



**Algebraic geometry.** — *Étale and crystalline beta and gamma functions via Fontaine’s periods*, by FRANCESCO BALDASSARRI, communicated on 10 March 2006.

ABSTRACT. — We compare the Ihara–Anderson theory of the  $p$ -adic étale beta function, which describes the Galois action on  $p$ -adic étale homology for the tower of Fermat curves over  $\mathbb{Q}$  of degree a power of  $p$ , with the crystalline theory of Dwork–Coleman, based on the calculation of the Frobenius action on  $p$ -adic de Rham cohomology of the same curves. The two constructions are easily related via a ramified extension of Fontaine’s period ring  $\mathbb{B}_{\text{crys}} = \mathbb{B}_{\text{crys},p}$  contained in  $\mathbb{B}_{\text{dR}} = \mathbb{B}_{\text{dR},p}$ , namely  $\mathbb{B}_p := \mathbb{B}_{\text{crys},p} \otimes_{\mathbb{Q}_p^{\text{ur}}} \overline{\mathbb{Q}}_p \subset \mathbb{B}_{\text{dR},p}$ . We propose, but do not carry out, a similar comparison for the  $p$ -adic étale gamma function of Anderson and the Morita–Dwork–Coleman  $p$ -adic crystalline gamma function.

KEY WORDS: Jacobi sums; Fermat curves;  $p$ -adic cohomology; Fontaine’s theory.

MATHEMATICS SUBJECT CLASSIFICATION (2000): Primary 14F30; Secondary 11S40.

#### INTRODUCTION

The classical beta and gamma functions are meromorphic functions of  $s, t \in \mathbb{C}$  defined, in suitable regions, by convergent integral formulas

$$(0.1) \quad B(s, t) = \int_0^1 x^{s-1} (1-x)^{t-1} dx,$$

$$(0.2) \quad \Gamma(s) = \int_0^\infty x^s e^{-x} \frac{dx}{x}.$$

They are related by

$$(0.3) \quad B(s, t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}.$$

Moreover,

$$(0.4) \quad \Gamma(s+1) = s\Gamma(s),$$

$$(0.5) \quad \Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s},$$

and

$$(0.6) \quad \prod_{i=1}^{n-1} \Gamma\left(\frac{s+i}{n}\right) = \left(\prod_{i=0}^{n-1} \Gamma\left(\frac{i}{n}\right)\right) n^{1-s} \Gamma(s).$$

A formula easier to handle, equivalent to (0.1), replaces the simplex  $[0, 1]$  by the Pochhammer contour [21, §12.43] and holds for any  $s, t \in \mathbb{C}$ :

$$(0.7) \quad -4 \sin(\pi s) \sin(\pi t) B(s, t) = \int_{\eta} x^{s-1} (1-x)^{t-1} dx.$$

The advantage here is that  $\eta \in \pi_1(\mathbb{C} \setminus \{0, 1\}, 1/2)$  is the commutator  $[\gamma_0, \gamma_1]$ , where  $\gamma_0$  (resp.  $\gamma_1$ ) is the simple circular loop through  $1/2$ , encircling  $0$  (resp.  $1$ ) in the positive direction. As such,  $\eta$  induces an element of the singular homology on any abelian cover of  $\mathbb{P}_{\mathbb{C}}^1$ , unramified but over  $0, 1$ , and  $\infty$ .

The previous analytic formulas all have their origin in algebraic geometry. Consider (0.2) first. It shows that  $\Gamma(s)$  is a period of a rank one algebraic differential module  $(\mathcal{E}_{\mathbb{C}}, \nabla_{\mathbb{C}})$  over the complex torus  $\mathbb{G}_{m, \mathbb{C}}$ , an algebraic group defined over  $\mathbb{Q}$ , namely the differential module whose (analytic or formal) solution is  $f_s(x) = x^s e^{-x}$ . The parameter  $s$  may be identified here as an exponent of local monodromy at  $0$  of  $(\mathcal{E}_{\mathbb{C}}, \nabla_{\mathbb{C}})$  (viewed as a logarithmic differential module at  $0$ ). An even more algebraic way of interpreting the variable  $s$  is to say that  $s$  is the isomorphism class of  $(\mathcal{E}_{\mathbb{C}}, \nabla_{\mathbb{C}})$ !

So, we regard  $s$  as a complex-valued character of the fundamental group of a tannakian category, namely the category of vector bundles on  $\mathbb{A}_{\mathbb{Q}}^1$ , endowed with an integrable connection with a logarithmic singularity and rational exponent at  $0$ . Notice that the singularity at  $\infty$  is instead *irregular*.

We express here, incidentally, our hope that this natural interpretation of the variable of the gamma function will be useful to relate the  $p$ -adic and complex incarnations of the  $L$ -function associated to a motive over  $\mathbb{Q}$ , say of the Riemann zeta function.

The case of (0.1) is easier. Here, the base space is  $\mathbb{P}_{\mathbb{Q}}^1$ , and the tannakian category consists of vector bundles with integrable connection having a logarithmic singularity at  $0, 1$  and  $\infty$  and rational exponents. Again,  $(s, t)$  is a complex-valued character of a pro-algebraic group defined over  $\mathbb{Q}$ . A further step is needed to interpret (0.5) and (0.7). The r.h.s. of (0.5) is a ratio of two terms. The denominator,  $s \mapsto \sin \pi s$ , takes integral algebraic values when the character  $s$  has finite period; so, it can be described purely algebraically. As for Archimedes' number  $\pi$ , it is algebraically related to a period of (the constant differential module on)  $\mathbb{G}_{m, \mathbb{C}}$ : for the generator  $\gamma_0$  of  $H_1(\mathbb{G}_{m, \mathbb{C}}, \mathbb{Z})$ , we have  $\int_{\gamma_0} \frac{dx}{x} = 2\pi\sqrt{-1}$ , with no sign ambiguity, if we regard  $\overline{\mathbb{Q}}$  as contained in  $\mathbb{C}$ .

When  $s$  and  $t$  are rational numbers with common denominator  $m$ , the integral formula (0.7) shows that  $-4 \sin(\pi s) \sin(\pi t) B(s, t)$  is the complex period of

$$(0.8) \quad \tau_{s,t} = x^{s-1} y^{t-1} dx, \quad x = X/Z, \quad y = Y/Z,$$

a differential form of the second kind on the Fermat curve  $F_m$  with homogeneous equation over  $\mathbb{Q}$

$$(0.9) \quad X^m + Y^m = Z^m,$$

along the inverse image  $\eta_m$  in  $F_m(\mathbb{C})$  of the Pochhammer path.

Now, for a rational prime  $p$  which does not divide  $m$ , the extension over  $\mathbb{Q}_p$  of the de Rham cohomology  $H_{\text{dR}}(F_m/\mathbb{Q})$  of  $F_m$  may be identified with the *rigid cohomology*

of the reduction modulo  $p$  of  $F_m$ . This type cohomology was first introduced by Dwork, Monsky and Washnitzer, and later, in full generality, by Berthelot. These finite-dimensional  $\mathbb{Q}_p$ -vector spaces carry, by functoriality in characteristic  $p$ , an action of the Frobenius element  $F$  of  $G_{\mathbb{F}_p} = \text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$ . One of the main themes of Dwork's theory is that the matrix describing the action of  $F$  on a rational basis of the space of rigid cohomology classes should be regarded as analog to the classical period matrix of  $(F_m)_{\mathbb{C}}$ . From this philosophy, Dwork deduced the main  $p$ -adic properties of solutions of classical Picard–Fuchs differential equations, and motivated cohomologically the  $p$ -adic beta function naturally constructed from Morita's  $p$ -adic gamma function. We refer to this  $p$ -adic beta and gamma functions as “crystalline”. The  $p$ -adic crystalline beta (resp. gamma) function provides a  $p$ -adic interpolation of Jacobi (resp. Gauß) sums over finite fields of characteristic  $p$ .

Similarly, the Galois action of  $G_{\mathbb{Q}} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  on the first  $p$ -adic étale homology group of  $(F_{p^m})_{\mathbb{C}}$  leads to the “étale”  $p$ -adic analog of beta due to Ihara [17]. The  $p$ -adic étale beta function provides an interpolating series for Jacobi sums over a finite field of characteristic  $\ell \neq p$ .

As they stand, the two theories of  $p$ -adic beta functions have disjoint ranges of application, the crystalline one dealing with the tower of Fermat curves  $(F_m)_{(m,p)=1}$ , in characteristic  $p$ , for  $m$  prime to  $p$ , while the étale one with the tower of Fermat curves  $(F_{p^m})_m$ , in characteristic 0 (say, over  $\mathbb{Q}_p$ ). Relating the two theories is not immediate, since  $F_{p^m}$  has bad reduction over  $\mathbb{Q}_p$ . However,  $F_{p^m}$  has potentially semistable reduction, and Coleman was able to compute the action of the Weil group on the de Rham cohomology of  $F_m$ , for any  $m$ , in his admirable paper [9]. We still refer to this theory as “crystalline” (even if “potentially semistable” would probably be a more appropriate choice). We recall, however, that the Jacobian variety  $J_m$  of  $F_m$  has potential good reduction at every  $p$ . The étale theory of the beta function was greatly improved by Anderson [1], [2], who considered the relative homology  $H_1(U_m(\mathbb{C}), Y_m(\mathbb{C}); \mathbb{Z}/n\mathbb{Z})$ , where  $U_m$  is the affine part of  $F_m$  and  $Y_m$  is its closed subscheme of equation  $xy = 0$ . He then considered the action of  $G_{\mathbb{Q}}$  on  $\varprojlim_{m,n} H_1(U_m(\mathbb{C}), Y_m(\mathbb{C}); \mathbb{Z}/n\mathbb{Z})$ , for unrestricted  $m, n \in \mathbb{N}$ . This has the advantage of carrying a special homology class  $\kappa_{m,n}$  corresponding to the real lifting of the path  $[0, 1]$ , and permits the discussion of (0.2) as well as (0.1). If one uses formula (0.7), however, it is enough to consider  $\varprojlim_{m,n} (J(F_m)_{\overline{\mathbb{Q}}}[n])$ , for unrestricted  $m, n \in \mathbb{N}$ , where  $J(F_m)_{\overline{\mathbb{Q}}}[n]$  denotes the subgroup of  $n$ -torsion points in the abelian variety  $J(F_m)$ , which permits one to avoid discussing 1-motives.

This being the situation, it was clearly possible to relate the two theories,  $p$ -adic étale and  $p$ -adic crystalline, of the beta function, suitably extended to treat the full tower of Fermat curves over  $\mathbb{Q}_p$ , via Fontaine's theory of  $p$ -adic periods.

Fontaine's theory represents one of the main achievements of mathematics in the twentieth century, in that it leads to very general comparison theorems between  $p$ -adic étale and  $p$ -adic de Rham realizations of motives and to a remarkable unification of theories. Granting a good formalism of algebraic homology [19], Fontaine's theory should provide a perfect “integration” pairing for motives defined over a  $p$ -adic number field, between de Rham cohomology and a suitable étale  $p$ -adic homology. This has been made explicit in the case of abelian varieties by Fontaine [16], Coleman [7] and, in the more general form we use, Colmez [12].

In this article, we deal with smooth projective curves  $X$  defined over  $\mathbb{Q}$ , and the identification of the  $p$ -adic étale homology

$$(0.10) \quad H_{1,\text{ét}}(X(\mathbb{C}), \mathbb{Z}_p) := \varprojlim_n H_{1,\text{sing}}(X(\mathbb{C}), \mathbb{Z}/p^n\mathbb{Z}) = \mathbb{Z}_p \otimes_{\mathbb{Z}} H_{1,\text{sing}}(X(\mathbb{C}), \mathbb{Z})$$

with

$$T_p(J(X)(\mathbb{C})) = \varprojlim_n \text{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, J(X)(\mathbb{C})),$$

where  $J(X)$  is the Jacobian variety of  $X$ , is classical [18, Lemma 9.2]. On the other hand,

$$(0.11) \quad T_p(J(X)(\mathbb{C})) = T_p(J(X)(\overline{\mathbb{Q}})),$$

and the identification is  $G_{\mathbb{Q}}$ -equivariant. In the previous formula, we should regard  $\overline{\mathbb{Q}}$  as contained in  $\mathbb{C}$ . If we also fix an embedding  $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ , the completion of the algebraic closure of  $\mathbb{Q}_p$ , we get a canonical  $G_{\mathbb{Q}}$ -equivariant isomorphism

$$(0.12) \quad \mathbb{Z}_p \otimes_{\mathbb{Z}} H_{1,\text{sing}}(X(\mathbb{C}), \mathbb{Z}) \cong T_p(J(X)(\overline{\mathbb{Q}}_p)).$$

On the other hand,  $H_{\text{dR}}^1(X/\mathbb{Q}) = H_{\text{dR}}^1(J(X)/\mathbb{Q})$ , and if  $J(X)$  has potential good reduction at  $p$ , and  $\mathcal{J}$  is a proper and smooth model of  $J(X)$  over the ring of integers of  $\overline{\mathbb{Q}}_p$ , we may use Colmez' explicit computation of  $p$ -adic periods [12]. We recall that  $\mathbb{B}_{\text{dR},p}$  is a complete discrete valuation field, with residue field  $\mathbb{C}_p$  and valuation ring  $\mathbb{B}_{\text{dR},p}^+$ . For  $u = (\dots, u_n, \dots) \in T_p(J(X)(\overline{\mathbb{Q}}_p))$  and  $\omega \in H_{\text{dR}}^1(X/\mathbb{Q})$ , identified with a differential form of the second kind on  $J(X)$ , Colmez defines

$$(0.13) \quad \int_u \omega = \lim_{n \rightarrow \infty} p^n (F_\omega(a_n) - F_\omega(a_n \oplus \hat{u}_n)).$$

In (0.13),  $\oplus$  denotes the addition law of  $J(X)$ ,  $F_\omega$  is a primitive of  $\omega$  and  $a_n, \hat{u}_n \in \mathcal{J}(\mathbb{B}_{\text{dR},p}^+)$  are suitably chosen, with  $\hat{u}_n$  a lifting of  $u_n \in J(X)(\mathbb{C}_p)$ . The value  $\int_u \omega$ , *a priori* in  $\mathbb{B}_{\text{dR},p}$ , in this particular case happens to live in the smaller ring  $\mathbb{B}_p := \mathbb{B}_{\text{crys},p} \otimes_{\mathbb{Q}_p^{\text{ur}}} \overline{\mathbb{Q}}_p$ .

We use in this article a variation of Anderson's construction in that we consider the Galois action (restricted to  $G_{\mathbb{Q}_p} = \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ ) on the "big"  $p$ -adic Tate module  $\mathbb{T}_p := \varprojlim_m (T_p(J(F_m)\overline{\mathbb{Q}}_p))$ , where  $m \in \mathbb{N}$  is unrestricted, as in Anderson's theory. Our main point is that  $\mathbb{T}_p$  is a free module of rank one on the profinite  $\mathbb{Z}_p$ -algebra  $\mathbb{Z}_p[[\hat{\mathbb{Z}}(1)^2]]$ , endowed with the natural cyclotomic action of  $G_{\mathbb{Q}_p}$ .

It is only natural to forget about the previous two definitions of  $p$ -adic beta functions, and to decree that, for  $s, t \in \mathbb{Q}$  with  $s, t, s+t \notin \mathbb{Z}$ ,

$$(0.14) \quad B_p^{\text{Font}}(s, t) := \int_{\eta^{(p)}} \tau_{s,t} \in \mathbb{B}_p$$

is *Fontaine's  $p$ -adic beta function*. The integral in (0.14), for rational  $s$  and  $t$  of common denominator  $m$ , is Fontaine's period of the differential form  $\tau_{s,t}$  against the image  $\eta_m^{(p)} \in T_p(J(F_m))$  of the lifting  $\eta_m \in H_{1,\text{sing}}(F_m(\mathbb{C}), \mathbb{Z})$  of the Pochhammer path to  $F_m$ . Our choice of a generator of the  $\mathbb{Z}_p[[\hat{\mathbb{Z}}(1)^2]]$ -module  $\mathbb{T}_p$  is precisely  $\eta^{(p)} := \varprojlim_m \eta_m^{(p)}$ ,

canonically corresponding to Pochhammer’s path, via the choice of two simultaneous embeddings of  $\overline{\mathbb{Q}}$  in  $\mathbb{C}$  and  $\mathbb{C}_p$ .

We do not understand at present the function-theoretic properties of (0.14). It is clear that it is not  $p$ -adically continuous as a function of  $\mathbb{Q}^2$  to  $\mathbb{B}_p$  (nor to  $\mathbb{B}_{\text{dR},p}$ ). It is also clear that this function has an application to the interpolation of Jacobi sums. Namely, if we denote by  $x \mapsto J_{a,b,c}(x) \in \mathbb{Q}(\zeta_m)$  the continuous character of the idèle group  $J_{\mathbb{Q}(\zeta_m)}$ , corresponding to the Jacobi sum Grössencharakter of [20], for  $a, b, c \in \mathbb{Q}/\mathbb{Z}$  and  $ma, mb, mc \in \mathbb{Z}$ , Fontaine’s beta function provides a  $p$ -adic interpolation of  $J_{a,b,c}(x) \in \mathbb{Q}(\zeta_m) \subset \overline{\mathbb{Q}_p} \subset \mathbb{C}_p$  as a simultaneous function of  $a, b, c \in \mathbb{Q}/\mathbb{Z}$  and  $x \in \bigcup_m J_{\mathbb{Q}(\zeta_m)}$ . Even though this function is not  $p$ -adically continuous, our final formulas (6.6), (6.7) show that  $B_p^{\text{Font}}(s, t)$  contains both the étale interpolation of Ihara [17], [1], and the crystalline (resp. potentially semistable) interpolation of Dwork (resp. Coleman) [8].

We only sketch here the main ideas and constructions, deferring to further papers actual computations and some questions which seem worth studying.

In the first place, we expect a *geometric* understanding of the link, via Fontaine’s periods, between an Anderson-style  $p$ -adic étale gamma  $G_{\mathbb{Q}_p}$ -cocycle, and Coleman’s extension to  $\mathbb{Q}_p$  of the  $p$ -adic Morita gamma function, which forms a cocycle of the Weil group  $W(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ . This seems to require Fontaine’s theory for  $p$ -adic realizations of 1-motives with potentially semistable reduction (like  $H_1(U_m, Y_m)$ ): this seems to be the subject of present investigations. At our request, Coleman kindly proved, during a week-end in October 2004, the local analyticity of his  $p$ -adic gamma cocycle, in line with Dwork’s *Boyarsky Principle*.

Furthermore, we have completely left out, much to our regret, the relation of the étale beta function with class field theory, via CM factors of the Jacobians of Fermat curves [2], [12]. In that perspective one should relate Anderson’s étale adelic beta and gamma functions to crystalline adelic beta and gamma functions, via some Fontaine’s “hyperadelic” beta and gamma functions taking values in the ring  $\prod_p \mathbb{B}_p$ .

It is also our hope to examine more closely the powerful computations of Coleman [9] and Coleman–McCallum [10], which should lead to a generalization of the notion of  $F$ -isocrystal into that of a  $p$ -adic differential equation on a  $p$ -adic algebraic variety, endowed with the action of a Weil group.

## 1. IHARA–ANDERSON BETA FUNCTION

1.1. From now on, we fix the prime number  $p$  and an embedding  $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$  (while  $\overline{\mathbb{Q}}$  is viewed as a subfield of  $\mathbb{C}$ ), and identify  $G_{\mathbb{Q}_p}$  with the decomposition group of  $G_{\overline{\mathbb{Q}}}$ , for the prime associated to  $\iota_p$ . For any  $m \in \mathbb{Z}_{>0}$ , let  $F_{m,\mathbb{Q}_p}$  denote the Fermat curve (0.9), viewed as a variety over  $\mathbb{Q}_p$ . Let  $J_m$  denote the Jacobian of  $F_m$ , and  $J_{m,\mathbb{Q}_p}$  that of  $F_{m,\mathbb{Q}_p}$ . We have natural maps  $\pi_{m,n} : F_m \rightarrow F_n$  whenever  $n \mid m$ , and corresponding maps of their Jacobians. We consider the projective system “Fermat tower”

$$(1.1.1) \quad \dots \rightarrow F_m \xrightarrow{\pi_{m,n}} F_n \xrightarrow{\pi_n} \mathbb{P}_{\mathbb{Q}}^1 \setminus \{x = 0, 1, \infty\}.$$

In particular, each  $F_m$  is, via  $\pi_m := \pi_{m,1}$ , an abelian covering of  $F_1 = \mathbb{P}_{\mathbb{Q}}^1 \setminus \{x = 0, 1, \infty\}$  with group  $\mu_m^3/\Delta$ , where  $\mu_m$  is the group of  $m$ -th roots of unity in  $\overline{\mathbb{Q}}^\times$ ,  $\Delta = \{(\zeta, \zeta, \zeta) \in$

$\mu_m^3$  is the diagonal subgroup, and  $(\zeta, \zeta', \zeta'') \in \mu_m^3$  acts on  $(x, y, z) \in F_m(\overline{\mathbb{Q}})$  via  $(x, y, z) \mapsto (\zeta x, \zeta' y, \zeta'' z)$ . The projection  $(\zeta, \zeta', \zeta'') \mapsto (\zeta, \zeta')$  identifies  $\mu_m^3/\Delta$  and  $\mu_m^2$  as  $G_{\mathbb{Q}}$ -modules. For  $\zeta \in \mu_m$ , we denote by  $[\zeta]_0$  (resp.  $[\zeta]_1$ ) the automorphism of  $F_m/F_1$  associated to  $(\zeta, 1, 1)$  (resp.  $(1, \zeta, 1)$ ), and  $[\zeta, \zeta'] := [\zeta]_0[\zeta']_1$ . So,  $\hat{\mathbb{Z}}(1)^2 = \varprojlim_m \mu_m^2$  acts on the tower of Fermat curves in a compatible way, namely if  $(z_0, z_1) \in \hat{\mathbb{Z}}(1)^2$  is represented by the projective system  $(\zeta_m, \zeta'_m)_m \in \varprojlim_m \mu_m^2$ , we set  $[z_1, z_2] = [\zeta_m, \zeta'_m]$  on  $F_m$ .

1.2. We are interested in the  $G_{\mathbb{Q}_p}$ -module

$$(1.2.1) \quad \mathbb{T}_p := \varprojlim_m T_p(J_m, \mathbb{Q}_p) = \varprojlim_m T_p(J_m(\overline{\mathbb{Q}_p}))$$

where  $T_p(-)$  denotes the Tate module. It will be viewed at the same time as a  $\mathbb{Z}_p[[\hat{\mathbb{Z}}(1)^2]]$ -module, where  $\hat{\mathbb{Z}}(1)^2$  is also viewed as a  $G_{\mathbb{Q}_p}$ -module.

The comparison isomorphisms recalled in the introduction imply that the Pochhammer paths  $\eta_m \in H_{1, \text{sing}}(F_m(\mathbb{C}), \mathbb{Z})$ , which are compatible with the maps  $F_m \rightarrow F_n$  for  $n|m$ , determine an element  $\eta^{(p)} = \varprojlim_m \eta_m^{(p)} \in \mathbb{T}_p$ . Our main point is

**THEOREM 1.2.2.**  $\mathbb{T}_p$  is a free  $\mathbb{Z}_p[[\hat{\mathbb{Z}}(1)^2]]$ -module of rank one, generated by  $\eta^{(p)}$ .

This follows from Anderson's results [2], [1, §5]. Anderson considers the affine part  $U_m$  of  $F_m$  and its closed subset  $Y_m$  of equation  $xy = 0$ . His theory is based on the study of the Galois structure of the relative (classical) homology groups  $H_1(U_m(\mathbb{C}), Y_m(\mathbb{C}); \mathbb{Z}/n\mathbb{Z})$ . He proves that this  $G_{\mathbb{Q}}$ -module is also a  $\mathbb{Z}/n\mathbb{Z}[\mu_m^2]$ -module of rank one, freely generated by the image  $\kappa_{m,n}$  of the path  $[0, 1]$ . But he also shows [2, (5.4.5)] that the Galois module  $J_m[n]$  is a quotient of the submodule of  $H_1(U_m(\mathbb{C}), Y_m(\mathbb{C}); \mathbb{Z}/n\mathbb{Z})$  generated by the image of the Pochhammer path

$$\eta_{m,n} = (1 - [\zeta_m]_{0*})(1 - [\zeta_m]_{1*})\kappa_{m,n},$$

where  $\zeta_m = e^{2\pi i/m}$ . The statement follows directly from this.

**REMARK 1.2.3.**  $H_1(U_m, Y_m; \mathbb{Z}/n\mathbb{Z}) = T_{\mathbb{Z}/n\mathbb{Z}}(H_1(U_m, Y_m))$  can be viewed as the  $n$ -torsion of the 1-motive  $H_1(U_m, Y_m)$  (over  $\mathbb{Q}$ ). It can be explicitly constructed from a generalized Jacobian of  $F_m$  [2, (4.2)]. So, it would be possible to pursue Anderson's approach throughout this paper. We prefer the more classical use of torsion points on the classical Jacobian, to have Colmez' results [11] at our disposal. In fact, to compare Anderson's theory of the gamma function with the theory of Dwork–Coleman, consideration of 1-motives will certainly be necessary.

We recall that Ihara [17] defined the  $G_{\mathbb{Q}}$ -modules

$$(1.2.4) \quad \mathbb{I}_p := \varprojlim_m T_p(J_{p^m}(\overline{\mathbb{Q}}))$$

and

$$(1.2.5) \quad \mathbb{Z}_p[[\mathbb{Z}_p(1)^2]] := \varprojlim_m (\mathbb{Z}/p^m\mathbb{Z})[\mu_{p^m}^2].$$

He showed that  $\mathbb{I}_p$  is a free  $\mathbb{Z}_p[[\mathbb{Z}_p(1)^2]]$ -module of rank one, generated by the image of the Pochhammer path.

So, our definition (1.2.1) differs slightly from both Anderson's and Ihara's.

1.3. We define a continuous cocycle  $G_{\mathbb{Q}_p} \rightarrow \mathbb{Z}_p[[\hat{\mathbb{Z}}(1)]]$ ,  $\sigma \mapsto B_\sigma^{\text{ét}}$ , called the *beta cocycle*, via

$$(1.3.1) \quad \sigma \eta = B_\sigma^{\text{ét}} \eta.$$

We may regard the elements of  $\mathbb{Z}_p[[\hat{\mathbb{Z}}(1)^r]]$ , for any  $r$ , as functions  $(\mathbb{Q}/\mathbb{Z})^r \rightarrow \overline{\mathbb{Q}_p}$  [1, (1.4)]. Namely, given  $a \in (m^{-1}\mathbb{Z}/\mathbb{Z})^r$  and  $f \in \mathbb{Z}_p[[\hat{\mathbb{Z}}(1)^r]]$ , one can define  $f(a) \in \overline{\mathbb{Q}_p}$  by first considering the image  $f_m$  of  $f$  in  $\mathbb{Z}_p[[\mu_m^r]]$ , and then using the identification  $\mu_m = \text{Hom}(m^{-1}\mathbb{Z}/\mathbb{Z}, \overline{\mathbb{Q}_p}^\times)$ , and  $\mathbb{Z}_p$ -linearity in  $f_m$ , to obtain an element  $f(a) = f_m(a) \in \mathcal{O}_{\overline{\mathbb{Q}_p}}$ . The cyclotomic character

$$\chi : G_{\mathbb{Q}_p} \rightarrow \hat{\mathbb{Z}}^\times$$

we consider, and an action of  $G_{\mathbb{Q}_p}$  on  $\mathbb{Q}/\mathbb{Z}$ , are defined by the formulas

$$(1.3.2) \quad f(\sigma a) = f(\chi(\sigma)a) = f(a)^\sigma$$

for  $f \in \mu_m(\overline{\mathbb{Q}_p})$ ,  $a \in m^{-1}\mathbb{Z}/\mathbb{Z}$ , and  $\sigma \in G_{\mathbb{Q}_p}$ . According to our previous definition, formulas (1.3.2) actually hold for  $f \in \mathbb{Z}_p[[\hat{\mathbb{Z}}(1)^r]]$ ,  $a \in (\mathbb{Q}/\mathbb{Z})^r$ , any  $r$ , and  $\sigma \in G_{\mathbb{Q}_p}$ . The cocycle property of  $B_\sigma^{\text{ét}}$  reads

$$(1.3.3) \quad B_{\sigma\tau}^{\text{ét}}(a) = B_\sigma^{\text{ét}}(a)B_\tau^{\text{ét}}(\chi(\sigma)a)$$

for  $a \in (\mathbb{Q}/\mathbb{Z})^2$  and  $\sigma, \tau \in G_{\mathbb{Q}_p}$ .

For  $\sigma \in G_{\mathbb{Q}_p}$  and  $a, b \in (\mathbb{Q}/\mathbb{Z})^2$  such that  $\sigma b = a$ , we set

$$(1.3.4) \quad B_p^{\text{ét}}(a; b) = B_\sigma^{\text{ét}}(a).$$

For  $\sigma \in G_{\mathbb{Q}_p}$ , let

$$(1.3.5) \quad \overline{\mathcal{H}}_\sigma := \{(a; b) \in (\mathbb{Q}/\mathbb{Z})^2 \times (\mathbb{Q}/\mathbb{Z})^2 \mid \sigma b = a\}$$

and

$$(1.3.6) \quad \overline{\mathcal{H}} := \bigcup_{\sigma \in G_{\mathbb{Q}_p}} \overline{\mathcal{H}}_\sigma$$

So,  $B_p^{\text{ét}}$  may be viewed as a function  $B_p^{\text{ét}} : \overline{\mathcal{H}} \rightarrow \mathcal{O}_{\overline{\mathbb{Q}_p}}$  such that

$$(1.3.7) \quad B_p^{\text{ét}}(a; c) = B_p^{\text{ét}}(a; b)B_p^{\text{ét}}(b; c)$$

for any  $a, b, c \in \overline{\mathcal{H}}$ . The function  $B_p^{\text{ét}}(a; b)$  will later be regarded as a function of  $(a; b) = (a_1, a_2; b_1, b_2) \in \mathbb{Q}^4$ , constant on classes mod  $\mathbb{Z}^4$ .

1.4. We now take the attitude outlined in the introduction, that the natural “global” variable for the beta function should be a line bundle (which will here be taken to be trivial) on  $\mathbb{P}^1 = \mathbb{P}_{\mathbb{Q}}^1$ , endowed with a connection with logarithmic singularities and rational exponents along the reduced divisor  $D$  with support  $\{0, 1, \infty\}$ . We define  $\mathcal{L}_{s,t} := (\mathcal{O}_{\mathbb{P}^1}, \nabla_{s,t})$ , where

$$(1.4.1) \quad \nabla_{s,t} : \mathcal{O}_{\mathbb{P}^1} \rightarrow \Omega_{\mathbb{P}^1/\mathbb{Q}}^1(\log D), \quad 1 \mapsto s \frac{dx}{x} + t \frac{dx}{x-1}.$$

We will consider the *logarithmic de Rham cohomology groups* of (1.4.1), namely the hypercohomology groups (as complexes of abelian sheaves for the étale topology)

$$(1.4.2) \quad H_{\log\text{-dR}}^i(\mathbb{P}^1/\mathbb{Q}, \nabla_{s,t}) := \mathbb{H}^i(\mathbb{P}^1/\mathbb{Q}, \mathcal{O}_{\mathbb{P}^1} \xrightarrow{\nabla_{s,t}} \Omega_{\mathbb{P}^1/\mathbb{Q}}^1(\log D)).$$

Let  $[\frac{dx}{x(x-1)}]_{s,t} \in H_{\log\text{-dR}}^1(\mathbb{P}^1/\mathbb{Q}, \nabla_{s,t})$  denote the cohomology class of the differential form  $\frac{dx}{x(x-1)}$ . The restriction of the map  $\nabla_{s,t}$  of (1.4.1) to  $\mathbb{P}^1 \setminus \{0, 1, \infty\}$  induces a logarithmic connection on any line bundle of the form  $\mathcal{O}_{\mathbb{P}^1} x^m (1-x)^n$  for  $m, n \in \mathbb{Z}$ . Multiplication by  $x^{-m} (1-x)^{-n}$  identifies this line bundle with logarithmic connection with  $(\mathcal{O}_{\mathbb{P}^1}, \nabla_{s+m, t+n})$ , so that  $H_{\log\text{-dR}}^1(\mathbb{P}^1/\mathbb{Q}, \nabla_{s+m, t+n}) = H_{\log\text{-dR}}^1(\mathbb{P}^1/\mathbb{Q}, \nabla_{s,t})$ . In this sense

$$(1.4.3) \quad \left[ \frac{dx}{x(x-1)} \right]_{s+m, t+n} = \left[ x^m (1-x)^n \frac{dx}{x(x-1)} \right]_{s,t} = \frac{B(s+m, t+n)}{B(s, t)} \left[ \frac{dx}{x(x-1)} \right]_{s,t}.$$

Let  $m \in \mathbb{N}$  be a common denominator of  $s, t \in \mathbb{Q}$ ,  $s = S/m, t = T/m$ , with  $S, T \in \mathbb{Z}$ . On the affine part  $U_m$  of  $F_m$  (see (0.9)) we consider the functions  $x_m$  and  $y_m$  induced by the coordinate projections  $X/Z$  and  $Y/Z$ . We also set  $x = x_1$  and  $y = y_1$ , both functions on  $\mathbb{P}^1$  and on  $F_m$  via the natural projection  $F_m \rightarrow F_1 = \mathbb{P}^1$ . So,  $x_m$  (resp.  $y_m$ ) is a branch of  $x^{1/m}$  (resp.  $(1-x)^{1/m}$ ), and  $x_m, y_m$  are linked by the equation

$$(1.4.4) \quad x_m^m + y_m^m = 1.$$

In this way we single out a choice of a branch of the algebraic function  $x^s (1-x)^t$  on  $\mathbb{P}^1$ , namely the rational function  $x_m^S y_m^T$  on  $F_m$ . The connection (1.4.1), when pulled back to  $F_m$ , becomes the unique logarithmic connection on  $\mathcal{O}_{F_m}$ , with logarithmic singularities along the inverse image  $D_m$  of  $D$  in  $F_m$ , and the rational solution  $x_m^{-S} y_m^{-T} = x^{-s} (1-x)^{-t}$ . On the other hand,  $\pi_{m*} \mathcal{O}_{F_m} = \bigoplus \mathcal{O}_{\mathbb{P}^1} x_m^S y_m^T$ , where the direct sum is taken over  $0 \leq S < m, 0 \leq T < m$ . Since the map  $(F_m, D_m)_{\mathbb{Q}(\mu_m)} \rightarrow (\mathbb{P}^1, D)_{\mathbb{Q}(\mu_m)}$  is a log-étale Galois covering with group  $\mu_m^2$ , the direct image of the logarithmic complex

$$\pi_{m*}(\mathcal{O}_{F_m} \otimes \mathbb{Q}(\mu_m)) \xrightarrow{d_{F_m} \otimes \mathbb{Q}(\mu_m)} \Omega_{F_m/\mathbb{Q}}^1(\log D_m) \otimes \mathbb{Q}(\mu_m)$$

decomposes into the direct sum

$$(\mathcal{O}_{\mathbb{P}^1} x_m^S y_m^T \xrightarrow{\pi_{m*}(d_{F_m})} \Omega_{\mathbb{P}^1/\mathbb{Q}}^1(\log D) x_m^S y_m^T) \otimes \mathbb{Q}(\mu_m)$$

which is isomorphic, via multiplication by  $x_m^S y_m^T$ , to

$$(\mathcal{O}_{\mathbb{P}^1} \xrightarrow{\nabla_{s,t}} \Omega_{\mathbb{P}^1/\mathbb{Q}}^1(\log D)) \otimes \mathbb{Q}(\mu_m).$$

Since everything descends to  $\mathbb{Q}$ , we conclude that

$$\pi_{m*}(\mathcal{O}_{F_m} \xrightarrow{d_{F_m}} \Omega_{F_m/\mathbb{Q}}^1(\log D_m))$$

decomposes into the direct sum

$$\mathcal{O}_{\mathbb{P}^1} x_m^S y_m^T \xrightarrow{\pi_{m*}(d_{F_m})} \Omega_{\mathbb{P}^1/\mathbb{Q}}^1(\log D) x_m^S y_m^T$$

which is isomorphic, via multiplication by  $x_m^S y_m^T$ , to

$$\mathcal{O}_{\mathbb{P}^1} \xrightarrow{\nabla_{s,t}} \Omega_{\mathbb{P}^1/\mathbb{Q}}^1(\log D).$$

In particular,

$$(1.4.5) \quad H_{\log\text{-dR}}^1((F_m, D_m)/\mathbb{Q}) = \bigoplus_{s,t} H_{\log\text{-dR}}^1(\mathbb{P}^1/\mathbb{Q}, \nabla_{s,t})$$

where the sum runs over  $s, t \in m^{-1}\mathbb{Z}$ ,  $0 \leq s, t < 1$ . For any  $s, t \in \mathbb{Q}$  of common denominator  $m \in \mathbb{N}$ , we may identify  $[\frac{dx}{x(x-1)}]_{s,t} \in H_{\log\text{-dR}}^1(\mathbb{P}^1/\mathbb{Q}, \nabla_{s,t})$  with the cohomology class  $[\tau_{s,t}]$  in  $H_{\log\text{-dR}}^1((F_m, D_m)/\mathbb{Q})$ , of the Fermat differential

$$\tau_{s,t} = x^{s-1} y^{t-1} dx, \quad x = X/Z, \quad y = Y/Z,$$

on the Fermat curve  $F_m$ . For  $s, t, s+t \notin \mathbb{Z}$ ,  $[\tau_{s,t}]$  is a differential form of the second kind on  $F_m$ , and in fact these forms generate  $H_{\text{dR}}^1(F_m/\mathbb{Q}) \subset H_{\log\text{-dR}}^1((F_m, D_m)/\mathbb{Q})$ . So,

$$(1.4.6) \quad H_{\text{dR}}^1(F_m/\mathbb{Q}) = \bigoplus_{s,t} H_{\log\text{-dR}}^1(\mathbb{P}^1/\mathbb{Q}, \nabla_{s,t})$$

where the sum runs over  $s, t \in m^{-1}\mathbb{Z}$  with  $0 < s, t < 1$  and  $s+t \notin \mathbb{Z}$ .

We define

$$(1.4.7) \quad H_{\log\text{-dR}}^1((F_\bullet, D_\bullet)/\mathbb{Q}) := \varinjlim H_{\log\text{-dR}}^1((F_m, D_m)/\mathbb{Q}) = \bigoplus_{s,t} H_{\log\text{-dR}}^1(\mathbb{P}^1/\mathbb{Q}, \nabla_{s,t}),$$

where the sum runs over  $s, t \in \mathbb{Q}$ ,  $0 \leq s, t < 1$ , and its subspace

$$(1.4.8) \quad H_{\text{dR}}^1(F_\bullet/\mathbb{Q}) := \varinjlim H_{\text{dR}}^1(F_m/\mathbb{Q}) = \bigoplus_{s,t} H_{\log\text{-dR}}^1(\mathbb{P}^1/\mathbb{Q}, \nabla_{s,t}),$$

where the sum is restricted to  $0 < s, t < 1$  with  $s+t \notin \mathbb{Z}$ .

The one-dimensional  $\mathbb{Q}$ -vector subspace of  $H_{\log\text{-dR}}^1((F_m, D_m)/\mathbb{Q})$  generated by the cohomology class  $[\tau_{s,t}] \in H_{\log\text{-dR}}^1((F_m, D_m)/\mathbb{Q})$ , for  $m \in \mathbb{N}$  such that  $ms, mt \in \mathbb{Z}$ , is

determined as the  $G_{\mathbb{Q}}$ -invariant subspace of  $H_{\log\text{-dR}}^1((F_m, D_m)/\mathbb{Q}) \otimes \overline{\mathbb{Q}}$  on which  $\mu_m^2$  acts via the character

$$(1.4.9) \quad \chi_{s,t} : \mu_m^2 \rightarrow \mu_m, \quad (\zeta, \zeta') \mapsto \zeta^S \zeta'^T.$$

For any  $\zeta, \zeta' \in \mu_m$ , and  $s, t \in \mathbb{Q}$ , of common denominator  $m$ , we then have

$$(1.4.10) \quad [\zeta, \zeta']^*[\tau_{s,t}] = \chi_{s,t}(\zeta, \zeta')[\tau_{s,t}].$$

Our functional interpretation is based on the map

$$(1.4.11) \quad \mathbb{Q}^2 \rightarrow H_{\log\text{-dR}}^1((F_{\bullet}, D_{\bullet})/\mathbb{Q}), \quad (s, t) \mapsto [\tau_{s,t}].$$

For any character  $\chi : \hat{\mathbb{Z}}(1) \rightarrow \overline{\mathbb{Q}}_p^{\times}$  of finite order, we denote by  $H_{\log\text{-dR}}^1((F_{\bullet}, D_{\bullet})/\mathbb{Q})_{\chi}$  (resp.  $H_{\text{dR}}^1(F_{\bullet}/\mathbb{Q})_{\chi}$ ) the  $\mathbb{Q}$ -vector subspace of  $H_{\log\text{-dR}}^1((F_{\bullet}, D_{\bullet})/\mathbb{Q})$  (resp.  $H_{\text{dR}}^1(F_{\bullet}/\mathbb{Q})$ ) of  $G_{\mathbb{Q}}$ -invariant classes  $\alpha \in H_{\log\text{-dR}}^1((F_{\bullet}, D_{\bullet})/\mathbb{Q}) \otimes \overline{\mathbb{Q}}$  (resp.  $H_{\text{dR}}^1(F_{\bullet}/\mathbb{Q}) \otimes \overline{\mathbb{Q}}$ ) such that

$$(1.4.12) \quad [z_1, z_2]_1^* \alpha = \chi(z_1, z_2) \alpha.$$

For  $\chi = \chi_{s,t}$ ,  $s, t$  as in (1.4.10), we get  $H_{\log\text{-dR}}^1((F_{\bullet}, D_{\bullet})/\mathbb{Q})_{\chi_{s,t}} = \mathbb{Q}[\tau_{s,t}]$ . So, the elements of  $\hat{\mathbb{Z}}(1)^2 = \varprojlim_m \mu_m^2$  are naturally functions on  $\mathbb{Q}^2$  with values in  $\overline{\mathbb{Q}}_p^{\times}$ , namely we set

$$(1.4.13) \quad (\zeta, \zeta')(s, t) = \chi_{s,t}(\zeta, \zeta') = \zeta^{ms} \zeta'^{mt}$$

for any  $m \in \mathbb{N}$  such that  $ms, mt \in \mathbb{Z}$  and  $\zeta, \zeta' \in \mu_m$ . This extends to an interpretation of the elements of  $\mathbb{Z}_p[[\hat{\mathbb{Z}}(1)^2]]$  as functions  $\mathbb{Q}^2 \rightarrow \overline{\mathbb{Q}}_p$ , constant on classes mod  $\mathbb{Z}^2$ , compatibly with our previous functional interpretation (see Section 1.3).

It is now natural to define, for  $\sigma \in G_{\mathbb{Q}_p}$ ,

$$(1.4.14) \quad \mathcal{H}_{\sigma} := \{(a; b) \in \mathbb{Q}^2 \times \mathbb{Q}^2 \mid (a; b) \bmod \mathbb{Z}^4 \in \overline{\mathcal{H}}_{\sigma}\}$$

and

$$(1.4.15) \quad \mathcal{H} := \bigcup_{\sigma \in G_{\mathbb{Q}_p}} \mathcal{H}_{\sigma}.$$

Then,  $B_p^{\text{ét}}$  may be viewed as a function (constant on classes mod  $\mathbb{Z}^4$ )

$$B_p^{\text{ét}} : \mathcal{H} \rightarrow \mathcal{O}_{\overline{\mathbb{Q}}_p}$$

such that the relation (1.3.7) holds for any  $a, b, c \in \mathcal{H}$ .

## 2. DWORK–MORITA CRYSTALLINE BETA FUNCTION

2.1. The *Boyarsky Principle* of Dwork [6], [13], [15] is the following vaguely stated conjecture:

*If cohomology is parametrized by a character then the Frobenius operation will vary continuously [locally analytically] with the character.*

The first problem in approaching the Boyarsky Principle consists in giving a precise meaning to the character mentioned in it. Of which group should it be a character? In this article we take the attitude outlined in the introduction, that the group involved should be the fundamental group of a suitable tannakian category of differential modules. We illustrate this by a typical example, in the classical language of overconvergent  $F$ -isocrystals [4, (2.3.7)].

Let  $X$  be a smooth variety defined over  $\mathbb{F}_p$ ,  $x_0 \in X(\mathbb{F}_p)$ , and let

$$(2.1.1) \quad \cdots \rightarrow X_n \rightarrow X_m \xrightarrow{\pi_m} X$$

be a projective system of abelian étale coverings, geometrically irreducible, with group  $H_m = G(X_m/X)$ , of order prime to  $p$ . Let  $\sigma \in G_{\mathbb{Q}_p}$  be a lifting of the Frobenius automorphism  $\sigma_0$  of  $G_{\mathbb{F}_p}$  and  $F_X : X \rightarrow X$  be the absolute Frobenius of  $X$ . We assume to be given, for some Galois extension  $K$  of  $\mathbb{Q}_p$ , an overconvergent  $F$ -isocrystal  $\mathcal{L}$  on  $X/(K, \sigma|_K)$ . Then  $\pi_{m*}\pi_m^*\mathcal{L}$  will decompose into a direct sum  $\bigoplus_{\chi \in \hat{H}_m} \mathcal{L}_\chi$ , where  $\hat{H}_m = \text{Hom}(H_m, \overline{\mathbb{Q}_p}^\times)$  denotes the group of characters of  $H_m$ . Each factor  $\mathcal{L}_\chi$  is an example of an  $F$ -isocrystal parametrized by the character  $\chi$  of  $H = \varprojlim_m H_m$ . The group  $H$  is an abelian quotient of  $\pi_1^{(\text{prime to } p)}(X_{\mathbb{F}_p}, x_0)$ , and it is endowed with a canonical action of  $G_{\mathbb{F}_p}$ . We have a horizontal morphism

$$(2.1.2) \quad \Phi_\chi(\sigma) : F_X^* \mathcal{L}_\chi^{(\sigma)} \xrightarrow{\sim} \mathcal{L}_{\chi \circ \sigma_0},$$

where  $\mathcal{L}_\chi^{(\sigma)}$  is the overconvergent isocrystal on  $X/K$ , deduced from  $\mathcal{L}_\chi$  by the base change  $\sigma|_K$ .

If  $X_m = (F_m \setminus D_m)_{\mathbb{F}_p}$  over  $X = X_1 \simeq \mathbb{P}_{\mathbb{F}_p}^1 \setminus \{0, 1, \infty\}$ , then  $H \cong \hat{\mathbb{Z}}^{(\text{prime to } p)}(1)^2$ , and the group of characters is  $\hat{H} \cong (\mathbb{Z}_{(p)}/\mathbb{Z})^2$ , with the cyclotomic action of  $G_{\mathbb{F}_p}$ . So, the Frobenius automorphism of  $G_{\mathbb{F}_p}$  acts by multiplication by  $p$  on  $\hat{H}$ . We have a horizontal morphism

$$(2.1.3) \quad \Phi_\chi(\sigma) : F_X^* \mathcal{L}_\chi^{(\sigma)} \xrightarrow{\sim} \mathcal{L}_{p\chi}.$$

Passing to the rigid cohomology of  $\pi_m^*\mathcal{L}$ ,  $H_{\text{rig}}(X_m/K, \pi_m^*\mathcal{L})$ , we see that it also decomposes into  $\bigoplus_{\chi \in \hat{H}_m} H_\chi$ ,  $H_\chi = H_{\text{rig}}(X_m/K, \pi_m^*\mathcal{L}_\chi)$  interchanged by Frobenius

$$(2.1.4) \quad \Phi_{\chi, \chi'}(\sigma) : H_{\chi'}^{(\sigma)} \rightarrow H_\chi,$$

where  $\chi' \circ \sigma_0 = \chi$  in  $\hat{H}_m$ . More general examples of  $F$ -isocrystals parametrized by a character may be obtained by adding to the previous data a smooth morphism  $f : X \rightarrow S$ , and studying the direct image of the isocrystals  $\mathcal{L}_\chi$  under  $f$  [14], [15].

2.2. To make sense of the Boyarsky Principle, which vaguely asserts the analyticity of the map (2.1.4) as a function of  $\chi$ , we need to lift the previous data to characteristic zero, and to take the logarithmic de Rham viewpoint of the previous section. We illustrate the statement in the particular case of the tower of Fermat curves  $F_m$ , of degree prime to  $p$ ,

over  $\mathbb{Q}_p$ . So,  $X = \mathbb{P}_{\mathbb{F}_p}^1 \setminus \{0, 1, \infty\}$  and  $X_m$  is the special fiber of  $F_m \setminus D_m$  over  $\mathbb{F}_p$ , for  $(m, p) = 1$ . Then  $K = \mathbb{Q}_p$  and the  $F$ -isocrystal  $\mathcal{L}$  on  $X/\mathbb{Q}_p$  is represented by the  $p$ -adic rigid analytification of the algebraic connection  $(\mathcal{O}_{\mathbb{P}_{\mathbb{Q}_p}^1 \setminus \{0, 1, \infty\}}, d)$ . We take our logarithmic viewpoint; then, under the covering  $\pi_m : F_m \rightarrow F_1$ ,  $\pi_{m*}\pi_m^*\mathcal{L}$  decomposes into a direct sum of factors represented by  $\mathcal{L}_{s,t}^{\text{an}} = (\mathcal{O}_{\mathbb{P}_{\mathbb{Q}_p}^1}^{\text{an}}, \nabla_{s,t}^{\text{an}})$ , with  $s, t \in m^{-1}\mathbb{Z}$ ,  $0 \leq s, t < 1$ . Also, via well-known comparison theorems [3],

$$(2.2.1) \quad H_{\text{dR}}^1(\mathbb{P}^1 \setminus \{0, 1, \infty\}/\mathbb{Q}, \nabla_{s,t}) \otimes \mathbb{Q}_p \cong H_{\text{rig}}^1(X/\mathbb{Q}_p, \mathcal{L}_{s,t}^{\text{an}}),$$

so, if  $0 < s, t < 1$  and  $s + t \neq 1$ , we also have

$$(2.2.2) \quad H_{\text{rig}}^1(X/\mathbb{Q}_p, \mathcal{L}_{s,t}^{\text{an}}) \cong H_{\text{log-dR}}^1(\mathbb{P}^1/\mathbb{Q}, \mathcal{L}_{s,t}) \otimes \mathbb{Q}_p.$$

The advantage of this viewpoint is that we have naturally parametrized characters of the (abelian quotient of the) geometric fundamental group  $\pi_1^{(\text{prime to } p)}(X_{\overline{\mathbb{F}_p}}, x_0)$ , for some  $x_0 \in \mathbb{P}^1(\mathbb{F}_p) \setminus \{0, 1, \infty\}$ , (corresponding to the Fermat tower of degrees prime to  $p$ ) by pairs  $(s, t) \in \mathbb{Z}_{(p)}^2$ , or equivalently by differential classes  $[\tau_{s,t}]$ . We are going to express the Frobenius matrix in terms of this natural parametrization. To avoid minor complications, we restrict ourselves to forms  $[\tau_{s,t}]$  which correspond to differentials of the second kind on some Fermat covering, i.e. to  $s, t, s + t \notin \mathbb{Z}$ .

Let us consider a subset of the set  $\mathcal{H}$  defined in (1.4.15), namely

$$(2.2.3) \quad \mathcal{H}_W := \bigcup_{\sigma \in \mathbf{W}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)} \mathcal{H}_\sigma.$$

We also define

$$(2.2.4) \quad \mathcal{H}_\sigma^{\text{non-deg}} = \{(a, b) = (a_1, a_2; b_1, b_2) \in \mathcal{H}_\sigma \mid a_1, a_2, b_1, b_2, a_1 + a_2, b_1 + b_2 \notin \mathbb{Z}\},$$

$$(2.2.5) \quad \mathcal{H}_W^{\text{non-deg}} := \bigcup_{\sigma \in \mathbf{W}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)} \mathcal{H}_\sigma^{\text{non-deg}}.$$

The *Dwork–Morita crystalline beta function* is the function

$$(2.2.6) \quad B_p^{\text{crys}} : \mathcal{H}_W^{\text{non-deg}} \cap \mathbb{Z}_{(p)}^4 \rightarrow \mathbb{Q}_p^\times$$

defined by

$$(2.2.7) \quad \Phi_{\chi_{s,t}, \chi_{s',t'}}(\sigma)(F_X^*[\tau_{s',t'}]) = p B_p^{\text{crys}}(s, t; s', t')^{-1}[\tau_{s,t}]$$

for  $p(s', t') - (s, t) \in \mathbb{Z}^2$ , and in general (writing  $B_p^{\text{crys}}(s, t; s', t') = B_p^{\text{crys}}((s, t); (s', t'))$ ) by the condition

$$(2.2.8) \quad B_p^{\text{crys}}(a; c) = B_p^{\text{crys}}(a; b) B_p^{\text{crys}}(b; c)$$

for any  $a, b, c \in \mathcal{H}_W^{\text{non-deg}}$ .

The action of Frobenius on  $[\tau_{s,t}]$ , for  $s, t \in \mathbb{Z}_{(p)}$  and  $s, t, s + t \notin \mathbb{Z}$ , is carefully computed in [14, (22.2.4)]. We now pause to discuss in detail Dwork's theory of the  $p$ -adic crystalline gamma function and its application to  $p$ -adic interpolation of Gauß sums.

3. EXAMPLE:  $p$ -ADIC INTERPOLATION OF GAUSS SUMS

3.1. We describe Dwork's approach to  $p$ -adic interpolation of functions arising from "variation of cohomology". To make this section completely independent of the previous ones, we start by detailing notation.

Let  $\psi_p$  be a non-trivial additive character of the field  $\mathbb{F}_p$  with  $p$  elements; let  $q = p^f$ , and  $\mathbb{F}_q$  be the field with  $q$  elements. Let  $\text{Teich}$  be the Teichmüller character of  $\mathbb{F}_q^\times$ . For  $a \in \frac{1}{q-1}\mathbb{Z}$ , define

$$(3.1.1) \quad \chi_a = \text{Teich}^{a(1-q)},$$

a multiplicative character of  $\mathbb{F}_q$  (extended by  $\chi_a(0) = 0 \forall a$ ). We recall the *Gauß sum*

$$(3.1.2) \quad G_q(a) = \sum_{x \in \mathbb{F}_q} \chi_a(x) \psi_p(\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(x)).$$

Let  $|\cdot|$  be the (ultrametric) absolute value of  $\mathbb{C}_p$ , normalized by  $|p| = p^{-1}$ . Let  $\pi_{\text{Dwork}} \in \mathbb{C}_p$ ,  $\pi_{\text{Dwork}}^{p-1} = -p$ , closest to  $\psi_p(1) - 1$ . We denote by  $\text{ord } x = -\log_p |x|$  the  $p$ -adic valuation of  $\mathbb{C}_p$ .

Y. Morita defined a  $p$ -adic analytic function  $\Gamma_p$  on the union of the  $p$  disks:

$$(3.1.3) \quad \bigcup_{\mu=0}^{p-1} D(-\mu, \rho^-)$$

with<sup>1</sup>  $p^{-1} < \rho = p^{-\frac{1}{p} - \frac{1}{p-1}} < 1$  such that

$$(3.1.4) \quad \begin{aligned} \Gamma_p(0) &= 1, \\ \frac{\Gamma_p(1+x)}{\Gamma_p(x)} &= \begin{cases} -x & \text{if } |x| = 1, \\ -1 & \text{if } |x| < 1. \end{cases} \end{aligned}$$

In particular,

$$(3.1.5) \quad \Gamma_p(n) = (-1)^n \prod_{\substack{j=0 \\ (j,p)=1}}^{n-1} j.$$

3.2. If we put

$$\Theta_p(x) = \exp \pi_{\text{Dwork}}(x - x^p) = \sum_{j=0}^{\infty} c_j x^j,$$

then

$$\text{ord } c_j \geq j \frac{p-1}{p^2} \quad \forall j = 0, 1, \dots,$$

and for  $\mu \in [0, p-1]$  we have

$$\Gamma_p(-\mu + py) = \pi^{-\mu} \sum_{i=0}^{\infty} \frac{c_{pi+\mu}(y)_i}{(-\pi_{\text{Dwork}})^i}$$

<sup>1</sup> This estimate is due to Dwork.

where, as usual,

$$(y)_i = \frac{\Gamma(y+i)}{\Gamma(y)} = y(y+1)\cdots(y+i-1)$$

for  $i = 1, 2, \dots$ . Dwork introduced the more flexible function  $\gamma_p(x, y)$  meromorphic in

$$(3.2.1) \quad \mathcal{D}^{(t)}(p\rho) = \{(x, y) \in \mathbb{C}_p^2 \mid py - x = t, |y| < p\rho = p^{1-\frac{1}{p}-\frac{1}{p-1}}\}$$

for any fixed  $t \in \mathbb{Z}$ , defined by

$$(3.2.2) \quad \gamma_p(x, y) = \sum_{pi+t \geq 0} c_{pi+t} \frac{\Gamma(y+i)}{\Gamma(y)} / (-\pi)^i.$$

The modular property of  $\gamma_p$  is then

$$(3.2.3) \quad \gamma_p(x+m, y+n) = \gamma_p(x, y) (-\pi)^{n-m} \frac{\Gamma(x+m)}{\Gamma(x)} \frac{\Gamma(y)}{\Gamma(y+n)}$$

for any  $(m, n)$  in  $\mathbb{Z}^2$ . The meromorphic function  $\gamma_p(x, y)$  has in  $\mathcal{D}^{(t)}$  only a finite number of poles, namely the solutions  $(x, y)$  of  $py - x = t$  with  $y \in \mathbb{Z}_{>0}$  and  $x \in \mathbb{Z}_{\leq 0}$ .

3.3. The *Gross–Koblitz formula* for interpolation of the Gauß sum  $G_q(a)$ ,  $a \in \frac{1}{q-1}\mathbb{Z}$ , is

$$(3.3.1) \quad -G_q(a) = \prod_{i=1}^f \gamma_p(a^{(i-1)}, a^{(i)}),$$

where  $a = a^{(0)}, a^{(1)}, \dots, a^{(f)} = a$  ( $a^{(i)} \in \mathbb{Z}_p$ ) are defined by

$$(3.3.2) \quad pa^{(i)} - a^{(i-1)} \in \mathbb{Z} \quad \text{for } i = 1, \dots, f-1,$$

which is possible since  $a \in \frac{1}{p^f-1}\mathbb{Z}$ .

We now explain the relation to the integral formula (0.2). The modular property of  $\gamma_p$  holds since it is the matrix of Frobenius on the cohomology of  $\mathbb{G}_{m, \mathbb{F}_q}$  with coefficients in the (overconvergent) differential module on  $\mathbb{G}_{m, \mathbb{C}_p}$ :

$$(3.3.3) \quad \nabla_a : \mathbb{C}_p[x, x^{-1}] \rightarrow \mathbb{C}_p[x, x^{-1}]dx, \quad f \mapsto df + f(\pi x + a) \frac{dx}{x} \\ = \left( x \frac{d}{dx} + a + \pi x \right) (f) \frac{dx}{x}.$$

One can  $p$ -adically complete:

$$(3.3.4) \quad \nabla_a^\dagger : \mathcal{R}^\dagger \rightarrow \mathcal{R}^\dagger \frac{dx}{x}, \quad f \mapsto \left( x \frac{d}{dx} + a + \pi x \right) (f) \frac{dx}{x},$$

where  $\mathcal{R}^\dagger$  denotes the ring of functions analytic for  $1 - \epsilon < x < 1 + \epsilon$  with unspecified  $\epsilon > 0$ . The rigid cohomology space, for  $a \in \mathbb{Z}_p \cap \overline{\mathbb{Q}}$ ,

$$H_{\text{rig}}^1(\mathbb{G}_{m, \mathbb{F}_q} / \mathbb{C}_p, (\mathcal{O}_{\mathbb{G}_{m, \mathbb{C}_q}^{\text{an}}}, \nabla_a^\dagger)) := H^1(\nabla_a^\dagger) = \mathcal{R}^\dagger \frac{dx}{x} / \nabla_a^\dagger \mathcal{R}^\dagger$$

is one-dimensional; if  $a \notin \mathbb{Z}$ , it is spanned by the class  $\left[\frac{dx}{x}\right]_a$  of  $\frac{dx}{x}$ . For  $m \in \mathbb{Z}$  we have the “reduction formulas”

$$(3.3.5) \quad \left[x^m \frac{dx}{x}\right]_a = \left[\frac{dx}{x}\right]_{a+m} = \frac{\Gamma(a+m)(-\pi)^{-m}}{\Gamma(a)} \left[\frac{dx}{x}\right]_a.$$

The *Frobenius map*  $F(a, b)$  is defined as the horizontal morphism, for  $a, b \in \mathbb{Z}_p \cap \overline{\mathbb{Q}}$  and  $pb - a = \mu \in \mathbb{Z}$ ,

$$(3.3.6) \quad F(a, b) : (\mathcal{R}^\dagger, \nabla_b^\dagger) \rightarrow (\mathcal{R}^\dagger, \nabla_a^\dagger), \quad f(x) \mapsto f(x^p)x^\mu / \Theta(x).$$

To understand this definition, keep in mind that, formally

$$(3.3.7) \quad \nabla_b^\dagger(x^{-b} \exp(-\pi x)) = 0 \quad \text{and} \quad \nabla_a^\dagger(x^{-a} \exp(-\pi x)) = 0$$

while

$$\begin{aligned} x^{-pb} \exp(-\pi x^p)x^\mu / \Theta(x) &= x^{-pb} \exp(-\pi x^p)x^{pb-a} \exp(-\pi x + \pi x^p) \\ &= x^{-a} \exp(-\pi x). \end{aligned}$$

Computations are formal, but the functions  $x^\mu$  and  $\Theta(x)$  are in  $\mathcal{R}^\dagger$ .

It is convenient to use simultaneously a left inverse of Frobenius, namely the *Dwork map*

$$(3.3.8) \quad D(b, a) : (\mathcal{R}^\dagger, \nabla_a^\dagger) \rightarrow (\mathcal{R}^\dagger, \nabla_b^\dagger), \quad f(x) \mapsto \psi(f(x)x^{-\mu} \Theta(x)),$$

where

$$(3.3.9) \quad \psi : \mathcal{R}^\dagger \rightarrow \mathcal{R}^\dagger, \quad (\psi \xi)(t) = \frac{1}{p} \sum_{x^p=t} \xi(x).$$

The Frobenius and Dwork maps induce inverse isomorphisms

$$\text{Frob}^1(a, b) : H^1(\nabla_b^\dagger) \rightarrow H^1(\nabla_a^\dagger), \quad \text{Dw}^1(b, a) : H^1(\nabla_a^\dagger) \rightarrow H^1(\nabla_b^\dagger).$$

A simple computation using the reduction formulas gives

$$(3.3.10) \quad \text{Dw}^1(b, a) \left( \left[ \frac{dx}{x} \right]_a \right) = p^{-1} \gamma_p(a, b) \left[ \frac{dx}{x} \right]_b,$$

or

$$(3.3.11) \quad \text{Frob}^1(a, b) \left( \left[ \frac{dx}{x} \right]_b \right) = p \gamma_p(a, b)^{-1} \left[ \frac{dx}{x} \right]_a.$$

Then the modular property of  $\gamma_p$  comes from the horizontal morphisms

$$(\mathcal{R}^\dagger, \nabla_{a+m}^\dagger) \xrightarrow{x^m} (\mathcal{R}^\dagger, \nabla_a^\dagger), \quad f \mapsto x^m f,$$

and the commutative diagrams

$$\begin{array}{ccc} H^1(\nabla_b^\dagger) & \xrightarrow{\text{Frob}^1(a,b)} & H^1(\nabla_a^\dagger) \\ x^n \uparrow & & \uparrow x^m \\ H^1(\nabla_{b+n}^\dagger) & \xrightarrow{\text{Frob}^1(a+m,b+n)} & H^1(\nabla_{a+m}^\dagger) \end{array}$$

coming from

$$\begin{array}{ccc} x^n f(x) & \xrightarrow{F(a,b)} & x^{pn} f(x^p) x^{pb-a} / \Theta(x) \\ x^n \uparrow & & \uparrow x^m \\ f(x) & \xrightarrow{F(a+m,b+n)} & f(x^p) x^{p(b+n)-a-m} / \Theta(x) \end{array}$$

for  $f(x) \in \mathcal{R}^\dagger$  and  $pb-a, m, n \in \mathbb{Z}$ , together with the base-change formulas given before.

3.4. The Gross–Koblitz formula is then a special case of the trace formula in rigid cohomology. Namely, for  $a \in \frac{1}{q-1}\mathbb{Z}$  as before, we set

$$G_q(r, a) = \sum_{x \in \mathbb{F}_{q^r}} \chi_a(\mathbb{N}_{\mathbb{F}_{q^r}/\mathbb{F}_q} x) \psi_p(\text{Tr}_{\mathbb{F}_{q^r}/\mathbb{F}_p}(x))$$

and define the corresponding  $L$ -function as

$$L(a, q, T) = \exp\left(\sum_{r=1}^{\infty} G_q(r, a) \frac{T^r}{r}\right).$$

We consider iterates of the previous operators:

$$(3.4.1) \quad F_q(a) = F(a, a^{(1)}) \circ F(a^{(1)}, a^{(2)}) \circ \dots \circ F(a^{(f-1)}, a),$$

$$F_q(a) : (\mathcal{R}^\dagger, \nabla_a^\dagger) \rightarrow (\mathcal{R}^\dagger, \nabla_a^\dagger),$$

and their left inverses

$$(3.4.2) \quad D_q(a) = D(a, a^{(f-1)}) \circ D(a^{(f-1)}, a^{(f-2)}) \circ \dots \circ D(a^{(1)}, a).$$

They induce inverse isomorphisms

$$(3.4.3) \quad \text{Frob}_q^1(a) : H^1(\nabla_a^\dagger) \rightarrow H^1(\nabla_a^\dagger)$$

and

$$(3.4.4) \quad \text{Dw}_q^1(a) : H^1(\nabla_a^\dagger) \rightarrow H^1(\nabla_a^\dagger).$$

The trace formula in this situation says that

$$L(a, q, T) = \det(1 - qT \text{Dw}_q^1(a)),$$

from which the Gross–Koblitz formula follows.

3.5. We can be more explicit. Assume  $0 < a < 1$ , where

$$a = \frac{\mu_0 + \mu_1 p + \cdots + \mu_{f-1} p^{f-1}}{p^f - 1}, \quad 0 \leq \mu_i \leq p - 1.$$

Then

$$a^{(1)} = \frac{\mu_1 + \mu_2 p + \cdots + \mu_{f-1} p^{f-2} + \mu_0 p^{f-1}}{p^f - 1}, \quad 0 \leq \mu_i \leq p - 1,$$

since  $pa^{(1)} - a = \mu_0$ . Then

$$(3.5.1) \quad \gamma_p(a^{(i-1)}, a^{(i)}) = \pi_{\text{Dwork}}^{\mu_{i-1}} \Gamma_p(-\mu_{i-1} + pa^{(i)})$$

and

$$(3.5.2) \quad \begin{aligned} -G_q(a) &= \pi^{\mu_0 + \mu_1 + \cdots + \mu_{f-1}} \prod_{i=1}^f \Gamma_p(-\mu_{i-1} + pa^{(i)}) \\ &\equiv \pi_{\text{Dwork}}^{\mu_0 + \mu_1 + \cdots + \mu_{f-1}} \prod_{i=0}^{f-1} \frac{1}{\mu_i!} \end{aligned}$$

modulo  $1 + (\pi)$  in  $\mathbb{Q}_p(\zeta_p, \zeta_{q-1})^\times$ , a multiplicative congruence due to Stickelberger.

3.6. Dwork's computation of Chapter 22 of [14] shows that, for  $((s, t), (s', t')) \in \mathcal{H}_W^{\text{non-deg}} \cap \mathbb{Z}_{(p)}^4$  and  $p(s', t') - (s, t) \in \mathbb{Z}^2$ ,

$$(3.6.1) \quad B_p^{\text{crys}}(s, t; s', t') = \frac{\gamma_p(s, s') \gamma_p(t, t')}{\gamma_p(s + t, s' + t')}.$$

It follows that  $B_p^{\text{crys}}(s, t; s', t')$  may be extended as a meromorphic function on each of the domains

$$(3.6.2) \quad \mathcal{D}^{(i)}(p\rho) \times \mathcal{D}^{(j)}(p\rho)$$

for fixed  $i, j \in \mathbb{Z}$ . We warn the reader that  $B_p^{\text{crys}}(s, t; s', t')$  is not meromorphic on the union  $\bigcup_{i, j \in \mathbb{Z}} \mathcal{D}^{(i)}(p\rho) \times \mathcal{D}^{(j)}(p\rho)$  of the previous 2-dimensional analytic domains. We point out that at an irrational point  $(s, t; s', t')$ ,  $B_p^{\text{crys}}(s, t; s', t')$  loses its cohomological meaning as matrix of the Dwork operator in rigid cohomology (rigid cohomology may be infinite-dimensional at such a point!). Similarly, the function  $B_p^{\text{crys}}(a, b)$  of (2.2.8) can be extended to a meromorphic function of  $b$ , for  $(a, b)$  in a countable union of 2-dimensional  $p$ -adic analytic domains containing  $\mathcal{H}_\sigma \cap \mathbb{Z}_{(p)}^4$ , for any  $\sigma \in W_+(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$  and any fixed pair of integers

$$(3.6.3) \quad p^{\deg \sigma} b_i - a_i = \mu_i \in \mathbb{Z}, \quad i = 1, 2.$$

This is then the statement (and essentially, the proof) of the Boyarsky Principle in this simple unramified case.

## 4. FROBENIUS MATRICES OF FERMAT CURVES (AFTER COLEMAN)

4.1. The  $p$ -adic crystalline computations of the previous two sections apply to Fermat curves  $F_m$ , when they have good reduction at  $p$ , i.e. to the prime-to- $p$  part of the tower (1.1.1). So, when  $(m, p) = 1$ , one obtains a natural  $p$ -adic definition of the beta function of  $(s, t)$ , for  $s$  and  $t$  in  $m^{-1}\mathbb{Z}$ , in terms of the action of Frobenius on the rigid differential class  $[\tau_{s,t}]$  in  $H_{\text{rig}}^1((F_m)_{\mathbb{F}_p}/\mathbb{Q}_p) = H_{\text{dR}}^1(F_m/\mathbb{Q}) \otimes \mathbb{Q}_p$ . The situation is much more complicated when  $p \mid m$ . The complete analysis of this situation is due to Coleman [9]. We just sketch the principles of Coleman's computations.

4.2. Let  $\mathbb{Q}_p^{\text{ur}}$  be the maximal unramified subextension of  $\overline{\mathbb{Q}_p}/\mathbb{Q}_p$ , and recall the Weil group  $W(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \subset \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ , which consists of the  $\sigma \in \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$  whose restriction to  $\mathbb{Q}_p^{\text{ur}}$  is an integral power ( $=: \deg \sigma$ , the *degree* of  $\sigma$ ) of the absolute Frobenius  $\varphi$  of  $\mathbb{Q}_p^{\text{ur}}$  (we identify  $\text{Gal}(\mathbb{Q}_p^{\text{ur}}/\mathbb{Q}_p)$  with  $\text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$ , for  $\mathbb{F}_p$  the prime field of characteristic  $p$ , and  $\overline{\mathbb{F}_p}$  its algebraic closure). Notice that the subgroup  $I(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$  of elements of degree 0 of  $W(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$  coincides with the inertia subgroup of  $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ . Then  $I(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$  carries the (profinite) topology of a subgroup of  $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ , while  $W(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$  will be equipped with the group topology for which  $I(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$  is an open subgroup. We also define

$$W_+(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) = \{\sigma \in W(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \mid \deg \sigma \geq 0\}.$$

We will identify  $\mathbb{Q}/\mathbb{Z}$  with the group of roots of unity in  $\overline{\mathbb{Q}_p}$ , via

$$(4.2.1) \quad q \bmod \mathbb{Z} \mapsto \iota_p(e^{2\pi\sqrt{-1}q}).$$

The point of the next subsection is to construct local liftings of the action of  $\mathbb{Q}_p^\times$ , identified via local class field theory to  $W(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)^{\text{ab}}$ , on  $\mathbb{Q}/\mathbb{Z}$ , to an action on  $\mathbb{Q}$ . The reason for that need is that we regard  $[\tau_{s,t}]$  for  $(s, t) \in \mathbb{Q}^2$  as our variable, while  $W(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)^{\text{ab}}$  (as was the case for  $G_{\mathbb{Q}_p}$ ) only acts on the characters  $\chi_{s,t}$ , which depend upon  $(s, t) \bmod \mathbb{Z}^2$ . The action is simply  $\chi_{s,t}^\sigma = \sigma \circ \chi_{s,t}$ . We can make this more explicit via local class field theory, but the problem of lifting this action to an action on  $(s, t)$  itself remains.

We recall that the Jacobian  $J_m$  has potential good reduction over  $\mathbb{Q}_p$ . By [5] there is a natural semi-linear action  $\rho_{\text{crys}}$  of  $W(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)^{\text{ab}}$  on  $H_{\text{dR}}^1(F_\bullet/\mathbb{Q}) \otimes \overline{\mathbb{Q}_p}$ . On the other hand, in this particular case [9], the action descends to a linear action of  $W(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)^{\text{ab}}$  on  $H_{\text{dR}}^1(F_\bullet/\mathbb{Q}) \otimes \mathbb{Q}_p$ . This action respects the isotypical decomposition of  $H_{\text{dR}}^1(F_\bullet/\mathbb{Q})$  via the characters of  $\hat{\mathbb{Z}}(1)^2$ . So, if  $\chi, \chi'$  are characters of finite order of  $\hat{\mathbb{Z}}(1)^2$ , related by  $\sigma \circ \chi' = \chi$ , then

$$(4.2.2) \quad \rho_{\text{crys}}(\sigma)(H_{\text{dR}}^1(F_\bullet/\mathbb{Q})_{\chi'}) \subset H_{\text{dR}}^1(F_\bullet/\mathbb{Q})_\chi \otimes \mathbb{Q}_p.$$

Now, in order to make sense of our contention, generalizing the classical Boyarsky Principle, that the action of the Weil group should be  $p$ -adically locally analytic, we need some way to relate the choice of basis elements in the previous eigenspaces, as was done

in the previous section in the unramified case. In the next subsection, we will indicate a uniform way to choose, for  $\chi, \chi'$  as in (4.2.2),  $(s, t; s', t')$  in  $\mathbb{Q}^4$  so that  $\chi = \chi_{s,t}$  and  $\chi' = \chi_{s',t'}$ .

4.3. The action of  $G_{\mathbb{Q}_p}$  on  $\mathbb{Q}/\mathbb{Z}$  factors through the maximal abelian quotient  $G_{\mathbb{Q}_p}^{\text{ab}} = G(\mathbb{Q}_p^{\text{mc}}/\mathbb{Q}_p)$ , where  $\mathbb{Q}_p^{\text{mc}}$  is the maximal cyclotomic extension of  $\mathbb{Q}_p$  and  $\mathbb{Q}_p^{\text{ur}}$  is its maximal unramified subextension. We have a diagram

$$(4.3.1) \quad \begin{array}{ccc} & \mathbb{Q}_p^{\text{mc}} & \\ & \swarrow \quad \searrow & \\ \mathbb{Q}_p(\mu_{p^\infty}) & & \mathbb{Q}_p^{\text{ur}} \\ & \swarrow \quad \searrow & \\ & \mathbb{Q}_p & \end{array}$$

where  $\mathbb{Q}_p(\mu_{p^\infty})$  is the field generated over  $\mathbb{Q}_p$  by all roots of unity of order a power of  $p$ . So,  $G(\mathbb{Q}_p^{\text{mc}}/\mathbb{Q}_p)$  is a direct product of  $G(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p) \xrightarrow{\sim} \mathbb{Z}_p^\times$  and  $G(\mathbb{Q}_p^{\text{ur}}/\mathbb{Q}_p) \xrightarrow{\sim} G_{\mathbb{F}_p} \xrightarrow{\sim} \hat{\mathbb{Z}}$ . The reciprocity map of local class field theory

$$\mathbb{Q}_p^\times \rightarrow G(\mathbb{Q}_p^{\text{mc}}/\mathbb{Q}_p)$$

identifies  $\mathbb{Q}_p^\times$  with the Weil group  $W(\mathbb{Q}_p^{\text{mc}}/\mathbb{Q}_p) = W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)^{\text{ab}}$ , and maps  $\alpha = p^n u$ , where  $u \in \mathbb{Z}_p^\times$  and  $n \in \mathbb{Z}$ , to  $\sigma_\alpha \in G(\mathbb{Q}_p^{\text{mc}}/\mathbb{Q}_p)$  described as follows:

- (i) on  $\mathbb{Q}_p^{\text{ur}}$ ,  $\sigma_\alpha$  induces the  $n$ -th power of the Frobenius automorphism,
- (ii) on  $\mathbb{Q}_p(\mu_{p^\infty})$ ,  $\sigma_\alpha$  acts as

$$\zeta \mapsto \zeta^{u^{-1}} = 1 + \sum_{n=1}^{\infty} \binom{u^{-1}}{n} (\zeta - 1)^n.$$

For any  $a \in \mathbb{Q}$ , let  $\langle a \rangle_p \in \mathbb{Z}[1/p]$ ,  $0 \leq \langle a \rangle_p < 1$ , be such that  $[a]_p := a - \langle a \rangle_p \in \mathbb{Z}_{(p)}$ , and let  $-r := \min(v_p(a), 0) \leq 0$ . For  $\alpha = p^n u \in \mathbb{Q}_p^\times$  and  $\sigma = \sigma_\alpha$ , as above, let  $u_r \in \mathbb{N}$ ,  $0 < u_r < p^r$ , be such that  $u_r u \equiv 1 \pmod{p^r \mathbb{Z}}$ . Then any  $b \in \mathbb{Q}$  such that  $\sigma(b + \mathbb{Z}) = a + \mathbb{Z}$  satisfies  $\min(v_p(b), 0) = -r$  and

$$(4.3.2) \quad u_r \langle b \rangle_p - \langle a \rangle_p \in \mathbb{Z}$$

and

$$(4.3.3) \quad p^n [b]_p - [a]_p = N_\sigma(a, b) \in \mathbb{Z}.$$

We define an extension of the Dwork–Morita crystalline beta function (2.2.6) (originally only defined on  $\mathcal{H}_\sigma^{\text{non-deg}} \cap \mathbb{Z}_{(p)}^4$ ) to the entire  $\mathcal{H}_\sigma^{\text{non-deg}}$ , for any  $\sigma \in W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ ; namely, for  $(a_1, a_2; b_1, b_2) \in \mathcal{H}_\sigma^{\text{non-deg}}$  we set

$$(4.3.4) \quad \rho_{\text{crys}}(\sigma)([\tau_{b_1, b_2}]) = p^{\deg \sigma} B_p^{\text{crys}}(a_1, a_2; b_1, b_2)^{-1} [\tau_{a_1, a_2}].$$

The function  $B_p^{\text{crys}}(a_1, a_2; b_1, b_2)$ , defined for  $(a_1, a_2; b_1, b_2) \in \mathcal{H}_\sigma^{\text{non-deg}}$ , extends to a locally meromorphic function of  $(a_1, a_2; b_1, b_2)$ , for any fixed pair of integers  $n_1 = N_\sigma(a_1, b_1)$  and  $n_2 = N_\sigma(a_2, b_2)$ . So,  $B_p^{\text{crys}}$  extends (with easily described poles) to a countable union of 2-dimensional analytic domains containing  $\mathcal{H}_W$ . This is the generalized Boyarsky Principle in this case. Coleman shows in fact [9] that  $B_p^{\text{crys}}$  can be written in terms of an extension of the Morita gamma function to  $\mathbb{Q}_p$ . He proved recently, at the author's request, the local analyticity of his extended gamma function. We plan to give full details on this point elsewhere.

## 5. FONTAINE'S PERIODS

5.1. Fontaine introduced certain topological  $\mathbb{Q}_p$ -algebras  $\mathbb{B}_{\text{crys}, p} \subset \mathbb{B}_{\text{dR}, p}$  endowed with a continuous action of  $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$

$$(5.1.1) \quad \rho_{\text{Gal}} : \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \times \mathbb{B}_{\text{dR}, p} \rightarrow \mathbb{B}_{\text{dR}, p}, \quad (\sigma, \eta) \mapsto \rho_{\text{Gal}}(\sigma, \eta) = \rho_{\text{Gal}}(\sigma)(\eta),$$

satisfying  $\rho_{\text{Gal}}(\sigma)(a\eta) = \sigma(a)\rho_{\text{Gal}}(\sigma)(\eta)$  for  $a \in \overline{\mathbb{Q}_p}$  and  $\eta \in \mathbb{B}_{\text{dR}, p}$ .

The embedding  $\mathbb{B}_{\text{crys}, p} \subset \mathbb{B}_{\text{dR}, p}$  is equivariant and continuous, but the natural topology of  $\mathbb{B}_{\text{crys}, p}$  is not that of a subspace of  $\mathbb{B}_{\text{dR}, p}$ . Now,  $\mathbb{B}_{\text{crys}, p}$  is a  $\mathbb{Q}_p^{\text{ur}}$ -algebra and there is a natural continuous embedding of  $\mathbb{B}_p := \mathbb{B}_{\text{crys}, p} \otimes_{\mathbb{Q}_p^{\text{ur}}} \overline{\mathbb{Q}_p}$  into  $\mathbb{B}_{\text{dR}}$ . The topological  $\overline{\mathbb{Q}_p}$ -algebra  $\mathbb{B}_p$  carries a further natural continuous operation

$$(5.1.2) \quad \rho_{\text{crys}} : \text{W}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \times \mathbb{B}_p \rightarrow \mathbb{B}_p, \quad (\sigma, \eta) \mapsto \rho_{\text{crys}}(\sigma, \eta) = \rho_{\text{crys}}(\sigma)(\eta),$$

satisfying  $\rho_{\text{crys}}(\sigma)(a\eta) = \sigma(a)\rho_{\text{crys}}(\sigma)(\eta)$  for  $a \in \overline{\mathbb{Q}_p}$  and  $\eta \in \mathbb{B}_p$ , which is *not induced* by the action of  $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ , and which *does not extend* to  $\mathbb{B}_{\text{dR}, p}$ . We call this operation *crystalline* to distinguish it from the *Galois* operation of  $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ . We recall that  $\mathbb{B}_{\text{dR}, p}$  is a complete discretely valued field, with valuation ring  $\mathbb{B}_{\text{dR}, p}^+$  and residue field the  $p$ -adic completion  $\mathbb{C}_p$  of  $\overline{\mathbb{Q}_p}$ . In particular, for any choice of a uniformizer  $t$ ,  $\mathbb{B}_{\text{dR}, p}$  carries a natural filtration  $\text{Fil}^i(\mathbb{B}_{\text{dR}, p}) = \mathbb{B}_{\text{dR}, p}^+ t^i$ ,  $i \in \mathbb{Z}$ .

We briefly describe Fontaine's theory in the case of a smooth and proper variety  $X$  defined over  $\mathbb{Q}_p$ , having potentially good reduction at  $p$ .

Under the previous assumptions, the finite-dimensional  $\overline{\mathbb{Q}_p}$ -vector space  $H_{\text{dR}}^i(X_{\overline{\mathbb{Q}_p}}/\overline{\mathbb{Q}_p}) = \overline{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} H_{\text{dR}}^i(X/\mathbb{Q}_p)$  carries the Hodge filtration, an obvious Galois action  $\rho_{\text{Gal}}$  of  $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ , and a natural action  $\rho_{\text{crys}}$  of  $\text{W}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$  [5]. On the other hand, the finite-dimensional  $\mathbb{Q}_p$ -vector space  $H_{\text{ét}}^i(X_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p)$  carries a non-trivial action of  $\rho_{\text{Gal}}$  of  $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ . Then Fontaine's theory says that  $\mathbb{B}_p \otimes_{\mathbb{Q}_p} H_{\text{dR}}^i(X/\mathbb{Q}_p)$  can be identified with  $\mathbb{B}_p \otimes_{\mathbb{Q}_p} H_{\text{ét}}^i(X_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p)$ , when equipped with (the tensor product of) all the previous structures. In particular

$$(5.1.3) \quad \text{Fil}^j H_{\text{dR}}^i(X/\mathbb{Q}_p) = (\text{Fil}^j \mathbb{B}_p \otimes_{\mathbb{Q}_p} H_{\text{ét}}^i(X_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p))^{\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)},$$

$$(5.1.4) \quad H_{\text{ét}}^i(X_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p) = (\text{Fil}^0(\mathbb{B}_p \otimes_{\mathbb{Q}_p} H_{\text{dR}}^i(X/\mathbb{Q}_p)))^{\text{W}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)}.$$

Equivalently, we may regard Fontaine's theory for a variety  $X$  as before, as a pairing,  $\mathbb{Q}_p$ -linear (resp.  $\overline{\mathbb{Q}}_p$ -linear) in the first (resp. second) variable and non-degenerate,

$$(5.1.5) \quad H_{i,\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \times H_{\text{dR}}^i(X_{\overline{\mathbb{Q}}_p}/\overline{\mathbb{Q}}_p) \rightarrow \mathbb{B}_p, \quad (\delta, \omega) \mapsto \langle \delta, \omega \rangle = \int_{\delta} \omega,$$

satisfying

$$(5.1.6) \quad \int_{\rho_{\text{Gal}}(\sigma)(\delta)} \omega = \rho_{\text{Gal}}(\sigma) \left( \int_{\delta} \omega \right)$$

for  $\sigma \in \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ , and

$$(5.1.7) \quad \int_{\delta} \rho_{\text{crys}}(\sigma)(\omega) = \rho_{\text{crys}}(\sigma) \left( \int_{\delta} \omega \right)$$

for  $\sigma \in \text{W}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ , and such that, for any  $\delta \in H_{i,\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ ,

$$(5.1.8) \quad \int_{\delta} \text{Fil}^j H_{\text{dR}}^i(X_{\overline{\mathbb{Q}}_p}/\overline{\mathbb{Q}}_p) \subset \text{Fil}^j \mathbb{B}_p.$$

The pairing (5.1.5) satisfies a natural functoriality with respect to morphisms  $f : X \rightarrow Y$  of proper and smooth  $\mathbb{Q}_p$ -varieties, namely

$$(5.1.9) \quad \int_{f_*(\delta)} \omega = \int_{\delta} f^* \omega.$$

5.2. The simple case of  $X = \mathbb{G}_{m,\mathbb{Q}}$  is not covered by the above scheme, still the results hold true, due to the existence of the nice compactification  $\mathbb{P}_{\mathbb{Q}}^1$  of  $\mathbb{G}_{m,\mathbb{Q}}$ .

The first  $p$ -adic étale homology group coincides with the Tate module  $T_p : H_{1,\text{ét}}(\mathbb{G}_{m,\mathbb{C}}, \mathbb{Z}_p) = T_p(\mathbb{G}_m(\mathbb{C})) = T_p(\mathbb{G}_m(\overline{\mathbb{Q}})) = \mathbb{Z}_p(1)$ , via  $\overline{\mathbb{Q}} \subset \mathbb{C}$ , with the cyclotomic Galois action of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ . By restricting the cyclotomic action to  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ , this in turn may be identified, via  $t_p$ , with  $T_p(\mathbb{G}_m(\mathbb{C}_p)) = H_{1,\text{ét}}(\mathbb{G}_{m,\overline{\mathbb{Q}}_p}, \mathbb{Z}_p)$ .

The choice of a system of primitive  $p^n$ -th roots of unity in  $\mathbb{C}_p$ ,  $\epsilon = (\epsilon^{(n)})_{n \in \mathbb{N}}$ , with  $\epsilon^{(0)} = 1$ ,  $\epsilon^{(n)} \neq 1$ ,  $\epsilon^{(n)} = (\epsilon^{(n+1)})^p$ , in our case  $\epsilon^{(n)} = t_p(e^{2\pi\sqrt{-1}/n})$ , determines a canonical uniformizer  $t_p = \log([\epsilon])$  (in a suitable sense) of  $\mathbb{B}_{\text{dR},p}$ , which actually belongs to  $\mathbb{B}_{\text{crys},p}$ . Then  $\mathbb{Z}_p t_p \subset \mathbb{B}_{\text{dR},p}$  is isomorphic to  $\mathbb{Z}_p(1)$  as a Galois module over  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$  and is a Dieudonné module via the crystalline action of  $\text{W}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ :

$$(5.2.1) \quad \rho_{\text{Gal}}(\sigma)t_p = \chi_p(\sigma)t_p \quad \text{and} \quad \rho_{\text{crys}}(\sigma)t_p = p^{\deg \sigma} t_p,$$

where  $\chi_p : \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \rightarrow \mathbb{Z}_p^\times$  is the cyclotomic character. Finally, for  $\gamma_0$  as in the introduction,  $H_{1,\text{sing}}(\mathbb{G}_{m,\mathbb{C}}, \mathbb{Z}) = \mathbb{Z}\gamma_0$  is canonically embedded in  $\mathbb{Z}_p(1)$ , by identifying  $\gamma$  with the vector  $\epsilon = (\epsilon^{(n)})_n$  defined before.

Fontaine's ring  $\mathbb{B}_{\text{crys},p}$  and its element  $t_p$  of (5.2.1) play for the  $p$ -adic cohomology theories of the algebraic group  $\mathbb{G}_{m,\mathbb{Q}}$  the role played by the complex field  $\mathbb{C}$  and Archimedes'  $\pi$  in the archimedean theories. In fact,

$$\int_{\epsilon} \frac{dx}{x} = t_p.$$

5.3. The integration pairing (5.1.5) acquires a more concrete description when  $X$  is an abelian variety (or a formal group) over  $\mathbb{Q}_p$  via *Colmez integration* [11], which we briefly recalled in the introduction (see (0.13)). We do not give any details here, because our understanding of the analytic properties of the Colmez period mappings is very primitive. We expect that any further investigation of the Frobenius action on classical differential equations arising from variation of cohomology should be confronted with the Fontaine–Colmez construction. We hope to come back to this question very soon.

## 6. BETA COCYCLES

We now take (0.14) as our definition of *Fontaine's  $p$ -adic beta function*  $B_p^{\text{Font}}(s, t) \in \mathbb{B}_p$ , for any  $s, t \in \mathbb{Q}$  with  $s, t, s + t \notin \mathbb{Z}$ . We clearly have, for  $m, n \in \mathbb{Z}$ ,

$$(6.1) \quad B_p^{\text{Font}}(s + m, t + n) = \frac{B(s + m, t + n)}{B(s, t)} B_p^{\text{Font}}(s, t).$$

We point out a consequence of formula (5.1.9). We consider  $\zeta, \zeta' \in \mu_m$  and the automorphism  $[\zeta, \zeta']$  of  $J_m(\overline{\mathbb{Q}_p})$ . Let  $s, t \in m^{-1}\mathbb{Z}$  with  $s, t, s + t \notin \mathbb{Z}$ , and let  $\delta \in T_p(J_{m, \overline{\mathbb{Q}_p}})$ . We have

$$(6.2) \quad \int_{[\zeta, \zeta']_*(\delta)} \tau_{s, t} = \int_{\delta} [\zeta, \zeta']^* \tau_{s, t} = \chi_{s, t}(\zeta, \zeta') \int_{\delta} \tau_{s, t}.$$

More generally, let  $(\eta_m^{(p)})_m$  be the compatible system of  $\eta_m^{(p)} \in T_p(J_{m, \overline{\mathbb{Q}_p}})$ , which defines  $\eta^{(p)}$ , and let  $f = (f_m)_m \in \mathbb{Z}_p[[\hat{\mathbb{Z}}(1)^2]] = \varprojlim_m \mathbb{Z}_p[\mu_m^2]$ . Then, for  $s, t$  as before,

$$(6.3) \quad \int_{f \eta^{(p)}} \tau_{s, t} = \int_{\eta_m^{(p)}} f_m(s, t) \tau_{s, t} = f(s, t) \int_{\eta^{(p)}} \tau_{s, t}.$$

Notice that, for  $f \in \mathbb{Z}_p[[\hat{\mathbb{Z}}(1)^2]]$ ,  $\sigma \in G_{\mathbb{Q}_p}$  and  $(s, t, s', t') \in \mathcal{H}_{\sigma}$ , we have

$$f(s', t')^{\sigma} = f(s, t).$$

Formula (5.1.6), together with the previous remarks, shows that, for any  $\sigma \in G_{\mathbb{Q}_p}$  and  $s, t \in \mathbb{Q}$  with  $s, t, s + t \notin \mathbb{Z}$ ,

$$\begin{aligned} \rho_{\text{Gal}}(\sigma)(B_p^{\text{Font}}(s, t)) &= \rho_{\text{Gal}}(\sigma) \left( \int_{\eta^{(p)}} \tau_{s, t} \right) = \int_{\rho_{\text{Gal}}(\sigma)(\eta^{(p)})} \tau_{s, t} = \int_{B_{\sigma}^{\text{ét}} \eta^{(p)}} \tau_{s, t} \\ &= B_{\sigma}^{\text{ét}}(s, t) \int_{\eta^{(p)}} \tau_{s, t} = B_{\sigma}^{\text{ét}}(s, t) B_p^{\text{Font}}(s, t). \end{aligned}$$

Therefore, for  $(a; b) = (a_1, a_2; b_1, b_2) \in \mathcal{H}_{\sigma}^{\text{non-deg}}$ ,

$$(6.4) \quad \rho_{\text{Gal}}(\sigma)(B_p^{\text{Font}}(b_1, b_2)) = B_p^{\text{ét}}(a_1, a_2; b_1, b_2) B_p^{\text{Font}}(a_1, a_2).$$

If now  $\sigma \in W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ , we also have the crystalline action on  $\mathbb{B}_p$ . For  $(a; b) = (a_1, a_2; b_1, b_2) \in \mathcal{H}_\sigma^{\text{non-deg}}$ , we have

$$\begin{aligned} \rho_{\text{crys}}(\sigma)(B_p^{\text{Font}}(b_1, b_2)) &= \rho_{\text{crys}}(\sigma) \left( \int_{\eta^{(p)}} \tau_{b_1, b_2} \right) = \int_{\eta^{(p)}} \rho_{\text{crys}}(\sigma)(\tau_{b_1, b_2}) \\ &= \int_{\eta^{(p)}} B_p^{\text{crys}}(a_1, a_2; b_1, b_2) \tau_{a_1, a_2} \\ &= B_p^{\text{crys}}(a_1, a_2; b_1, b_2) \int_{\eta^{(p)}} \tau_{a_1, a_2} \\ &= B_p^{\text{crys}}(a_1, a_2; b_1, b_2) B_p^{\text{Font}}(a_1, a_2). \end{aligned}$$

Summarizing, we have defined, for  $s, t \in \mathbb{Q}$  with  $s, t, s+t \notin \mathbb{Z}$ , a function

$$(6.5) \quad B_p^{\text{Font}}(s, t) = \int_{\eta^{(p)}} \tau_{s, t}$$

such that, for any  $\sigma \in \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q})$  and  $(a; b) \in \mathcal{H}_\sigma^{\text{non-deg}}$ ,

$$(6.6) \quad \rho_{\text{Gal}}(\sigma)(B_p^{\text{Font}}(b)) = B_p^{\text{ét}}(a; b) B_p^{\text{Font}}(a),$$

and, if  $\sigma \in W(\overline{\mathbb{Q}}_p/\mathbb{Q})$ ,

$$(6.7) \quad \rho_{\text{crys}}(\sigma)(B_p^{\text{Font}}(b)) = B_p^{\text{crys}}(a; b) B_p^{\text{Font}}(a).$$

The functions  $B_p^{\text{ét}}$  and  $B_p^{\text{crys}}$  enjoy strong continuity and analyticity properties and naturally interpolate Jacobi sums.

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