

A Remark on Extensions of CR Functions from Hyperplanes

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Abstract. In the characterization of the range of the Radon transform, one encounters the problem of the holomorphic extension of functions defined on $\mathbb{R}^2 \setminus \Delta_{\mathbb{R}}$ (where $\Delta_{\mathbb{R}}$ is the diagonal in $\mathbb{R}^2$) and which extend as “separately holomorphic” functions of their two arguments. In particular, these functions extend in fact to $\mathbb{C}^2 \setminus \Delta_{\mathbb{C}}$ where $\Delta_{\mathbb{C}}$ is the complexification of $\Delta_{\mathbb{R}}$. We take this theorem from the integral geometry and put it in the more natural context of the CR geometry where it accepts an easier proof and a more general statement. In this new setting it becomes a variant of the celebrated “edge of the wedge” theorem of Ajrapetyan and Henkin.

Let $x = (x_1, x_2)$ and $z = (z_1, z_2)$, with $z = x + iy$, be variables in $\mathbb{R}^2$ and $\mathbb{C}^2$, respectively. We will use the notations

$$T = (\mathbb{R} \times \{0\}) \cup (\{0\} \times \mathbb{R}), \quad \dot{T} = T \setminus \{0\}.$$

We consider a C^1 -curve without boundary L in $\mathbb{R}^2$ and focus our attention on the CR manifold $(\mathbb{R}^2 + iT) \setminus L$. We note that the set $(\mathbb{R}^2 + iT) \setminus L$ is rather unusual in several complex variables, in the sense that it is neither a manifold with boundary nor a wedge-like domain. In this set there is a distinguished subset $E = \mathbb{R}^2 \setminus L$, which plays the role of edge, and four manifolds issuing from it, namely the ones which are defined by $y_1 \geq 0$, $y_2 \geq 0$, $y_1 \leq 0$ and $y_2 \leq 0$. We denote by Q_j , $j = 1, \dots, 4$, the four quadrants of $\mathbb{R}^2$ with vertex at 0. The aim of this paper is to prove the following extension result.

Theorem 1 *Assume that for any $x \in \mathbb{R}^2 \setminus L$ and a suitable j we have $(x + Q_j) \cap L = \emptyset$. Let $f: \mathbb{R}^2 \setminus L \rightarrow \mathbb{C}$ be a continuous function which extends, as a separately holomorphic continuous function (i.e., a continuous CR function) to the set $(\mathbb{R}^2 + iT) \setminus L$. Then f extends as a holomorphic function to $\mathbb{C}^2$ unless L is a straight line, in which case f extends to $\mathbb{C}^2 \setminus L^{\mathbb{C}}$, the complement of the complexification of L .*

Before giving the proof, let us make some comments. When L is a line, defined say by $l(x) = 0$, then the function $f(z) = \frac{1}{l(z)}$ exhibits an example of a function which extends to $\mathbb{C}^2 \setminus L^{\mathbb{C}}$ but not to the whole $\mathbb{C}^2$; we are thus in the second instance of the statement of Theorem 1. We point out that the above theorem generalizes results from [1, 5] in the context of the characterization of the range of the exponential Radon transform. In those statements L was assumed to be the diagonal of $\mathbb{R}^2$, which can be easily generalized to any straight line. Our improvement consists in treating the case of general curves L . It is worth noticing that this is the first time that this

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extension problem is treated in the framework of the CR geometry. More precisely, it is reduced to an “edge of the wedge” theorem of the type from [2] in the presence of a singular set such as L . Also, let us notice that an initial holomorphic extension to a neighborhood of $(\mathbb{R}^2 \setminus L) + iT$ is guaranteed by [2]. Our goal is to continue in this process of extension, first reaching the set $(\mathbb{R}^2 \setminus L) + i\mathbb{R}^2$, and then eventually the whole set $\mathbb{C}^2$ or $\mathbb{C}^2 \setminus L^c$ according as L is curved or straight.

Proof of Theorem 1 We begin by noticing that any CR function on $(\mathbb{R}^2 + iT) \setminus L$ extends holomorphically to a neighborhood of $(\mathbb{R}^2 \setminus L) + iT$. In fact, fix any point $x \in \mathbb{R}^2$ outside L . Then locally around x , $(\mathbb{R}^2 + iT) \setminus L$ is the union of the two hyperplanes $y_1 = 0$ and $y_2 = 0$. By the edge of the wedge theorem from [2], f extends as a holomorphic function to the wedges

$$\begin{aligned} Q_1 &= \{y_1 \geq 0, y_2 \geq 0\}, & Q_2 &= \{y_1 \geq 0, y_2 \leq 0\}, \\ Q_3 &= \{y_1 \leq 0, y_2 \leq 0\}, & Q_4 &= \{y_1 \leq 0, y_2 \geq 0\}. \end{aligned}$$

Then f extends to a full neighborhood of x . We observe next that by [4] the analyticity of f in x propagates along the complex lines $\gamma_1(z_1) = (z_1, x_2)$ and $\gamma_2(z_2) = (x_1, z_2)$. Since all the points of $(\mathbb{R}^2 \setminus L) + iT$ lie on a line of type γ_1 or γ_2 , we conclude that f extends to a neighborhood of $(\mathbb{R}^2 \setminus L) + iT$ which we will denote by V .

At this point our plan is to use the continuity principle to gain extendibility to other points of $\mathbb{C}^2$. Now we will be able to extend f to those points z for which there exists a continuous family of analytic discs attached to $(\mathbb{R}^2 + iT) \setminus L$, that is, having their boundaries in $(\mathbb{R}^2 + iT) \setminus L$ starting from an initial disc entirely contained in V and ending up with a disc passing through z . The hard part of this task is to find discs attached to $(\mathbb{R}^2 + iT) \setminus L$. Let us recall some standard notations. The symbol Δ denotes the standard disc in $\mathbb{C}$ and A an analytic disc in $\mathbb{C}^2$, that is, an analytic mapping $A(\zeta) = (z_1(\zeta), z_2(\zeta))$ for $\zeta \in \Delta$ which is of class $C^{1,\alpha}$ (i.e., differentiable with α -Hölder continuous derivative up to $\partial\Delta$). We denote by the same notation A both the disc $A(\Delta)$ and its parametrization $\zeta \mapsto A(\zeta)$. The disc A is said to be attached to $(\mathbb{R}^2 + iT) \setminus L$ when $A(\partial\Delta) \subset (\mathbb{R}^2 + iT) \setminus L$. The set $(\mathbb{R}^2 + iT) \setminus L$ is contained in the set defined by $y_1 y_2 = 0$, and hence a disc A is attached to $(\mathbb{R}^2 + iT) \setminus L$ if $y_1(\zeta) y_2(\zeta) = 0 \forall \zeta \in \partial\Delta$ and if for $y_1(\zeta)$ and $y_2(\zeta)$ simultaneously 0 we have $(x_1(\zeta), x_2(\zeta)) \notin L$. To check this condition it is convenient to look for a representation formula for analytic discs which involves only the imaginary part. Let K be the Cauchy transform, i.e.,

$$K(g)(\zeta) = \frac{1}{2\pi i} \int_{\partial\Delta} \frac{g(\tau)}{\tau - \zeta} d\tau.$$

Then if we have a holomorphic function $h(\zeta) = u(\zeta) + iv(\zeta)$, it is easily verified that

$$(1) \quad h(\zeta) = 2iK(v)(\zeta) + u(0) - iv(0).$$

Let us point out that (1) gives a holomorphic function h starting from its arbitrary imaginary part v .

Let $\eta_1(\zeta)$ and $\eta_2(\zeta)$ for $\zeta \in \partial\Delta$ be two positive $C^{1,\alpha}$ functions with unitary mean whose support is contained in $\{\operatorname{Re} \zeta \geq 0\}$ and $\{\operatorname{Re} \zeta \leq 0\}$, respectively, and which are simultaneously 0 only at $\pm i$. (Their choice will be further specified in the course of the proof.) We write $\eta(\zeta) = (\eta_j(\zeta))_j$, and take $x^o = (x_1^o, x_2^o)$ and $y^o = (y_1^o, y_2^o)$ in $\mathbb{R}^2$. The analytic disc

$$A_{x^o, y^o, \eta}(\zeta) = (2iK(y_j^o \eta_j)(\zeta) + x_j^o - iy_j^o)_{j=1,2}$$

is attached to the set defined by $y_1 y_2 = 0$, and its center is $(x_j^o + iy_j^o)_{j=1,2}$. To get the disc attached to $(\mathbb{R}^2 + iT) \setminus L$, we need $A(\zeta) \notin L$ whenever $\operatorname{Im} A(\zeta) = 0$. But we see that the only case when $\operatorname{Im} A(\zeta) = 0$ is when $\eta_1(\zeta)$ and $\eta_2(\zeta)$ are simultaneously 0, that is, for $\zeta = +i$ and $-i$. By (1) we have

$$\begin{aligned} A(i) &= \left(x_j^o - \frac{1}{2\pi} \int_0^{2\pi} y_j^o \eta_j(e^{i\theta}) \frac{\cos(\theta)}{1 - \sin(\theta)} d\theta \right)_j, \\ A(-i) &= \left(x_j^o + \frac{1}{2\pi} \int_0^{2\pi} y_j^o \eta_j(e^{i\theta}) \frac{\cos(\theta)}{1 + \sin(\theta)} d\theta \right)_j. \end{aligned}$$

We call

$$\begin{aligned} a &= -\frac{1}{2\pi} \int_0^{2\pi} \eta_1(e^{i\theta}) \frac{\cos(\theta)}{1 - \sin(\theta)} d\theta, & b &= -\frac{1}{2\pi} \int_0^{2\pi} \eta_2(e^{i\theta}) \frac{\cos(\theta)}{1 - \sin(\theta)} d\theta, \\ c &= \frac{1}{2\pi} \int_0^{2\pi} \eta_1(e^{i\theta}) \frac{\cos(\theta)}{1 + \sin(\theta)} d\theta, & d &= \frac{1}{2\pi} \int_0^{2\pi} \eta_2(e^{i\theta}) \frac{\cos(\theta)}{1 + \sin(\theta)} d\theta. \end{aligned}$$

With our notations we have

$$A(i) = (x_1^o + y_1^o a, x_2^o + y_2^o b), \quad A(-i) = (x_1^o + y_1^o c, x_2^o + y_2^o d).$$

To carry on our proof we need the following.

Proposition 2 *For every point $z^o = (z_1^o, z_2^o)$ such that $\operatorname{Re} z^o \notin L$, we can choose the functions η_j , $j = 1, 2$ in such a way that*

$$(2) \quad A_{x^o, y^o, \eta}(\pm i) \notin L \quad \text{for any } 0 \leq t \leq 1.$$

Proof We recall that the functions η_1 and η_2 must be chosen with support in the half-circles $\{e^{i\theta} : -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\}$ and $\{e^{i\theta} : \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}\}$, respectively, vanishing only at the points $\pm i$, and with unit mean value. For the rest, we can play freely with the displacement of their masses in order to achieve (2). To this end we will make a choice which depends on x^o and y^o . We assume without loss of generality that it is for the first quadrant Q_1 that we have $(x^o + Q_1) \cap L = \emptyset$. Suppose first, $y_1^o > 0$, $y_2^o > 0$. We take any η_2 and choose η_1 with its mass so close to $-i$ that a is small (negative) and c is big (positive) and so that (ay_1^o, by_2^o) and (cy_1^o, dy_2^o) are close to the

$x_2 \geq 0$ and $x_1 \geq 0$ axes, respectively. We get, therefore, for $0 \leq t \leq 1$,

$$(3) \quad (x_1^o - ty_1^o a, x_2^o - ty_2^o b) \notin L,$$

$$(4) \quad (x_1^o + ty_1^o c, x_2^o + ty_2^o d) \in x^o + Q_1 \subset \mathbb{R}^2 \setminus L.$$

Suppose now $y_1^o > 0$ and $y_2^o < 0$. In this case, we take η_1 and η_2 both with most of their masses at $-i$ so that ay_1^o and by_2^o are small and therefore (3) follows. In this situation cy_1^o and dy_2^o are both positive, and hence (4) also trivially holds. In both the above cases, the discs obtained by the described choices of η_1 and η_2 do not intersect L at either of the two points where they meet $\mathbb{R}^2$, namely $\pm i$, and this is the case for all values of the parameter t for $0 \leq t \leq 1$; thus they are attached to $(\mathbb{R}^2 \setminus L) + iT$. It is clear that all other choices of signs for the y_j can be handled likewise which concludes the proof of the proposition. ■

End of proof of Theorem 1 It follows easily from Proposition 2 that any CR function f on $(\mathbb{R}^2 + iT) \setminus L$ extends to any $z^o \in (\mathbb{R}^2 \setminus L) + i\mathbb{R}^2$. In fact, for every such z^o with real part $x^o = (x_1^o, x_2^o) \notin L$, we can find a continuous family of analytic discs A , depending on the parameter t with $0 \leq t \leq 1$, attached to $(\mathbb{R}^2 + iT) \setminus L$ and such that for $t = 0$ the disc A reduces to the single point x^o , and for $t = 1$, the center of A reaches the point z^o . Then by applying the continuity principle to the function f , which was already known to extend holomorphically to a neighborhood V of $(\mathbb{R}^2 \setminus L) + iT$, f extends, in fact, to the whole family of discs A for any t in $[0, 1]$, hence in particular to z^o . We shall denote by F the holomorphic extension of f .

Denote now by M the hypersurface $L + i\mathbb{R}^2$ of $\mathbb{C}^2$. We suppose first that M contains no complex curve, that is, it is “minimal” in the sense of Tumanov. This implies that M contains no complex straight line and hence L is not a line; in particular, it is not a line parallel to the axes. In this case the complex lines $z_1 = x_1^o$ or $z_2 = x_2^o$ for $x^o \notin L$ cover the full set $(L + iT) \setminus L$, and hence by the propagation theorem from [4] already used, it extends through M over $L + iT$. But according to the theory by Tumanov [6], it also extends by minimality at the points of L . Thus F is holomorphic on the whole $\mathbb{C}^2$.

The other case is when M contains a complex curve, say γ . For any $z^o = x^o + iy^o \in \gamma$, we have $T_{z^o}\gamma = T_{x^o}L + iT_{x^o}L$. Hence γ contains the straight line issued from z^o in direction $iT_{x^o}L$, and therefore γ is in fact the straight complex line $\gamma = z^o + (T_{x^o}L + iT_{x^o}L)$ and L is the real line $L = x^o + T_{x^o}L$. At this point, we switch from the notation γ to L^C . We then prove that F extends also to those points in $L + i\mathbb{R}^2$ which are not in L^C . We denote by $l(z) = 0$ an equation for L^C and notice that M is foliated by the complex lines $\{z : l(z) = it\}$ for all values of the real parameter t . All these lines meet $\mathbb{R}^2$ outside L except for the line L^C which corresponds to $t = 0$. We also notice that the boundary values of F on M define a CR function on M . We then apply [4] and conclude that the analyticity of F propagates along the above lines L_t for $t \neq 0$ to reach all points in M except for those which belong to L^C . This completes the proof of Theorem 1. ■

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