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The classification of surfaces with $p_g = q = 1$ isogenous to a product of curves

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Abstract. A smooth, projective surface S is said to be *isogenous to a product* if there exist two smooth curves C, F and a finite group G acting freely on $C \times F$ so that $S = (C \times F)/G$. In this paper we classify all surfaces with $p_g = q = 1$ which are isogenous to a product.

Key words. Surfaces of general type, isotrivial fibrations, actions of finite groups.

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0 Introduction

The classification of smooth, complex surfaces S of general type with small birational invariants is quite a natural problem in the framework of algebraic geometry. For instance, one may want to understand the case where the Euler characteristic $\chi(\mathcal{O}_S)$ is 1, that is, when the geometric genus $p_g(S)$ is equal to the irregularity $q(S)$. All surfaces of general type with these invariants satisfy $p_g \leq 4$. In addition, if $p_g = q = 4$ then the self-intersection K_S^2 of the canonical class of S is equal to 8 and S is the product of two genus 2 curves, whereas if $p_g = q = 3$ then $K_S^2 = 6$ or 8 and both cases are completely described ([11], [17], [24]). On the other hand, surfaces of general type with $p_g = q = 0, 1, 2$ are still far from being classified. We refer the reader to the survey paper [3] for a recent account on this topic and a comprehensive list of references.

A natural way of producing interesting examples of algebraic surfaces is to construct them as quotients of known ones by the action of a finite group. For instance Godeaux constructed in [15] the first example of surface of general type with vanishing geometric genus taking the quotient of a general quintic surface of $\mathbb{P}^3$ by a free action of $\mathbb{Z}_5$. In line with this Beauville proposed in [4, p. 118] the construction of a surface of general type with $p_g = q = 0$, $K_S^2 = 8$ as the quotient of a product of two curves C and F by the free action of a finite group G whose order is related to the genera $g(C)$ and $g(F)$ by the equality $|G| = (g(C) - 1)(g(F) - 1)$. Generalizing Beauville's example we say that a surface S is *isogenous to a product* if $S=(C\times F)/G$, for C and F smooth curves and G a finite group acting freely on $C\times F$. A systematic study of these surfaces has been carried out in [8]. They are of general type if and only if both $g(C)$ and $g(F)$ are greater than or equal to 2 and in this case S admits a unique minimal realization where they are as small as possible. From now on, we tacitly assume that such a realization is chosen, so that the genera of the curves and the group G are invariants of S. The action of G can be seen to respect the product structure on $C\times F$. This means that such actions fall in two cases: the *mixed* one, where there exists some element in G exchanging the two factors (in this situation C and F must be isomorphic) and the *unmixed* one, where G acts faithfully on both C and F and diagonally on their product.

After [4], examples of surfaces isogenous to a product with $p_g=q=0$ appeared in [22] and [1], and their complete classification was obtained in [2].

The next natural step is therefore the analysis of the case $p_g=q=1$. Surfaces of general type with these invariants are the irregular ones with the lowest geometric genus and for this reason it would be important to provide their complete description. So far, this has been obtained only in the cases $K_S^2=2,3$ ([7], [9], [10], [25], [12]).

The goal of the present paper is to give the full list of surfaces with $p_g=q=1$ that are isogenous to a product. Our work has to be seen as the sequel to the article [26], which describes all unmixed cases with G abelian and some unmixed examples with G nonabelian. Apart from the complete list of the genera and groups occurring, our paper contains the first examples of surfaces of mixed type with $q=1$. The mixed cases turn out to be much less frequent than the unmixed ones and, as when $p_g=q=0$, they occur for only one value of the order of G. However, in contrast with what happens when $p_g=q=0$, the mixed cases do not correspond to the maximum value of $|G|$ but appear for a rather small order, namely $|G|=16$.

Our classification procedure involves arguments from both geometry and computational group theory. We will give here a brief account on how the result is achieved.

If S is any surface isogenous to a product and satisfying $p_g=q$ then $|G|$, $g(C)$, $g(F)$ are related as in Beauville's example and we have $K_S^2=8$. Besides, if $p_g=q=1$ then such surfaces are necessarily minimal and of general type (Lemma 2.1).

If $S=(C\times F)/G$ is of unmixed type, then the two projections $\pi_C\colon C\times F\longrightarrow C$, $\pi_F\colon C\times F\longrightarrow F$ induce two morphisms $\alpha\colon S\longrightarrow C/G$, $\beta\colon S\longrightarrow F/G$, whose smooth fibres are isomorphic to F and C, respectively. Moreover, the geometry of S is encoded in the geometry of the two coverings $h\colon C\longrightarrow C/G$, $f\colon F\longrightarrow F/G$ and the invariants of S impose strong restrictions on $g(C)$, $g(F)$ and $|G|$. Indeed we have $1=q(S)=g(C/G)+g(F/G)$ so we may assume that $E:=C/G$ is an elliptic curve and $F/G\cong\mathbb{P}^1$. Then $\alpha\colon S\longrightarrow E$ is the Albanese morphism of S and the genus g_{alb} of the general Albanese fibre equals $g(F)$. It is proven in [26, Proposition 2.3] that $3\leq g(F)\leq 5$; in particular this allows us to control $|G|$. The covers f and h are determined by two suitable systems of generators for G, that we call $\mathcal{V}$ and $\mathcal{W}$, respectively. Besides, in order to obtain a free action of G on $C\times F$ and a quotient S with the desired invariants, $\mathcal{V}$ and $\mathcal{W}$ are subject to strict conditions of combinatorial nature (Proposition 2.2). The geometry imposes also strong restrictions on the possible $\mathcal{W}$ and the genus of C, so the existence of $\mathcal{V}$ and $\mathcal{W}$ and the compatibility conditions can be verified by a computer search. It is worth mentioning that the classification of finite groups of automorphisms acting on curves of genus less than or equal to 5 could have also been retrieved from the existing literature ([6], [19], [20], [21]).

If $S = (C \times C)/G$ is of mixed type, then the index two subgroup G° of G corresponding to transformations that do not exchange the coordinates in $C \times C$ acts faithfully on C. The quotient $E = C/G^\circ$ is isomorphic to the Albanese variety of S and $g_{\mathrm{alb}} = g(C)$ (Proposition 2.5). Moreover $g(C)$ may only be 5, 7 or 9, hence $|G|$ is at most 64 (Proposition 2.10). The cover $h \colon C \longrightarrow E$ is determined by a suitable system of generators $\mathcal{V}$ for G° and since the action of G on $C \times C$ is required to be free, combinatorial restrictions involving the elements of $\mathcal{V}$ and those of $G \backslash G^\circ$ have to be imposed (Proposition 2.6). Our classification is obtained by first listing those groups G° for which $\mathcal{V}$ exists and then by looking at the admissible extensions G of G°. We find that the only possibility occurring is for $g(C) = 5$ so that $|G|$ is necessarily 16 (Propositions 4.1, 4.2, 4.3).

In the last part of the paper we examine the structure of the subset of the moduli space corresponding to surfaces isogenous to a product with $p_g = q = 1$. It can be explicitly described by calculating the number of orbits of the direct product of certain mapping class groups with $\mathrm{Aut}(G)$ acting on the set (of pairs) of systems of generators (Proposition 5.1). In particular it is possible to determine the number of irreducible connected components and their respective dimensions, see the forthcoming article [23].

Our computations were carried out by using the computer algebra program `GAP4`, whose database includes all groups of order less than 2000, with the exception of 1024 (see [16]). For the reader's convenience we included the scripts in the Appendix.

Now let us state the main result of this paper.

Main Theorem. *Let $S = (C \times F)/G$ be a surface with $p_g = q = 1$, isogenous to a product of curves. Then S is minimal of general type and the occurrences for $g(F)$, $g(C)$, G, the dimension D of the moduli space and the number N of its connected components are precisely those in Table* 1.

In the table `IdSmallGroup`(G) denotes the label of the group G in the `GAP4` database of small groups. The calculation of N is due to Penegini and Rollenske, see [23], except for the cases marked with $(*)$, which were already studied in [26]. The cases marked with $(**)$ also appeared in [26], but the computation of N was missing.

This work is organized as follows. In Section 1 we collect the basic facts about surfaces isogenous to a product, following the treatment given by Catanese in [8] and we fix the algebraic setup. In Section 2 we apply the structure theorems of Catanese to the case $p_g = q = 1$ and this leads to Propositions 2.2 and 2.6, that provide the translation of our classification problem from geometry to algebra. All these results are used in Sections 3 and 4, which are the core of the paper and give the complete lists of the occurring groups and genera in the unmixed and mixed cases, respectively. Finally, Section 5 is devoted to the description of the moduli spaces.

Notations and conventions. All varieties, morphisms, etc. in this article are defined over $\mathbb{C}$. By "surface" we mean a projective, non-singular surface S, and for such a surface K_S denotes the canonical class, $p_g(S) = h^0(S, K_S)$ is the *geometric genus*, $q(S) = h^1(S, K_S)$ is the *irregularity* and $\chi(\mathcal{O}_S) = 1 - q(S) + p_g(S)$ is the *Euler characteristic*.

$g(F) = g_{\mathrm{alb}}$	$g(C)$	G	$\mathtt{IdSmallGroup}(G)$	Type	D	N
3	3	$(\mathbb{Z}_2)^2$	$G(4,2)$	unmixed $(*)$	5	1
3	5	$(\mathbb{Z}_2)^3$	$G(8,5)$	unmixed $(*)$	4	1
3	5	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$G(8,2)$	unmixed $(*)$	3	2
3	9	$\mathbb{Z}_2 \times \mathbb{Z}_8$	$G(16,5)$	unmixed $(*)$	2	1
3	5	D_4	$G(8,3)$	unmixed	3	1
3	7	D_6	$G(12,4)$	unmixed $(**)$	3	1
3	9	$\mathbb{Z}_2 \times D_4$	$G(16,11)$	unmixed	3	1
3	13	$D_{2,12,5}$	$G(24,5)$	unmixed	2	1
3	13	$\mathbb{Z}_2 \times A_4$	$G(24,13)$	unmixed	2	1
3	13	S_4	$G(24,12)$	unmixed	2	1
3	17	$\mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_8)$	$G(32,9)$	unmixed	2	1
3	25	$\mathbb{Z}_2 \times S_4$	$G(48,48)$	unmixed	2	1
4	3	S_3	$G(6,1)$	unmixed $(**)$	4	1
4	5	D_6	$G(12,4)$	unmixed	3	1
4	7	$\mathbb{Z}_3 \times S_3$	$G(18,3)$	unmixed	2	2
4	7	$\mathbb{Z}_3 \times S_3$	$G(18,3)$	unmixed	1	1
4	9	S_4	$G(24,12)$	unmixed $(**)$	2	1
4	13	$S_3 \times S_3$	$G(36,10)$	unmixed	1	1
4	13	$\mathbb{Z}_6 \times S_3$	$G(36,12)$	unmixed	1	1
4	13	$\mathbb{Z}_4 \ltimes (\mathbb{Z}_3)^2$	$G(36,9)$	unmixed	1	2
4	21	A_5	$G(60,5)$	unmixed $(**)$	1	1
4	25	$\mathbb{Z}_3 \times S_4$	$G(72,42)$	unmixed	1	1
4	41	S_5	$G(120,34)$	unmixed	1	1
5	3	D_4	$G(8,3)$	unmixed $(**)$	4	1
5	4	A_4	$G(12,3)$	unmixed $(**)$	2	2
5	5	$\mathbb{Z}_4 \ltimes (\mathbb{Z}_2)^2$	$G(16,3)$	unmixed	2	3
5	7	$\mathbb{Z}_2 \times A_4$	$G(24,13)$	unmixed	2	2
5	7	$\mathbb{Z}_2 \times A_4$	$G(24,13)$	unmixed	1	1
5	9	$\mathbb{Z}_8 \ltimes (\mathbb{Z}_2)^2$	$G(32,5)$	unmixed	1	1
5	9	$\mathbb{Z}_2 \ltimes D_{2,8,5}$	$G(32,7)$	unmixed	1	1
5	9	$\mathbb{Z}_4 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_2)$	$G(32,2)$	unmixed	1	1
5	9	$\mathbb{Z}_4 \ltimes (\mathbb{Z}_2)^3$	$G(32,6)$	unmixed	1	1
5	13	$(\mathbb{Z}_2)^2 \times A_4$	$G(48,49)$	unmixed	1	1
5	17	$\mathbb{Z}_4 \ltimes (\mathbb{Z}_2)^4$	$G(64,32)$	unmixed	1	2
5	21	$\mathbb{Z}_5 \ltimes (\mathbb{Z}_2)^4$	$G(80,49)$	unmixed	1	2
5	5	$D_{2,8,3}$	$G(16,8)$	mixed	2	1
5	5	$D_{2,8,5}$	$G(16,6)$	mixed	2	3
5	5	$\mathbb{Z}_4 \ltimes (\mathbb{Z}_2)^2$	$G(16,3)$	mixed	2	1

Table 1.

Throughout the paper we use the following notation for groups:

- $\mathbb{Z}_n$: cyclic group of order n.
- $D_{p,q,r} = \mathbb{Z}_p \ltimes \mathbb{Z}_q = \langle x, y | x^p = y^q = 1, xyx^{-1} = y^r \rangle$: split metacyclic group of order pq. The group $D_{2,n,-1}$ is the dihedral group of order $2n$ and it will be denoted by D_n.
- S_n, A_n: symmetric, alternating group on n symbols.
- If $x, y \in G$, their commutator is defined as $[x, y] = xyx^{-1}y^{-1}$.
- If $x \in G$ we denote by Int_x the inner automorphism of G defined as $\mathrm{Int}_x(g) = xgx^{-1}$.
- $\texttt{IdSmallGroup}(G)$ indicates the label of the group G in the `GAP4` database of small groups. For instance $\texttt{IdSmallGroup}(D_4) = G(8, 3)$ and this means that D_4 is the third in the list of groups of order 8.

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1 Basics on surfaces isogenous to a product

In this section we collect for the reader's convenience some basic results on groups acting on curves and surfaces isogenous to a product, referring to [8] for further details.

Definition 1.1. A complex surface S of general type is said to be *isogenous to a product* if there exist two smooth curves C, F and a finite group G acting freely on $C \times F$ so that $S = (C \times F)/G$.

There are two cases: the *unmixed* one, where G acts diagonally, and the *mixed* one, where there exist elements of G exchanging the two factors (then C and F are isomorphic).

In both cases, since the action of G on $C \times F$ is free, we have

$$\begin{aligned} K_S^2 &= \frac{K_{C\times F}^2}{|G|} = \frac{8(g(C)-1)(g(F)-1)}{|G|}, \\ \chi(\mathcal{O}_S) &= \frac{\chi(\mathcal{O}_{C\times F})}{|G|} = \frac{(g(C)-1)(g(F)-1)}{|G|}, \end{aligned} \tag{1}$$

hence $K_S^2 = 8\chi(\mathcal{O}_S)$.

Let C, F be curves of genus ≥ 2. Then the inclusion $\mathrm{Aut}(C \times F) \supset \mathrm{Aut}(C) \times \mathrm{Aut}(F)$ is an equality if C and F are not isomorphic, whereas $\mathrm{Aut}(C \times C) = \mathbb{Z}_2 \ltimes (\mathrm{Aut}(C) \times \mathrm{Aut}(C))$, the $\mathbb{Z}_2$ being generated by the involution exchanging the two coordinates. If $S = (C \times F)/G$ is a surface isogenous to a product, we will always consider its unique *minimal realization*. This means that

- in the unmixed case, we have $G \subset \mathrm{Aut}(C)$ and $G \subset \mathrm{Aut}(F)$ (i.e. G acts faithfully on both C and F);
- in the mixed case, where $C \cong F$, we have $G^{\circ} \subset \mathrm{Aut}(C)$, for $G^{\circ} := G \cap (\mathrm{Aut}(C) \times \mathrm{Aut}(C))$.

(See [8, Corollary 3.9 and Remark 3.10].)

Definition 1.2. Let G be a finite group and let $\mathfrak{g}' \geq 0$, and $m_r \geq m_{r-1} \geq \cdots \geq m_1 \geq 2$ be integers. A *generating vector* for G of type $(\mathfrak{g}'|m_1, \dots, m_r)$ is a $(2\mathfrak{g}' + r)$-tuple of elements

$$\mathcal{V} = \{g_1, \dots, g_r; h_1, \dots, h_{2\mathfrak{g}'}\}$$

such that: the set $\mathcal{V}$ generates G; $|g_i| = m_i$ and $g_1 g_2 \dots g_r \prod_{i=1}^{\mathfrak{g}'} [h_i, h_{i+\mathfrak{g}'}] = 1$. If such a $\mathcal{V}$ exists, then G is said to be $(\mathfrak{g}'|m_1, \dots, m_r)$*-generated.*

For convenience we make abbreviations such as $(4|2^3, 3^2)$ for $(4|2, 2, 2, 3, 3)$ when we write down the type of the generating vector $\mathcal{V}$.

By Riemann's existence theorem a finite group G acts as a group of automorphisms of some compact Riemann surface X of genus $\mathfrak{g}$ with quotient a Riemann surface Y of genus $\mathfrak{g}'$ if and only if there exist integers $m_r \geq m_{r-1} \geq \cdots \geq m_1 \geq 2$ such that G is $(\mathfrak{g}'|m_1, \dots, m_r)$-generated and $\mathfrak{g}$, $\mathfrak{g}'$, $|G|$ and the m_i are related by the Riemann–Hurwitz formula. Moreover, if $\mathcal{V} = \{g_1, \dots, g_r; h_1, \dots, h_{2\mathfrak{g}'}\}$ is a generating vector for G, the subgroups $\langle g_i \rangle$ and their conjugates are precisely the nontrivial stabilizers of the G-action ([6, Section 2], [5, Chapter 3], [18]). The description of surfaces isogenous to a product can be therefore reduced to finding suitable generating vectors. Requiring that S has given invariants p_g and q imposes numerical restrictions on the order of the group G and the genus of the curves C and F. Our goal is to classify all surfaces with $p_g = q = 1$ isogenous to a product. The aim of the next section is to translate this classification problem from geometry to algebra.

2 The case $p_g = q = 1$. Building data

Lemma 2.1. *Let $S = (C \times F)/G$ be a surface isogenous to a product with $p_g = q = 1$. Then*

(i) $K_S^2 = 8$.
(ii) $|G| = (g(C) - 1)(g(F) - 1)$.
(iii) *S is a minimal surface of general type.*

Proof. Claims (i) and (ii) follow from formulas (1). Now let us consider (iii). Since $C \times F$ is minimal and the cover $C \times F \longrightarrow S$ is étale, S is minimal as well. Moreover (ii) implies either $g(C) = g(F) = 0$ or $g(C) \geq 2$, $g(F) \geq 2$. The first case is impossible otherwise $S = \mathbb{P}^1 \times \mathbb{P}^1$ and $p_g = q = 0$; thus the second case occurs, hence S is of general type. $\square$

2.1 Unmixed case. If $S = (C \times F)/G$ is a surface with $p_g = q = 1$, isogenous to an unmixed product, then $g(C) \geq 3$, $g(F) \geq 3$ and up to exchanging F and C one may assume $F/G \cong \mathbb{P}^1$ and $C/G \cong E$, where E is an elliptic curve. Moreover $\alpha\colon S \longrightarrow C/G$ is the Albanese morphism of S and $g_{\mathrm{alb}} = g(F)$, see [26, Proposition 2.2]. This leads to

Proposition 2.2 ([26, Proposition 3.1]). *Let G be a finite group which is both $(0|m_1, \dots, m_r)$ and $(1|n_1, \dots, n_s)$-generated, with generating vectors $\mathcal{V} = \{g_1, \dots, g_r\}$ and $\mathcal{W} = \{\ell_1, \dots, \ell_s; h_1, h_2\}$, respectively. Let $g(F), g(C)$ be the positive integers defined by the Riemann–Hurwitz relations*

$$\begin{aligned} 2g(F) - 2 &= |G|\Bigg(-2 + \sum_{i=1}^{r} \Big(1 - \frac{1}{m_i}\Big)\Bigg), \\ 2g(C) - 2 &= |G| \sum_{j=1}^{s} \Big(1 - \frac{1}{n_j}\Big). \end{aligned} \tag{2}$$

Assume moreover that $g(C) \geq 3$, $g(F) \geq 3$, $|G| = (g(C)-1)(g(F)-1)$ and

$$\Bigg(\bigcup_{\sigma \in G} \bigcup_{i=1}^{r} \langle \sigma g_i \sigma^{-1} \rangle\Bigg) \cap \Bigg(\bigcup_{\sigma \in G} \bigcup_{j=1}^{s} \langle \sigma \ell_j \sigma^{-1} \rangle\Bigg) = \{1_G\}. \tag{U}$$

Then there is a free, diagonal action of G on $C \times F$ such that the quotient $S = (C \times F)/G$ is a minimal surface of general type with $p_g = q = 1$, $K_S^2 = 8$. Conversely, every surface with $p_g = q = 1$, isogenous to an unmixed product, arises in this way.

Here, condition (U) ensures that the G-action on $C \times F$ is free.

Set $\mathbf{m} := (m_1, \dots, m_r)$ and $\mathbf{n} := (n_1, \dots, n_s)$; if $S = (C \times F)/G$ is a surface with $p_g = q = 1$ which is constructed by using the recipe in Proposition 2.2, it will be called an *unmixed surface of type* $(G, \mathbf{m}, \mathbf{n})$.

Proposition 2.3 ([26, Proposition 2.3]). *Let $S = (C \times F)/G$ be an unmixed surface of type $(G, \mathbf{m}, \mathbf{n})$. Then there are exactly the following possibilities:*

- *$g(F) = 3$, $\mathbf{n} = (2^2)$;*
- *$g(F) = 4$, $\mathbf{n} = (3)$;*
- *$g(F) = 5$, $\mathbf{n} = (2)$.*

The following lemma gives a restriction on $\mathbf{m}$ instead.

Lemma 2.4. *Let $S = (C \times F)/G$ be an unmixed surface of type $(G, \mathbf{m}, \mathbf{n})$. Then every m_i divides $|G|/(g(F)-1)$.*

Proof. Since $\langle g_i \rangle$ is a stabilizer for the G-action on F and since G acts freely on $C \times F$, the subgroup $\langle g_i \rangle \cong \mathbb{Z}_{m_i}$ acts freely on C. By the Riemann–Hurwitz formula applied to the cover $C \longrightarrow C/\langle g_i \rangle$ we have $g(C) - 1 = m_i(g(C/\langle g_i \rangle) - 1)$. Thus m_i divides $g(C) - 1 = |G|/(g(F)-1)$. $\square$

2.2 Mixed case.

Proposition 2.5. *Let $S=(C\times C)/G$ be a surface with $p_g=q=1$ isogenous to a mixed product. Then $E:=C/G^\circ$ is an elliptic curve isomorphic to the Albanese variety of S.*

Proof. We have (see [8, Proposition 3.15])

$$\begin{aligned}\mathbb{C}=H^0(\Omega^1_S)&=(H^0(\Omega^1_C)\oplus H^0(\Omega^1_C))^G=(H^0(\Omega^1_C)^{G^\circ}\oplus H^0(\Omega^1_C)^{G^\circ})^{G/G^\circ}\\ &=(H^0(\Omega^1_E)\oplus H^0(\Omega^1_E))^{G/G^\circ}.\end{aligned}$$

Since S is of mixed type, the quotient $\mathbb{Z}_2=G/G^\circ$ exchanges the last two summands, whence $h^0(\Omega^1_E)=1$. Thus E is an elliptic curve and there is a commutative diagram

$$\begin{tikzcd} C\times C \arrow[r, "\rho"] \arrow[d, "\pi"] & E\times E \arrow[d, "\varepsilon"] \\ S \arrow[r, "\hat{\rho}"] \arrow[dr, "\alpha"'] & E^{(2)} \arrow[d, "\hat{\alpha}"] \\ & E \end{tikzcd} \tag{3}$$

showing that the Albanese morphism α of S factors through the Abel–Jacobi map $\hat{\alpha}$ of the double symmetric product $E^{(2)}$ of E. $\square$

By Lemma 2.1 we have $|G|=(g(C)-1)^2$. In this case [8, Proposition 3.16] becomes

Proposition 2.6. *Assume that G° is a $(1|n_1,\dots,n_s)$-generated finite group with generating vector $\mathcal{V}=\{\ell_1,\dots,\ell_s;h_1,h_2\}$ and that there is a nonsplit extension*

$$1\longrightarrow G^\circ\longrightarrow G\longrightarrow \mathbb{Z}_2\longrightarrow 1 \tag{4}$$

which gives an involution $[\varphi]$ in $\mathrm{Out}(G^\circ)$. Let $g(C)\in\mathbb{N}$ be defined by the Riemann–Hurwitz relation $2g(C)-2=|G^\circ|\sum_{j=1}^s(1-\frac{1}{n_j})$. Assume, in addition, that $|G|=(g(C)-1)^2$ and that

(M1) *for all $g\in G\setminus G^\circ$ we have $\{\ell_1,\dots,\ell_s\}\cap\{g\ell_1g^{-1},\dots,g\ell_sg^{-1}\}=\emptyset$;*

(M2) *for all $g\in G\setminus G^\circ$ we have $g^2\notin\bigcup_{j=1}^s\bigcup_{\sigma\in G^\circ}\langle\sigma\ell_j\sigma^{-1}\rangle$.*

Then there is a free, mixed action of G on $C\times C$ such that the quotient $S=(C\times C)/G$ is a minimal surface of general type with $p_g=q=1$, $K_S^2=8$.

Conversely, every surface S with $p_g=q=1$, isogenous to a mixed product, arises in this way.

Here, conditions (M1) and (M2) ensure that the G-action on $C\times C$ is free.

Remark 2.7. The surface S is not covered by elliptic curves because it is of general type (Lemma 2.1), so the map $C\longrightarrow C/G^\circ=E$ is ramified. Therefore condition (M1) implies that G is not abelian.

Remark 2.8. The exact sequence (4) is nonsplit if and only if the number of elements of order 2 in G equals the number of elements of order 2 in G°.

Proposition 2.9. *Let $S = (C \times C)/G$ be a surface with $p_g = q = 1$, isogenous to a mixed product. Then $g_{\mathrm{alb}} = g(C)$.*

Proof. Let us look at diagram (3). The Abel–Jacobi map $\hat{\alpha}$ gives to $E^{(2)}$ the structure of a $\mathbb{P}^1$-bundle over E ([10]); let $\mathfrak{f}$ be the generic fibre of this bundle and $F^* := \rho^* \varepsilon^*(\mathfrak{f})$. If F_{alb} is the generic Albanese fibre of S we have $F_{\mathrm{alb}} = \pi(F^*)$. Let $\mathbf{n} = (n_1, \dots, n_s)$ be such that G° is $(1|n_1, \dots, n_s)$-generated and $2g(C) - 2 = |G^\circ| \sum_{j=1}^{s}(1 - \frac{1}{n_j})$. The $(G^\circ \times G^\circ)$-cover ρ is branched exactly along the union of s "horizontal" copies of E and s "vertical" copies of E; moreover for each i there are one horizontal copy and one vertical copy whose branching number is n_i. Since $\varepsilon^*(\mathfrak{f})$ is an elliptic curve that intersects all these copies of E transversally in one point, by the Riemann–Hurwitz formula applied to $F^* \longrightarrow \varepsilon^*(\mathfrak{f})$ we obtain

$$2g(F^*) - 2 = |G^\circ|^2 \cdot \sum_{j=1}^{s} 2\left(1 - \frac{1}{n_j}\right).$$

On the other hand the G-cover π is étale, so we have

$$2g(F_{\mathrm{alb}}) - 2 = \frac{1}{|G|}(2g(F^*) - 2) = |G^\circ| \sum_{j=1}^{s}\left(1 - \frac{1}{n_j}\right) = 2g(C) - 2,$$

whence $g_{\mathrm{alb}} = g(C)$. □

If $S = (C \times C)/G$ is a surface with $p_g = q = 1$ which is constructed by using the recipe of Proposition 2.6, it will be called a *mixed surface of type* $(G, \mathbf{n})$. The analogue of Proposition 2.3 in the mixed case is

Proposition 2.10. *Let $S = (C \times C)/G$ be a mixed surface of type $(G, \mathbf{n})$. Then there are at most the following possibilities:*

- $g(C) = 5$, $\mathbf{n} = (2^2)$, $|G| = 16$;
- $g(C) = 7$, $\mathbf{n} = (3)$, $|G| = 36$;
- $g(C) = 9$, $\mathbf{n} = (2)$, $|G| = 64$.

Proof. By Proposition 2.6 we have $2g(C) - 2 = |G^\circ| \sum_{j=1}^{s}(1 - 1/n_j)$ and $|G^\circ| = \frac{1}{2}(g(C) - 1)^2$, so $g(C)$ must be odd and we obtain $4 = (g(C) - 1)\sum_{j=1}^{s}(1 - 1/n_j)$. Therefore $4 \geq \frac{1}{2}(g(C) - 1)$ and the only possibilities are $g(C) = 3, 5, 7, 9$. The case $g(C) = 3$ is ruled out because G cannot be abelian by Remark 2.7.

If $g(C) = 5$ then $\sum_{j=1}^{s}(1 - 1/n_j) = 1$, so $\mathbf{n} = (2^2)$ and $|G| = 16$.

If $g(C) = 7$ then $\sum_{j=1}^{s}(1 - 1/n_j) = \frac{2}{3}$, so $\mathbf{n} = (3)$ and $|G| = 36$.

If $g(C) = 9$ then $\sum_{j=1}^{s}(1 - 1/n_j) = \frac{1}{2}$, so $\mathbf{n} = (2)$ and $|G| = 64$. □

We will see in Section 2.10 that only the case $g(C) = 5$ actually occurs.

3 The unmixed case

The classification of surfaces of general type with $p_g = q = 1$ isogenous to an unmixed product is carried out in [26] when the group G is abelian. Therefore in this section we assume that G is nonabelian.

Following [2, Section 1.2], for an r-tuple $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{N}^r$ we set

$$\Theta(\mathbf{m}) := -2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right), \qquad \alpha(\mathbf{m}) := \frac{2}{\Theta(\mathbf{m})}.$$

If S is an unmixed surface of type $(G, \mathbf{m}, \mathbf{n})$ then we have $2 \leq m_1 \leq \cdots \leq m_r$ and $\Theta(\mathbf{m}) > 0$. Besides, by Proposition 2.2 we have $\alpha(\mathbf{m}) = \frac{|G|}{g(F)-1} = g(C) - 1 \in \mathbb{N}$ and by Lemma 2.4 each integer m_i divides $\alpha(\mathbf{m})$. Then we get

Proposition 3.1. *Let $S = (C \times F)/G$ be a surface with $p_g = q = 1$ isogenous to an unmixed product of type $(G, \mathbf{m}, \mathbf{n})$. Then $\mathbf{m}$ and $\alpha(\mathbf{m})$, written in the format $\mathbf{m}_{\alpha(\mathbf{m})}$, lie in the set $\mathcal{T}$ whose elements are:*

$$\begin{array}{cccccc}
(2,3,7)_{84}, & (2,3,8)_{48}, & (2,4,5)_{40}, & (2,3,9)_{36}, & (2,3,10)_{30}, & (2,3,12)_{24}, \\
(2,4,6)_{24}, & (3^2,4)_{24}, & (2,5^2)_{20}, & (2,3,18)_{18}, & (2,4,8)_{16}, & (3^2,5)_{15}, \\
(2,4,12)_{12}, & (2,6^2)_{12}, & (3^2,6)_{12}, & (3,4^2)_{12}, & (2,5,10)_{10}, & (3^2,9)_{9}, \\
(2,8^2)_{8}, & (4^3)_{8}, & (3,6^2)_{6}, & (5^3)_{5}, & (2^3,3)_{12}, & (2^3,4)_{8}, \\
(2^3,6)_{6}, & (2^2,3^2)_{6}, & (2^2,4^2)_{4}, & (3^4)_{3}, & (2^5)_{4}, & (2^6)_{2}.
\end{array}$$

Proof. This follows combining [2, Proposition 1.4] with Lemma 2.4. □

By abuse of notation, we write $\mathbf{m} \in \mathcal{T}$ instead of $\mathbf{m}_{\alpha(\mathbf{m})} \in \mathcal{T}$.

Now we analyze the three cases in Proposition 2.3 separately, according to the value of $g(F)$. Note that if $g(F) = 3, 4, 5$ then $|\operatorname{Aut}(F)| \leq 168, 120, 192$, respectively ([5, p. 91]).

Proposition 3.2. *If $g(F) = 3$ we have precisely the following possibilities.*

G	`IdSmallGroup`(G)	$\mathbf{m}$
D_4	$G(8,3)$	$(2^2,4^2)$
D_6	$G(12,4)$	$(2^3,6)$
$\mathbb{Z}_2 \times D_4$	$G(16,11)$	$(2^3,4)$
$D_{2,12,5}$	$G(24,5)$	$(2,4,12)$
$\mathbb{Z}_2 \times A_4$	$G(24,13)$	$(2,6^2)$
S_4	$G(24,12)$	$(3,4^2)$
$\mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_8)$	$G(32,9)$	$(2,4,8)$
$\mathbb{Z}_2 \times S_4$	$G(48,48)$	$(2,4,6)$

Proof. Since $\mathbf{n} = (2^2)$ it follows that G is $(1|2^2)$-generated and by the second relation in (2) we have $|G| = 2(g(C) - 1)$. So we must describe all unmixed surfaces of type $(G, \mathbf{m}, \mathbf{n})$ with $\mathbf{m} \in \mathcal{T}$, $\mathbf{n} = (2^2)$ and $|G| = 2\alpha(\mathbf{m})$. By a computer search through the r-tuples in Proposition 3.1 we can list all possibilities, proving our statement. See the `GAP4` script 1 in the Appendix to see how this procedure applies to an explicit example. □

Proposition 3.3. *If* $g(F) = 4$ *we have precisely the following possibilities.*

G	`IdSmallGroup`(G)	$\mathbf{m}$
S_3	$G(6,1)$	(2^6)
D_6	$G(12,4)$	(2^5)
$\mathbb{Z}_3 \times S_3$	$G(18,3)$	$(2^2,3^2)$
$\mathbb{Z}_3 \times S_3$	$G(18,3)$	$(3,6^2)$
S_4	$G(24,12)$	$(2^3,4)$
$S_3 \times S_3$	$G(36,10)$	$(2,6^2)$
$\mathbb{Z}_6 \times S_3$	$G(36,12)$	$(2,6^2)$
$\mathbb{Z}_4 \ltimes (\mathbb{Z}_3)^2$	$G(36,9)$	$(3,4^2)$
A_5	$G(60,5)$	$(2,5^2)$
$\mathbb{Z}_3 \times S_4$	$G(72,42)$	$(2,3,12)$
S_5	$G(120,34)$	$(2,4,5)$

Proof. Since $\mathbf{n} = (3)$ it follows that G is $(1|3)$-generated and by the second relation in (2) we have $|G| = 3(g(C) - 1)$. Therefore our statement can be proven searching by computer calculation all unmixed surfaces of type $(G, \mathbf{m}, \mathbf{n})$ with $\mathbf{m} \in \mathcal{T}$, $\mathbf{n} = (3)$, $|G| = 3\alpha(\mathbf{m})$ and $\alpha(\mathbf{m}) \leq 40$. □

Proposition 3.4. *If* $g(F) = 5$ *we have precisely the following possibilities.*

G	`IdSmallGroup`(G)	$\mathbf{m}$
D_4	$G(8,3)$	(2^6)
A_4	$G(12,3)$	(3^4)
$\mathbb{Z}_4 \ltimes (\mathbb{Z}_2)^2$	$G(16,3)$	$(2^2,4^2)$
$\mathbb{Z}_2 \times A_4$	$G(24,13)$	$(2^2,3^2)$
$\mathbb{Z}_2 \times A_4$	$G(24,13)$	$(3,6^2)$
$\mathbb{Z}_8 \ltimes (\mathbb{Z}_2)^2$	$G(32,5)$	$(2,8^2)$
$\mathbb{Z}_2 \ltimes D_{2,8,5}$	$G(32,7)$	$(2,8^2)$
$\mathbb{Z}_4 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_2)$	$G(32,2)$	(4^3)
$\mathbb{Z}_4 \ltimes (\mathbb{Z}_2)^3$	$G(32,6)$	(4^3)
$(\mathbb{Z}_2)^2 \times A_4$	$G(48,49)$	$(2,6^2)$
$\mathbb{Z}_4 \ltimes (\mathbb{Z}_2)^4$	$G(64,32)$	$(2,4,8)$
$\mathbb{Z}_5 \ltimes (\mathbb{Z}_2)^4$	$G(80,49)$	$(2,5^2)$

Proof. Since $\mathbf{n} = (2)$, it follows that G is $(1|2)$-generated and by the second relation in (2) we have $|G| = 4(g(C) - 1)$. Therefore our statement can be proven searching by computer calculation all unmixed surfaces of type $(G, \mathbf{m}, \mathbf{n})$ with $\mathbf{m} \in \mathcal{T}$, $\mathbf{n} = (2)$, $|G| = 4\alpha(\mathbf{m})$ and $\alpha(\mathbf{m}) \leq 48$. $\square$

4 The mixed case

In this section we use Proposition 2.6 in order to classify the surfaces with $p_g = q = 1$ isogenous to a mixed product. By Proposition 2.10 we have $g(C) = 5$, 7 or 9. Let us consider the three cases separately.

4.1 The case $g(C) = 5, |G| = 16$.

Proposition 4.1. *If* $g(C) = 5, |G| = 16$ *we have precisely the following possibilities.*

G°	`IdSmallGroup`(G°)	G	`Group`(G)
D_4	$G(8,3)$	$D_{2,8,3}$	$G(16,8)$
$\mathbb{Z}_2 \times \mathbb{Z}_4$	$G(8,2)$	$D_{2,8,5}$	$G(16,6)$
$(\mathbb{Z}_2)^3$	$G(8,5)$	$\mathbb{Z}_4 \ltimes (\mathbb{Z}_2)^2$	$G(16,3)$

Proof. In this case $\mathbf{n} = (2^2)$, so our first task is to find all nonsplit sequences of type (4) for which G° is a $(1|2^2)$-generated group of order 8. The three abelian groups of order 8 and D_4 are $(1|2^2)$-generated whereas the quaternion group Q_8 is not.

Since $\mathbb{Z}_8$ has only one element ℓ of order 2, condition (M1) in Proposition 2.6 cannot be satisfied for any choice of $\mathcal{V}$. By Remark 2.7 we are left to analyze the possible embeddings of $\mathbb{Z}_2 \times \mathbb{Z}_4$, D_4 and $(\mathbb{Z}_2)^3$ in nonabelian groups of order 16. The groups $\mathbb{Z}_2 \times \mathbb{Z}_4$, D_4 and $(\mathbb{Z}_2)^3$ have 3, 5 and 7 elements of order 2, respectively. Therefore if n_2 denotes the number of elements of order 2 in G, by Remark 2.8 we must consider only those groups G of order 16 with $n_2 \in \{3, 5, 7\}$. The nonabelian groups of order 16 with $n_2 = 3$ are $D_{2,8,5}, \mathbb{Z}_2 \times Q_8$ and $D_{4,4,-1}$ and they all contain a copy of $\mathbb{Z}_2 \times \mathbb{Z}_4$. The only nonabelian group of order 16 with $n_2 = 5$ is $D_{2,8,3}$ and it contains a subgroup isomorphic to D_4. The nonabelian groups of order 16 with $n_2 = 7$ are $\mathbb{Z}_4 \ltimes (\mathbb{Z}_2)^2 = G(16,3)$ and $\mathbb{Z}_2 \ltimes Q_8$, and only the former contains a subgroup isomorphic to $(\mathbb{Z}_2)^3$ (cf. [28]).

Summarizing, we are left with the following cases:

G°	G
D_4	$D_{2,8,3}$
$\mathbb{Z}_2 \times \mathbb{Z}_4$	$D_{2,8,5}$
$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times Q_8$
$\mathbb{Z}_2 \times \mathbb{Z}_4$	$D_{4,4,-1}$
$(\mathbb{Z}_2)^3$	$\mathbb{Z}_4 \ltimes (\mathbb{Z}_2)^2$

Let us analyze them separately.

- $G^\circ = D_4, G = D_{2,8,3} = \langle x, y | x^2 = y^8 = 1, xyx^{-1} = y^3 \rangle$.
We consider the subgroup $G^\circ := \langle x, y^2 \rangle \cong D_4$. Set $\ell_1 = \ell_2 = x$ and $h_1 = h_2 = y^2$. Condition (M1) holds because $C_G(x) = \langle x, y^4 \rangle \subset G^\circ$. Condition (M2) is satisfied because the conjugacy class of x in G° is contained in the coset $x\langle y^2 \rangle$ while for every $g \in yG^\circ$ we have $g^2 \in \langle y \rangle$. Therefore this case occurs by Proposition 2.6.
- $G^\circ = \mathbb{Z}_2 \times \mathbb{Z}_4, G = D_{2,8,5} = \langle x, y | x^2 = y^8 = 1, xyx^{-1} = y^5 \rangle$.
We consider the subgroup $G^\circ := \langle x, y^2 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_4$. Set $\ell_1 = \ell_2 = x$ and $h_1 = h_2 = y^2$. Conditions (M1) and (M2) are verified as in the previous case, so this possibility occurs.
- $G^\circ = \mathbb{Z}_2 \times \mathbb{Z}_4, G = \mathbb{Z}_2 \times Q_8$ and $G^\circ = \mathbb{Z}_2 \times \mathbb{Z}_4, G = D_{4,4,-1}$.
All elements of order 2 in G are central, so condition (M1) cannot be satisfied and these cases do not occur.
- $G^\circ = (\mathbb{Z}_2)^3, G = \mathbb{Z}_4 \ltimes (\mathbb{Z}_2)^2 = \langle x, y, z | x^4 = y^2 = z^2 = 1, xyx^{-1} = yz, [x, z] = [y, z] = 1 \rangle$.
We consider the subgroup $G^\circ := \langle y, z, x^2 \rangle \cong (\mathbb{Z}_2)^3$. Set $\ell_1 = \ell_2 = y$ and $h_1 = z, h_2 = x^2$. Condition (M1) holds as G° is abelian and $[x, y] \neq 1$. Condition (M2) is satisfied because if $g \in xG^\circ$ then $g^2 \in \langle z, x^2 \rangle$. Therefore this case occurs. □

4.2 The case $g(C) = 7, |G| = 36$.

Proposition 4.2. *The case $g(C) = 7, |G| = 36$ does not occur.*

Proof. In this case $\mathbf{n} = (3)$, so G° is a group of order 18 which is $(1|3)$-generated. There are five groups of order 18 up to isomorphism. By computer search or direct calculation we see that the only one which is $(1|3)$-generated is $\mathbb{Z}_3 \times S_3 = G(18, 3)$. Thus G would fit into a short exact sequence

$$1 \longrightarrow \mathbb{Z}_3 \times S_3 \longrightarrow G \longrightarrow \mathbb{Z}_2 \longrightarrow 1. \tag{5}$$

A computer search shows that the only groups of order 36 containing a subgroup isomorphic to $\mathbb{Z}_3 \times S_3$ are $G(36, 10) = S_3 \times S_3$ and $G(36, 12) = \mathbb{Z}_6 \times S_3$ (see `GAP4` script 2 in the Appendix). They contain 15 and 7 elements of order 2, respectively. On the other hand $\mathbb{Z}_3 \times S_3$ contains 3 elements of order 2, so by Remark 2.8 all possible extensions of the form (5) are split and this case cannot occur. □

4.3 The case $g(C) = 9, |G| = 64$.

Proposition 4.3. *The case $g(C) = 9, |G| = 64$ does not occur.*

The proof will be a consequence of the results below. First notice that, since $\mathbf{n} = (2)$, the group G° must be $(1|2)$-generated.

Computational Fact 4.4. *There exist precisely* 8 *groups of order* 32 *which are* $(1|2)$-*generated, namely* $G(32, t)$ *for* $t \in \{2, 4, 5, 6, 7, 8, 12, 17\}$. *The number* n_2 *of their elements of order* 2 *is given in the following table:*

t	2	4	5	6	7	8	12	17
$n_2(G(32,t))$	7	3	7	11	11	3	3	3

Proof. Slightly modifying the first part of `GAP4` script 1 in the Appendix we easily find that the groups of order 32 which are $(1|2)$-generated are exactly those in the statement. The number of elements of order 2 in each case are found by a quick computer search: see again the Appendix, `GAP4` script 3. □

Computational Fact 4.5. *Let $t \in \{2,4,5,6,7,8,12,17\}$. A nonsplit extension of the form*

$$1 \longrightarrow G(32,t) \longrightarrow G(64,s) \longrightarrow \mathbb{Z}_2 \longrightarrow 1 \tag{6}$$

exists if and only if the pair (t,s) is one of the following:
$(2,9)$, $(2,57)$, $(2,59)$, $(2,63)$, $(2,64)$, $(2,68)$, $(2,70)$, $(2,72)$, $(2,76)$, $(2,79)$, $(2,81)$, $(2,82)$,
$(4,11)$, $(4,28)$, $(4,122)$, $(4,127)$, $(4,172)$, $(4,182)$,
$(5,5)$, $(5,9)$, $(5,112)$, $(5,113)$, $(5,114)$, $(5,132)$, $(5,164)$, $(5,165)$, $(5,166)$,
$(6,33)$, $(6,35)$,
$(7,33)$,
$(8,37)$,
$(12,7)$, $(12,13)$, $(12,14)$, $(12,15)$, $(12,16)$, $(12,126)$, $(12,127)$, $(12,143)$, $(12,156)$, $(12,158)$, $(12,160)$,
$(17,28)$, $(17,43)$, $(17,45)$, $(17,46)$.

Proof. Assume $t = 2$. Using the `GAP4` script 4 in the Appendix we find that the groups of order 64 containing a subgroup isomorphic to $G(32,2)$ are $G(64,s)$ for $s \in \{8,9,56,57,58,59,61,62,63,64,66,67,68,69,70,72,73,74,75,76,77,78,79,80,81,82\}$. By Remark 2.8 and Computational Fact 4.4, in order to detect all the groups $G(64,s)$ fitting in some nonsplit extension of type (6) with $t = 2$, it is sufficient to select from the previous list the groups containing exactly $n_2 = 7$ elements of order 2. This can be done with the `GAP4` script 5 in the Appendix, proving the claim in the case $t = 2$. The proof for the other values of t may be carried out exactly in the same way. □

Let us denote by $[G,G]_2$ and $[G^\circ,G^\circ]_2$ the subsets of elements of order 2 in $[G,G]$ and $[G^\circ,G^\circ]$, respectively.

Lemma 4.6. *Assume $g(C) = 9$ and that one of the following situations occurs:*

- *$[G,G]_2 \subseteq Z(G)$;*
- *there exists some element $y \in G \setminus G^\circ$ commuting with all elements in $[G^\circ,G^\circ]_2$.*

Then given any generating vector $\mathcal{V} = \{\ell_1; h_1, h_2\}$ of type $(1|2)$ for G°, condition (M1) in Proposition 2.6 cannot be satisfied.

Proof. Since $\ell_1 \in [G^\circ,G^\circ]_2 \subseteq [G,G]_2$, in any of the above situations $C_G(\ell_1)$ is not contained in G°, so (M1) cannot hold. □

Computational Fact 4.7. *Let $G = G(64, s)$ be one of the groups appearing in the list of Computational Fact* 4.5*. Then $[G, G]_2$ is not contained in $Z(G)$ if and only if $s = 5$, 33, 35, 37.*

Proof. See the `GAP4` script 6 in the Appendix. □

Computational facts 4.5, 4.7 and Lemma 4.6 imply that we only need to analyze the following pairs (G°, G):

G°	G
$G(32,5)$	$G(64,5)$
$G(32,6)$	$G(64,33)$
$G(32,7)$	$G(64,33)$
$G(32,6)$	$G(64,35)$
$G(32,8)$	$G(64,37)$

Proposition 4.8. *The case $G^\circ = G(32, 5)$ does not occur.*

Proof. A presentation for the group G° is

$$G^\circ = \langle x, y, z \mid x^8 = y^2 = z^2 = 1, [y, z] = [x, z] = 1, [x, y] = z\rangle.$$

Its derived subgroup contains exactly one element of order 2, namely z. It follows that if $\{\ell_1; h_1, h_2\}$ is any generating vector of type $(1|2)$ for G°, then $\ell_1 = z$. Since $[G^\circ, G^\circ]$ is characteristic in G°, condition (M1) cannot be satisfied for any embedding of G° into G. □

By using the two instructions `P:=PresentationViaCosetTable(G)` and `TzPrintRelators(P)` and setting in the output

$$x := \mathtt{f1}, y := \mathtt{f2}, z := \mathtt{f3}, w := \mathtt{f4}, v := \mathtt{f5}, u := \mathtt{f6}$$

one obtains the following presentations for $G(64, 33)$, $G(64, 35)$ and $G(64, 37)$.

$$\begin{aligned} G(64, 33) = \langle x, y, z, w, v, u \mid\ & z^2 = w^2 = v^2 = u^2 = 1, x^2 = w, y^2 = u, \\ & [x, zy] = z, [x, vz] = v, [x, vu] = u, \\ & [y, z] = [y, v] = [z, v] = [w, v] = [x, u] = 1\rangle, \end{aligned} \tag{7}$$

$$\begin{aligned} G(64, 35) = \langle x, y, z, w, v, u \mid\ & w^2 = v^2 = u^2 = 1, z^2 = y^2 = u, x^2 = w, \\ & [y, z] = [z, w] = u, [x, yz] = z, [x, z] = uv, \\ & [y, v] = [z, v] = [w, v] = [x, u] = 1\rangle, \end{aligned} \tag{8}$$

$$\begin{aligned} G(64, 37) = \langle x, y, z, w, v, u \mid\ & v^2 = u^2 = 1, w^2 = z^2 = y^2 = u, x^2 = w, \\ & [y, z] = [z, w] = u, [x, yz] = z, [x, z] = uv, \\ & [y, v] = [z, v] = [w, v] = 1\rangle. \end{aligned} \tag{9}$$

Computational Fact 4.9. *Referring to presentations* (7), (8) *and* (9)*, we have the following facts.*

- *The group* $G(64,33)$ *contains exactly one subgroup* N_1 *isomorphic to* $G(32,6)$ *and one subgroup* N_2 *isomorphic to* $G(32,7)$*, namely*

$$N_1 := \langle x, z, w, v, u \rangle, \quad N_2 := \langle xy, z, w, v, u \rangle.$$

- *The group* $G(64,35)$ *contains exactly two subgroups* N_3*,* N_4 *isomorphic to* $G(32,6)$*, namely*

$$N_3 := \langle x, z, w, v, u \rangle, \quad N_4 := \langle xy, z, w, v, u \rangle.$$

- *The group* $G(64,37)$ *contains exactly two subgroups* N_5*,* N_6 *isomorphic to* $G(32,8)$*, namely*

$$N_5 := \langle x, z, w, v, u \rangle, \quad N_6 := \langle xy, z, w, v, u \rangle.$$

In addition, for every $i \in \{1, \ldots, 6\}$ *we have*

(a) $[N_i, N_i] = \langle v, u \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.
(b) $y \notin N_i$ *and* y *commutes with all elements in* $[N_i, N_i]$.

Proof. See the `GAP4` script 7 in the Appendix. □

Proposition 4.10. *The cases* $G^\circ = G(32,6), G(32,7), G(32,8)$ *do not occur.*

Proof. By Lemma 4.6 and Computational Fact 4.9 it follows that, given any nonsplit extension of type (6) with G° as above, condition (M1) in Proposition 2.6 cannot be satisfied. □

Summing up, we finally obtain

Proof of Proposition 4.3. It follows from Propositions 4.8 and 4.10. □

5 Moduli spaces

Let $\mathfrak{M}_{a,b}$ be the moduli space of smooth minimal surfaces of general type with $\chi(\mathcal{O}_S) = a, K_S^2 = b$; by an important result of Gieseker, $\mathfrak{M}_{a,b}$ is a quasiprojective variety for all $a, b \in \mathbb{N}$ (see [14]). Obviously, our surfaces are contained in $\mathfrak{M}_{1,8}$ and we want to describe their locus there. We denote by $\mathfrak{M}(G, \mathbf{m}, \mathbf{n})$ the moduli space of unmixed surfaces of type $(G, \mathbf{m}, \mathbf{n})$ and by $\mathfrak{M}(G, \mathbf{n})$ the moduli space of mixed surfaces of type $(G, \mathbf{n})$. We know that $\mathbf{n} = (2^2)$, (3) or (2) in the unmixed case, whereas $\mathbf{n} = (2^2)$ in the mixed one. By a general result of Catanese ([8]), both $\mathfrak{M}(G, \mathbf{m}, \mathbf{n})$ and $\mathfrak{M}(G, \mathbf{n})$ consist of finitely many irreducible connected components of $\mathfrak{M}_{1,8}$, all of the same dimension. More precisely, we have

$$\dim \mathfrak{M}(G, \mathbf{m}, \mathbf{n}) = r + s - 3, \quad \dim \mathfrak{M}(G, \mathbf{n}) = s.$$

Consider the mapping class groups in genus zero and one:

$$\begin{aligned}
\mathrm{Mod}_{0,[r]} &:= \langle \sigma_1, \dots, \sigma_r \mid \ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \\
&\qquad\qquad\qquad\quad \sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{if } |i-j| \geq 2, \\
&\qquad\qquad\qquad\quad \sigma_{r-1}\sigma_{r-2}\dots\sigma_1^2\dots\sigma_{r-2}\sigma_{r-1} = 1\rangle, \\
\mathrm{Mod}_{1,1} &:= \langle t_\alpha, t_\beta, t_\gamma \mid t_\alpha t_\beta t_\alpha = t_\beta t_\alpha t_\beta, (t_\alpha t_\beta)^3 = 1\rangle, \\
\mathrm{Mod}_{1,[2]} &:= \langle t_\alpha, t_\beta, t_\gamma, \rho \mid \ t_\alpha t_\beta t_\alpha = t_\beta t_\alpha t_\beta, t_\alpha t_\gamma t_\alpha = t_\gamma t_\alpha t_\gamma, \\
&\qquad\qquad\qquad\qquad t_\beta t_\gamma = t_\gamma t_\beta, (t_\alpha t_\beta t_\gamma)^4 = 1, \\
&\qquad\qquad\qquad\qquad t_\alpha\rho = \rho t_\alpha, t_\beta\rho = \rho t_\beta, t_\gamma\rho = \rho t_\gamma\rangle.
\end{aligned}$$

One can prove that

$$\begin{aligned}
\mathrm{Mod}_{0,[r]} &:= \pi_0 \operatorname{Diff}^+(\mathbb{P}^1 - \{p_1, \dots, p_r\}), \\
\mathrm{Mod}_{1,1} &:= \pi_0 \operatorname{Diff}^+(\Sigma_1 - \{p\}), \\
\mathrm{Mod}_{1,[2]} &:= \pi_0 \operatorname{Diff}^+(\Sigma_1 - \{p, q\}),
\end{aligned}$$

where Σ_1 is the torus $S^1 \times S^1$ ([27], [13]). This implies that we can define actions of these groups on the set of generating vectors for G of type $(0|m_1, \dots, m_r)$, $(1|n)$ and $(1|n^2)$, respectively.

If $\mathcal{V} := \{g_1, \dots, g_r\}$ is of type $(0|m_1, \dots, m_r)$ then the action is given by

$$\sigma_i : \begin{cases} g_i & \longrightarrow g_{i+1} \\ g_{i+1} & \longrightarrow g_{i+1}^{-1} g_i g_{i+1} \\ g_j & \longrightarrow g_j \quad \text{if } j \neq i, i+1. \end{cases}$$

If $\mathcal{W} := \{\ell_1; h_1, h_2\}$ is of type $(1|n)$ then

$$t_\alpha \colon \begin{cases} \ell_1 & \longrightarrow \ell_1 \\ h_1 & \longrightarrow h_1 \\ h_2 & \longrightarrow h_2 h_1, \end{cases} \qquad t_\beta \colon \begin{cases} \ell_1 & \longrightarrow \ell_1 \\ h_1 & \longrightarrow h_1 h_2^{-1} \\ h_2 & \longrightarrow h_2. \end{cases}$$

If $\mathcal{W} := \{\ell_1, \ell_2; h_1, h_2\}$ is of type $(1|n^2)$ then

$$t_\alpha \colon \begin{cases} \ell_1 & \longrightarrow \ell_1 \\ \ell_2 & \longrightarrow \ell_2 \\ h_1 & \longrightarrow h_1 \\ h_2 & \longrightarrow h_2 h_1, \end{cases} \qquad t_\beta \colon \begin{cases} \ell_1 & \longrightarrow \ell_1 \\ \ell_2 & \longrightarrow \ell_2 \\ h_1 & \longrightarrow h_1 h_2^{-1} \\ h_2 & \longrightarrow h_2, \end{cases}$$

$$t_\gamma \colon \begin{cases} \ell_1 & \longrightarrow \ell_1 \\ \ell_2 & \longrightarrow h_1 h_2^{-1} h_1^{-1} \ell_2 h_1 h_2 h_1^{-1} \\ h_1 & \longrightarrow h_2^{-1} \ell_1 h_1 \\ h_2 & \longrightarrow h_2, \end{cases} \qquad \rho\colon \begin{cases} \ell_1 & \longrightarrow h_2^{-1} h_1^{-1} \ell_2 h_1 h_2 \\ \ell_2 & \longrightarrow h_1^{-1} h_2^{-1} \ell_1 h_2 h_1 \\ h_1 & \longrightarrow h_1^{-1} \\ h_2 & \longrightarrow h_2^{-1}. \end{cases}$$

These are called *Hurwitz moves* and the induced equivalence relation on generating vectors is the *Hurwitz equivalence* (see [1], [2], [26]).

Now let $\mathfrak{B}(G, \mathbf{m}, \mathbf{n})$ be the set of pairs of generating vectors $(\mathcal{V}, \mathcal{W})$ such that the assumptions in Proposition 2.2 are satisfied; then we denote by $\mathfrak{R}$ the equivalence relation on $\mathfrak{B}(G, \mathbf{m}, \mathbf{n})$ generated by Hurwitz moves on $\mathcal{V}$, Hurwitz moves on $\mathcal{W}$ and the simultaneous action of $\mathrm{Aut}(G)$ on $\mathcal{V}$ and $\mathcal{W}$. Similarly, let $\mathfrak{B}(G, \mathbf{n})$ be the set of generating vectors $\mathcal{V}$ such that the assumptions of Proposition 2.6 are satisfied; then we denote by $\mathfrak{R}$ the equivalence relation on $\mathfrak{B}(G, \mathbf{n})$ generated by the Hurwitz moves and the action of $\mathrm{Aut}(G)$ on $\mathcal{V}$.

Proposition 5.1. *The number of irreducible components in* $\mathfrak{M}(G, \mathbf{m}, \mathbf{n})$ *equals the number of* $\mathfrak{R}$*-classes in* $\mathfrak{B}(G, \mathbf{m}, \mathbf{n})$*. Analogously, the number of irreducible components in* $\mathfrak{M}(G, \mathbf{n})$ *equals the number of* $\mathfrak{R}$*-classes in* $\mathfrak{B}(G, \mathbf{n})$.

Proof. We can repeat exactly the same argument used in [2, Propositions 5.2 and 5.5]; we must just replace, where it is necessary, the mapping class group of $\mathbb{P}^1$ with the mapping class group of the elliptic curve E. □

Proposition 5.1 in principle allows us to compute the number of connected components of the moduli space in each case. In practice, this task may be too hard to be achieved by hand, but it is not out of reach if one uses the computer. Recently, M. Penegini and S. Rollenske developed a `GAP4` script that solves this problem in a rather short time. We put the result of their calculations in the Main Theorem (see Introduction), referring the reader to the forthcoming paper [23] for further details.

6 Appendix

In this appendix we include, for the reader's convenience, some of the `GAP4` scripts that we have used in our computations; all the others are similar and can be easily obtained modifying the ones below.

Let us show how the procedure in the proof of Proposition 3.2 applies to an explicit example, namely $\mathbf{m}_{\alpha(\mathbf{m})} = (2, 4, 12)_{12}$. First we find all the nonabelian groups of order 24 that are $(0|2, 4, 12)$-generated. This is done using `GAP4` as below; the output tells us that there is only one such a group, namely $G = G(24, 5)$.

```
gap> # ------------- SCRIPT 1 ------------------
gap> s:=NumberSmallGroups(24);; set:=[1..s];
[1..15]
gap> for t in set do
> c:=0; G:= SmallGroup(24,t);
> Ab:=IsAbelian(G);
> for g1 in G do
> for g2 in G do
> g3:=(g1*g2)^-1;
> H:= Subgroup(G, [g1,g2]);
```

```
> if Order(g1)=2 and Order(g2)=4 and Order(g3)=12 and
> Order(H)=Order(G) and
> Ab=false then
> c:=c+1; fi;
> if Order(g1)=2 and Order(g2)=4 and Order(g3)=12 and
> Order(H)=Order(G) and
> Ab=false and c=1 then
> Print(IdSmallGroup(G)," ");
> fi; od; od; od; Print("\n");
[24,5]
```

By using the two instructions `P:=PresentationViaCosetTable(G)` and `TzPrintRelators(P)` we see that G has the presentation $\langle x, y | x^2 = y^{12} = 1, xyx^{-1} = y^5 \rangle$, hence it is isomorphic to the metacyclic group $D_{2,12,5}$.

In order to speed up further computations, we define the sets $G2$, $G4$ given by the elements of G having order 2 and 4, respectively.

```
gap> G:=SmallGroup(24,5);;
gap> G2:=[];; G4:=[];;
gap> for g in G do
> if Order(g)=2 then Add(G2,g); fi;
> if Order(g)=4 then Add(G4,g); fi; od;
```

Then we check whether G is actually $(1|2^2)$-generated; if not, it should be excluded.

```
gap> c:=0;;
gap> for l2 in G2 do
> for h1 in G do
> for h2 in G do
> l1:=(l2*h1*h2*h1^-1*h2^-1)^-1;
> K:=Subgroup(G, [l2, h1, h2]);
> if Order(l1)=2 and Order(K)=Order(G) then
> Print(IdSmallGroup(G), " is (1 | 2,2)-generated", "\n");
> c:=1; fi;
> if c=1 then break; fi; od;
> if c=1 then break; fi; od;
> if c=1 then break; fi; od;
[24,5] is (1 | 2,2)-generated
```

We finish the proof by checking whether the surface S actually exists; the procedure is to look for a pair $(\mathcal{V}, \mathcal{W})$ of generating vectors for G satisfying the assumptions of Proposition 2.2.

```
gap> c:=0;;
gap> for g1 in G2 do
> for g2 in G4 do
> g3:=(g1*g2)^-1;
> H:=Subgroup(G, [g1, g2]);
> for l2 in G2 do
```

```
> for h1 in G do
> for h2 in G do
> l1:=(l2*h1*h2*h1^-1*h2^-1)^-1;
> K:=Subgroup(G, [l2, h1, h2]);
> Boole1:=l1 in ConjugacyClass(G, g1);
> Boole2:=l1 in ConjugacyClass(G, g2^2);
> Boole3:=l1 in ConjugacyClass(G, g3^6);
> Boole4:=l2 in ConjugacyClass(G, g1);
> Boole5:=l2 in ConjugacyClass(G, g2^2);
> Boole6:=l2 in ConjugacyClass(G, g3^6);
> if Order(g3)=12 and Order(l1)=2 and
> Order(H)=Order(G) and Order(K)=Order(G) and
> Boole1=false and Boole2=false and Boole3=false and
> Boole4=false and Boole5=false and Boole6=false then
> Print("The surface exists "); c:=1; fi;
> if c=1 then break; fi; od;
> if c=1 then break; fi; od;
> if c=1 then break; fi; od;
> if c=1 then break; fi; od;
> if c=1 then break; fi; od; Print("\n");
The surface exists
```

The script above can be easily modified in order to obtain the list of all admissible pairs $(\mathcal{V}, \mathcal{W})$; for instance, one of such pair is given by

$$\begin{aligned} &g_1 = x, \quad g_2 = xy^{-1}, \quad g_3 = y, \\ &\ell_1 = xy^2, \quad \ell_2 = xy^2, \quad h_1 = y, \quad h_2 = y. \end{aligned}$$

Finally, here are the `GAP4` scripts used in Section 4.

```
gap> # -------------- SCRIPT 2 ------------------
gap> s:=NumberSmallGroups(36);; set:=[1..s];
[1..14]
gap> for t in set do
> c:=0; G:=SmallGroup(36,t);
> N:=NormalSubgroups(G);
> for G0 in N do
> if IdSmallGroup(G0)=[18,3] then
> c:=c+1; fi;
> if IdSmallGroup(G0)=[18,3] and c=1 then
> Print(IdSmallGroup(G), " ");
> fi; od; od; Print("\n");
[36,10] [36,12]
```

```
gap> # -------------- SCRIPT 3 ------------------
gap> set:=[2,4,5,6,7,8,12,17];;
gap> for t in set do
```

```
> n2:=0;
> G0:=SmallGroup(32,t);
> for g in G0 do
> if Order(g)=2 then
> n2:=n2+1; fi; od;
> Print(IdSmallGroup(G0), " "); Print(n2, "    ");
> od; Print("\n");
[32,2] 7    [32,4] 3    [32,5] 7    [32,6] 11    [32,7] 11
[32,8] 3    [32,12] 3    [32,17] 3
```

```
gap> # -------------- SCRIPT 4 ------------------
gap> s:=NumberSmallGroups(64);; set:=[1..s];
[1..267]
gap> for t in set do
> c:=0; G:=SmallGroup(64,t);
> N:=NormalSubgroups(G);
> for G0 in N do
> if IdSmallGroup(G0)=[32,2] then
> c:=c+1; fi;
> if IdSmallGroup(G0)=[32,2] and c=1 then
> Print(IdSmallGroup(G), " ");
> fi; od; od; Print("\n");
[64,8] [64,9] [64,56] [64,57] [64,58] [64,59] [64,61]
[64,62] [64,63] [64,64] [64,66] [64,67] [64,68]
[64,69] [64,70] [64,72] [64,73] [64,74] [64,75]
[64,76] [64,77] [64,78] [64,79] [64,80] [64,81]
[64,82]
```

```
gap> # -------------- SCRIPT 5 ------------------
gap> set:=[8,9,56,57,58,59,61,62,63,64,66,67,68,69,70,
>72,73,74,75,76,77,78,79,80,81,82];;
gap> for t in set do
> n2:=0; G:=SmallGroup(64,t);
> for g in G do
> if Order(g)=2 then n2:=n2+1;
> fi; od;
> if n2=7 then
> Print(IdSmallGroup(G), " ");
> fi; od; Print("\n");
[64,9] [64,57] [64,59] [64,63] [64,64] [64,68] [64,70]
[64,72] [64,76] [64,79] [64,81] [64,82]
```

```
gap> # -------------- SCRIPT 6 ------------------
gap> set:=[5,7,9,11,13,,14,15,16,28,33,35,37,43,45,46,
>57,59,63,64,68,70,72,76,79,81,82,112,113,114, 122,126,
>127,132,143,156,158,160,164,165,166,172,182];;
gap> for t in set do
```

```
> c:=0; G:=SmallGroup(64,t);
> D:=DerivedSubgroup(G);
> for d in D do
> B:=d in Center(G);
> if Order(d)=2 and B=false then
> c:=c+1; fi;
> if Order(d)=2 and B=false and c=1 then
> Print(IdSmallGroup(G), " ");
> fi; od; od; Print("\n");
[64,5] [64,33] [64,35] [64,37]
```

```
gap> # -------------- SCRIPT 7 ------------------
gap> s:=[33, 35, 37];; I:=[1, 2, 3];;
gap> r:=[ [[32,6], [32,7]], [[32,6]], [[32,8]] ];;
> for i in I do
> G:=SmallGroup(64, s[i]); Print(IdSmallGroup(G), "\n");
> for N in NormalSubgroups(G) do
> if IdSmallGroup(N) in r[i] then
> Print(N, "="); Print(IdSmallGroup(N), "    ");
> Print(DerivedSubgroup(N), "\n");
> fi; od; Print("\n"); od;
[64,33]
Group( [ f1*f2, f3, f4, f5, f6 ] )=[32,7]
Group( [ f5, f6 ] )
Group( [ f1, f3, f4, f5, f6 ] )=[32,6]
Group( [ f5, f6 ] )

[64,35]
Group( [ f1*f2, f3, f4, f5, f6 ] )=[32,6]
Group( [ f5, f6 ] )
Group( [ f1, f3, f4, f5, f6 ] )=[32,6]
Group( [ f5, f6 ] )

[64,37]
Group( [ f1*f2, f3, f4, f5, f6 ] )=[32,8]
Group( [ f5, f6 ] )
Group( [ f1, f3, f4, f5, f6 ] )=[32,8]
Group( [ f5, f6 ] )
```

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