0$.

By Theorem 1.3 part (iii), $\lambda_p(-L)>0$ implies the uniqueness of bounded entire solutions of $\partial_t u-Lu=f$. Indeed, it is also a sufficient condition for the existence when f is bounded.

Corollary 2.2. *If $P=\partial_t-L$, with L periodic such that $\lambda_p(-L)>0$, and $f\in L^\infty(\mathbb{R}^{N+1})$ then* (2) *admits a unique bounded solution.*

Proof. A standard method to construct entire solutions in the whole space is to consider the limit as $r\to\infty$ of solutions u_r of (for instance) the Dirichlet problems

$$\begin{cases} Pu_r=f(x,t), & x\in B_r,\ t\in(-r,r),\\ u_r=0, & x\in\partial B_r,\ t\in(-r,r),\\ u_r=0, & x\in B_r,\ t=-r \end{cases}\tag{7}$$

(here, B_r denotes the ball in $\mathbb{R}^N$ with radius r and centre 0). The so obtained solution is bounded provided that the family $(u_r)_{r>0}$ is uniformly bounded. This will follow from the strict positivity of $\lambda_p(-L)$. Define the function

$$v(x) := \frac{\|f\|_{L^\infty(\mathbb{R}^{N+1})}}{\lambda_p(-L)\min_{\mathbb{R}^N}\varphi_p}\varphi_p(x).$$

Since $-v$ and v are respectively a sub and a supersolution of (7), the parabolic comparison principle yields

$$\forall r>0, \quad -v \leqslant u_r \leqslant v \quad \text{in } B_r \times (-r,r).$$

Thus, using interior estimates and the embedding theorem, we can find a diverging sequence $(r_n)_{n\in\mathbb{N}}$ such that $(u_{r_n})_{n\in\mathbb{N}}$ converges locally uniformly in $\mathbb{R}^{N+1}$ to a bounded solution of (2). The uniqueness result is an immediate consequence of Theorem 1.3 part (iii). □

We remark that if f is periodic then one can prove the existence result of Corollary 2.2 by a standard functional method: after regularizing the operator in order to write it in divergence form, one considers the problem in the space of periodic functions and, owing to Theorem 1.3 part (iii), applies the Fredholm alternative to the inverse operator. Also, note that by Theorem 1.3 part (ii), the equation $\partial_t u - \partial_{xx} u = 1$ does not admit entire bounded solutions and then the hypothesis $\lambda_p(-L) > 0$ is sharp for the existence result of Corollary 2.2.

3. Counterexample when L is almost periodic

This section is devoted to the construction of Counterexample 1. Note that, by the uniqueness of solutions of the Cauchy problem, any non-constant solution of (3) must be strictly monotone.

We first construct a discontinuous function σ, then we modify it to obtain a Lipschitz continuous limit periodic function b. Let us recall the definition of limit periodic functions, which are a proper subset of a.p. functions.

Definition 3.1. We say that a function $\phi \in C(\mathbb{R}^N)$ is limit periodic if there exists a sequence of continuous periodic functions converging uniformly to ϕ in $\mathbb{R}^N$.

We start defining σ on the interval $(-1,1]$:

$$\sigma(x) = \begin{cases} -1 & \text{if } -1 < x \leqslant 0, \\ 1 & \text{if } 0 < x \leqslant 1. \end{cases}$$

Then in $(-3,3]$ setting

$$\forall x \in (-3,-1], \quad \sigma(x) = \sigma(x+2) - 1,$$
$$\forall x \in (1,3], \quad \sigma(x) = \sigma(x-2) + 1,$$

and, by iteration,

$$\forall x \in \left(-3^{n+1}, -3^n\right], \quad \sigma(x) = \sigma\left(x + 2\cdot 3^n\right) - \frac{1}{(n+1)^2}, \tag{8}$$

$$\forall x \in \left(3^n, 3^{n+1}\right], \quad \sigma(x) = \sigma\left(x - 2\cdot 3^n\right) + \frac{1}{(n+1)^2}. \tag{9}$$

By construction, the function σ satisfies $\|\sigma\|_\infty = 1 + \sum_{n=1}^\infty n^{-2}$, and it is odd except for the set $\mathbb{Z}$, in the sense that $\sigma(-x) = -\sigma(x)$ for $x \in \mathbb{R}\setminus\mathbb{Z}$.

Proposition 3.2. *There exists a sequence of bounded periodic functions $(\phi_n)_{n\in\mathbb{N}}$ converging uniformly to σ in $\mathbb{R}$ and such that*

$$\forall n \in \mathbb{N}, \quad \phi_n \in C(\mathbb{R}\backslash\mathbb{Z}), \quad \phi_n \text{ has period } 2\cdot 3^n.$$

Proof. Fix $n \in \mathbb{N}$. For $x \in (-3^n, 3^n]$ set $\phi_n(x) := \sigma(x)$, then extend ϕ_n to the whole real line by periodicity, with period $2\cdot 3^n$. We claim that

$$\|\sigma - \phi_n\|_\infty \leqslant \sum_{k=n+1}^\infty \frac{1}{k^2},$$

which would conclude the proof. We prove our claim by a recursive argument, showing that the property

$$(\mathcal{P}_i) \qquad \forall x \in \left(-3^{n+i}, 3^{n+i}\right], \quad \left|\sigma(x) - \phi_n(x)\right| \leqslant \sum_{k=n+1}^{n+i} \frac{1}{k^2}$$

holds for every $i \in \mathbb{N}$. Let us check $(\mathcal{P}_1)$. By (8) and (9) we get

$$\sigma(x) = \begin{cases} \sigma(x + 2 \cdot 3^n) - \frac{1}{(n+1)^2} & \text{if } -3^{n+1} < x \leqslant -3^n, \\ \phi_n(x) & \text{if } -3^n < x \leqslant 3^n, \\ \sigma(x - 2 \cdot 3^n) + \frac{1}{(n+1)^2} & \text{if } 3^n < x \leqslant 3^{n+1}. \end{cases}$$

Property $(\mathcal{P}_1)$ then follows from the periodicity of ϕ_n.

Assume now that $(\mathcal{P}_i)$ holds for some $i \in \mathbb{N}$. Let $x \in (-3^{n+i+1}, 3^{n+i+1}]$. If $x \in (-3^{n+i}, 3^{n+i}]$ then

$$\left|\sigma(x) - \phi_n(x)\right| \leqslant \sum_{k=n+1}^{n+i} \frac{1}{k^2} \leqslant \sum_{k=n+1}^{n+i+1} \frac{1}{k^2}.$$

Otherwise, set

$$y := \begin{cases} x + 2 \cdot 3^{n+i} & \text{if } x < 0, \\ x - 2 \cdot 3^{n+i} & \text{if } x > 0. \end{cases}$$

Note that $y \in (-3^{n+i}, 3^{n+i}]$ and $|x - y| = 2 \cdot 3^{n+i}$. Thus, (8), (9), $(\mathcal{P}_i)$ and the periodicity of ϕ_n yield

$$\begin{aligned} \left|\sigma(x) - \phi_n(x)\right| &\leqslant \left|\sigma(x) - \sigma(y)\right| + \left|\sigma(y) - \phi_n(y)\right| \\ &\leqslant \frac{1}{(n+i+1)^2} + \sum_{k=n+1}^{n+i} \frac{1}{k^2} \\ &= \sum_{k=n+1}^{n+i+1} \frac{1}{k^2}. \end{aligned}$$

This means that $(\mathcal{P}_{i+1})$ holds and then the proof is concluded. $\square$

Note that σ is not limit periodic because it is discontinuous on $\mathbb{Z}$.

Proposition 3.3. *The function σ satisfies*

$$\forall x \geqslant 1, \quad \int_0^x \sigma(t)\,dt \geqslant \frac{x}{2(\log_3 x + 1)^2}. \tag{10}$$

Proof. For $y \in \mathbb{R}$, define $F(y) := \int_0^y \sigma(t)\,dt$. Let us preliminarily show that, for every $n \in \mathbb{N}$, the following formula holds:

$$\forall y \in \left[0, 3^n\right], \quad F(y) \geqslant \frac{y}{2n^2}. \tag{11}$$

We shall do it by iteration on n. It is immediately seen that (11) holds for $n = 1$. Assume that (11) holds for some $n \in \mathbb{N}$. We want to prove that (11) holds with n replaced by $n + 1$. If $y \in [0, 3^n]$ then

$$F(y) \geqslant \frac{y}{2n^2} \geqslant \frac{y}{2(n+1)^2}.$$

If $y \in (3^n, 2 \cdot 3^n]$ then, by computation,

$$F(y) = F\left(2 \cdot 3^n - y\right) + \int_{2 \cdot 3^n - y}^{y} \sigma(t)\,dt \geqslant \frac{2 \cdot 3^n - y}{2n^2} + \int_{-(y-3^n)}^{y-3^n} \sigma\left(\tau + 3^n\right)d\tau.$$

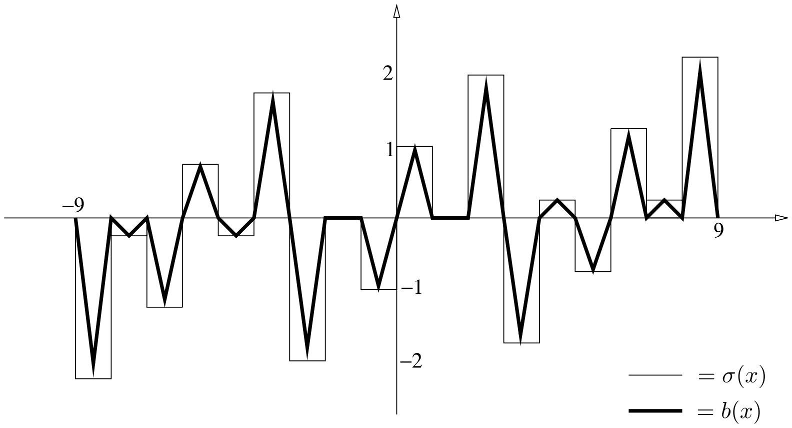

Fig. 1. Graphs of σ and b.

Using property (9), one sees that

$$\int_{-(y-3^n)}^{y-3^n} \sigma\left(\tau+3^n\right) d\tau = \int_{-(y-3^n)}^{0} \sigma\left(\tau+3^n\right) d\tau + \int_{0}^{y-3^n} \sigma\left(\tau-3^n\right) d\tau + \frac{y-3^n}{(n+1)^2}$$
$$= \frac{y-3^n}{(n+1)^2},$$

where the last equality holds because σ is odd except in the set $\mathbb{Z}$. Hence,

$$F(y) \geqslant \frac{2\cdot 3^n - y}{2n^2} + \frac{y-3^n}{(n+1)^2} \geqslant \frac{y}{2(n+1)^2}.$$

Let now $y \in (2\cdot 3^n, 3^{n+1}]$. Since $F(2\cdot 3^n) \geqslant 3^n(n+1)^{-2}$, as we have seen before, and (9) holds, it follows that

$$F(y) = F\left(2\cdot 3^n\right) + \int_{2\cdot 3^n}^{y} \sigma(t)\,dt \geqslant \frac{3^n}{(n+1)^2} + F\left(y-2\cdot 3^n\right) + \frac{y-2\cdot 3^n}{(n+1)^2}.$$

Using the hypothesis (11) we then get

$$F(y) \geqslant \frac{y-3^n}{(n+1)^2} + \frac{y-2\cdot 3^n}{2n^2} \geqslant \frac{y}{2(n+1)^2}.$$

We have proved that (11) holds for any $n \in \mathbb{N}$. Consider now $x \geqslant 1$. We can find an integer $n = n(x)$ such that $x \in [3^{n-1}, 3^n)$. Applying (11) we get $F(x) \geqslant x(2n^2)^{-1}$. Therefore, since $n \leqslant \log_3 x + 1$, we infer that

$$F(x) \geqslant \frac{x}{2(\log_3 x + 1)^2}. \qquad \square$$

In order to define the function b, we introduce the following auxiliary function $z \in C(\mathbb{R})$ vanishing on $\mathbb{Z}$: $z(x) := 2|x|$ if $x \in [-1/2, 1/2]$, and it is extended by periodicity with period 1 outside $[-1/2, 1/2]$. Then we set

$$b(x) := \sigma(x)z(x).$$

The definition of b is easier to understand by its graph (see Fig. 1).

Proposition 3.4. *The function b is odd and limit periodic.*

Proof. Let us check that b is odd. For $x \in \mathbb{Z}$ we find $b(-x) = 0 = -b(x)$, while, for $x \in \mathbb{R}\setminus\mathbb{Z}$,

$$b(-x) = \sigma(-x)z(-x) = -\sigma(x)z(x) = -b(x).$$

In order to prove that b is limit periodic, consider the sequence of periodic functions $(\phi_n)_{n\in\mathbb{N}}$ given by Proposition 3.2. Then define

$$\psi_n(x) := \phi_n(x)z(x).$$

Clearly, the functions ψ_n are continuous (because z vanishes on $\mathbb{Z}$) and periodic, with period $2 \cdot 3^n$ (because z has period 1). Also, for $n \in \mathbb{N}$,

$$|b - \psi_n| = |\sigma - \phi_n| z \leqslant |\sigma - \phi_n|.$$

Therefore, ψ_n converges uniformly to b as n goes to infinity. □

Proposition 3.5. *All solutions of* (3) *are bounded and they are generated by* $u_1 \equiv 1$ *and a non-a.p. function* u_2.

Proof. The two-dimensional space of solutions of (3) is generated by $u_1 \equiv 1$ and

$$u_2(x) := \int_0^x \exp\left(-\int_0^y b(t)\,dt\right) dy.$$

Since u_2 is strictly increasing, it cannot be a.p. So, to prove the statement it only remains to show that u_2 is bounded. By construction, it is clear that, for $m \in \mathbb{Z}$,

$$\int_0^m b(t)\,dt = \frac{1}{2}\int_0^m \sigma(t)\,dt.$$

Consequently, by (10), we get for $x \geqslant 1$

$$\int_0^x b(t)\,dt = \frac{1}{2}\int_0^{[x]} \sigma(t)\,dt + \int_{[x]}^x b(t)\,dt \geqslant \frac{x-1}{4(\log_3 x + 1)^2} - \|b\|_\infty$$

and then

$$\begin{aligned} 0 \leqslant u_2(x) &\leqslant e^{\|b\|_\infty} \int_0^x \exp\left(-\frac{y-1}{4(\log_3 y + 1)^2}\right) dy \\ &\leqslant e^{\|b\|_\infty} \int_0^{+\infty} \exp\left(-\frac{y-1}{4(\log_3 y + 1)^2}\right) dy. \end{aligned}$$

Since b is odd, it follows that u_2 is odd too and then it is bounded on $\mathbb{R}$. □

Remark 2. The function $b = \sigma z$ we have constructed before is uniformly Lipschitz continuous, with Lipschitz constant equal to $2\|\sigma\|_{L^\infty(\mathbb{R})}$. Actually, one could use a suitable C^∞ function instead of z in order to obtain a function $b \in C^\infty(\mathbb{R})$.

Remark 3. The reason why the LP fails to hold in the a.p. case is that, as shown by the previous counterexample, an a.p. linear equation with nonpositive zero order coefficient may admit non-a.p. bounded solutions in the whole space. Instead, the space of a.p. solutions of (1), with $c \leqslant 0$ and without any almost periodicity assumptions on L, has at most dimension one, that is, the LP holds if all bounded solutions are a.p. More precisely, the result of Corollary 1.2 holds true if one requires u to be a.p., even by dropping the periodicity assumption on L. To see this, consider an a.p. solution u of (1). Up to replacing u with $-u$, we can assume that $U := \sup u \geqslant 0$. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^{N+1}$ such that $u(X_n) \to U$. Then, up to subsequences, the functions $u_n(X) := u(X + X_n)$ converge locally uniformly in $X \in \mathbb{R}^{N+1}$ to a solution u_∞ of a linear equation $\tilde{P} = 0$ in $\mathbb{R}^{N+1}$, with nonpositive zero order term (see the arguments in the proof of Lemma 2.1). The strong maximum principle then yields $u_\infty \equiv U$ for $t \leqslant 0$. Since the convergence of a subsequence of u_n is also uniform in $\mathbb{R}^{N+1}$, by the almost periodicity of u, we find that $\lim_{t\to-\infty} u(x,t) = U$ uniformly in $x \in \mathbb{R}^N$. Arguing as in the proof of Theorem 1.5 part (ii) below, we infer that $u \equiv U$ and then the conclusion of Corollary 1.2 holds.

4. Sufficient conditions for the almost periodicity of solutions

Proof of Theorem 1.5. (i) consider an arbitrary sequence $(X_n)_{n\in\mathbb{N}} = ((x_n, t_n))_{n\in\mathbb{N}}$ in $\mathbb{R}^N \times \mathbb{R}$. Since a_{ij}, b_i, c and f are a.p. (because $a_{ij}, b_i, c \in C(\mathbb{R}^N)$ are periodic) there exists a subsequence of $(X_n)_{n\in\mathbb{N}}$ (that we still call $(X_n)_{n\in\mathbb{N}}$) such that $a_{ij}(x + x_n)$, $b_i(x + x_n)$, $c(x + x_n)$ and $f(x + x_n, t + t_n)$ converge uniformly in $x \in \mathbb{R}^N$, $t \in \mathbb{R}$. We claim that $u(X + X_n)$ converges uniformly in $X \in \mathbb{R}^{N+1}$ too. Assume by contradiction that this is not the case. Then, there exist $\varepsilon > 0$, a sequence $(Y_n)_{n\in\mathbb{N}} = ((y_n, \tau_n))_{n\in\mathbb{N}}$ in $\mathbb{R}^N \times \mathbb{R}$ and two subsequences $(X_n^1)_{n\in\mathbb{N}}$ and $(X_n^2)_{n\in\mathbb{N}}$ of $(X_n)_{n\in\mathbb{N}}$ such that

$$\forall n \in \mathbb{N}, \quad \left|u\left(Y_n + X_n^1\right) - u\left(Y_n + X_n^2\right)\right| > \varepsilon. \tag{12}$$

For $\sigma = 1, 2$ set $(Z_n^\sigma)_{n\in\mathbb{N}} := (Y_n + X_n^\sigma)_{n\in\mathbb{N}}$. Applying again the definition of almost periodicity, we can find a common sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that, for $\sigma = 1, 2$, the functions $f(X + Z_{n_k}^\sigma)$ converge to some functions f^σ uniformly in $X \in \mathbb{R}^{N+1}$. As $f(X + X_n)$ converges uniformly in $X \in \mathbb{R}^{N+1}$, we see that

$$\forall x \in \mathbb{R}^{N+1}, \quad f^1(X) = \lim_{k\to\infty} f\left(X + Y_{n_k} + X_{n_k}^1\right) = \lim_{k\to\infty} f\left(X + Y_{n_k} + X_{n_k}^2\right) = f^2(X).$$

Let $(\eta_k)_{k\in\mathbb{N}}$ be a sequence in $[0, l_1) \times \cdots \times [0, l_N)$ such that $y_{n_k} + x_{n_k}^1 - \eta_k \in \prod_{i=1}^N l_i\mathbb{Z}$ and let η be the limit of (a subsequence of) $(\eta_k)_{k\in\mathbb{N}}$. Owing to the periodicity and the uniform continuity of c, we get

$$c(x + \eta) = \lim_{k\to\infty} c(x + \eta_k) = \lim_{k\to\infty} c\left(x + y_{n_k} + x_{n_k}^1\right) = \lim_{k\to\infty} c\left(x + y_{n_k} + x_{n_k}^2\right),$$

uniformly in $x \in \mathbb{R}^N$. Analogously, for $\sigma = 1, 2$,

$$\lim_{k\to\infty} a_{ij}\left(x + y_{n_k} + x_{n_k}^\sigma\right) = a_{ij}(x + \eta), \qquad b_i\left(x + y_{n_k} + x_{n_k}^\sigma\right) = b_i(x + \eta),$$

uniformly with respect to $x \in \mathbb{R}^N$. By standard parabolic estimates and compact injection theorem, it follows that there exists a subsequence of $(n_k)_{k\in\mathbb{N}}$ (that we still call $(n_k)_{k\in\mathbb{N}}$) such that, for $\sigma = 1, 2$, the sequences $u_k^\sigma := u(\cdot + Z_{n_k}^\sigma)$ converge locally uniformly in $\mathbb{R}^{N+1}$ to some functions u^σ and $\partial_t u_k^\sigma \to \partial_t u^\sigma$, $\partial_i u_k^\sigma \to \partial_i u^\sigma$, $\partial_{ij} u_k^\sigma \to \partial_{ij} u^\sigma$ weakly in $L^p_{loc}(\mathbb{R}^{N+1})$, for $1 < p < \infty$. Passing to the weak limit in the equations satisfied by the u_k^σ, we infer that the u^σ satisfy

$$\inf_{\mathbb{R}^{N+1}} u \leqslant u^\sigma \leqslant \sup_{\mathbb{R}^{N+1}} u, \qquad \partial_t u^\sigma - L_\eta u^\sigma = f^1 \quad \text{in } \mathbb{R}^{N+1}, \tag{13}$$

where

$$L_\eta := a_{ij}(\cdot + \eta)\partial_{ij} + b_i(\cdot + \eta)\partial_i + c(\cdot + \eta).$$

Clearly, if φ_p is the periodic principal eigenfunction of $-L$, then $\varphi_p(\cdot + \eta)$ is the periodic principal eigenfunction of $-L_\eta$. This shows that $\lambda_p(-L_\eta) = \lambda_p(-L) \geqslant 0$. As $\partial_t(u^1 - u^2) - L_\eta(u^1 - u^2) = 0$ in $\mathbb{R}^{N+1}$, statements (ii) and (iii) of Theorem 1.3 yield

$$\forall x \in \mathbb{R}^N,\ t \in \mathbb{R}, \quad u^1(x,t) - u^2(x,t) = k\varphi_p(x + \eta), \tag{14}$$

for some $k \in \mathbb{R}$. In order to show that $k = 0$, we come back to the original equation. For $\sigma = 1, 2$, the following limits hold uniformly in $X = (x, y) \in \mathbb{R}^N \times \mathbb{R}$:

$$\lim_{k\to\infty} f^1\left(X - Z_{n_k}^\sigma\right) = \lim_{k\to\infty} f\left(\left(X - Z_{n_k}^\sigma\right) + Z_{n_k}^\sigma\right) = f(X),$$

$$\lim_{k\to\infty} a_{ij}\left(x + \eta - y_{n_k} - x_{n_k}^\sigma\right) = a_{ij}(x), \qquad \lim_{k\to\infty} b_i\left(x + \eta - y_{n_k} - x_{n_k}^\sigma\right) = b_i(x),$$

$$\lim_{k\to\infty} c\left(x + \eta - y_{n_k} - x_{n_k}^\sigma\right) = c(x).$$

Therefore, with usual arguments, we see that, for $\sigma = 1, 2$, $u^\sigma(\cdot - Z^\sigma_{n_k})$ converges (up to subsequences) locally uniformly to a function v^σ satisfying

$$\inf_{\mathbb{R}^{N+1}} u^\sigma \leqslant v^\sigma \leqslant \sup_{\mathbb{R}^{N+1}} u^\sigma, \qquad P v^\sigma = f \quad \text{in } \mathbb{R}^{N+1}. \tag{15}$$

Hence, $P(u - v^\sigma) = 0$ and then, again by Theorem 1.3 parts (ii)–(iii), there exists a constant $h^\sigma \in \mathbb{R}$ such that $u - v^\sigma \equiv h^\sigma \varphi_p$. Since $\inf_{\mathbb{R}^{N+1}} u \leqslant v^\sigma \leqslant \sup_{\mathbb{R}^{N+1}} u$ by (13) and (15), we infer that $h^1 = h^2 = 0$, that is, $v^1 \equiv v^2 \equiv u$. Consequently,

$$\inf_{\mathbb{R}^{N+1}} u^1 = \inf_{\mathbb{R}^{N+1}} u^2 = \inf_{\mathbb{R}^{N+1}} u, \qquad \sup_{\mathbb{R}^{N+1}} u^1 = \sup_{\mathbb{R}^{N+1}} u^2 = \sup_{\mathbb{R}^{N+1}} u,$$

and then, by (14), $u_1 \equiv u_2$. This is a contradiction because, by (12), $|u^1(0) - u^2(0)| \geqslant \varepsilon$.

(ii) Up to replacing u with $-u$, we can assume that $f \leqslant 0$. Set

$$k := \sup_{\substack{x \in \mathbb{R}^N \\ t \in \mathbb{R}}} \frac{u(x,t)}{\varphi_p(x)}$$

and $v(x,t) := k\varphi_p(x) - u(x,t)$. Thus, $v \geqslant 0$ and there exists a sequence $(X_n)_{n\in\mathbb{N}} = ((x_n, t_n))_{n\in\mathbb{N}}$ in $\mathbb{R}^N \times \mathbb{R}$ such that $\lim_{n\to\infty} v(X_n) = 0$. Arguing as above, we find that (up to subsequences) $v(\cdot + X_n)$ converges locally uniformly to a nonnegative function $\tilde{v}$ satisfying

$$\tilde{v}(0) = 0, \qquad \partial_t \tilde{v} - a_{ij}(\cdot + \eta)\partial_{ij}\tilde{v} - b_i(\cdot + \eta)\partial_i \tilde{v} - c(\cdot + \eta)\tilde{v} \geqslant 0 \quad \text{in } \mathbb{R}^{N+1},$$

for some $\eta \in [0, l_1) \times \cdots \times [0, l_N)$. Applying the strong maximum principle, we get $\tilde{v}(x,t) = 0$ for $x \in \mathbb{R}^N$, $t \leqslant 0$. As v is a.p. by part (i), we infer that $\lim_{n\to\infty} v(X + X_n) = \tilde{v}(X)$ uniformly with respect to $X \in \mathbb{R}^{N+1}$. Thus, $\lim_{t\to-\infty} v(x,t) = 0$ uniformly in $x \in \mathbb{R}^N$. Again by the almost periodicity, we can find a sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ tending to $-\infty$ and such that $v(x, t + t_n)$ converges uniformly with respect to $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. Since

$$\forall x \in \mathbb{R}^N,\ t \in \mathbb{R}, \quad \lim_{n\to\infty} v(x, t + t_n) = 0,$$

we derive $v \equiv 0$. □

Corollary 2.2 and Theorem 1.5 part (i) imply the existence of a unique a.p. solution of (2) when $P = \partial_t - L$, L is periodic, $\lambda_p(-L) > 0$ and f is a.p.

We conclude this section with a result concerning solutions of (2) when P is periodic and f is uniformly continuous (UC) and a.p. in just one variable, i.e. there exists $m \in \{1, \dots, N+1\}$ such that, for any $(X_1, \dots, X_{m-1}, X_{m+1}, \dots, X_{N+1}) \in \mathbb{R}^N$, $X_m \mapsto f(X_1, \dots, X_m, \dots, X_{N+1})$ is a.p.

Theorem 4.1. *Let u be a bounded solution of* (2), *with $f \in UC(\mathbb{R}^{N+1})$ a.p. in the m-th variable and either P periodic in the m-th variable, $c \leqslant 0$, or $P = \partial_t - L$, L periodic, $\lambda_p(-L) \geqslant 0$. Then, u is a.p. in the m-th variable.*

Owing to the next consideration, the proof of Theorem 4.1 is essentially the same as that of Theorem 1.5 part (i).

Lemma 4.2. *Let $\phi \in UC(\mathbb{R}^{N+1})$ be a.p. in the m-th variable. Then, from any real sequence $(s_n)_{n\in\mathbb{N}}$ can be extracted a subsequence $(s_{n_k})_{k\in\mathbb{N}}$ such that, for all $(X_1, \dots, X_{m-1}, X_{m+1}, \dots, X_{N+1}) \in \mathbb{R}^N$, the sequence $(\phi(X_1, \dots, X_m + s_{n_k}, \dots, X_{N+1}))_{k\in\mathbb{N}}$ converges uniformly in $X_m \in \mathbb{R}$.*

Proof. The proof is similar to that of the Arzela–Ascoli theorem. For simplicity, consider the case $m = N + 1$. Let $(s_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. As for any $q \in \mathbb{Q}^N$ there exists a subsequence $(s_n^q)_{n\in\mathbb{N}}$ of $(s_n)_{n\in\mathbb{N}}$ such that $(\phi(q, t + s_n^q))_{n\in\mathbb{N}}$ converges uniformly in $t \in \mathbb{R}$, using a diagonal method we can find a common subsequence $(s_{n_k})_{k\in\mathbb{N}}$ such that $(\phi(q, t + s_{n_k}))_{k\in\mathbb{N}}$ converges uniformly in $t \in \mathbb{R}$, for all $q \in \mathbb{Q}^N$. Fix $x \in \mathbb{R}^N$. By the uniform continuity of ϕ, for any $\varepsilon > 0$ there exists $q \in \mathbb{Q}^N$ such that

$$\forall t \in \mathbb{R}, \quad \left|\phi(x,t) - \phi(q,t)\right| < \frac{\varepsilon}{3}.$$

Therefore,

$$\left|\phi(x,t+s_{n_k})-\phi(x,t+s_{n_h})\right|<\frac{2}{3}\varepsilon+\left|\phi(q,t+s_{n_k})-\phi(q,t+s_{n_h})\right|<\varepsilon$$

for h, k big enough, independent of $t\in\mathbb{R}$. $\square$

Remark 4. Statement (i) of Theorem 1.5 does not follow from Theorem 4.1 because being a.p. separately in each variable does not imply the almost periodicity in the sense of Definition 1.4. For example, the function $\phi(x,y)=\sin(xy)$ is periodic in each variable but it is not a.p., because it is known that any a.p. function is uniformly continuous (see e.g. [2]).

5. General periodic domains

Henceforth, Ω denotes a **uniformly smooth** domain in $\mathbb{R}^N$. The symbol ν stands for the outer unit normal vector field to Ω.

We fix $l_1,\dots,l_{N+1}>0$. The domain $\Omega\subset\mathbb{R}^N$ is said to be periodic in the direction e_m, $m\in\{1,\dots,N\}$, if $\Omega+\{l_m e_m\}=\Omega$. If Ω is periodic in all directions, we simply say that it is periodic. From now on, when we say that a function or an operator is periodic (resp. periodic in the m-th variable with $m\in\{1,\dots,N+1\}$) we mean that its period is $(l_1,\dots,l_{N+1})$ (resp. l_m).

Besides the assumptions of Section 1.1, we will sometimes require in the sequel that the coefficients of P and the function f are uniformly Hölder continuous.[2] This is because, in some proofs, we need the solutions to be Lipschitz continuous. In the elliptic case, this property follows from $W^{2,p}$ estimates, for $p>N$, and embedding theorem and indeed the Hölder continuity assumption is not necessary.

5.1. Dirichlet boundary conditions

We deal with the Dirichlet problem

$$\begin{cases} Pu=f(x,t), & x\in\Omega,\ t\in\mathbb{R},\\ u=g(x,t), & x\in\partial\Omega,\ t\in\mathbb{R}, \end{cases}\tag{16}$$

with f measurable and $g\in C^0(\partial\Omega\times\mathbb{R})$. The boundary condition in (16) is understood in classical sense: $u\in C^0(\overline{\Omega}\times\mathbb{R})$ and $u=g$ on $\partial\Omega\times\mathbb{R}$.

If L is a periodic elliptic operator (as defined in Section 1.1), then $\lambda_{p,D}(-L)$ and $\varphi_{p,D}$ denote respectively the periodic principal eigenvalue and eigenfunction of $-L$ in Ω, with Dirichlet boundary conditions. That is, $\lambda_{p,D}(-L)$ is the unique real number such that the problem

$$\begin{cases} -L\varphi_{p,D}=\lambda_{p,D}(-L)\varphi_{p,D} & \text{in } \Omega,\\ \varphi_{p,D}=0 & \text{on } \partial\Omega \end{cases}$$

admits a solution $\varphi_{p,D}$ (unique up to a multiplicative constant) which is positive in Ω and periodic.

The next result is the analogue of Theorem 1.1.

Theorem 5.1. *Let u be a bounded solution of* (16)*, with P, f, g periodic in the m-th variable, as well as Ω if $m\neq N+1$, and with $c\leqslant 0$. Then, u is periodic in the m-th variable.*

Proof. We use the same method as in the proof of Lemma 2.1, with $v\equiv 1$. As before, it is sufficient to show that the function

$$\psi(X):=u(X+l_m e_m)-u(X)$$

[2] We denote by $C^{2n+\gamma,n+\frac{\gamma}{2}}$, with $n\in\{0,1\}$ and $\gamma\in[0,1)$, the space of functions whose space derivatives up to order $2n$ and time derivative, if $n=1$, are **locally** Hölder continuous with exponent γ with respect to x and with exponent $\gamma/2$ with respect to t. If these derivatives are **uniformly** Hölder continuous then we say that the function belongs to $C_b^{2n+\gamma,n+\frac{\gamma}{2}}$.

is nonpositive. Assume by contradiction that $k := \sup_{\Omega\times\mathbb{R}} \psi > 0$. Let $(X_n)_{n\in\mathbb{N}} = ((x_n, t_n)_n)_{n\in\mathbb{N}}$ in $\Omega \times \mathbb{R}$ be such that $\psi(X_n) \to k$. We consider the translated $\psi_n(X) := \psi(X + X_n)$. The problem is that, in principle, these functions are well defined only at $\{(0, \dots, 0)\} \times \mathbb{R}$. As ψ is a solution of (16) with $f \equiv g \equiv 0$ and Ω is uniformly smooth, parabolic estimates up to the boundary together with embedding theorem yield $\psi \in UC(\Omega \times \mathbb{R})$. Hence, there exists $r > 0$ such that $\psi_n > 0$ in $B_r \times (-r, r)$ for n large enough. In particular, the set

$$\mathcal{R} := \big\{r > 0\colon\ B_r + \{x_n\} \subset \Omega \text{ for } n \text{ large enough}\big\}$$

is not empty. We claim that $\mathcal{R} = \mathbb{R}^+$. Let $r \in \mathcal{R}$. We know that, for n large enough, the ψ_n are well defined and uniformly bounded in $B_r \times \mathbb{R}$. Moreover, again by the estimates up to the boundary, for any $1 < p < \infty$,

$$\|\partial_t \psi_n\|_{L^p(B_r\times(-r,r))},\ \|\partial_i \psi_n\|_{L^p(B_r\times(-r,r))},\ \|\partial_{ij} \psi_n\|_{L^p(B_r\times(-r,r))} \leqslant C,$$

where $C > 0$ is independent of n. Therefore, by the compact injection of L^p in C^0, we infer that the ψ_n converge (up to subsequences) to a bounded solution ψ_∞ of

$$\partial_t \psi_\infty - \tilde{a}_{ij}(x,t)\partial_{ij}\psi_\infty - \tilde{b}_i(x,t)\partial_i\psi_\infty - \tilde{c}(x,t)w_\infty = 0, \quad x \in B_r,\ t \in (-r, r),$$

where $\tilde{a}_{ij} = \lim_{n\to\infty} a_{ij}(\cdot + X_n)$, $\tilde{b}_i = \lim_{n\to\infty} b_i(\cdot + X_n)$ uniformly in $B_r \times (-r, r)$ and $\tilde{c} = \lim_{n\to\infty} c(\cdot + X_n)$ weakly in $L^p(B_r \times (-r, r))$. We know that ψ_∞ attains its maximum value k at 0 and then the strong maximum principle yields $\psi_\infty(x, t) = k$ for $x \in B_r$, $t \in (-r, 0]$ (note that $\tilde{c} \leqslant 0$). As a consequence, for n large, $\psi_n \geqslant k/2$ in $B_r \times (-r, 0]$ and then, by the uniform continuity, there exists $\delta > 0$ independent of r and n such that $\psi_n > 0$ in $B_{r+\delta} \times (-r, 0]$. This shows that $\mathcal{R} = \mathbb{R}^+$. We then get a contradiction exactly as in the proof of Lemma 2.1. □

From Theorem 5.1 it follows immediately the following uniqueness result (which implies in particular the LP).

Corollary 5.2. *If Ω and P are periodic and $c \leqslant 0$ then problem* (16) *admits at most a unique bounded solution.*

Proof. Suppose that u_1, u_2 solve (16). Applying Theorem 5.1 we infer that $v := u_1 - u_2$ is periodic and then it has a global maximum and minimum in $\overline{\Omega}$. Since either $\max v \geqslant 0$ or $\min v \leqslant 0$, the strong maximum principle implies that v is constant. But it vanishes on $\partial\Omega$ and then $v \equiv 0$. □

In order to prove the LP when $P = \partial_t - L$ and $\lambda_D(-L) \geqslant 0$, we will make use of the following consideration.

Lemma 5.3. *Let $v_1 \in W^{1,\infty}(\Omega \times \mathbb{R})$ and $v_2 \in C^1(\overline{\Omega} \times \mathbb{R})$ be such that*

$$v_1 \leqslant v_2 \quad \text{on } \partial\Omega \times \mathbb{R}, \qquad \nabla v_2 \in UC(\overline{\Omega} \times \mathbb{R}),$$

$$\sup_{\partial\Omega\times\mathbb{R}} \big(v_1 - v_2 + \min(\partial_\nu v_2, 0)\big) < 0, \tag{17}$$

$$\forall \varepsilon > 0, \quad \inf\big\{v_2(x,t)\colon \operatorname{dist}\big(x, \Omega^c\big) > \varepsilon,\ t \in \mathbb{R}\big\} > 0. \tag{18}$$

Then, there exists a positive constant k such that $kv_2 \geqslant v_1$ in $\Omega \times \mathbb{R}$.

Proof. Assume by way of contradiction that there exist two sequences $(x_n)_{n\in\mathbb{N}}$ in Ω and $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that $nv_2(x_n, t_n) < v_1(x_n, t_n)$. Hence, $\lim_{n\to\infty} v_2(x_n, t_n) = 0$ and then $\operatorname{dist}(x_n, \partial\Omega) \to 0$ by (18). For $n \in \mathbb{N}$, let y_n denote a projection of x_n on $\partial\Omega$. We find that

$$\begin{aligned} 0 &\leqslant \lim_{n\to\infty}\big(v_2(y_n, t_n) - v_1(y_n, t_n)\big) = \lim_{n\to\infty}\big(v_2(x_n, t_n) - v_1(x_n, t_n)\big) \\ &\leqslant \lim_{n\to\infty}\big(v_2(x_n, t_n) - nv_2(x_n, t_n)\big) = 0. \end{aligned}$$

Therefore, by (17),

$$\limsup_{n\to\infty} \partial_\nu v_2(y_n, t_n) < 0.$$

As $\nabla v_2 \in UC(\overline{\Omega} \times \mathbb{R})$, we then infer that

$$\begin{aligned}\lim_{n\to\infty}\frac{v_1(x_n,t_n)-v_1(y_n,t_n)}{|x_n-y_n|} &\geqslant \lim_{n\to\infty}\frac{nv_2(x_n,t_n)-v_2(y_n,t_n)}{|x_n-y_n|}\\ &\geqslant \lim_{n\to\infty} n\frac{v_2(x_n,t_n)-v_2(y_n,t_n)}{|x_n-y_n|}\\ &=+\infty,\end{aligned}$$

which is a contradiction. □

We need the following uniform Hölder continuity assumptions:

$$a_{ij}, b_i, c \in C_b^{\gamma}(\Omega), \tag{19}$$

$$f \in C_b^{\gamma,\frac{\gamma}{2}}(\Omega\times\mathbb{R}), \qquad g \in C_b^{2+\gamma,1+\frac{\gamma}{2}}(\partial\Omega\times\mathbb{R}), \tag{20}$$

for some $0<\gamma<1$.

Theorem 5.4. *Let $P=\partial_t - L$ with coefficients satisfying* (19) *and let Ω, L, f, g be periodic. If u is a bounded solution of* (16) *we have that:*

(i) *if $\lambda_{p,D}(-L)\geqslant 0$ then u is periodic, with the same period as f, g;*
(ii) *if $\lambda_{p,D}(-L)=0$ and f, g satisfy* (20) *and either $f,g\leqslant 0$ or $f,g\geqslant 0$ then $u\equiv k\varphi_{p,D}$, for some $k\in\mathbb{R}$, and $f,g\equiv 0$;*
(iii) *if $\lambda_{p,D}(-L)>0$ and $f,g\equiv 0$ then $u\equiv 0$.*

Proof. (i) Fix $m\in\{1,\dots,N+1\}$ and set

$$\psi(X):=u(X+l_me_m)-u(X).$$

Let us check that the functions $v_1=\psi$ and $v_2=\varphi_{p,D}$ fulfill the hypotheses of Lemma 5.3. Parabolic and elliptic estimates up to the boundary yield $v_1, v_2\in C_b^{2+\gamma,1+\frac{\gamma}{2}}(\Omega\times\mathbb{R})$. Moreover,

$$v_1=v_2=0, \qquad \partial_\nu v_2<0, \quad \text{on } \partial\Omega\times\mathbb{R},$$

the last inequality following from the Hopf lemma. Therefore, the hypotheses of Lemma 5.3 are satisfied owing to the periodicity of $\varphi_{p,D}$. As a consequence, there exists $k>0$ such that $k\varphi_{p,D}\geqslant\psi$. Define

$$k^*:=\inf\{k>0\colon\ k\varphi_{p,D}\geqslant\psi\}.$$

Assume by contradiction that $k^*>0$. The function $w(x,t):=k^*\varphi_{p,D}(x)-\psi(x,t)$ is nonnegative by the definition of k^*. We distinguish two different cases.

Case 1: w satisfies (18).

If $\sup_{\partial\Omega\times\mathbb{R}}\partial_\nu w\geqslant 0$ then there exist a sequence $(Z_n)_{n\in\mathbb{N}}$ in $\mathbb{Z}l_1\times\cdots\times\mathbb{Z}l_{N+1}$ and a sequence $(Y_n)_{n\in\mathbb{N}}$ in $\partial\Omega\times\mathbb{R}$ converging to some $(y,\tau)\in\partial\Omega\times\mathbb{R}$ such that

$$\limsup_{n\to\infty}\partial_\nu w(Y_n+Z_n)\geqslant 0.$$

The sequence $w(\cdot+Z_n)$ converges (up to subsequences) in $C_b^{2+\tilde\gamma,1+\frac{\tilde\gamma}{2}}(K\times(-r,r))$, for any $0<\tilde\gamma<\gamma$, compact set $K\subset\overline{\Omega}$ and $r>0$, to a nonnegative function w_∞ satisfying

$$Pw_\infty\geqslant 0 \quad \text{in } \Omega\times\mathbb{R}, \qquad w_\infty=0 \quad \text{on } \partial\Omega\times\mathbb{R}, \qquad \partial_\nu w_\infty(y,\tau)\geqslant 0.$$

By Hopf's lemma it follows that $w_\infty=0$ in $\Omega\times(-\infty,\tau]$, which is impossible because w satisfies (18). This shows that $\sup_{\partial\Omega\times\mathbb{R}}\partial_\nu w<0$ and then (17) holds with $v_1=\psi$ and $v_2=w$. Therefore, by Lemma 5.3 we can find another positive constant k' such that $k'w\geqslant\psi$ in $\Omega\times\mathbb{R}$. That is,

$$\frac{k'}{k'+1}k^*\varphi_{p,D}\geqslant\psi,$$

which contradicts the definition of k^*. This case is ruled out.

Case 2: w does not satisfies (18).

There exist then a sequence $(Z_n)_{n\in\mathbb{N}}$ in $\mathbb{Z}l_1 \times \cdots \times \mathbb{Z}l_{N+1}$ and a sequence $(Y_n)_{n\in\mathbb{N}}$ in $\Omega \times \mathbb{R}$ converging to some $(y, \tau) \in \Omega \times \mathbb{R}$ such that

$$\lim_{n\to\infty} w(Y_n + Z_n) = 0.$$

With usual arguments, we find that (a subsequence of) the sequence $w(\cdot + Z_n)$ converges to 0 locally uniformly in $\Omega \times (0, \tau]$. Defining the bounded sequence $(\zeta_h)_{h\in\mathbb{N}}$ as at the end of the proof of Theorem 5.1, we get the following contradiction:

$$\forall h \in \mathbb{N}, \quad \zeta_h - \zeta_{h+1} = k^* \varphi_{p,D}(y).$$

In both Cases 1 and 2, we have shown that $k^* = 0$, that is, $u(X + l_m e_m) \leqslant u(X)$. The converse inequality is obtained in analogous way by replacing l_m with $-l_m$.

(ii) Up to replacing u with $-u$, it is not restrictive to assume that $f, g \leqslant 0$. Hence, $u(x, t) \leqslant \varphi_{p,D}(x)$ for $x \in \partial\Omega$, $t \in \mathbb{R}$. Note that by parabolic estimates up to the boundary, $u \in C_b^{2+\gamma,1+\frac{\gamma}{2}}(\Omega \times \mathbb{R})$. Applying Lemma 5.3 with $v_1 = u$ and $v_2 = \varphi_{p,D}$, we find a positive constant k such that $k\varphi_{p,D} \geqslant u$. Set

$$k^* := \inf\{k \in \mathbb{R}\colon\ k\varphi_{p,D} \geqslant u\}.$$

The function $w := k^*\varphi_{p,D} - u$ is nonnegative, periodic, by (i), and satisfies

$$Pw = -f \geqslant 0 \quad \text{in } \Omega \times \mathbb{R}.$$

If w vanishes somewhere in $\Omega \times \mathbb{R}$ then the parabolic strong maximum principle and the time-periodicity of w yields $w \equiv 0$, which concludes the proof of the statement. Otherwise, for any $x \in \partial\Omega$, $t \in \mathbb{R}$ such that $w(x, t) = 0$, the Hopf lemma yields $\partial_\nu w(x, t) < 0$. Consequently,

$$\forall x \in \partial\Omega,\ t \in \mathbb{R}, \quad -w(x, t) + \min\bigl(\partial_\nu w(x, t), 0\bigr) < 0.$$

As $\varphi_{p,D}$ and w are periodic, we see that the hypotheses of Lemma 5.3 are satisfied by $v_1 = \varphi_{p,D}$ and $v_2 = w$ and then we there exists another positive constant h such that $hw \geqslant \varphi_{p,D}$. Hence, $(k^* - h^{-1})\varphi_{p,D} \geqslant u$ which contradicts the definition of k^*.

(iii) It is not restrictive to assume that $\sup u \geqslant 0$ (if not, replace u with $-u$). We proceed exactly as in the proof of (ii). Now, the constant k^* is nonnegative and then the function $w := k^*\varphi_{p,D} - u$ satisfies

$$Pw = k^*\lambda_{p,D}(-L)\varphi_{p,D} \geqslant 0.$$

Then, as before, we derive $w \equiv 0$. From the above expression we see that $k^* = 0$ and then $u \equiv 0$. □

Remark 5. Theorem 5.4 part (i) when $\lambda_{p,D}(-L) > 0$ and part (iii) hold without the additional assumption (19). In fact, the latter is only used to have the Lipschitz continuity of solutions required to apply Lemma 5.3. But this can be avoided by approximating Ω by a sequence of domains $(\mathcal{O}_m)_{m\in\mathbb{N}}$ which contain Ω. Then, one argues as before, with $\varphi_{p,D}$ replaced by the periodic principal eigenfunction associated with $\mathcal{O}_m$. This function is strictly positive in $\overline{\Omega}$ and is still a supersolution of $-L = 0$ provided that m is large enough, because $\lambda_{p,D}(-L) > 0$ (see the proof of Corollary 5.5 below). This allows one to define the function w – which does not satisfy (18) – and obtain the same contradiction as before.

Corollary 5.5. *If $P = \partial_t - L$, the domain Ω and L are periodic, $\lambda_{p,D}(-L) > 0$ and $f \in L^\infty(\Omega \times \mathbb{R})$, $g \in W_\infty^{2,1}(\Omega \times \mathbb{R})$,*[3] *then problem* (16) *admits a unique bounded solution.*

[3] $W_\infty^{2,1}$ denotes the space of functions u such that $u, \partial_i u, \partial_{ij} u, \partial_t u \in L^\infty$.

Proof. Note that, up to replacing f with $f - Pg$, it is not restrictive to assume that $g \equiv 0$. As in the proof of Corollary 2.2, we find a solution u as the limit as $n \to \infty$ of solutions u_n of the problems

$$\begin{cases} Pu_n = f(x,t), & x \in \Omega_n,\ t \in (-n,n), \\ u_n = 0, & x \in \partial\Omega_n,\ t \in (-n,n), \\ u_n(x,-n) = 0, & x \in \Omega_n, \end{cases} \tag{21}$$

where $(\Omega_n)_{n\in\mathbb{N}}$ is a family of bounded domains recovering Ω (defined below). The proof of the uniform boundedness of the u_n is now more delicate, because $\varphi_{p,D}$ is not bounded from below away from zero and then we cannot take as sub- and supersolution of (21) the functions $-k\varphi_{p,D}$ and $k\varphi_{p,D}$ with k large enough. We overcome this difficulty by extending a_{ij}, b_i, c to the whole space and by considering a domain which is slightly larger than Ω. Let $(\mathcal{O}_m)_{m\in\mathbb{N}}$ be a uniformly smooth family of periodic domains satisfying

$$\forall m \in \mathbb{N}, \quad \mathcal{O}_m \supset \mathcal{O}_{m+1} \supset \overline{\Omega}, \qquad \bigcap_{m\in\mathbb{N}} \mathcal{O}_m = \overline{\Omega}.$$

For any $m \in \mathbb{N}$ let λ_m and φ_m be the periodic principal eigenvalue and eigenfunction of $-L$ in $\mathcal{O}_m$, with Dirichlet boundary conditions, such that $\|\varphi_m\|_{L^\infty(\mathcal{O}_m)} = 1$. It follows from the maximum principle that the sequence $(\lambda_m)_{m\in\mathbb{N}}$ is increasing and bounded from above by $\lambda_{p,D}(-L)$. Owing to the uniform smoothness of the $\mathcal{O}_m$, elliptic estimates up to the boundary imply that the φ_m converge (up to subsequences) uniformly in Ω to a nonnegative periodic solution φ of $-L\varphi = \lambda\varphi$ in Ω, where $\lambda = \lim_{n\to\infty} \lambda_m$. Moreover, since for any $\varepsilon > 0$ there exists δ such that

$$\forall m \in \mathbb{N}, \quad \operatorname{dist}(x, \partial\mathcal{O}_m) \leqslant \delta \quad \Rightarrow \quad \varphi_m(x) \leqslant \varepsilon$$

(by gradient estimates up to the boundary), we see that φ vanishes on $\partial\Omega$ and that $\|\varphi\|_{L^\infty(\Omega)} = 1$. Hence, $\varphi > 0$ in Ω by the strong maximum principle. This shows that $\lambda = \lambda_{p,D}(-L)$. Thus, there exists $m^* \in \mathbb{N}$ such that $\lambda_{m^*} > 0$. The function

$$v(x) := \frac{\|f\|_{L^\infty(\Omega\times\mathbb{R})}}{\lambda_{m^*} \min_{\overline{\Omega}} \varphi_{m^*}} \varphi_{m^*}(x)$$

is the strictly positive supersolution we need to show that the solutions u_n of (21) are uniformly bounded for $n \in \mathbb{N}$. The smooth domains Ω_n are defined in such a way that, for $n \in \mathbb{N}$, $\Omega_n \subset B_{n+1}$ and $\Omega_n \cap B_n$ coincides with the connected component of $\Omega \cap B_n$ containing 0 (which can be assumed to belong to Ω). It is easily seen that for any compact $K \subset \mathbb{R}^N$ there exists $n_0 \in \mathbb{N}$ such that $\Omega \cap K \subset \Omega_n$ for $n \geqslant n_0$. Then, we proceed exactly as in the proof of Corollary 2.2, with B_r replaced by Ω_n. The uniqueness result follows from Theorem 5.4 part (iii) and Remark 5. □

Remark 6. If $c \leqslant 0$ then $\lambda_{p,D}(-L) > 0$ and then the existence and uniqueness result of Corollary 5.5 applies (in contrast with the whole space case, cf. Remark 1). This is easily seen by applying the strong maximum principle to the periodic principal eigenfunction $\varphi_{p,D}$.

5.2. Robin boundary conditions

We consider now the Robin problem

$$\begin{cases} Pu = f(x,t), & x \in \Omega,\ t \in \mathbb{R}, \\ \mathcal{N}u = h(x,t), & x \in \partial\Omega,\ t \in \mathbb{R}, \end{cases} \tag{22}$$

where

$$\mathcal{N}u = \alpha(x,t)u + \beta(x,t) \cdot \nabla u,$$

with α, β bounded and satisfying

$$\alpha \geqslant 0, \qquad \inf_{x\in\partial\Omega} \beta(x) \cdot \nu(x) > 0.$$

We always assume in this section that

$$a_{ij}, b_i, c \in C_b^{\gamma,\frac{\gamma}{2}}(\Omega \times \mathbb{R}),$$

for some $0 < \gamma < 1$, and solutions of (22) are understood in classical sense. Hence, (22) admits solution only if f and h satisfy some regularity conditions, but for our uniqueness results we do not need to impose them.

If $P = \partial_t - L$,

$$\alpha = \alpha(x), \qquad \beta = \beta(x), \quad \alpha, \beta \in C_b^{1+\gamma}(\partial\Omega), \tag{23}$$

and $\Omega, L, \mathcal{N}$ are periodic then $\lambda_{p,\mathcal{N}}$ and $\varphi_{p,\mathcal{N}}$ denote respectively the periodic principal eigenvalue and eigenfunction of $-L$ in Ω, under Robin boundary conditions. That is, $\lambda_{p,\mathcal{N}}$ is the unique (real) number such that the eigenvalue problem

$$\begin{cases} -L\varphi_{p,\mathcal{N}} = \lambda_{p,\mathcal{N}}\varphi_{p,\mathcal{N}} & \text{in } \Omega, \\ \mathcal{N}\varphi_{p,\mathcal{N}} = 0 & \text{on } \partial\Omega \end{cases}$$

admits a positive periodic solution $\varphi_{p,\mathcal{N}}$ (unique up to a multiplicative constant).

The strategy used to prove our results is exactly the same as in Section 2, the following lemma being the analogue of Lemma 2.1. While in the whole space case we used interior estimates for strong solutions, here we need Hölder estimates up to the boundary (see [22,25], or [17,33] for the elliptic case).

Lemma 5.6. *Assume that Ω is periodic and that the operators P, $\mathcal{N}$ and the functions f, h are periodic in the m-th variable. If there exists a function $v \in C_b^{2,1}(\Omega \times \mathbb{R})$ satisfying*

$$\inf_{\Omega} v > 0, \qquad \begin{cases} Pv \geqslant 0, & x \in \Omega,\ t \in \mathbb{R}, \\ \mathcal{N}v \geqslant 0, & x \in \partial\Omega,\ t \in \mathbb{R}, \end{cases}$$

then any bounded solution of (22) *is periodic in the m-th variable.*

Proof. The proof is similar to that of Lemma 2.1 and we will skip some details. But now we translate the functions ψ, v and the coefficients of the equation by Z_n instead of X_n, where $(Z_n)_{n\in\mathbb{N}}$ is the sequence in $\mathbb{Z}l_1 \times \cdots \times \mathbb{Z}l_{N+1}$ such that $Y_n := X_n - Z_n \in [0, l_1) \times \cdots \times [0, l_{N+1})$. Then, the only situation which is not covered by the arguments in the whole space is when $w_\infty < k$ in Ω and Y_n converges (up to subsequences) to some $Y_\infty = (y_\infty, \eta_\infty) \in \partial\Omega \times [0, l_{N+1}]$. Let us show that this cannot occur. Let α^* and β^* be the limits of (subsequences of) $\alpha(Y_\infty + Z_n)$ and $\beta(Y_\infty + Z_n)$ respectively. Clearly,

$$\alpha^* \geqslant 0, \qquad \beta^* \cdot \nu(y_\infty) > 0.$$

Thus, since w_∞ is a solution of a linear parabolic equation with nonpositive zero order term achieving a positive maximum at Y_∞, the Hopf lemma yields $\beta^* \cdot \nabla w_\infty(Y_\infty) > 0$. This is impossible, because

$$\begin{aligned} 0 &= \alpha^*\psi_\infty(Y_\infty) + \beta^* \cdot \nabla\psi_\infty(Y_\infty) \\ &= \alpha^*(w_\infty v_\infty)(Y_\infty) + \beta^* \cdot \nabla(w_\infty v_\infty)(Y_\infty) \\ &= k\big(\alpha^* v_\infty(Y_\infty) + \beta^* \cdot \nabla v_\infty(Y_\infty)\big) + v_\infty(Y_\infty)\beta^* \cdot \nabla w_\infty(Y_\infty) \\ &\geqslant v_\infty(Y_\infty)\beta^* \cdot \nabla w_\infty(Y_\infty) \\ &> 0. \qquad \square \end{aligned}$$

Applying Lemma 5.6 with $v \equiv 1$ we immediately get

Theorem 5.7. *Let u be a bounded solution of* (22)*, with Ω periodic, P, $\mathcal{N}$, f, h periodic in the m-th variable and $c \leqslant 0$. Then, u is periodic in the m-th variable.*

Compare the previous statement with Theorem 5.1, which holds for domains periodic just in the direction e_m. In the case of Robin boundary conditions, we are only able to deal with domains periodic in all directions.

Corollary 5.8. *Let u be a bounded solution of*

$$\begin{cases} Pu = 0, & x \in \Omega,\ t \in \mathbb{R}, \\ \mathcal{N}u = 0, & x \in \partial\Omega,\ t \in \mathbb{R}, \end{cases} \tag{24}$$

with Ω, P, $\mathcal{N}$ periodic and $c \leqslant 0$. Then, two possibilities occur:

(1) $c \equiv 0$, $\alpha \equiv 0$ *and u is constant*;
(2) $\|c\|_{L^\infty(\Omega)} + \|\alpha\|_{L^\infty(\partial\Omega)} \neq 0$ *and* $u \equiv 0$.

Proof. By Theorem 5.7 we know that u is periodic in every space/time variables and then it has global maximum and minimum in $\overline{\Omega} \times \mathbb{R}$. Let $M = \max u = u(x_0, t_0)$. Up to replacing u with $-u$, we can assume without loss of generality that $M \geqslant 0$. Thus, by the strong maximum principle, either $u = M$ in $\Omega \times (-\infty, t_0]$, or $u < M$ in $\Omega \times (-\infty, t_0]$ and $\bar{x} \in \partial\Omega$. The second case is ruled out because, by Hopf's lemma we would have

$$0 < \beta(x_0, t_0) \cdot \nabla u(x_0, t_0) \leqslant \mathcal{N}u(x_0, t_0) = 0.$$

Therefore, $u = M$ in $\Omega \times (-\infty, t_0]$ and then, by periodicity, in $\Omega \times \mathbb{R}$. The statement follows. □

Theorem 5.9. *Let $P = \partial_t - L$, the functions α, β satisfy* (23) *and Ω, L, $\mathcal{N}$, f, h be periodic. If u is a bounded solution of* (22) *we have that*

(i) *if $\lambda_{p,\mathcal{N}}(-L) \geqslant 0$ then u is periodic*;
(ii) *if $\lambda_{p,\mathcal{N}}(-L) = 0$ and either $f, h \leqslant 0$ or $f, h \geqslant 0$ then $u \equiv k\varphi_{p,\mathcal{N}}$, for some $k \in \mathbb{R}$, and $f, h \equiv 0$*;
(iii) *if $\lambda_{p,\mathcal{N}}(-L) > 0$ and $f, h \equiv 0$ then $u \equiv 0$.*

Proof. First, we show that

$$\inf_{\Omega} \varphi_{p,\mathcal{N}} > 0,$$

no matter what the sign of $\lambda_{p,\mathcal{N}}(-L)$ is. Indeed, if $\inf_\Omega \varphi_{p,\mathcal{N}} = 0$, then the periodicity and the positivity of $\varphi_{p,\mathcal{N}}$ in Ω yield $\varphi_{p,\mathcal{N}}(y) = 0$ for some $y \in \partial\Omega$. Hence,

$$0 = \mathcal{N}\varphi_{p,\mathcal{N}}(y) = \beta(y) \cdot \nabla\varphi_{p,\mathcal{N}}(y),$$

which contradicts the Hopf lemma.

(i) The statement follows by applying Lemma 5.6 with $v = \varphi_{p,\mathcal{N}}$.

(ii)–(iii) We can argue exactly as in the proof of Theorem 1.3 parts (ii) and (iii). The only different situation is if $w > 0$ in $\Omega \times \mathbb{R}$ and vanishes at $(x_0, t_0) \in \partial\Omega \times \mathbb{R}$. In this case, we get

$$\beta(x_0) \cdot \nabla w(x_0, t_0) = \mathcal{N}w(x_0, t_0) = -\mathcal{N}u(x_0, t_0) = -h(x_0, t_0) \geqslant 0$$

(we recall that it is not restrictive to assume that $f, h \leqslant 0$). Once again, this is in contradiction with the Hopf lemma. □

We conclude with the existence and uniqueness result for (22). We assume that

$$f \in C_b^{\gamma, \frac{\gamma}{2}}(\Omega \times \mathbb{R}), \qquad h \in C_b^{2+\gamma, 1+\frac{\gamma}{2}}(\partial\Omega \times \mathbb{R}), \tag{25}$$

and we strengthen the regularity condition on β in (23):

$$\alpha = \alpha(x), \qquad \beta = \beta(x), \quad \alpha \in C_b^{1+\gamma}(\partial\Omega),\ \beta \in C_b^{2+\gamma}(\partial\Omega). \tag{26}$$

Theorem 5.10. *If $P = \partial_t - L$, conditions* (25)–(26) *hold, L, $\mathcal{N}$ are periodic and $\lambda_{p,\mathcal{N}}(-L) > 0$, then problem* (22) *admits a unique bounded solution u. If in addition f and h are also periodic, then u is periodic.*

Proof. From the uniform smoothness of Ω it follows that there exists $\delta > 0$ such that each point in $\Omega^\delta := \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) < \delta\}$ admits a unique projection $\pi(x)$ on $\partial\Omega$. Hence, the function $\operatorname{dist}(x, \partial\Omega)$ is well defined and smooth in Ω^δ. Let $\chi \in C^\infty(\mathbb{R})$ be a cut-off function such that $\chi = 1$ in $(0, \delta/2)$, $\chi = 0$ in $(\delta, +\infty)$. The function

$$\psi(x, t) := \frac{h(\pi(x), t)}{\beta(\pi(x)) \cdot \nu(\pi(x))} \operatorname{dist}(x, \partial\Omega)\chi\big(\operatorname{dist}(x, \partial\Omega)\big)$$

belongs to $C^{2+\gamma,1+\frac{\gamma}{2}}(\Omega\times\mathbb{R})$ and satisfies $\mathcal{N}\psi=h$ on $\partial\Omega$. Therefore, replacing f by $f-P\psi$, we can take $h\equiv 0$ in (22). Define the domains $(\Omega_n)_{n\in\mathbb{N}}$ as in the proof of Corollary 5.5. Consider a family of cut-off functions $(\chi_n)_{n\in\mathbb{N}}$ uniformly bounded in $C_b^{2+\gamma,1+\frac{\gamma}{2}}(\mathbb{R}^N)$ such that, for $n>1$,

$$\chi_n=1 \quad \text{in } B_{n-1}, \qquad \chi_n>0 \quad \text{in } B_n\backslash B_{n-1}, \qquad \chi_n=0 \quad \text{in } \mathbb{R}^N\backslash B_n.$$

Proceeding as in the proof of Corollary 2.2, with B_R replaced by Ω_n and φ_p by $\varphi_{p,\mathcal{N}}$ (which has positive infimum), we see that, as $n\to\infty$, the unique solution of

$$\begin{cases} Pu_n=f(x,t), & x\in\Omega_n,\ t\in(-n,n),\\ \big(\chi_n\mathcal{N}+(1-\chi_n)\big)u_n=0, & x\in\partial\Omega_n,\ t\in(-n,n),\\ u_n(x,-n)=0, & x\in\Omega_n,\end{cases}$$

converges (up to subsequences) in $C_b^{2,1}(\Omega\cap K,(-r,r))$, for any compact $K\subset\mathbb{R}^N$ and any $r>0$, to a bounded solution of (22). The uniqueness result is a consequence of Theorem 5.9 part (iii). □

Using the Hopf lemma, one can readily check that if $L,\mathcal{N}$ are periodic, $c\leqslant 0$ and α, c are not identically equal to zero, then $\lambda_{p,\mathcal{N}}(-L)>0$. Therefore, the result of Theorem 5.10 applies in this case.

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