NON-EXISTENCE OF POSITIVE SOLUTIONS OF FULLY NONLINEAR ELLIPTIC EQUATIONS IN UNBOUNDED DOMAINS

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1. Introduction

Consider the following ordinary differential equation:

$$u'' + bu' + cu = 0, \qquad \text{in } \mathbb{R}, \tag{1}$$

where b and c are constants. Obviously, the above equation admits positive (exponential) solutions if and only if $b^2 - 4c \geq 0$. Therefore, if $4c - b^2 > 0$ then the unique nonnegative solution of (1) is $u \equiv 0$, even if we replace $\mathbb{R}$ with an unbounded interval.

The paper [2], which deals with semilinear elliptic problems in $\mathbb{R}^N$, contains a generalization of the above result to partial differential equations of the type

$$-\Delta u - b \cdot Du - cu = 0, \quad \text{in } \mathbb{R}^N, \tag{2}$$

with $b \in \mathbb{R}^N$ and $c \in \mathbb{R}$. Indeed, Berestycki, Hamel and Nadirashvili implicitly proved that if $4c - |b|^2 > 0$, then the unique nonnegative solution of (2) is $u \equiv 0$. We will review the results of [2] and other related in the next section. In the joint work [4] with Berestycki and Hamel, we have extended the results of [2] to elliptic equations with non-constant coefficients. The arguments of [4] imply that, if $A(x)$ is a smooth uniformly elliptic matrix field, $b : \mathbb{R}^N \to \mathbb{R}^N$ and $c : \mathbb{R}^N \to \mathbb{R}$ are bounded and smooth and the following condition holds:

$$\liminf_{|x|\to\infty} (4\lambda(x)c(x) - |b(x)|^2) > 0, \tag{3}$$

where

$$\lambda(x) := \min_{|\xi|=1} A(x)\xi \cdot \xi,$$

then the only nonnegative function u satisfying

$$-\mathrm{tr}(A(x)D^2u) - b(x) \cdot Du - c(x)u \geq 0, \qquad x \in \mathbb{R}^N,$$

in the classical sense is $u \equiv 0$. Hence, roughly speaking, the condition $4c - |b|^2 > 0$ which guarantees the uniqueness result in the case of constant coefficients is only required at infinity.

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2. Main results

In the present paper, we improve the previous uniqueness result to lower semi-continuous functions u satisfying in the viscosity sense

$$F(x,u,Du,D^2u) \geq 0, \qquad x \in \Omega,$$

that is to viscosity super-solutions of the fully nonlinear equation

$$F(x,u,Du,D^2u) = 0, \qquad x \in \Omega, \tag{4}$$

(see Definition 4.1 below) where Ω is an unbounded domain in $\mathbb{R}^N$ satisfying some conditions we will precise later. We do not require any boundary condition. We will always assume that the function $F(x,t,p,M) : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}_N \to \mathbb{R}$ is measurable and (uniformly) elliptic, in the sense that there exist two constants $0 < \lambda \leq \Lambda$ such that

$$\lambda \mathrm{tr}(Q) \leq F(x,t,p,M) - F(x,t,p,M+Q) \leq \Lambda \mathrm{tr}(Q),$$

for all $x \in \Omega,\ t \in \mathbb{R},\ p \in \mathbb{R}^N$ and $M, Q \in \mathcal{S}_N$, Q nonnegative definite.

An important example of fully nonlinear second order elliptic operator is the Bellman operator, which arises in the control theory of diffusion processes (see [10] for a comprehensive treatment of the subject). It is defined by

$$F(x,u,Du,D^2u) := \sup_{\alpha \in \mathcal{A}} (-L_\alpha u(x)), \tag{5}$$

where $(-L_\alpha)_{\alpha \in \mathcal{A}}$ is a family of linear elliptic operators of the form

$$L_\alpha u(x) := \mathrm{tr}(A_\alpha(x) D^2 u) + b_\alpha(x) \cdot Du + c_\alpha(x) u,$$

such that

$$\forall\, \alpha \in \mathcal{A},\ x \in \Omega, \quad \lambda \leq \lambda_\alpha(x) := \min_{|\xi|=1} A_\alpha(x)\xi \cdot \xi \leq \max_{|\xi|=1} A_\alpha(x)\xi \cdot \xi \leq \Lambda. \tag{6}$$

It is quite natural to ask if the uniqueness result for linear operators mentioned in the introduction holds true for the Bellman operator, provided that all the L_α satisfy the assumption (3). We will show that the answer is affirmative if (3) holds uniformly in $\alpha \in \mathcal{A}$, in the sense that

$$\liminf_{|x| \to \infty} \left(\inf_{\alpha \in \mathcal{A}} \left(4\lambda_\alpha(x) c_\alpha(x) - |b_\alpha(x)|^2 \right) \right) > 0. \tag{7}$$

Furthermore, this is true not only for super-solutions in the whole space, but also in domains Ω containing balls of arbitrary large radius, i.e. such that

$$\sup_{x \in \Omega} \mathrm{dist}(x, \partial\Omega) = +\infty. \tag{8}$$

Condition (8) is fulfilled for example by (domains containing) the half-space, open cones, but also more general domains, such as the following spiral domain in $\mathbb{R}^2$:

$$\{(\rho\cos\theta, \rho\sin\theta) \mid \theta > 0,\ \theta^2 < \rho < \theta^2 + \theta\}.$$

Other assumptions we need on the coefficients of the operators L_α are

$$\sup_{\substack{x \in \Omega \\ \alpha \in \mathcal{A}}} |b_\alpha(x)| < +\infty, \qquad \inf_{\substack{x \in \Omega \\ \alpha \in \mathcal{A}}} c_\alpha(x) > -\infty. \tag{9}$$

Theorem 2.1. *Let F be the Bellman operator given by (5) and assume that (7)-(9) hold. Then the only nonnegative viscosity super-solution of (4) is $u \equiv 0$.*

Another classical example of fully nonlinear operator - arising in differential games - is the Isaacs operator

$$F(x,u,Du,D^2u) := \inf_{\beta\in\mathcal{B}} \sup_{\alpha\in\mathcal{A}} (-L_{\alpha,\beta}u(x)),$$

where the $-L_{\alpha,\beta}$ are linear elliptic operators, with the same elliptic parameters. Since $F = \inf_{\beta\in\mathcal{B}} F_\beta$, with F_β Bellman operators, any super-solution of $F = 0$ is also a super-solution of $F_\beta = 0$, $\forall\, \beta \in \mathcal{B}$. Therefore, the conclusion of Theorem 2.1 holds for the Isaacs operator provided that the assumptions there are satisfied at least by one of the F_β.

In order to extend the uniqueness result to more general fully nonlinear elliptic operators F, we first require that

$$\text{(10)}\qquad \begin{cases} (p,M) \mapsto F(x,t,p,M) \text{ is continuous in } (0,0), \\ \text{uniformly in } x\in\Omega,\ t\in\mathbb{R}_+, \end{cases}$$

$$\text{(11)}\qquad \forall\, t>0, \qquad \sup_{\substack{x\in\Omega \\ s\geq t}} F(x,s,0,0) < 0.$$

Condition (11) yields that (4) does not admit positive constant super-solutions. We need to translate condition (3) to fully nonlinear operators. A way to do it is to assume that there exist two bounded functions $b, c : \mathbb{R}^N \to \mathbb{R}$ and a positive constant δ such that

$$\text{(12)}\qquad \forall\, (x,t,p) \in \Omega\times[0,\delta]\times[-\delta,\delta]^N, \qquad F(x,t,p,0) \leq b(x)|p| - c(x)t,$$

and

$$\text{(13)}\qquad \liminf_{\substack{x\in\Omega \\ |x|\to\infty}} \big(4\lambda(x)c(x) - b^2(x)\big) > 0,$$

where

$$\text{(14)}\qquad \lambda(x) := \inf_{\substack{t\in\mathbb{R},\ p\in\mathbb{R}^N \\ M,Q\in\mathcal{S}_N \\ Q>0}} \frac{F(x,t,p,M) - F(x,t,p,M+Q)}{\mathrm{tr}Q}.$$

Notice that, since $\lambda(x) \geq \lambda$ for every $x \in \Omega$, we could simplify condition (13) by replacing $\lambda(x)$ with the elliptic parameter λ. We use $\lambda(x)$ instead because we want to obtain a stronger result which contains that for linear operators (as well as that of [4] for semilinear operators, cfr. Remark 2 below). Indeed, if F is a linear elliptic operator of the type $F(x,t,p,M) = -\mathrm{tr}(A(x)M) - b(x)\cdot p - c(x)t$, then (12) holds with $b(x) = |b(x)|$, $\lambda(x)$ coincides with the smallest eigenvalue of the matrix $A(x)$ and (13) reduces to (3). Instead, the result we obtain in the general setting does not contain that for Bellman operators - Theorem 2.1 - because condition (7) is weaker than (12)-(13) (see Proposition 2 below).

In this general setting, we are not able to deal with any positive super-solution u of (4), but only with those satisfying a prescribed maximal growth condition at infinity, which depends on the geometry of Ω. This condition reads:

$$\text{(15)}\qquad \inf_{x\in\Omega} \frac{u(x)+1}{\mathrm{dist}(x,\partial\Omega)} = 0.$$

Since the function $x \mapsto \mathrm{dist}(x,\partial\Omega)$ grows at most as $|x|$, it follows that u satisfies (15) only if it is strictly sublinear in an unbounded subset of Ω, in the sense that $\liminf_{|x|\to\infty} u(x)/|x| = 0$. On the other hand, if Ω contains an open cone, then it is possible to find a sequence $(x_n)_{n\in\mathbb{N}}$ in Ω and a constant $\alpha > 0$ such that $\mathrm{dist}(x_n, \partial\Omega) \geq \alpha|x_n|$. Therefore, in this case (15) holds for any strictly sublinear function u. Our result is

Theorem 2.2. *Assume that Ω satisfies (8) and F satisfies (10)-(13). Let u be a nonnegative viscosity super-solution of (4) satisfying (15). Then $u \equiv 0$ and $F(x, 0, 0, 0) = 0$ for $x \in \Omega$.*

A particular class of elliptic operators satisfying assumptions (12)-(13) is $F(x, t, p, M) = H(x, M) + b(x) \cdot p - c(x)t$, with H uniformly elliptic (with parameters λ, Λ) and b, c bounded and such that $H(x, 0) = 0$ and $\liminf_{|x|\to\infty}(4\lambda c(x) - |b(x)|^2) > 0$. Under these assumptions, the equation $H(x, D^2u) + b(x) \cdot Du - c(x)u = 0$ does not admit positive super-solutions in a domain Ω satisfying (8) (see Remark 3 below).

If $\Omega = \mathbb{R}^N$ we have that, for any $x \in \Omega$, $\mathrm{dist}(x, \partial\Omega) = +\infty$ and then - at least formally - condition (15) is always fulfilled. Indeed, when $\Omega = \mathbb{R}^N$ we are able to prove the uniqueness of nonnegative viscosity super-solutions u without requiring any growth condition of the type (15). Furthermore, we can relax (10)-(11) by

$$\left\{ \begin{array}{l} (p, M) \mapsto F(x, t, p, M) \text{ is continuous in } (0, 0), \\ \text{uniformly in } x \in \Omega, \text{ locally uniformly in } t \in \mathbb{R}_+, \end{array} \right. \tag{16}$$

$$\exists\, T > 0, \quad \forall\, t \in (0, T), \quad \sup_{\substack{x\in\Omega \\ s\in[t,T]}} F(x, s, 0, 0) < 0. \tag{17}$$

Theorem 2.3. *Assume that $\Omega = \mathbb{R}^N$ and that F satisfies (12)-(13) and (16)-(17). Let u be a viscosity super-solution of (4) such that $\inf_{\mathbb{R}^N} u \in [0, T)$. Then $u \equiv 0$ and $F(x, 0, 0, 0) = 0$ for $x \in \mathbb{R}^N$.*

If $\Omega \neq \mathbb{R}^N$ then conditions (16)-(17) do not yield the non-existence of positive super-solutions of (4), but only of those lying in $(0, T]$. The following result extends those of [2] and [4] for semilinear equations (see Section 3).

Proposition 1. *Assume that Ω satisfies (8) and F satisfies (12)-(13) and (16)-(17). Let u be a viscosity super-solution of (4) such that $0 \leq u \leq T$. Then $u \equiv 0$ and $F(x, 0, 0, 0) = 0$ for $x \in \Omega$.*

The assumption $u \leq T$ in Proposition 1 is sharp. Consider in fact the following operator in dimension one:

$$F(x, u, u', u'') = -u'' - \frac{2e^x}{\cosh x} u(1 - u), \qquad x \in (0, +\infty).$$

The domain $\Omega = (0, +\infty)$ satisfies (8), F is elliptic with parameters $\lambda = \Lambda = 1$ and conditions (12)-(13) hold with $\delta = 1/2$, $b \equiv 0$ and $c(x) = \frac{e^x}{\cosh x}$. Since the function $u(x) = \tanh x$ is a positive solution of $F(x, u, u', u'') = 0$ in Ω, the uniqueness result does not hold. Indeed, we are not under the hypotheses of Proposition 1 because (17) holds for any $T \in (0, 1)$, but does not hold for $T = 1 = \sup_\Omega u$.

Remark 1. Let us show that condition (8) in Theorem 2.2 is necessary and that the non-existence of positive super-solutions (even bounded) does not hold in general unbounded domains. Consider the domain $\Omega = \{\underline{x} = (x, y) \in \mathbb{R}^2 \mid x \in \mathbb{R},\ y \in (0, 1)\}$ and the operator

$$F(\underline{x}, t, p, M) = -\mathrm{tr}M - \frac{2t}{2 - y^2}, \qquad \forall\ \underline{x} = (x, y) \in \Omega.$$

It is easily seen that F is elliptic and satisfies (10)-(13), but the function $u(x, y) = 1 - y^2/2$ is a positive bounded solution of $F(\underline{x}, u, Du, D^2u) = 0$.

3. Previous results

In this section, we review some of the known results concerning the uniqueness of nonnegative solutions of elliptic equations. When one considers only bounded solutions, this type of results are often called Liouville type results, in analogy to the classical Liouville theorem for harmonic functions.

The result of [2] about semilinear equations mentioned in Section 1 is the following: assume that $b \in \mathbb{R}^N$ and $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(0) = f(1) = 0,\ f > 0$ in $(0, 1)$. If $4f'(0) - |b|^2 > 0$ then the unique solutions of

$$-\Delta u - b \cdot Du - f(u) = 0, \quad \text{in } \mathbb{R}^N, \tag{18}$$

satisfying $0 \le u \le 1$ are $u \equiv 0$ and $u \equiv 1$. Conversely, under the additional assumption $0 < f(s) \le f'(0)s$, if $4f'(0) - b^2 \le 0$ then a classical result of Kolmogorov, Petrovskiĭ and Piskunov [13] asserts that the equation $-u'' - bu' - f(u) = 0$ in $\mathbb{R}$ admits infinite many heteroclinic solutions with values in $(0, 1)$. Therefore, the result of [2] is sharp.

A first generalization of the previous Liouville type result to semilinear equations with non-constant coefficients has been given in [3]. There, the authors showed that the existence and uniqueness of positive bounded solutions of

$$-D \cdot (A(x)Du) - f(x, u) = 0, \qquad x \in \mathbb{R}^N,$$

with $A(x)$ and $x \mapsto f(x, t)$ *periodic*, with the same period, depend on the sign of the periodic principal eigenvalue of an associated linear operator.

In [4], we have extended the result of [2] to semilinear operators in non-divergence form with non-constant coefficients, without any periodicity assumptions. We considered the equation

$$-\operatorname{tr}(A(x)D^2u) - b(x) \cdot Du - f(x, u) = 0, \qquad x \in \mathbb{R}^N, \tag{19}$$

with A, b and f smooth. One of our results is that if

$$\forall\, x \in \mathbb{R}^N, \quad f(x, 0) = f(x, 1) = 0, \qquad \forall\, t \in (0, 1), \quad \inf_{x \in \mathbb{R}^N} f(x, t) > 0, \tag{20}$$

and (3) holds with $c(x) := f_t(x, 0)$, then the only classical super-solutions u of (19) satisfying $0 \le u \le 1$ in $\mathbb{R}^N$ are $u \equiv 0$ and $u \equiv 1$ (cfr. Theorem 3.7 in [4]). Actually, the hypothesis $f(x, 1) = 0$ was only needed to have the solution $u \equiv 1$.

Remark 2. Proposition 1 above completely extends Theorem 3.7 in [4], and then the result of [2]. Indeed, the equation (19) is a particular case of (4) with

$$F(x, t, p, M) = -\operatorname{tr}(A(x)M) - b(x) \cdot p - f(x, t).$$

If $f(x, \cdot) \in C^1(\mathbb{R}_+)$, uniformly in $x \in \Omega$, and (3) holds, then there exist $\delta, \varepsilon > 0$ such that (12)-(13) hold with $c(x) = f_t(x, 0) - \varepsilon$. Conversely, it is easy to check that (12)-(13) imply (3).

The first Liouville type results for viscosity solutions of fully nonlinear elliptic equations are due to Cabré and Caffarelli [5], in the case of equation $F(D^2u) = f(x)$ in the whole space. Under the assumption $F(x, 0) = 0$, Cutrì and Leoni [9] proved that the Liouville property still holds if we add a lower order perturbation term to the operator. More precisely, they showed that there exists $p_0 > 1$, depending on

λ/Λ and the dimension N, such that the only nonnegative viscosity super-solution u of

$$F(x, D^2u) - u^\alpha \geq 0, \qquad \text{in } \mathbb{R}^N,$$

with $\alpha \in (0, p_0)$, is $u \equiv 0$. In order to deal with general operators $F(x, u, Du, D^2u)$, Capuzzo Dolcetta and Cutrì introduced in [6] the following sublinear first-order dependence assumption on F:

$$\forall\ (x,t,p) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N, \qquad F(x,t,p,0) \leq b(|x|)|p|, \tag{21}$$

with b bounded and such that

$$\frac{-\Lambda(N-1)}{|x|} \leq b(|x|) \leq \frac{\lambda - \Lambda(N-1)}{|x|}, \qquad \text{for } |x| \text{ large.} \tag{22}$$

The authors proved that, under these assumptions, any nonnegative viscosity super-solution of (4) must be constant.

Our results are independent from those of [6]. Indeed, (21) yields (12) with $c \equiv 0$, but (13) holds only if $c > 0$. On the other hand, since we require (12) only for t, p small, (12) does not imply (21). Furthermore, (22) yields $\lim_{|x|\to\infty} b(|x|) = 0$, while on the contrary (13) may hold even if b does not vanish at infinity - provided that c is big enough.

4. Plan of the paper and preliminary results

In order to prove the above results we establish, in the next section, a version of the strong maximum principle (see [12], [1] for related results). This result yields that, under our assumptions, any nonnegative viscosity super-solution u of (4) is either identically equal to zero, or it is strictly positive. Then, we proceed following the same ideas as in [4]. In Section 6, we explicitly construct a family of C^2 sub-solutions ψ which are strictly positive in a ball and vanish on its boundary. Using the functions ψ as test functions, we are able to show that u has a positive infimum in a suitable subset of Ω. Finally, with the aid of other test functions and using the definition of viscosity super-solution, we get a contradiction with (11). The proofs of our main results - Theorems 2.2, 2.3 and 2.1 - are essentially based on the same ideas and are presented in the last three sections.

The starting point of our study is that, thanks to (12), we can replace the operator F with the following fully nonlinear operator:

$$F^+(x,t,p,M) := \mathcal{M}^+_{\lambda(x),\Lambda}(M) + b(x)|p| - c(x)t, \tag{23}$$

where $\mathcal{M}^+_{\lambda(x),\Lambda}$ denotes the Pucci's maximal operator associated with $\lambda(x), \Lambda$. The Pucci's maximal operator is a fundamental tool in the viscosity solutions theory. It is defined by:

$$\forall\ M \in \mathcal{S}_N, \qquad \mathcal{M}^+_{\lambda,\Lambda}(M) := -\lambda \sum_{e_i>0} e_i - \Lambda \sum_{e_i<0} e_i,$$

where $e_1, \ldots, e_N$ are the eigenvalues of the matrix M. The operator $\mathcal{M}^+_{\lambda,\Lambda}$ is, in some sense, the "biggest" elliptic operator with parameters λ, Λ. Consider an elliptic operator F and the associated function $\lambda(x)$ defined by (14). Then, we have that

$$\forall\ (x,t,p,M) \in \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}_N, \qquad F(x,t,p,M) \leq F(x,t,p,0) + \mathcal{M}^+_{\lambda(x),\Lambda}(M). \tag{24}$$

This is easily seen by decomposing M in the following way: $M = M^+ - M^-$, where $M^+, M^- \geq 0$ and $M^+M^- = 0$ (this decomposition exists for every $M \in \mathcal{S}_N$ and it is unique).

Throughout the paper, we will always denote with F^+ the operator given by (23), associated with an elliptic operator F satisfying (12). From (24), it follows that if the elliptic operator F satisfies (12), then

$$\forall\ (x,t,p,M) \in \Omega \times [0,\delta] \times [-\delta,\delta]^N \times \mathcal{S}_N, \qquad F(x,t,p,M) \leq F^+(x,t,p,M). \tag{25}$$

Therefore, any classical super-solution u of (4) is also a super-solution of $F^+ = 0$ in the set where u and Du are small, and this property extends to viscosity super-solutions. The advantage of using the operator F^+ instead of F is that $F^+(x,kt,kp,kM) = kF^+(x,t,p,M)$, for any $k > 0$.

Let us recall the definition of *viscosity solution*, which is the standard notion of weak solution for fully nonlinear elliptic equations.

Definition 4.1. We say that a function $u : \Omega \to \mathbb{R}$ is a *viscosity super-solution* (resp. *sub-solution*) of (4) if it is lower (upper) semi-continuous and, for any $\phi \in C^2(\Omega)$ and $x_0 \in \Omega$ such that $u - \phi$ has a local minimum (maximum) in x_0 one has:

$$F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \geq 0 \qquad (\leq 0).$$

If u is both a viscosity super- and sub-solution of (4), then we say that it is a viscosity solution of (4).

The definition of (continuous) viscosity solution dates back to Crandall and Lions [8]. The Definition 4.1 for semi-continuous functions can be found for instance in [11] and [7].

5. The strong maximum principle

The first step for proving our results consists in the derivation of a strong maximum principle for viscosity super-solutions of (4). We state it in a generic (eventually bounded) domain $\Omega \subset \mathbb{R}^N$, requiring that the following property holds:

$$\forall\ (x,t,p) \in \Omega \times [0,\delta] \times [-\delta,\delta]^N, \qquad F(x,t,p,0) \leq k|p| + lt, \tag{26}$$

for some constants $k, l \geq 0$. Since $F(x,t,p,M)$ is not assumed to be Lipschitz-continuous in t, p, we can not apply standard strong maximum principles for viscosity solutions such as the one of Kawohl and Kutev [12]. Nevertheless, the Lipschitz-continuity of $F(t,p,0,0)$ at $(0,0)$ given by (26) suffices to prove our result.

Note that, if one requires (26) with $\delta = \infty$, then any viscosity super-solution of (4) is also a super-solution of $\mathcal{M}^+_{\lambda,\Lambda}(M) + k|p| + lt$. Then, one could apply the strong maximum principle of [12], or of Bardi and Da Lio [1].

Lemma 5.1. *Let Ω be a general domain and assume that F is an elliptic operator satisfying (26). Let u be a nonnegative viscosity super-solution of (4). If there exists $\underline{x} \in \Omega$ such that $u(\underline{x}) = 0$, then $u \equiv 0$ and $F(x,0,0,0) = 0$.*

Proof. It is quite similar to the classical proof of the strong maximum principle for C^2 functions. Suppose, by a contradiction, that u is not identically equal to zero and that there exists $\underline{x} \in \Omega$ such that $u(\underline{x}) = 0$. Since the set $\{x \in \Omega \mid u(x) = 0\}$ can not be open (otherwise Ω would be disconnected) we infer that there exists $x_0 \in \Omega$ such that $u(x_0) = 0$ and $B_\gamma(x_0) \cap \{u > 0\} \neq \emptyset$ for all $\gamma > 0$. Call $\eta := \mathrm{dist}(x_0, \partial\Omega)$ and consider a point $x_1 \in B_{\eta/3}(x_0)$ such that $u(x_1) > 0$. Define

$$\beta := \max\{\gamma > 0 \mid u > 0 \text{ in } B_\gamma(x_1)\}.$$

Clearly $\beta \leq \eta/3$ and then $\overline{B_\beta(x_1)} \subset \Omega$. Furthermore, by the lower semi-continuity of u, we have that there exists $y \in \partial B_\beta(x_1)$ such that $u(y) = 0$. Set now $\rho := \beta/2$ and $z := (x_1 + y)/2$. Then, $y \in \partial B_\rho(z)$ and $\overline{B_\rho(z)} \setminus \{y\} \subset B_\beta(x_1)$. Resuming, we have that

$$u(y) = 0, \qquad u > 0 \text{ in } \overline{B_\rho(z)} \setminus \{y\}. \tag{27}$$

Define the function

$$\zeta(x) := e^{-\alpha|x-z|^2} - e^{-\alpha\rho^2}, \qquad x \in \mathbb{R}^N,$$

where $\alpha > 0$ will be chosen later. Consider k, l given by (26) and set $H(x, t, p, M) := \mathcal{M}^+_{\lambda,\Lambda}(M) + k|p| + lt$, for $(x, t, p, M) \in \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}_N$. By expressing the matrix $D^2\zeta(y)$ in any orthonormal basis containing the unit vector $(y - z)/|y - z|$, one finds that it has one eigenvalue equal to $(4\alpha^2\rho^2 - 2\alpha)e^{-\alpha\rho^2}$, and the others equal to $-2\alpha e^{-\alpha\rho^2}$. Consequently,

$$H(y, \zeta, D\zeta, D^2\zeta) \leq 2\alpha e^{-\alpha\rho^2}\big[-2\rho^2\lambda\alpha + N\Lambda + k\rho\big],$$

and then we can choose α big enough in order to have $H(y, \zeta, D\zeta, D^2\zeta) < 0$. Since H is continuous, there exists $\tau > 0$ such that $B_\tau(y) \subset \Omega$ and

$$H(x, \zeta, D\zeta, D^2\zeta) < 0, \qquad x \in B_\tau(y). \tag{28}$$

Call $K := \partial B_\tau(y) \cap \overline{B_\rho(z)}$ and

$$m := \min\left\{\min_K u\,,\ \delta\right\},$$

where δ is the positive constant in (26). By (27) we know that u is positive in the compact set $K$and then $m > 0$. Take $\varepsilon > 0$ such that

$$\forall\, x \in \overline{B_\tau(y)}, \qquad \varepsilon\zeta(x) < m, \qquad \varepsilon D\zeta(x) \in [-\delta, \delta]^N. \tag{29}$$

Since $\varepsilon\zeta < 0 < u$ in $\partial B_\tau(y) \setminus K$, we have that $\varepsilon\zeta < u$ in $\partial B_\tau(y)$. Moreover, $u(y) = 0 = \varepsilon\zeta(y)$ and then there exists $\tilde{x} \in B_\tau(y)$ such that

$$(u - \varepsilon\zeta)(\tilde{x}) = \min_{\overline{B_\tau(y)}}(u - \varepsilon\zeta) \leq 0.$$

Thus, the fact that u is a viscosity super-solution of $F = 0$ in Ω implies that $F(\tilde{x}, u(\tilde{x}), \varepsilon D\zeta(\tilde{x}), \varepsilon D^2\zeta(\tilde{x})) \geq 0$. Finally, by (24) and (26), we know that

$$\forall\, (x, t, p, M) \in \Omega \times [0, \delta] \times [-\delta, \delta]^N \times \mathcal{S}_N, \qquad F(x, t, p, M) \leq H(x, t, p, M).$$

Consequently, from (29), the fact that $\varepsilon\zeta(\tilde{x}) \geq u(\tilde{x}) \geq 0$ and that $H(x, t, p, M)$ is increasing in t, it follows

$$0 \leq H(\tilde{x}, \varepsilon\zeta(\tilde{x}), \varepsilon D\zeta(\tilde{x}), \varepsilon D^2\zeta(\tilde{x})) = \varepsilon H(\tilde{x}, \zeta(\tilde{x}), D\zeta(\tilde{x}), D^2\zeta(\tilde{x})),$$

which is in contradiction with (28). Therefore, $u \equiv 0$ and then $F(x, 0, 0, 0) \geq 0$. Since the reverse inequality also holds, by (26), we find that $F(x, 0, 0, 0) = 0$. $\square$

6. Construction of a family of positive sub-solutions of $F^+ = 0$

This section is devoted to the construction of a family of functions $\psi \in C^2(\Omega)$ which are strictly positive inside a ball, where they satisfy $F^+(x,\psi,D\psi,D^2\psi) < 0$, and vanish outside. The inequality (25) yields that the functions ψ - once opportunely normalized - are sub-solutions of (4). In the next sections, we will compare them to the super-solutions u in order to prove our uniqueness results.

We first construct a function in dimension one, then we rotate it and obtain the desired function ψ. This construction is essentially the same as in [4] (see the Appendix there), even if here we deal with the Pucci's maximum operator instead of a linear one.

For us, the symbol $\mathcal{M}^+_{\lambda,\Lambda}$ represents the Pucci's maximal operator in the N-dimensional case as well as in the 1-dimensional case, depending on the fact that its argument is a $N \times N$ matrix or a real number. Thus, if $a \in \mathbb{R}$,

$$\mathcal{M}^+_{\lambda,\Lambda}(a) = \begin{cases} -\lambda a & \text{if } a \geq 0 \\ -\Lambda a & \text{if } a < 0. \end{cases}$$

Lemma 6.1. *Let Λ, ξ, μ be three positive constants. Then there exist a nonnegative function $h \in C^2(\mathbb{R};\mathbb{R})$ and a positive number τ such that*

$$\begin{array}{rcl} h(\rho) = 0 & \textit{for} & \rho \leq 0, \\ h'(\rho) > 0 & \textit{for} & 0 < \rho < \tau, \\ h(\rho) = 1 & \textit{for} & \rho \geq \tau, \end{array}$$

and

$$\forall\, \rho \in \mathbb{R}_+, \qquad \mathcal{M}^+_{\eta,\Lambda}(h''(\rho)) + Bh'(\rho) - Ch(\rho) < 0,$$

for any positive constants η, B, C satisfying

$$\eta \leq \Lambda, \qquad B \leq \xi, \qquad 4\eta C - B^2 \geq \mu. \tag{30}$$

Proof. We will explicitly construct the desired function h. We start with defining it on the interval $(-\infty, r]$:

$$h(\rho) = \begin{cases} 0 & \text{for } \rho \leq 0 \\ \rho^n & \text{for } \rho \in (0,r] \end{cases}$$

where the integer $n \geq 3$ and the real $r > 0$ are to be chosen. Let η, B, C be three positive numbers satisfying (30). Consider the operator $H : \mathbb{R}^3 \to \mathbb{R}$ defined by $H(t,p,m) := \mathcal{M}^+_{\eta,\Lambda}(m) + Bp - Ct$. For $\rho \in (0,r]$, we have that

$$\begin{aligned} H(h(\rho), h'(\rho), h''(\rho)) &= \left[\eta(n-n^2) + Bn\rho - C\rho^2\right]\rho^{n-2} \\ &\leq \left[\eta(n-n^2) + \frac{B^2}{4C}n^2\right]\rho^{n-2} \\ &= \left(\eta - \frac{4\eta C - B^2}{4C}n\right) n\rho^{n-2}. \end{aligned}$$

From (30) it follows that, if $B^2 < \eta C$, then the last quantity is less than

$$\left(\eta - \frac{4\eta C - \eta C}{4C}n\right) n\rho^{n-2} \leq \left(1 - \frac{3}{4}n\right)\eta n\rho^{n-2} < 0,$$

else it is less than or equal to

$$\left(1 - \frac{\mu}{4\xi^2}n\right)\eta n\rho^{n-2}.$$

In both cases, choosing $n \in \mathbb{N}$ big enough (dependently only on ξ and μ) we have that $H(h, h', h'') < 0$ in $(0, r]$. Now that we have fixed n, choose $r > 0$ in such a way that the following inequalities hold:

$$\left(\xi n - \frac{\mu}{4\Lambda} r\right) r^{n-1} \leq -\Lambda - 1, \tag{31}$$

$$h''(r) = n(n-1)r^{n-2} > 1. \tag{32}$$

We then extend h in $(r, s]$ by setting $h''(\rho) = h''(r) - a(\rho - r)$ and requiring that $h \in C^2((-\infty, s))$. Again, the positive constants s, a will be chosen later. We impose $h''(s) = -1$, which yields $s = r + \frac{h''(r)+1}{a}$. The function h' reaches its maximum in $[r, s]$ at the point $r + h''(r)/a$. Then,

$$\forall\, \rho \in [r, s], \qquad h'(\rho) \leq h'\left(r + \frac{h''(r)}{a}\right) = h'(r) + \frac{[h''(r)]^2}{2a}.$$

Furthermore, $h'(s) = h'(r) + \frac{h''(r)+1}{2a}(h''(r) - 1) > h'(r)$, by (32), and then the concavity of h' in the interval $[r, s]$ yields $h' \geq h'(r) > 0$ in $[r, s]$. Consequently, h is strictly increasing in $[r, s]$. Using these inequalities, together with $4\eta C \geq \mu + B^2 \geq \mu$, we derive, for $\rho \in (r, s]$,

$$\begin{aligned} H(h(\rho), h'(\rho), h''(\rho)) &\leq \Lambda + \xi\left(h'(r) + \frac{[h''(r)]^2}{2a}\right) - \frac{\mu}{4\eta} h(r) \\ &\leq \Lambda + \left(\xi n - \frac{\mu}{4\Lambda} r\right) r^{n-1} + \frac{\xi}{2a}[h''(r)]^2. \end{aligned}$$

Therefore, by (31), $H(h, h', h'') \leq -1 + \frac{\xi}{2a}[h''(r)]^2$ in $(r, s]$, and then we can choose a big enough to have $H(h, h', h'') < 0$ in $(r, s]$. Finally, for $\rho \in (s, \tau]$, we set $h''(\rho) = -1 + d(\rho - s)$, with

$$\tau = s + 2h'(s)\,, \qquad d = \frac{1}{2h'(s)}\,.$$

It follows that $h'(\tau) = h''(\tau) = 0$. Since $h'' < 0$ in (s, τ), we have that h' is decreasing and positive in (s, τ), and then h is increasing in (s, τ). This allows to conclude that, for $\rho \in (s, \tau]$,

$$\begin{aligned} H(h(\rho), h'(\rho), h''(\rho)) &= -\Lambda h''(\rho) + Bh'(\rho) - Ch(\rho) \\ &\leq -\Lambda h''(s) + Bh'(s) - Ch(s) \\ &= H(h(s), h'(s), h''(s)) \\ &< 0. \end{aligned}$$

Extending h by $h(\rho) := h(\tau)$ for $\rho > \tau$, we have that $h \in C^2(\mathbb{R})$. Therefore, to obtain the desired function it only remains to divide h by $h(\tau)$. □

Now, turn to the N-dimensional case. By use of Lemma 6.1, we prove the following

Lemma 6.2. *Let F be an elliptic operator satisfying (12)-(13). There are then three positive constants R_0, ρ, γ such that, for every $R \geq R_0$ and $y \in \Omega$ such that $B_R(y) \subset \Omega \setminus B_\rho$, there exists $\psi \in C^2(\Omega)$ satisfying*

$$\psi = 0 \;\; in \;\; \Omega \setminus B_R(y), \qquad \psi = 1 \;\; in \;\; B_{R/2}(y), \qquad \|\psi\|_{C^1(\Omega)} \leq \gamma,$$

$$F^+(x, \psi, D\psi, D^2\psi) < 0 \;\; in \;\; B_R(y).$$

Proof. By (13), we can find two positive constants ρ and μ such that

$$\forall\, x \in \Omega \setminus B_\rho, \qquad 4\lambda(x)c(x) - b^2(x) \geq 2\mu. \tag{33}$$

Consider the function h and the constant $\tau > 0$ given by Lemma 6.1, associated to the positive constants Λ, μ and $\xi := \sup_{x\in\Omega} |b(x)| + 1$. Let $R_0 > 2\tau$ be such that

$$\frac{N-1}{R_0}\Lambda \leq \frac{1}{2}, \qquad 4\frac{N-1}{R_0}\Lambda\left(\frac{N-1}{R_0}\Lambda + \xi\right) \leq \mu. \tag{34}$$

Suppose that there exist $R \geq R_0$ and $y \in \Omega$ such that $B_R(y) \subset \Omega \setminus B_\rho$. For $x \in \Omega$ define

$$\psi(x) := h(R - |x-y|).$$

The function ψ belongs to $C^2(\Omega)$ because $h = 1$ in $(\tau, +\infty)$ and $R > \tau$. Moreover, $\|\psi\|_{C^1(\Omega)}$ is less than a positive constant γ only depending on $\|h\|_{C^1(\mathbb{R})}$ and N, and not on R and y. Let us compute $F^+(x, \psi, D\psi, D^2\psi)$. For $x \in B_{R/2}(y)$, we have that $R - |x-y| > \tau$ and then $\psi(x) = 1$. Hence, $F^+(x, \psi, D\psi, D^2\psi) = -c(x) < 0$, because, by (33), $c > 0$ in $B_{R/2}(y) \subset \Omega \setminus B_\rho$. Fix now $x \in B_R(y) \setminus B_{R/2}(y)$ and denote for brief $\rho := R - |x-y|$. We have that $\rho > 0$ and

$$\psi(x) = h(\rho), \qquad D\psi(x) = -\frac{x-y}{|x-y|}h'(\rho),$$

$$D^2\psi(x) = \left(\frac{(x-y)\otimes(x-y)}{|x-y|^2} - I\right)\frac{h'(\rho)}{|x-y|} + \frac{(x-y)\otimes(x-y)}{|x-y|^2}h''(\rho),$$

where $\otimes$ denotes the vector direct product and I the $N \times N$ identity matrix. The matrix $D^2\psi(x)$ can be diagonalized by expressing it in any orthonormal basis containing the vector $(x-y)/|x-y|$. This shows, after some computations, that one of its eigenvalues is equal to $h''(\rho)$ and the others $N-1$ are equal to $-h'(\rho)/|x-y|$. Consequently,

$$F^+(x, \psi, D\psi, D^2\psi) \leq \mathcal{M}^+_{\lambda(x),\Lambda}(h''(\rho)) + \left(|b(x)| + \frac{N-1}{|x-y|}\Lambda\right)h'(\rho) - c(x)h(\rho).$$

Set $\eta := \lambda(x)$, $C := c(x)$ and

$$B := |b(x)| + \frac{N-1}{|x-y|}\Lambda.$$

We have that

$$|b(x)| \leq B \leq |b(x)| + 2\frac{N-1}{R}\Lambda.$$

Furthermore, from (33) it follows that

$$4\eta C - B^2 \geq 2\mu - 4\frac{(N-1)^2}{R^2}\Lambda^2 - 4\xi\frac{N-1}{R}\Lambda.$$

Therefore, by (34), we have that $B \leq |b(x)| + 1 \leq \xi$ and $4\eta C - B^2 \geq \mu$, that is (30) holds. Lemma 6.1 then yields

$$F^+(x, \psi, D\psi, D^2\psi) \leq \mathcal{M}^+_{\eta,\Lambda}(h''(\rho)) + Bh'(\rho) - Ch(\rho) < 0.$$

□

7. The case $\Omega \neq \mathbb{R}^N$

Lemma 7.1. *Let Ω be an unbounded domain satisfying (8) and F an elliptic operator satisfying (12)-(13). Let u be a positive viscosity super-solution of (4). Then, there are two positive constants R_0 and ε such that*

$$\inf_{B_{R/2}(y)} u \geq \varepsilon,$$

for every $R \geq R_0$ and $y \in \Omega$ satisfying $B_R(y) \subset \Omega$.

Proof. Consider the positive constants R_0, ρ and γ given by Lemma 6.2. To prove the statement, consider first $y \in \Omega$ and $R \geq R_0$ such that $B_R(y) \subset \Omega \setminus B_\rho$. We claim that $\inf_{B_{R/2}(y)} u \geq \delta/\gamma$, where δ is the positive constant in (12). In order to prove this, consider the function ψ given by Lemma 6.2, associated to y and R. Define

$$k^* := \sup\{k \geq 0 \mid k\psi \leq u \text{ in } B_R(y)\}.$$

We assume, by a contradiction, that $k^* < \delta/\gamma$. From the definition of k^*, the lower semi-continuity of u and the fact that $\psi = 0$ on $\partial B_R(y)$, it follows that $u - k^*\psi \geq 0$ and there exists $\underline{x} \in B_R(y)$ such that $u(\underline{x}) = k^*\psi(\underline{x})$. Since u is a viscosity super-solution of (4), it follows that $F(\underline{x}, k^*\psi(\underline{x}), k^*D\psi(\underline{x}), k^*D^2\psi(\underline{x})) \geq 0$. Hence, we can apply (25), because

$$\forall\, x \in B_R(y), \qquad k^*\psi(x) \in [0, \delta], \qquad k^*D\psi(x) \in [-\delta, \delta]^N,$$

and we get

$$0 \leq F^+(\underline{x}, k^*\psi(\underline{x}), k^*D\psi(\underline{x}), k^*D^2\psi(\underline{x})) = k^*F^+(\underline{x}, \psi(\underline{x}), D\psi(\underline{x}), D^2\psi(\underline{x})).$$

This is a contradiction, because $F^+(x, \psi, D\psi, D^2\psi) < 0$ in $B_R(y)$. Therefore, $k^* \geq \delta/\gamma$, that is $u \geq \frac{\delta}{\gamma}\psi$ in $B_R(y)$. Then the claim is proved, because $\psi = 1$ in $B_{R/2}(y)$.

Consider now an arbitrary ball $B_R(y) \subset \Omega$, with $R \geq R_0$. Set $K := \{x \in \Omega \cap \overline{B}_{\rho+3R_0} \mid \text{dist}(x, \partial\Omega) \geq R_0/2\}$. Since $K \subset\subset \Omega$ and u is positive and lower semi-continuous, it follows that $\mu := \min_K u > 0$. If $|y| \geq R + \rho$ then $B_R(y) \subset \Omega \setminus B_\rho$. Hence, we are in the case considered before and then $\inf_{B_{R/2}(y)} u \geq \delta/\gamma$. If $|y| < R + \rho$, consider $x \in B_{R/2}(y)$. If $x \in \overline{B}_{\rho+3R_0}$ then $x \in K$, because $\text{dist}(x, \partial\Omega) \geq R/2 \geq R_0/2$. Hence, in this case $u(x) \geq \mu$. If, on the contrary, $x \notin \overline{B}_{\rho+3R_0}$ then we have

$$\frac{R}{2} > |x - y| \geq |x| - |y| > 3R_0 - R.$$

Thus, $R_0 < R/2$ and then $B_{R_0}(x) \subset (B_R(y) \setminus B_\rho) \subset (\Omega \setminus B_\rho)$. It follows that $u(x) \geq \delta/\gamma$. The statement is then proved, with $\varepsilon = \min\{\delta/\gamma, \mu\}$. □

Remark 3. In the proof of Lemma 7.1, we have shown that - with the same assumptions and notation as there - there exists a positive constant γ such that any nonnegative viscosity super-solution u of (4) is either identically equal to zero, or satisfies $\inf_{B_{R/2}(y)} u \geq \delta/\gamma$, for any $R \geq R_0$ and y such that $B_R(y) \subset \Omega \setminus B_\rho$. Consequently, if Ω satisfies (8), F is elliptic and satisfies (12)-(13) with $\delta = +\infty$, then the unique nonnegative viscosity super-solution of (4) is $u \equiv 0$. Thus, in this case, the conclusion of Theorem 2.2 holds for any nonnegative viscosity super-solution u without prescribing any maximal growth.

Proof of Theorem 2.2. Suppose that u vanishes at some point in Ω. By (12), we have that (26) holds with $k = \|b\|_\infty$ and $l = \|c\|_\infty$. Hence, we can apply Lemma 5.1 and infer that $u \equiv 0$ and $F(x, 0, 0, 0) = 0$ in Ω.

Assume, by a contradiction, that u is strictly positive. Let us denote with $\sigma : \Omega \to \mathbb{R}_+$ the distance function from $\partial\Omega$, that is $\sigma(x) := \mathrm{dist}(x, \partial\Omega)$. By (15), there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in Ω such that

$$|x_n| \to \infty, \qquad \lim_{n\to\infty} \frac{u(x_n)}{\sigma(x_n)} = 0, \qquad \lim_{n\to\infty} \sigma(x_n) = +\infty. \tag{35}$$

Let $n_0 \in \mathbb{N}$ be such that $\sigma(x_n) \geq R_0$ for $n \geq n_0$, where R_0 is the positive constant in Lemma 7.1. Then, Lemma 7.1 yields:

$$\forall\, x \in \bigcup_{n\geq n_0} B_{\sigma(x_n)/2}(x_n), \qquad u(x) \geq \varepsilon > 0.$$

For $n \in \mathbb{N}$, define

$$\phi_n(x) := -4u(x_n)\frac{|x - x_n|^2}{\sigma^2(x_n)}.$$

Since $(u-\phi_n)(x_n) = u(x_n)$ and, for $x \in \partial B_{\sigma(x_n)/2}(x_n)$, $(u-\phi_n)(x) = u(x)+u(x_n) \geq u(x_n)$, we infer that there exists a point $z_n \in B_{\sigma(x_n)/2}(x_n)$ of local minimum for the function $u - \phi_n$. Therefore,

$$F(z_n, u(z_n), D\phi_n(z_n), D^2\phi_n(z_n)) \geq 0. \tag{36}$$

For $n \geq n_0$ we have that

$$u(z_n) \geq \varepsilon, \qquad |D\phi_n(z_n)| \leq \frac{4u(x_n)}{\sigma(x_n)}, \qquad D^2\phi_n(z_n) = 8\frac{u(x_n)}{\sigma^2(x_n)} I.$$

Consequently, letting n go to infinity in (36) and using (10) and (35), we find that $\sup_{\substack{x\in\Omega \\ s\geq\varepsilon}} F(x, s, 0, 0) \geq 0$, which is in contradiction with (11). □

Proof of Proposition 1. As in the proof of Theorem 2.2, we have that the strong maximum principle implies that either $u \equiv 0$, or $u > 0$ in Ω. In the second case, Lemma 7.1 yields the existence of two positive constants ε, R_0 such that

$$\inf_{B_{R/2}(y)} u \geq \varepsilon,$$

for every $R \geq R_0$ and $y \in \Omega$ satisfying $B_R(y) \subset \Omega$. For $n \in \mathbb{N}$ let $x_n \in \Omega$ be such that $\mathrm{dist}(x_n, \partial\Omega) \geq 2n$. Then, consider the family of paraboloids $\phi_n(x) = -u(x_n)|x - x_n|^2/n^2$. We have that $(u - \phi_n)(x_n) = u(x_n)$ and, for any $x \in \partial B_n(x_n)$,

$$(u - \phi_n)(x) = u(x) + u(x_n) \geq u(x_n).$$

Hence, $u - \phi_n$ admits a local minimum at some point $z_n \in B_n(x_n)$. Applying the definition of viscosity super-solution we get

$$F(z_n, u(z_n), D\phi_n(z_n), D^2\phi_n(z_n)) \geq 0.$$

Notice that $u(z_n) \geq \varepsilon$ for $n \geq R_0/2$ and that $D\phi_n(z_n), D^2\phi_n(z_n)$ go to zero as n goes to infinity. Consequently, from (16) it follows that

$$\inf_{\substack{x\in\Omega \\ t\in[\varepsilon, \sup u]}} F(x, s, 0, 0) \geq 0,$$

which contradicts (17). □

8. The case $\Omega = \mathbb{R}^N$

The proof of Theorem 2.3 is similar to that of Theorem 2.2, and relies on the same result, Lemma 7.1. Here, no growth condition on the viscosity super-solution u is needed because, roughly speaking, the fact that $\Omega = \mathbb{R}^N$ allows to focus our attention on the points where u is small. More precisely, Lemma 7.1 implies that any positive viscosity super-solution of (4) is bounded from below away from zero. Then, we can apply the same arguments of the proof of Theorem 2.2, using a family of test functions $(\phi)_{n\in\mathbb{N}}$ given by paraboloids centered at a minimizing sequence of u.

Proof of Theorem 2.3. Let u be a viscosity super-solution of (4) such that $\inf u \in [0,T)$. The strong maximum principle (Lemma 5.1) yields that either u is strictly positive, or $u \equiv 0$ and $F(x,0,0,0) \equiv 0$. Assume, by a contradiction, that $u > 0$ in $\mathbb{R}^N$. We apply Lemma 7.1 that, in the case $\Omega = \mathbb{R}^N$, yields $\inf_{\mathbb{R}^N} u \geq \varepsilon$. Set $m := \inf_{\mathbb{R}^N} u > 0$ and let $(y_n)_{n\in\mathbb{N}}$ be such that $\lim_{n\to\infty} u(y_n) = m$. For $n \in \mathbb{N}$, define the functions

$$\phi_n(x) := u(y_n) - \frac{1}{n}|x - y_n|^2.$$

For every $n \in \mathbb{N}$, we have that $(u - \phi_n)(y_n) = 0$ and $\lim_{|x|\to\infty}(u - \phi_n)(x) = +\infty$. Hence, there exists $z_n \in \mathbb{R}^N$ minimum point for $u - \phi_n$ in $\mathbb{R}^N$. Therefore, $F(z_n, u(z_n), D\phi_n(z_n), D^2\phi_n(z_n)) \geq 0$. Moreover, since

$$0 \geq u(z_n) - \phi_n(z_n) = u(z_n) - u(y_n) + \frac{1}{n}|z_n - y_n|^2,$$

it follows that $u(z_n) \leq u(y_n)$ and, for n large enough, $|z_n - y_n| \leq \sqrt{n(m+1)}$. This shows that $u(z_n) \to m$ and $D\phi_n(z_n),\ D^2\phi_n(z_n) \to 0$ as n goes to infinity. In particular, there exists $n_0 \in \mathbb{N}$ such that $u(z_n) \in [m,T)$ for any $n \geq n_0$. Then, the continuity assumption (16) yields that for any $\varepsilon > 0$ there exists $n = n(\varepsilon) \geq n_0$ such that $F(z_n, u(z_n), 0, 0) \geq -\varepsilon$. Consequently,

$$\sup_{\substack{x\in\mathbb{R}^N \\ s\in[m,T)}} F(x,s,0,0) \geq 0,$$

which contradicts (17). □

9. The Bellman operator

In this section, we deal with the Bellman operator (5) and we give the proof of Theorem 2.1. To do that, we follow exactly the same ideas of the proofs of Theorems 2.2 and 2.3.

Theorem 2.1 is not contained in Theorems 2.2 and 2.3 for two reasons. First, we do not assume that the Bellman operator satisfies condition (13), but only that (13) holds for all the operators $-L_\alpha$, uniformly in $\alpha \in \mathcal{A}$ (which is a weaker assumption, see Proposition 2 below). Second, using the fact that $F(x,kt,kp,kM) = kF(x,t,p,M)$ for any $k > 0$, we are able to prove the uniqueness result in any unbounded domain Ω satisfying (8) without prescribing any maximal growth of the super-solutions.

Throughout this section, F will denote the Bellman operator, as we defined it in Section 2.

Proof of Theorem 2.1. Let u be a nonnegative viscosity super-solution of (4). For $x \in \Omega,\ t \in \mathbb{R}_+$ and $p \in \mathbb{R}^N$, we have that

$$F(x,t,p,0) \le \sup_{\alpha\in\mathcal{A}} |b_\alpha(x)||p| - \inf_{\alpha\in\mathcal{A}} c_\alpha(x)t \le \sup_{\substack{z\in\Omega\\ \alpha\in\mathcal{A}}} |b_\alpha(z)||p| - \inf_{\substack{z\in\Omega\\ \alpha\in\mathcal{A}}} c_\alpha(z)t.$$

Then, thanks to (9), we can apply Lemma 5.1 and infer that either $u > 0$ in Ω, or $u \equiv 0$ and $F(x,0,0,0) \equiv 0$.

In order to prove that u can not be strictly positive, we construct a function ψ which is positive in a ball and is a sub-solution of (4). The construction is almost the same as in Lemma 6.2. By (7), there exist two constants $\rho, \mu > 0$ such that

$$\forall\, x \in \Omega \setminus B_\rho,\quad \forall\, \alpha \in \mathcal{A},\qquad 4\lambda_\alpha(x)c_\alpha(x) - |b_\alpha(x)|^2 \ge 3\mu. \tag{37}$$

Consider the function h and the positive constant τ given by Lemma 6.1, associated to Λ, $\xi := \sup_{\substack{\alpha\in\mathcal{A}\\ x\in\Omega}} |b_\alpha(x)| + 1$ and μ. Let $R \ge 2\tau$ be such that

$$2\frac{N-1}{R}\Lambda \le 1, \qquad 4\frac{N-1}{R}\Lambda\left(\frac{N-1}{R}\Lambda + \xi\right) \le \mu, \tag{38}$$

and $y \in \Omega$ be such that $B_R(y) \subset \Omega \setminus B_\rho$. Define the function $\psi(x) := h(R - |x-y|)$. We claim that

$$F(x,\psi,D\psi,D^2\psi) < 0, \qquad x \in B_R(y).$$

Take $x \in B_{R/2}(y)$. We have that

$$F(x,\psi,D\psi,D^2\psi) = -\inf_{\alpha\in\mathcal{A}} c_\alpha(x) \le -\frac{3\mu}{4\Lambda},$$

where the last inequality follows from (37) and (6). Consider now $x \in B_R(y) \setminus B_{R/2}(y)$. For any $\gamma > 0$ there exists $\alpha = \alpha_{\gamma,x}$ such that $F(x,\psi,D\psi,D^2\psi) \le (-L_\alpha + \gamma)\psi(x)$. Then, after the usual computations, we find:

$$F(x,\psi,D\psi,D^2\psi) \le \mathcal{M}^+_{\eta,\Lambda}(h''(R-|x-y|)) + Bh'(R-|x-y|) - C_\gamma h(R-|x-y|),$$

with $\eta = \lambda_\alpha(x)$,

$$B = |b_\alpha(x)| + 2\frac{N-1}{R}\Lambda$$

and $C_\gamma = c_\alpha(x) - \gamma$. By (38) and (37), we have that $B \le \xi$ and $4\eta C_\gamma - B^2 \ge 2\mu - 4\Lambda\gamma$. Therefore, for $\gamma \le \frac{\mu}{4\Lambda}$, the quantities $\eta,\ B$ and $C = C_\gamma$ satisfy (30) and then Lemma 6.1 yields $F(x,\psi,D\psi,D^2\psi) < 0$. The claim is then proved.

Now, we assume by a way of contradiction that $u > 0$. Since the Bellman operator F satisfies $F(x,kt,kp,kM) = kF(x,t,p,M)$, for any positive constant k, we can use the functions $k\psi$ as test functions as we did in the proof of Lemma 7.1, with F^+ replaced by F. Indeed, set

$$k^* := \inf_{x\in B_R(y)} \frac{u(x)}{\psi(x)}.$$

Clearly, $k^* > 0$ and $u - k^*\psi \ge 0$ in $B_R(y)$. Furthermore, since $\psi = 0$ on $\partial B_R(y)$, there exists $\underline{x} \in B_R(y)$ such that $(u - k^*\psi)(\underline{x}) = 0$. Hence,

$$0 \le F(\underline{x}, k^*\psi(\underline{x}), k^*D\psi(\underline{x}), k^*D^2\psi(\underline{x})) = k^*F(\underline{x}, \psi(\underline{x}), D\psi(\underline{x}), D^2\psi(\underline{x})).$$

This is in contradiction with the fact that $F(x,\psi,D\psi,D^2\psi) < 0$ for $x \in B_R(y)$. $\square$

Let us conclude with showing that the assumption (7) is weaker than (12)-(13).

Proposition 2. *If the Bellman operator F satisfies (12)-(13) then (7) holds. On the contrary, there are examples of Bellman operators satisfying (7) but not (12)-(13).*

Proof. Assume that F satisfies (12), that is

$$\forall\ (x,t,p)\in\Omega\times[0,\delta]\times[-\delta,\delta]^N,\quad \sup_{\alpha\in\mathcal{A}}(-b_\alpha(x)\cdot p-c_\alpha(x)t)\leq b(x)|p|-c(x)t.$$

It follows that

$$c(x)\leq\inf_{\alpha\in\mathcal{A}}c_\alpha(x),\qquad b(x)\geq\sup_{\alpha\in\mathcal{A}}|b_\alpha(x)|.$$

Moreover, we claim that $\lambda(x)\leq\inf_{\alpha\in\mathcal{A}}\lambda_\alpha(x)$, for $x\in\Omega$, where $\lambda(x)$ is given by (14) and $\lambda_\alpha(x)$ is the smallest eigenvalue of $A_\alpha(x)$. Indeed, we have that

$$\lambda(x)\leq\inf_{\substack{M,Q\in\mathcal{S}_N\\ Q>0}}\frac{\sup_{\alpha\in\mathcal{A}}(-\mathrm{tr}(A_\alpha(x)M))-\sup_{\alpha\in\mathcal{A}}(-\mathrm{tr}(A_\alpha(x)(M+Q)))}{\mathrm{tr}Q}.$$

Thus, if we take in particular $M=Q$, we obtain

$$\begin{aligned}\lambda(x)&\leq\inf_{\substack{Q\in\mathcal{S}_N\\ Q>0}}\frac{-\inf_{\alpha\in\mathcal{A}}\mathrm{tr}(A_\alpha(x)Q)+2\inf_{\alpha\in\mathcal{A}}\mathrm{tr}(A_\alpha(x)Q)}{\mathrm{tr}Q}\\ &=\inf_{\substack{Q\in\mathcal{S}_N,\ Q>0\\ \alpha\in\mathcal{A}}}\frac{\mathrm{tr}(A_\alpha(x)Q)}{\mathrm{tr}Q}\\ &=\inf_{\alpha\in\mathcal{A}}\lambda_\alpha(x).\end{aligned}$$

Therefore, if F satisfies (13) then (7) holds.

Consider now the Bellman operator

$$F(x,u,u',u'')=\sup(-L_1u,-L_2u),\quad\text{in }\mathbb{R},$$

with $-L_1u=-u''-u$ and $-L_2u=-u''+3u'-3u$. We have that F satisfies (7). Assume that (12) holds for some $b,c\in L^\infty(\mathbb{R})$. Thus, taking $t=1$ and $p=0$ in (12), we derive

$$c(x)\leq-\sup(-1,-3)=1,$$

while, taking $t=0$ and $p=1$,

$$b(x)\geq\sup(0,3)=3.$$

Since $\lambda(x)\equiv1$, it follows that $4\lambda(x)c(x)-b^2(x)\leq-5$, i. e. (13) does not hold. □

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