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A characterization of the maximally almost periodic abelian groups[☆]

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Abstract

We introduce a categorical closure operator $\mathfrak{g}$ in the category of topological abelian groups (and continuous homomorphisms) as a Galois closure with respect to an appropriate Galois correspondence defined by means of the Pontryagin dual of the underlying group. We prove that a topological abelian group G is maximally almost periodic if and only if every cyclic subgroup of G is $\mathfrak{g}$-closed. This generalizes a property characterizing the circle group from (Studia Sci. Math. Hungar. 38 (2001) 97–113, A characterization of the circle group and the p-adic integers via sequential limit laws, preprint), and answers an appropriate version of a question posed in (A characterization of the circle group and the p-adic integers via sequential limit laws, preprint).

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1. Introduction

For an abelian topological group G and a sequence $\underline{u} = (u_n) \in \mathbb{Z}^{\mathbb{N}}$ one can consider the subgroup $t_{\underline{u}}(G) := \{x \in G : u_n x \to 0 \text{ in } G\}$ of all *topologically* $\underline{u}$*-torsion* elements of G [10,13]. If H is a subgroup of G, then we define $\mathfrak{t}_G(H) := \bigcap\{t_{\underline{u}}(G) : \underline{u} \in \mathbb{Z}^{\mathbb{N}}, H \leqslant t_{\underline{u}}(G)\}$. This allows us to distinguish between $\mathfrak{t}$-closed subgroups of G (i.e., $H \leqslant G$ such that $\mathfrak{t}_G(H) = H$) and $\mathfrak{t}$-dense ones (i.e., $H \leqslant G$ such that $\mathfrak{t}_G(H) = G$). The subgroups of G of the form $t_{\underline{u}}(G)$, $\underline{u} \in \mathbb{Z}^{\mathbb{N}}$, are called *basic* $\mathfrak{t}$-closed subgroups of G.

Clearly, every finite subgroup of the circle group $\mathbb{T}$ is a basic $\mathfrak{t}$-closed subgroup. It is proved in [19] that for every irrational number α with bounded continued fraction coefficients, the cyclic subgroup of $\mathbb{T}$ generated by α (mod $\mathbb{Z}$) has the form $t_{\underline{u}}(\mathbb{T})$ where $\underline{u}$ is the sequence of best approximations denominators of α (for another proof of this fact see [4]). This was extended to *all* cyclic subgroups of the circle group in [7, Theorem 2] where the following stronger result is proved.

Theorem 1.1. *Let H be a countable subgroup of $\mathbb{T}$. Then H is a basic $\mathfrak{t}$-closed subgroup of $\mathbb{T}$.*

Let us mention that for cyclic subgroups $\langle\alpha\rangle$ of $\mathbb{T}$ the authors provide an explicit construction of the sequence $\underline{u} \in \mathbb{Z}^{\mathbb{N}}$ such that $\langle\alpha\rangle = t_{\underline{u}}(\mathbb{T})$ in terms of the continued fraction development of α without any assumption on boundedness of its continued fraction coefficients (cf. [7, Theorem 1]). Moreover, the authors conjecture that this property can be extended to compact abelian groups G different from $\mathbb{T}$. It was established in [11] (see also [10, Theorem 4.7]) that this cannot be done by means of $\mathfrak{t}$-closedness. More precisely, these authors prove that for every non-discrete locally compact abelian group $G \ncong \mathbb{T}$, there exists a cyclic subgroup of G that is not even $\mathfrak{t}$-closed (in other words, the property of having all cyclic subgroups $\mathfrak{t}$-closed characterizes $\mathbb{T}$ among *all non-discrete locally compact* groups). These papers offer also a short independent proof of the $\mathfrak{t}$-closedness of the countable subgroups of $\mathbb{T}$.

Motivated by these results, we positively answer in this paper an appropriate version of the question posed in [7].

Let us recall that the above mentioned $\mathfrak{t}$-closure arose in [11] in the framework of an appropriate Galois correspondence between subgroups of abelian topological groups G and subgroups of $\mathbb{Z}^{\mathbb{N}}$. The special characterizing property of the circle group $\mathbb{T}$ (in terms of this $\mathfrak{t}$-closure) is due to the fact that the sequences of integers $\underline{u} \in \mathbb{Z}^{\mathbb{N}}$ arise as a natural object via the Pontryagin duality as $\mathbb{Z}$ is the dual group of $\mathbb{T}$. This suggests to modify appropriately the domain of the sequences $\underline{u}$ for arbitrary abelian topological groups (G, τ) considering sequences $\underline{u}$ in the Pontryagin dual $\widehat{G}$ of G and replacing the subgroups $t_{\underline{u}}(G)$, with $\underline{u} \in \mathbb{Z}^{\mathbb{N}}$, by the analogously defined (see Definition 2.1) subgroups $s_{\underline{u}}(G)$, where $\underline{u}$ is a sequence with values in $\widehat{G}$. This approach necessarily leads to a different Galois correspondence and consequently, to a different Galois closure $\mathfrak{g}_G(H)$ for a subgroup H of a topological abelian group G that coincides with $\mathfrak{t}(H)$ in the case of $\mathbb{T}$ (cf. Definition 2.4). In the sequel we say that a topological group G has the $\mathfrak{g}$*-closure property* if every cyclic subgroup of G is Galois-closed (shortly, $\mathfrak{g}$-closed) for this new Galois correspondence.

The natural question arises of whether it is possible to classify in this setting the abelian topological groups G having the $\mathfrak{g}$-closure property. This will give a counterpart of [10, Theorem 4.7] and prove an appropriate version of the conjecture from [7] in the setting of abelian topological groups. In resolving this question for the important class of *maximally almost periodic* (MAP for short) abelian groups we obtain in fact a characterization of the latter class:

Theorem 1.2. *A topological abelian group G is MAP if and only if G has the* $\mathfrak{g}$*-closure property*.

In particular, the topological abelian groups that are either locally compact or precompact have the $\mathfrak{g}$-closure property.

Of course, in Theorem 1.2 we consider the zero subgroup as a cyclic one. The larger class of abelian topological groups in which *all non-zero* cyclic subgroups are $\mathfrak{g}$-closed is completely described in Corollary 4.9.

It is easy to see that the topological groups having the $\mathfrak{g}$-closure property are necessarily MAP (see Proposition 2.9). Hence the proof of Theorem 1.2 consists of establishing that *all MAP groups have the* $\mathfrak{g}$*-closure property*. It splits into several steps. In Section 2 we point out the basic properties of the Galois closure $\mathfrak{g}_G(-)$ in the class of topological abelian groups (cf. Propositions 2.6 and 2.14). Then we reduce the problem to precompact groups (see Proposition 2.9). In Section 4 we consider first the case of totally disconnected compact abelian groups. A key role in the proof is to show that the group of p-adic integers $\mathbb{Z}_p$, $p \in \mathbb{P}$, and countably infinite products of finite cyclic groups have the $\mathfrak{g}$-closure property (cf. Lemmas 4.1 and 4.4). This allows us to prove, via Corollaries 4.5 and 4.6, that *every compact* abelian group has the $\mathfrak{g}$-closure property (cf. Theorem 4.8). Now Theorem 1.2 easily follows from the above results and immediate categorical properties of closure operators.

The fact that the compact group $\mathbb{Z}_p$ has the $\mathfrak{g}$-closure property follows from a specific characterization of its subgroups $s_{\underline{c}^*}(\mathbb{Z}_p)$ where $\underline{c}^*$ is a subsequence of the canonic set of generators $\underline{c} = (c_n)$ of the Prüfer group $\mathbb{Z}(p^\infty)$ (hence a sequence of characters of $\mathbb{Z}_p$). In Section 3 we analyze the structure of the subgroups $s_{\underline{c}^*}(\mathbb{Z}_p)$ of $\mathbb{Z}_p$ and obtain some consequences in terms of convergence of sequences of $\mathbb{Z}(p^\infty)$ with respect to precompact group topologies.

Indeed, following [3] and motivated by [20], we define a sequence $\underline{u} = (u_n)$ in a group G to be a *TB-sequence* if there exists a precompact group topology τ on G such that $u_n \to 0$ in (G, τ). The first examples of *TB*-sequences of weight $\mathfrak{c}$ in $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ were given in [20]. Extending [3, Proposition 2.5] one can show that $\underline{u}$ is a *TB*-sequence if and only if the subgroup $s_{\underline{u}}(\widehat{G})$ is dense in the dual group $\widehat{G}$ of G (cf. Lemma 3.1). More generally, for a discrete abelian group G one can study the *TB*-sequences $\underline{u}$ of G by studying the subgroups $s_{\underline{u}}(\widehat{G})$ of the compact dual of G. Analogously to the case of $G = \mathbb{Z}$, there is a *finest totally bounded* group topology $\sigma_{\underline{u}}$ on G that makes $\underline{u}$ converge to 0 and the weight of $\sigma_{\underline{u}}$coincides with the size of the subgroup $s_{\underline{u}}(\widehat{G})$ of $\widehat{G}$ (cf. Proposition 3.2). Clearly, $\sigma_{\underline{u}}$ is a Hausdorff group topology if and only if $\underline{u}$ is a *TB*-sequence.

The fact that $\underline{c}$ is a *TB*-sequence of $\mathbb{Z}(p^\infty)$ follows from the following property proved in [22, Example 6]: $\mathbb{Z}(p^\infty)$, *endowed with the finest group topology that makes* c_n *converge to 0, has the same continuous characters as* $(\mathbb{Z}(p^\infty), \tau)$, *where* τ *is the* (*precompact*) *group* *topology induced by* $\mathbb{T}$ *on* $\mathbb{Z}(p^\infty)$ (in particular, this entails $\sigma_{\underline{c}} = \tau$). This follows also from our Theorem 3.3 where we prove the following sharper result: for every subsequence $\underline{c}^*$ of $\underline{c}$ the following are equivalent:

(a) $s_{\underline{c}^*}(\mathbb{Z}_p) = \mathbb{Z}$;
(b) the set of differences $n_{k+1} - n_k$ is bounded;
(c) $\{n_k : k \in \mathbb{N}\}$ is a large subset of $\mathbb{N}$ (i.e., there exists a finite set $F \subseteq \mathbb{N}$ such that $\mathbb{N} \subseteq F \cup (F + \{n_k : k \in \mathbb{N}\})$;
(d) the finest precompact group topology $\sigma_{\underline{c}^*}$ on $\mathbb{Z}(p^\infty)$ that makes c_{n_k} converge to 0 is metrizable;
(e) the finest precompact group topology $\sigma_{\underline{c}^*}$ on $\mathbb{Z}(p^\infty)$ that makes c_{n_k} converge to 0 has weight $< \mathfrak{c}$.

Further information on *TB*-sequences and $\mathfrak{t}$-dense subgroups can be found in [6] and [5], respectively.

Notation and terminology. We denote by $\mathbb{N}$ and $\mathbb{P}$ the sets of positive naturals and primes, respectively; by $\mathbb{Z}$ the integers, by $\mathbb{T}$ the circle group $\mathbb{R}/\mathbb{Z}$ in the additive notation and by $\mathbb{Z}_p$ the group of *p*-adic integers ($p \in \mathbb{P}$). For $n \in \mathbb{N}$ we denote by $\mathbb{Z}(n)$ the cyclic group of order *n*.

For a topological group (G, τ) we denote by $w(G)$ the *weight* of *G* (i.e., the minimal cardinality of a base for the topology on *G*). All topological groups, unless otherwise specified, satisfy Hausdorff's separation axiom. Completeness of topological groups is intended with respect to the two sided uniformity, so that every topological group has a (Raĭkov) completion denoted by $\widetilde{G}$.

A topological group *G* is said to be *totally bounded* if for every non-empty open set *U* of *G* there exists a finite subset $F \subseteq G$ of *G* such that $G = U + F$. Hausdorff totally bounded topological groups *G* are also called *precompact* because they are determined by the property to be (isomorphic to) subgroups of compact groups [21].

For any abelian group *G* let $Hom(G, \mathbb{T})$ be the group of all homomorphisms from *G* to the circle group $\mathbb{T}$. When (G, τ) is an abelian topological group, the set of τ-continuous homomorphisms $\chi : G \to \mathbb{T}$ (*characters*) is a subgroup of $Hom(G, \mathbb{T})$ and is denoted by $\widehat{G}$.

For an abelian group *G* and a subgroup $H \leqslant Hom(G, \mathbb{T})$, let T_H be the weakest topology on *G* that makes all characters of *H* continuous with respect to T_H. One easily shows that T_H is a totally bounded group topology on *G*, called the *topology generated* by *H*.

A subgroup $H \leqslant Hom(G, \mathbb{T})$ is said to *separate* the points of *G* if for every $g \in G \setminus \{0\}$ there exists $h \in H$ such that $h(g) \neq 0$.

It was proved by Comfort and Ross that every precompact group topology τ on an abelian group *G* is generated by some suitable point-separating subgroup of characters $H \leqslant Hom(G, \mathbb{T})$ of *G*. Conversely, every such topology is precompact (cf. [9, Theorem 1.2]).

Given an abelian topological group *G*, we denote by $G^\sharp$ the group *G* endowed with the initial topology of all continuous characters $\chi \in \widehat{G}$. The topology of $G^\sharp$ is called the *Bohr topology* of *G*. Clearly, the groups *G* and $G^\sharp$ have the same continuous characters. Notice that $G^\sharp$ is a totally bounded topological group that need not be Hausdorff.

The group G is said to be MAP (*maximally almost periodic*) when $G^\sharp$ is Haudorff. Therefore, a topological group G is MAP iff the continuous characters of G separates the points of G.

For a topological abelian group G let us denote by $n(G)$ the intersection of the kernels of all characters in $\widehat{G}$. $n(G)$ is called the *von Neumann's kernel* of G and is characterized by the property that $G/n(G)$ is the largest MAP quotient of G (note that the completion of $(G/n(G))^\#$ is the so called *Bohr compactification* of G). In particular, G itself is MAP iff $n(G)$ is trivial.

For undefined terms see [8,13,15,16].

2. The Galois correspondence on topological abelian groups

In this section we shall consider the following Galois correspondence between subgroups of a topological abelian group G and subgroups of the power $\widehat{G}^{\mathbb{N}}$.

Definition 2.1. Let G be a topological abelian group. Then:

(a) for $\underline{u} \in \widehat{G}^{\mathbb{N}}$ set $s_{\underline{u}}(G) := \{x \in G : u_n(x) \to 0 \text{ in } \mathbb{T}\}$,
(b) for $x \in G$ set $S_x(\widehat{G}^{\mathbb{N}}) := \{\underline{u} = (u_n) \in \widehat{G}^{\mathbb{N}} : u_n(x) \to 0 \text{ in } \mathbb{T}\}$.

We naturally extend these definition also to subgroups $H \leqslant G$ and $K \leqslant \widehat{G}^{\mathbb{N}}$ by letting

$$s_{\underline{u}}(H) := \{x \in H : u_n(x) \to 0 \text{ in } \mathbb{T}\} \quad \text{and} \quad S_x(K) := \{\underline{u} \in K : u_n(x) \to 0 \text{ in } \mathbb{T}\}.$$

One can easily obtain

Lemma 2.2. *Following the above notation*:

(a) $S_0(\widehat{G}^{\mathbb{N}}) = \widehat{G}^{\mathbb{N}}$ *and* $s_{\underline{0}}(G) = G$,
(b) $\underline{u} \in S_x(\widehat{G}^{\mathbb{N}})$ *if and only if* $x \in s_{\underline{u}}(G)$.

Let G and H be topological abelian groups. Given a continuous homomorphism $f : G \to H$, we denote by $\widehat{f} : \widehat{H} \to \widehat{G}$ the transposed homomorphism defined by $\chi \mapsto \chi \circ f$. Both $s_{\underline{u}}$ and S_x are functorial in the following sense:

Lemma 2.3. *Let G, H be topological abelian groups and $f : G \to H$ be a continuous homomorphism. Given $\underline{u} \in \widehat{G}^{\mathbb{N}}$ and $x \in G$, we have*:

(a) *$s_{\underline{u}}(G)$ and $S_x(\widehat{G}^{\mathbb{N}})$ are subgroups of G and $\widehat{G}^{\mathbb{N}}$, respectively,*
(b) *if J is a topological subgroup of G and K is a subgroup of $\widehat{G}^{\mathbb{N}}$, then*

$$s_{\underline{u}}(J) = J \cap s_{\underline{u}}(G) \quad \textit{and} \quad S_x(K) = K \cap S_x(\widehat{G}^{\mathbb{N}});$$

(c) $\widehat{f}^{\mathbb{N}}(S_{f(x)}(\widehat{H}^{\mathbb{N}})) \subseteq S_x(\widehat{G}^{\mathbb{N}})$ *for every* $x \in G$,

(d) $f(s_{\hat{f}^{\mathbb{N}}(\underline{v})}(G)) \subseteq s_{\underline{v}}(H)$ *for every* $\underline{v} \in \widehat{H}^{\mathbb{N}}$.

In particular, if f is a topological isomorphism, then both inclusions are equalities.

Proof. (a) and (b) are obvious.

(c) Let $x \in G$ and $\underline{v} \in S_{f(x)}(\widehat{H}^{\mathbb{N}})$. Then $v_n(f(x)) = (v_n \circ f)(x) \to 0$ in $\mathbb{T}$, i.e. $\widehat{f}^{\mathbb{N}}(\underline{v}) \in S_x(\widehat{G}^{\mathbb{N}})$.

(d) If $x \in s_{\hat{f}^{\mathbb{N}}(\underline{v})}(G)$, then one has $(v_n \circ f)(x) \to 0$ in $\mathbb{T}$, i.e., $f(x) \in s_{\underline{v}}(H)$. $\square$

In analogy with $\mathfrak{t}_G(-)$ we define now the $\mathfrak{g}$-closure replacing the subgroups $t_{\underline{u}}(G)$ by $s_{\underline{u}}(G)$.

Definition 2.4. Let H and K be topological subgroups of G and $\widehat{G}^{\mathbb{N}}$, respectively. We set

$$\mathfrak{g}_G(H) := \bigcap\{s_{\underline{u}}(G) : \ \underline{u} \in \widehat{G}^{\mathbb{N}}, \ H \leqslant s_{\underline{u}}(G)\} \quad \text{and}$$
$$\mathfrak{G}_{\widehat{G}}(K) := \bigcap\{S_x(\widehat{G}^{\mathbb{N}}) : \ x \in G, \ K \leqslant S_x(\widehat{G}^{\mathbb{N}})\}.$$

We say that

- H (resp. K) is $\mathfrak{g}$-*closed* (resp. $\mathfrak{G}$-*closed*) if $H = \mathfrak{g}_G(H)$ (resp. $K = \mathfrak{G}_{\widehat{G}}(K)$);
  H (resp. K) is $\mathfrak{g}$-*dense* (resp. $\mathfrak{G}$-*dense*) if $\mathfrak{g}_G(H) = G$ (resp. $\mathfrak{G}_{\widehat{G}}(K) = \widehat{G}^{\mathbb{N}}$).

In this paper we will concentrate on the operator $\mathfrak{g}$: we will see that it defines a closure operator in the category of topological abelian groups in the sense of [12] and [14].

When no confusion is possible, we write simply $\mathfrak{g}(H)$ instead of $\mathfrak{g}_G(H)$ and $\mathfrak{g}(x)$ instead of $\mathfrak{g}(\langle x\rangle)$ for every element $x \in G$. For every sequence $\underline{u} \in \widehat{G}^{\mathbb{N}}$, $s_{\underline{u}}(G)$ is called a *basic* $\mathfrak{g}$-*closed* subgroup of G.

Remark 2.5. (a) One can easily prove that any finite intersection of basic $\mathfrak{g}$-closed subgroups of a group G is a basic $\mathfrak{g}$-closed subgroup of G.

(b) Let K, H be topological subgroups of G such that K is $\mathfrak{g}$-closed and H is a basic $\mathfrak{g}$-closed subgroup of G. If $K \leqslant H$ and K is of finite index in H, then K is a basic $\mathfrak{g}$-closed subgroup of G. $N = \cap_{i=1}^{n_0}$.

(c) Of course, the kernel of a continuous character χ of an abelian group G is a basic $\mathfrak{g}$-closed subgroup (since $\ker\chi$ coincides with $s_{\underline{\chi}}(G)$, where $\underline{\chi}$ is the constant sequence $\chi, \chi, \chi, \ldots$). Actually, any countable intersection of kernels of continuous characters is a basic $\mathfrak{g}$-closed subgroup. Since the kernels of characters are closed G_δ-subgroups of G, countable intersections of kernels of continuous characters are closed G_δ-subgroups of G.

Let us show that the $\mathfrak{g}$-closure is functorial.

Proposition 2.6. *Let* $f : G \to H$ *be a continuous homomorphism of groups. Then for every subgroup N of G one has* $f(\mathfrak{g}_G(N)) \subseteq \mathfrak{g}_H(f(N))$.

Proof. Given $\underline{v} \in \widehat{H}^{\mathbb{N}}$ such that $f(N) \leqslant s_{\underline{v}}(H)$, it is sufficient to prove that $f(\mathfrak{g}_G(N)) \subseteq s_{\underline{v}}(H)$. Clearly we have $N \leqslant s_{\hat{f}^{\mathbb{N}}(\underline{v})}(G)$; therefore $\mathfrak{g}_G(N) \leqslant s_{\hat{f}^{\mathbb{N}}(\underline{v})}(G)$ and hence

$$f(\mathfrak{g}_G(N)) \leqslant f(s_{\hat{f}^{\mathbb{N}}(\underline{v})}(G)) \leqslant s_{\underline{v}}(H)$$

by Lemma 2.3 (d). □

Corollary 2.7. *The* $\mathfrak{g}$*-closure defines a closure operator in the category of topological abelian groups.*

Proof. To prove that the assignment $H \mapsto \mathfrak{g}_G(H)$ defines a closure operator in the category of topological abelian groups we have to verify the three properties characterizing closure operators (see [12] and [14]). Let G be a topological abelian group and $J \leqslant H$ be topological subgroups of G. Since $J \leqslant \mathfrak{g}_G(J)$ and $\mathfrak{g}_G(J) \leqslant \mathfrak{g}_G(H)$, the extension and the monotonicity properties are satisfied (see [12] and [14]). The functoriality of $\mathfrak{g}_G(-)$ established in Proposition 2.6 is precisely the third property required for closure operators. □

Corollary 2.8. *Let* $\{G_i\}_{i\in I}$ *be a family of topological groups and let* $\{H_i\}_{i\in I}$ *be a family of subgroups* $H_i \leqslant G_i$. *Set* $G = \prod_{i\in I} G_i$ *and* $H = \prod_{i\in I} H_i$. *Then* $\mathfrak{g}_G(H) \subseteq \prod \mathfrak{g}_{G_i}(H_i)$.

In the next proposition we point out the relation between the $\mathfrak{g}_G$-closure and the $\mathfrak{g}_{G^\sharp}$-closure of a subgroup H of G.

Proposition 2.9. *Let G be a topological abelian group. Then for each subgroup* $0 \leqslant H \leqslant G$ *we have*

(a) $\mathfrak{g}_G(H) = \mathfrak{g}_{G^\sharp}(H)$,
(b) $\mathfrak{g}_G(0)$ *contains* $n(G)$.

In particular, G is MAP if and only if the trivial subgroup $\{0\}$ *is* $\mathfrak{g}$*-closed.*

Proof. (a) The topological groups G and $G^\sharp$ have the same continuous characters and hence the same basic $\mathfrak{g}$-closed subgroups. Then the required equality follows by definition.

(b) If $x \in n(G)$, then x belongs to $s_{\underline{u}}(G)$ for any sequence $\underline{u}$ in $\widehat{G}$. As $\mathfrak{g}_G(0) = \bigcap_{\underline{u}\in\widehat{G}^{\mathbb{N}}} s_{\underline{u}}(G)$, the assertion is proved. □

The conclusion of Proposition 2.9 implies one direction of our Theorem 1.2: indeed, if all the cyclic subgroups of G are $\mathfrak{g}$-closed, necessarily the group G is maximally almost periodic (in fact, it suffices to know that 0 is $\mathfrak{g}$-closed). On the other hand by (a) of the above proposition, to study properties of the $\mathfrak{g}$-closure it is not restrictive to assume G precompact:

Corollary 2.10. *If G is a MAP abelian group, then a cyclic subgroup of G is* $\mathfrak{g}$*-closed if and only if it is* $\mathfrak{g}$*-closed as a subgroup of* $G^\sharp$. *In particular, G has the* $\mathfrak{g}$*-closure property if and only if* $G^\sharp$ *does.*

This is why in the sequel we will work always with abelian precompact topological groups.

Remark 2.11. (a) Let G be a topological abelian group. Since each basic $\mathfrak{g}$-closed subgroup $s_{\underline{u}}(G)$, $\underline{u} \in \widehat{G}^{\mathbb{N}}$, contains obviously $\bigcap_n \ker u_n$, $s_{\underline{u}}(G)$ contains a closed G_δ-subgroup (cf. Remark 2.5 (c)). In particular this says that if G is not metrizable, then basic $\mathfrak{g}$-closed subgroup are *large*. Conversely, if G is compact, every closed G_δ-subgroup N of G is a countable intersection of kernels of characters of G and hence N is a closed basic $\mathfrak{g}$-closed subgroup. In particular, *the closed subgroups of any compact metric group are basic* $\mathfrak{g}$-*closed subgroups*.

(b) Let G be a precompact abelian group and let $x \in G$. If N is a $\mathfrak{g}$-closed subgroup of G containing x, then $\mathfrak{g}_G(x) = \mathfrak{g}_N(x)$. Indeed, by Lemma 2.3 $\mathfrak{g}_G(x) \cap N = \mathfrak{g}_N(x)$. On the other hand, since N is $\mathfrak{g}$-closed and contains $\langle x \rangle$, one obviously has $\mathfrak{g}_G(x) \leqslant N$.

Lemma 2.12. *Let G be a MAP abelian group. Then every closed subgroup of G that is closed in $G^\sharp$ is* $\mathfrak{g}$-*closed in G. Hence*:

(a) every finite subgroup of G is $\mathfrak{g}$-closed in G,
(b) every $\mathfrak{g}$-dense subgroup of G is dense in $G^\sharp$.

Proof. It suffices to observe that a subgroup of G is closed in $G^\sharp$ if and only if it is the intersection of kernels of continuous characters (recall that G and $G^\sharp$ have the same continuous characters), which are basic $\mathfrak{g}$-closed subgroups according to Remark 2.11 (a). For (a) note that the finite subgroups of G are certainly closed in $G^\sharp$ since the latter group is Hausdorff. □

Since in LCA groups and precompact abelian groups all closed subgroups H are also closed in $G^\sharp$ (this is equivalent to ask G/H to be MAP), this lemma implies that for such groups *closed subgroups are always* $\mathfrak{g}$-*closed*. However, one cannot extend this property to the larger class of the MAP groups as *a Hausdorff quotient of a MAP group need not be MAP* (see [2,17,18] for an example). This leaves open the following:

Question 2.13. Is every closed subgroup of a MAP abelian group also $\mathfrak{g}$-closed ?

Observe that Lemma 2.12, along with Remark 2.11 (b), shows that in order to establish that *all* precompact groups have the $\mathfrak{g}$-closure property, *it is not restrictive to consider only the monothetic ones*.

Let us consider now finite products of groups having the $\mathfrak{g}$-closure property.

Proposition 2.14. *Let G, H be MAP abelian groups. Then for every element $z = (x, y) \in G \times H$ such that $\langle x \rangle \leqslant G$ and $\langle y \rangle \leqslant H$ are* $\mathfrak{g}$-*closed subgroups, the cyclic subgroup $\langle z \rangle$ is* $\mathfrak{g}$-*closed in $G \times H$. In particular, the* $\mathfrak{g}$-*closure property is preserved by finite products of MAP groups*.

Proof. If both x, y are torsion, then also z is torsion, hence the subgroup $\langle z \rangle$ is finite and consequently $\mathfrak{g}$-closed by Lemma 2.12 (a). Hence assume that $o(x) = \infty$ or $o(y) = \infty$. Note that by Corollary 2.8 one immediately has $\langle z\rangle \leqslant \mathfrak{g}_{G\times H}(z) \leqslant \langle x\rangle \times \langle y\rangle$. To prove that $\langle z\rangle$ is $\mathfrak{g}$-closed it suffices to show that $\mathfrak{g}_{G\times H}(z) \cap \langle (0, y)\rangle = \{0\}$. Let us consider the following two cases.

(1) Assume that $o(x) = \infty = o(y)$. Then there exist countably many characters $\chi_1, \ldots, \chi_n, \ldots$ and $\psi_1, \ldots, \psi_n, \ldots$ of G and H respectively, such that $X := \langle \chi_n(x) : n \in \mathbb{N}\rangle$ and $Y := \langle \psi_n(x) : n \in \mathbb{N}\rangle$ are infinite subgroups of $\mathbb{T}$. Indeed, as x is non-torsion, $nx \neq 0$ for every $n \in \mathbb{N}$. Therefore, there exists a continuous character $\chi_n : G \to \mathbb{T}$ such that $\chi_n(nx) = n\chi_n(x) \neq 0$ in $\mathbb{T}$. In particular, $\chi_n(x) \notin \mathbb{Z}(n) \cong \mathbb{T}[n]$. Since this is true for every $n \in \mathbb{N}$, the subgroup $X := \langle \chi_n(x) : n \in \mathbb{N}\rangle$ has the desired properties. In particular, X and Y are dense subgroups of $\mathbb{T}$. If $a \in \mathbb{T}$ is a non-torsion element, then there exist two sequences $(u_n(x)) \in X^{\mathbb{N}}$ and $(v_n(y)) \in Y^{\mathbb{N}}$ such that $u_n(x) \to a$ and $v_n(y) \to -a$ in $\mathbb{T}$ (where u_n and v_n are linear combinations of characters of G and H, respectively). Set $\underline{w} = (u_n, v_n)$. Then $\underline{w} \in \widehat{G}^{\mathbb{N}} \times \widehat{H}^{\mathbb{N}}$ and $w_n(z) = u_n(x) + v_n(y) \to a - a = 0$ in $\mathbb{T}$ so that $z \in s_{\underline{w}}(G \times H)$.Note that, for every positive integer $d \in \mathbb{N}$, $w_n(0, dy) = v_n(dy) = dv_n(y) \to -da \neq 0$ as a is not torsion. Thus $(0, dy) \notin s_{\underline{w}}(G \times H)$. As $\langle z\rangle \leqslant \mathfrak{g}_{G\times H}(z) \leqslant \langle x\rangle \times \langle y\rangle$, this entails $\langle z\rangle = \mathfrak{g}_{G\times H}(z)$.

(2) Assume that $o(x) = \infty$ and $o(y) = m$. As before one can find continuous characters $\chi_1, \ldots, \chi_n, \ldots$ of G such that $X := \langle \chi_n(x) : n \in \mathbb{N}\rangle$ is an infinite subgroup of $\mathbb{T}$. Let $v : H \to \mathbb{T}$ be a continuous character of H such that $o(v(y)) = m$. In particular, $\langle v(y)\rangle \cong \mathbb{Z}(m)$. Since X is dense in $\mathbb{T}$, there exists a sequence $(u_n(x)) \in X^{\mathbb{N}}$ such that $u_n(x) \to -v(y)$ in $\mathbb{T}$, i.e., $u_n(x) + v(y) \to 0$ in $\mathbb{T}$. Set $\underline{w} = (u_n, v_n)$, where $v_n = v$ for every $n \in \mathbb{N}$. Then $\underline{w} \in \widehat{G}^{\mathbb{N}} \times \widehat{H}^{\mathbb{N}}$ and by definition $z \in s_{\underline{w}}(G \times H)$. If $(0, dy) \in \mathfrak{g}_{G\times H}(z)$ for some positive integer $d \in \mathbb{N}$, then $(0, dy) \in s_{\underline{w}}(G \times H)$, i.e., $v_n(dy) = v(dy) = dv(y) = 0$. Hence $m|d$ so that $dy = 0$. Again one concludes that $\langle z\rangle = \mathfrak{g}_{G\times H}(z)$. $\square$

Remark 2.15. One can easily show that if G_1, H_1 are basic $\mathfrak{g}$-closed subgroups of two topological abelian groups G, H respectively, then $G_1 \times H_1$ is a basic $\mathfrak{g}$-closed subgroup of $G \times H$. Therefore, if G, H are MAP abelian groups, then for every element $z = (x, y) \in G \times H$ such that $\langle x\rangle \leqslant G$ and $\langle y\rangle \leqslant H$ are basic $\mathfrak{g}$-closed subgroups, $\langle z\rangle$ is a basic $\mathfrak{g}$-closed subgroup of $G \times H$. Indeed, if both x, y are torsion, then Remark 2.5 (b) applies. If $o(x) = \infty$ or $o(y) = \infty$, then a careful reading of the proofs of Proposition 2.14 (1)-(2) shows that $s_{\underline{w}}(G \times H) \cap \langle x\rangle \times \langle y\rangle = \langle z\rangle$.

3. *TB*-sequences

3.1. TB-sequences : general background

In this section we study *TB*-sequences of discrete abelian groups G and in particular of the Prüfer group $\mathbb{Z}(p^\infty)$.

The following lemma generalizes [3, Proposition 2.3]:

Lemma 3.1. *Let G be a topological abelian group and let $H \leqslant \widehat{G}$. Then for a sequence $\underline{u} = (u_n) \in G^{\mathbb{N}}$ one has $u_n \to 0$ in (G, T_H) if and only if $H \leqslant s_{\underline{u}}(\widehat{G})$.*

Proof. Identifying every element u_n of $\underline{u}$ with the character $\widetilde{u_n}$ of $\widehat{G}$ given by the evaluation homomorphism, the definition of T_H implies that $u_n \to 0$ in (G, T_H) if and only if $\widetilde{u_n}(h) = h(u_n) \to 0$ in $\mathbb{T}$ for every $h \in H$ if and only if $H \leqslant s_{\underline{u}}(\widehat{G})$. □

Since the supremum of totally bounded group topologies is always totally bounded, it follows that for every sequence $\underline{u} = (u_n)$ of a topological abelian group G there exists a *finest* totally bounded group topology τ on G such that $u_n \to 0$ in (G, τ). More precisely, $\sigma_{\underline{u}} := T_{s_{\underline{u}}(\widehat{G})}$ is the *finest totally bounded group topology on G for which* (u_n) *converges to 0*.

The properties of the topology $\sigma_{\underline{u}}$ can be characterized in terms of corresponding properties of $s_{\underline{u}}(\widehat{G})$.

Proposition 3.2. *Let G be a topological abelian group and let* $\underline{u} \in G^{\mathbb{N}}$ *be a sequence. Then*:

(a) $w(G, \sigma_{\underline{u}}) = |s_{\underline{u}}(\widehat{G})|$,
(b) $\sigma_{\underline{u}}$ *is Hausdorff if and only if* $s_{\underline{u}}(\widehat{G})$ *is dense in* $\widehat{G}$,
(c) $\sigma_{\underline{u}}$ *is metrizable if and only if* $s_{\underline{u}}(\widehat{G})$ *is countable and dense in* $\widehat{G}$.

Proof. (a) follows from the property $w(G, T_H) = |H|$ of the topology T_H generated by any subgroup $H \leqslant \widehat{G}$.

(b) According to [9, Theorem 4.15], $\sigma_{\underline{u}}$ is Hausdorff if and only if $s_{\underline{u}}(\widehat{G})$ separates the points of G if and only if $s_{\underline{u}}(\widehat{G})$ is dense in $\widehat{G}$.

(c) If $\sigma_{\underline{u}}$ is metrizable, then $w(G, \sigma_{\underline{u}})$ is countable and the assertion follows from (a) and (b).

Conversely, if $w(G, \sigma_{\underline{u}})$ is infinite and countable, then $\sigma_{\underline{u}}$ is metrizable by Urysohn's metrization theorem. □

3.2. TB-sequences in $\mathbb{Z}(p^\infty)$

Let p be a prime number. We consider the Prüfer group $\mathbb{Z}(p^\infty)$ (let us recall that its dual group is the group of p-adic integers $\mathbb{Z}_p$).

Denote by c_n the element $1/p^n + \mathbb{Z}$ of $\mathbb{Z}(p^\infty)$, so that the sequence $\underline{c} = (c_n)$ is nothing else but the canonical set of generators of $\mathbb{Z}(p^\infty)$. (One can look at c_n also as the character $\mathbb{Z}_p \to \mathbb{Z}_p/p^n\mathbb{Z}_p \cong \mathbb{Z}(p^n) \leqslant \mathbb{T}$ obtained from the canonical map $\mathbb{Z}_p \to \mathbb{Z}_p/p^n\mathbb{Z}_p$.)

Theorem 3.3. *Let n_k be a strictly increasing sequence of naturals and let* $\underline{c}^* = (c_{n_k})$ *be a subsequence of* $\underline{c}$. *Then* $\underline{c}^*$ *is a TB-sequence of* $\mathbb{Z}(p^\infty)$ *and the following are equivalent*:

(a) $s_{\underline{c}^*}(\mathbb{Z}_p) = \mathbb{Z}$,
(b) $s_{\underline{c}^*}(\mathbb{Z}_p)$ *is countable*,
(c) $|s_{\underline{c}^*}(\mathbb{Z}_p)| < \mathfrak{c}$,
(d) *the differences* $n_{k+1} - n_k$ *are bounded*,
(e) $\{n_k : k \in \mathbb{N}\}$ *is a large subset of* $\mathbb{N}$ (*i.e., there exists a finite set* $F \subseteq \mathbb{N}$ *such that* $\mathbb{N} \subseteq F \cup (F + \{n_k : k \in \mathbb{N}\})$),

(f) *the finest precompact topology* $\sigma_{\underline{c}^*}$ *on* $\mathbb{Z}(p^\infty)$ *that makes* c_{n_k} *converge to* 0 *is metrizable,*
(g) *the finest precompact topology* $\sigma_{\underline{c}^*}$ *on* $\mathbb{Z}(p^\infty)$ *that makes* c_{n_k} *converge to* 0 *has weight* $< \mathfrak{c}$.

Proof. The first assertion follows from the fact that for $n_k = k$ the sequence $\underline{c}^*$ converges to 0 w.r.t the topology induced by $\mathbb{T}$ on $\mathbb{Z}(p^\infty)$.

The implications $(a) \Rightarrow (b) \Rightarrow (c)$ and $(f) \Rightarrow (g)$ are obvious.

To prove the implications $(c) \Rightarrow (d)$ assume that the differences $n_{k+1} - n_k$ are not bounded. Then there exists a subsequence k_r of k such that $n_{k_r+1} - n_{k_r} \to \infty$ and the sequence $n_{k_r+1} - n_{k_r}$ is strictly increasing with $n_{k_0+1} > n_{k_0}$. Let us show by induction on k that this subsequence has the following additional property

$$|\{k_s : k_s < k\}| \leqslant n_{k_r+1} - n_{k_r} \text{ for each } k > k_0 \text{ and}$$
$$\text{for the largest } r \in \mathbb{N} \text{ with } k_r < k. \tag{1}$$

For simplicity let $f(x) := |\{k_s : k_s < x\}|$. Start with the lowest non-trivial value $k = k_0 + 1$, where (1) is trivially true. Suppose now that $k > k_0 + 1$ and (1) is true for $k - 1$ and let $r \in \mathbb{N}$ be the largest positive integer with $k_r < k$. If $k > k_r + 1$, then $f(k) = f(k-1) = r+1$ and r is the largest $r \in \mathbb{N}$ with $k_r < k - 1$, so that (1) holds also for k. If $k = k_r + 1$, then note that $n_{k_r+1} - n_{k_r} > n_{k_{r-1}+1} - n_{k_{r-1}}$ by the choice of the subsequence k_r and apply the inductive hypothesis.

Now let $\alpha = (\alpha_s)$, where $0 \leqslant \alpha_s \leqslant p - 1$, and let

$$\xi_\alpha = \sum_{s=0}^{\infty} p^{n_{k_s}} \alpha_s.$$

Clearly, distinct α's give rise to distinct elements ξ_α of $\mathbb{Z}_p$. Hence to prove that $|s_{\underline{c}^*}(\mathbb{Z}_p)| = \mathfrak{c}$ it suffices to see that each ξ_α belongs to $s_{\underline{c}^*}(\mathbb{Z}_p)$. To this end we have to show that $c_{n_k}\xi_\alpha \to 0$ in $\mathbb{T}$.

In the sequel we denote elements of $\mathbb{T}$ (in particular, $c_{n_k}(\xi_\alpha)$) by their unique representative in the interval $[0, 1)$. This will allow us to write inequalities in $\mathbb{R}$ between such elements. Therefore, denoted by r the largest index such that $k_r < k$, we have

$$c_{n_k}\xi_\alpha = \frac{\sum_{s=0}^{r} p^{n_{k_s}} \alpha_s}{p^{n_k}} \leqslant \frac{(p-1)(n_{k_r+1} - n_{k_r}) p^{n_{k_r}}}{p^{n_k}} \leqslant (p-1)\, \frac{(n_{k_r+1} - n_{k_r})}{p^{n_k - n_{k_r}}}.$$

Since this implies $n_{k_r+1} \leqslant n_k$, we have $p^{n_k - n_{k_r}} \geqslant p^{n_{k_r+1} - n_{k_r}}$. Since

$$\lim_r \frac{n_{k_r+1} - n_{k_r}}{p^{n_{k_r+1} - n_{k_r}}} = 0,$$

we are through.

The equivalence $(d) \Leftrightarrow (e)$ is known (see for example [1]), while the equivalences $(b) \Leftrightarrow (f)$ and $(c) \Leftrightarrow (g)$ follow respectively from (c) and (a) of Proposition 3.2.

We are left with the proof of the implication $(d) \Rightarrow (a)$. Assume that for some natural number d one has

$$n_{k+1} - n_k \leqslant d \quad \text{for every } k \in \mathbb{N}. \tag{2}$$

We have to prove that $s_{\underline{c}^*}(\mathbb{Z}_p) = \mathbb{Z}$. The inclusion $\mathbb{Z} \leqslant s_{\underline{c}^*}(\mathbb{Z}_p)$ is obvious.

Let $\xi \in s_{\underline{c}^*}(\mathbb{Z}_p)$. Then ξ admits a representation of the form $\xi = \sum_{k=0}^{\infty} p^k \alpha_k$ where $0 \leqslant \alpha_k \leqslant p-1$ for every $k \in \mathbb{N}$. Note that, modulo $\mathbb{Z}$, $c_n(\xi) = c_n(\sum_{k=0}^{n-1} p^k \alpha_k)$ for every $n \in \mathbb{N}$. Hence, for a fixed integer $n > d$ and

$$c_n(\xi) = \frac{\alpha_{n-1}}{p} + \frac{\alpha_{n-2}}{p^2} + \cdots + \frac{\alpha_1}{p^{n-1}} + \frac{\alpha_0}{p^n},$$

we have

$$\frac{\alpha_{n-1}}{p} + \cdots + \frac{\alpha_1}{p^{n-1}} + \frac{\alpha_0}{p^n} \leqslant \frac{\alpha_{n-1}}{p} + \frac{\alpha_{n-2}}{p^2} + \cdots + \frac{\alpha_{n-d-1}}{p^{d+1}} + (p-1)\left(\frac{1}{p^{d+2}} + \cdots + \frac{1}{p^n}\right).$$

Since $(p-1)\left(\frac{1}{p^{d+2}} + \cdots + \frac{1}{p^n}\right) = \frac{1}{p^{d+1}} - \frac{1}{p^n} < \frac{1}{p^{d+1}}$, one obtains

$$\frac{\alpha_{n-1}}{p} + \frac{\alpha_{n-2}}{p^2} \cdots + \frac{\alpha_{n-d-1}}{p^{d+1}} \leqslant c_n(\xi) < \frac{\alpha_{n-1}}{p} + \frac{\alpha_{n-2}}{p^2} + \cdots + \frac{\alpha_{n-d-1}+1}{p^{d+1}}. \tag{3}$$

Now $c_{n_t}(\xi) \to 0$ in $\mathbb{T}$ by hypothesis, then there exists $t_0 > 2$ such that, for every $t \geqslant t_0$, one has

$$c_{n_t}(\xi) < \frac{1}{2p^{d+1}} \quad \text{or} \quad 1 - \frac{1}{2p^{d+1}} < c_{n_t}(\xi). \tag{4}$$

Let us see now that if

$$c_{n_t}(\xi) < \frac{1}{2p^{d+1}} \quad \text{for some } t \geqslant t_0, \tag{5}$$

then

$$c_{n_{t'}}(\xi) < \frac{1}{2p^{d+1}} \quad \text{for all } t' \geqslant t. \tag{6}$$

Indeed, we will see first that (5) implies

$$\alpha_{n_t-1} = \alpha_{n_t-2} = \cdots = \alpha_{n_t-d-1} = 0 \tag{7}$$

and then we shall see that (7) implies $c_{n_{t+1}}(\xi) < 1/2p^{d+1}$. Then by a simple induction one can see that (5) implies (6).

Indeed, assume that (5) holds and $\alpha_{n_t-s} \neq 0$ for some $0 < s \leqslant d+1$. Then

$$\frac{1}{p^{d+1}} \leqslant \frac{1}{p^s} \leqslant \frac{\alpha_{n_t-1}}{p} + \frac{\alpha_{n_t-2}}{p^2} + \cdots + \frac{\alpha_{n_t-d-1}}{p^{d+1}} \leqslant c_{n_t}(\xi) < \frac{1}{2p^{d+1}}$$

so that $1/p^{d+1} < 1/2p^{d+1}$, a contradiction. Hence (7) holds. Let us see now that $1 - 1/2p^{d+1} < c_{n_{t+1}}(\xi)$ cannot hold. Indeed, according to (3) and taking into account that by

$n_t - d - 1 \leqslant n_{t+1} - d - 1 \leqslant n_t - 1$ (7) yields $\alpha_{n_{t+1}-d-1} = 0$, one obtains

$$c_{n_{t+1}}(\xi) < \frac{\alpha_{n_{t+1}-1}}{p} + \cdots + \frac{\alpha_{n_{t+1}-d}}{p^d} + \frac{\alpha_{n_{t+1}-d-1}+1}{p^{d+1}}$$
$$\leqslant (p-1)\left(\frac{1}{p} + \cdots + \frac{1}{p^d}\right) + \frac{1}{p^{d+1}} \leqslant 1 - \frac{1}{p^d} + \frac{1}{p^{d+1}} \leqslant 1 - \frac{1}{2p^{d+1}}.$$

Now we can conclude that $c_{n_{t+1}}(\xi) < 1/2p^{d+1}$ holds by (4) applied to n_{t+1}. This yields (6) by induction. It is clear that (5) implies, through (6) and (7), that all α_n with sufficiently large n are zero. Hence $\xi \in \mathbb{Z}$.

Assume that (5) does not hold, then by (4) $1 - 1/2p^{d+1} < c_{n_t}(\xi)$ for all $t \geqslant t_0$. Then from

$$1 - \frac{1}{2p^{d+1}} < c_{n_t}(\xi) \leqslant \frac{\alpha_{n_t-1}}{p} + \cdots + \frac{\alpha_{n_t-d}}{p^d} + \frac{\alpha_{n_t-d-1}+1}{p^{d+1}}$$
$$\leqslant (p-1)\left(\frac{1}{p} + \cdots + \frac{1}{p^d}\right) + \frac{\alpha_{n_t-d-1}+1}{p^{d+1}}$$

for every $t \geqslant t_0$, one obtains

$$1 - \frac{1}{2p^{d+1}} \leqslant 1 - \frac{1}{p^d} + \frac{\alpha_{n_t-d-1}+1}{p^{d+1}}. \tag{8}$$

This implies $(2p-1)/2p^{d+1} \leqslant (\alpha_{n_t-d-1}+1)/p^{d+1}$ and consequently $2p-1 \leqslant 2\alpha_{n_t-d-1}+2$. Since $\alpha_{n_t-d-1} \leqslant p-1$ this yields $\alpha_{n_t-d-1} = p-1$. With the same argument we get also $\alpha_{n_t-1} = \alpha_{n_t-2} = \cdots = \alpha_{n_t-d} = p-1$ for all $t \geqslant t_0$. Then by (2) $\alpha_n = p-1$ for every $n \geqslant n_{t_0}$. Hence $\xi = \sum_{k=0}^{n_{t_0}-1} p^k\alpha_k + \sum_{k=n_{t_0}}^{\infty} p^k\alpha_k = \sum_{k=0}^{n_{t_0}-1} p^k\alpha_k - p^{n_{t_0}} \in \mathbb{Z}$. $\square$

Remark 3.4. Let p be a prime number, $p > 2$, and consider the sequence (a_n) of $\mathbb{Z}(p^\infty)$ defined by $a_n := b_n/p^n$ where $b_n = (p^n-1)/2$ for every $n \in \mathbb{N}$. Then $a_n \to \frac{1}{2}$ w.r.t. to the induced by $\mathbb{T}$ topology on $\mathbb{Z}(p^\infty)$. Therefore, also the subsequence $\underline{a} = (a_{n!}) \to \frac{1}{2}$ w.r.t. to the same topology. On the other hand, $(a_{n!})$ is a *TB*-sequence of $\mathbb{Z}(p^\infty)$. Indeed, let $\xi \in \mathbb{Z}_p$ be the character of $\mathbb{Z}(p^\infty)$ defined by $\xi := 1 + p + 2\sum_{k=2}^{\infty} p^{k!}$. Then, arguing as in the proof of Theorem 3.3, one can prove that $\xi(a_{n!}) \to 0$ in $\mathbb{T}$ so that $\xi \in s_{\underline{a}}(\mathbb{Z}_p)$. Since $\langle\xi\rangle$ is dense in $\mathbb{Z}_p$, Proposition 3.2 applies to conclude that $(a_{n!})$ is a *TB*-sequence.

4. Proof of Theorem 1.2

We show that Theorem 1.2 can be easily deduced from the fact that every compact abelian group has the $\mathfrak{g}$-closure property. This is proved in Theorem 4.8 and this section is dedicated to prove that theorem. The proof splits into several steps. Let us start examining the case of totally disconnected compact groups. Any totally disconnected monothetic compact group is isomorphic to $\prod_{p\in\pi}\mathbb{Z}_p \times \prod_{p\in\pi'} \mathbb{Z}(p^{k_p})$ where π and π' are disjoint subsets of $\mathbb{P}$. We begin considering $\mathbb{Z}_p$.

Lemma 4.1. *Let $p \in \mathbb{P}$. The group $\mathbb{Z}_p$ has the $\mathfrak{g}$-closure property.*

Proof. We prove first that for every positive integer $m \geqslant 1$, $m\mathbb{Z}$ is a $\mathfrak{g}$-closed subgroup of $\mathbb{Z}_p$. For $m=1$ apply Theorem 3.3. If $m>1$, then consider the subring $\mathbb{Z}_{(p)}$ of $\mathbb{Q}$ defined as $\mathbb{Z}_{(p)} := \{\frac{a}{p^n} : a, n \in \mathbb{Z}\}$. Since $\mathbb{Z}_{(p)}$ is dense in $\mathbb{Q}$ and $0<1/m<1$, there exists a sequence $(a_n/p^n) \in (0,1)$ such that $a_n/p^n \to 1/m$. Let $u_n : \mathbb{Z}_p \to \mathbb{Z}_p/p^n\mathbb{Z}_p \cong \mathbb{Z}(p^n) \leqslant \mathbb{T}$ be the character of $\mathbb{Z}_p$ defined as $u_n := a_n c_n$. Then $u_n(1) = a_n/p^n + \mathbb{Z}$, so that $u_n(d) \to d/m + \mathbb{Z} \neq 0$ in $\mathbb{T}$ for every $0<d<m$, while $u_n(m) \to 0$ in $\mathbb{T}$. This shows that $m \in s_{\underline{u}}(\mathbb{Z}_p)$ and $d \notin s_{\underline{u}}(\mathbb{Z}_p)$ for every $0<d<m$. Since $\mathfrak{g}_{\mathbb{Z}_p}(\langle m\rangle)$ is a subgroup of $\mathbb{Z}$ containing $m\mathbb{Z}$, one has $\mathfrak{g}_{\mathbb{Z}_p}(\langle m\rangle) = d\mathbb{Z}$ for some positive integer $d \in \mathbb{Z}$ with $d|m$. Since we have shown that $k \notin s_{\underline{u}}(\mathbb{Z}_p)$ for every $0<k<m$, we conclude that $d=m$.

Now let ξ be an arbitrary non-zero element of $\mathbb{Z}_p$. Then $\xi = p^n\varepsilon$ where $n \in \mathbb{N}$ and $\varepsilon \in \mathbb{Z}_p \setminus p\mathbb{Z}_p$. Let $m_\varepsilon : \mathbb{Z}_p \to \mathbb{Z}_p$ be the multiplication by ε. Then m_ε is a topological automorphism of $\mathbb{Z}_p$ with $m_\varepsilon(p^n) = p^n\varepsilon = \xi$. To finish the proof it suffices to note that by the above argument the cyclic subgroup $\langle p^n\rangle$ is $\mathfrak{g}$-closed and, by Lemma 2.3, the property to be $\mathfrak{g}$-closed is invariant for topological isomorphisms. □

Corollary 4.2. *Every cyclic subgroup of $\mathbb{Z}_p$ is a basic $\mathfrak{g}$-closed subgroup.*

Proof. Let us consider first the subgroups $m\mathbb{Z}$ for every positive integer $m \geqslant 1$. For $m=1$ apply Theorem 3.3. If $m>1$, then consider the sequence $\underline{u}$ of characters of $\mathbb{Z}_p$ defined as in Lemma 4.1. Then $m\mathbb{Z} = s_{\underline{c}}(\mathbb{Z}_p) \cap s_{\underline{u}}(\mathbb{Z}_p)$, hence $m\mathbb{Z}$ is a basic $\mathfrak{g}$-closed subgroup. Now let ξ be an arbitrary non-zero element of $\mathbb{Z}_p$. Then $\xi = p^m\varepsilon$ where $n \in \mathbb{N}$ and $\varepsilon \in \mathbb{Z}_p \setminus p\mathbb{Z}_p$. Since $\langle p^m\rangle = s_{\underline{w}}(\mathbb{Z}_p)$ for some $\underline{w} \in \widehat{\mathbb{Z}_p}^{\mathbb{N}}$, the sequence $\underline{v} := (w_n \circ m_{\varepsilon^{-1}})$ is in $\widehat{\mathbb{Z}_p}^{\mathbb{N}}$ and $\langle\xi\rangle = s_{\underline{v}}(\mathbb{Z}_p)$. □

Lemma 4.3. *Let $\pi \subseteq \mathbb{P}$ be a set of prime numbers. Then $G = \prod_{p\in\pi} \mathbb{Z}_p$ has the $\mathfrak{g}$-closure property.*

Proof. If π is a singleton, then Lemma 4.1 applies. Assume that $|\pi|>1$ and denote by z the element $(1_p)_{p\in\pi} \in G$. Then $\mathfrak{g}_G(z) \subseteq \prod_{p\in\pi}\langle 1_p\rangle$ by Corollary 2.8. If $x = (x_p)_{p\in\pi} \in \mathfrak{g}_G(z)$ is an arbitrary element, then $x_p = k_p 1_p$ for every $p \in \pi$ and for some $k_p \in \mathbb{Z}$. Let p, q be distinct prime numbers in π and let us consider the projection $f : G \to \mathbb{Z}_p \times \mathbb{Z}_q$. Propositions 2.6 and 2.14 imply that

$$(x_p, x_q) = f(x) \in f(\mathfrak{g}_G(z)) \subseteq \mathfrak{g}_{\mathbb{Z}_p\times\mathbb{Z}_q}(f(z)) = \langle f(z)\rangle.$$

Thus, $(x_p, x_q) \in \langle(1_p, 1_q)\rangle$ so that $k_p = k_q$. Since this is true for every pair (p,q) of distinct primes in π one concludes that $x \in \langle z\rangle$. Hence $\langle z\rangle$ is $\mathfrak{g}$-closed in G.

Let $\xi = (\xi_p)_{p\in\pi}$ be an arbitrary element of G. Without loss of generality we may assume that $\overline{\langle\xi\rangle} = G$. To finish the proof it suffices to take the topological automorphism $\phi : G \to G$ defined by $\phi(z) = \xi$ and use the fact that, by Lemma 2.3, the property to be $\mathfrak{g}$-closed is invariant under topological isomorphisms. This allows us to conclude that $\langle\xi\rangle$ is $\mathfrak{g}$-closed in G. □

By [10, Theorem 4.7] (see also [7, Theorem 1]) and Proposition 2.14, we have proved so far that every compact group of the form $G = \mathbb{T}^m \times \prod_{p\in\mathbb{P}} \mathbb{Z}_p^{n_p}$, where m is a positive integer and (n_p) is a bounded sequence, satisfies the $\mathfrak{g}$-closure property. It follows from Proposition 2.14 and Lemma 4.3 that the $\mathfrak{g}$-closure property holds also for groups of the form $\prod_{p\in\pi}\mathbb{Z}_p \times F$, where π is a set of primes and $F = \mathbb{Z}(q_1^{m_1}) \times \cdots \times \mathbb{Z}(q_k^{m_k})$, q_j are distinct prime numbers and $m_j \in \mathbb{N}$ for every $j = 1, \ldots, k$.

Next we take care of infinite products $\prod_{p\in\pi}\mathbb{Z}(p^{k_p})$.

Lemma 4.4. *Arbitrary countably infinite products of finite cyclic groups have the* $\mathfrak{g}$-*closure property*.

Proof. Let us consider an infinite product of cyclic groups $G=\prod_{k\in\mathbb{N}}\mathbb{Z}(n_k)$ and for $k \in \mathbb{N}$ let $\pi_k : G \to \mathbb{Z}(n_k)$ be the canonical projection. We have to prove that for every $x \in G$ the subgroup $\langle x\rangle$ is $\mathfrak{g}$-closed. Fix $x = (x_k)_{k\in\mathbb{N}}$ in G. The $\mathfrak{g}$-closure of x in G coincides with the $\mathfrak{g}$-closure of x in $G_0 = \prod_{k\in\mathbb{N}}\langle x_k\rangle$. Denote by m_k the order of x_k in $\mathbb{Z}(n_k)$. We can assume without loss of generality that m_k is a non-decreasing sequence of natural numbers. We think of each $\mathbb{Z}(n_k)$ as the subgroup of $\mathbb{T}$ of order n_k. There exists an automorphism ϕ of G_0 such that $z = \phi(x) = (1/m_k)_{k\in\mathbb{N}}$. Since the $\mathfrak{g}$-closure is invariant for topological isomorphisms, it is sufficient to prove that $\langle z\rangle$ is $\mathfrak{g}$-closed. If $\langle z\rangle$ is finite, then it is closed and hence $\mathfrak{g}$-closed by Lemma 2.12 (a). Then we can assume that z is a torsion-free element; therefore the increasing sequence of natural numbers $m_k, k \in \mathbb{N}$, is unbounded.

Let $y \in G_0$, $y \notin \langle z\rangle$: we will prove that y does not belong to the $\mathfrak{g}$-closure of $\langle z\rangle$. To this end we construct a sequence u_n, $n \in \mathbb{N}$, of characters of G_0 such that $u_n(z)$ converges to zero, while $u_n(y)$ does not. Our hypothesis $y \notin \langle z\rangle$ entails that only one of the following two cases occurs:

(i) Assume there exists a strictly increasing sequence k_i, $1 \leqslant i \in \mathbb{N}$, such that $y_{k_i} = a/m_{k_i}$ with $0 \leqslant |a| < m_{k_i}$ and there exists $k_0 < k_1$, such that $y_{k_0} = b/m_{k_0}$ with $b/m_{k_0} \neq a/m_{k_0}$. Let $l_n = [m_{k_n}/m_{k_0}]$ and note that $\lim_n l_n/m_{k_n} = 1/m_{k_0}$. Consider the sequence $\underline{u} = (u_n)$ of characters of G_0 defined by

$$u_n := l_n\pi_{k_n} - \pi_{k_0} \quad \text{for every} \quad n \in \mathbb{N}.$$

Clearly $u_n(z) = l_n/m_{k_n} - 1/m_{k_0}$ converges to zero for $n \to \infty$. Nevertheless $u_n(y) = l_n a/m_{k_n} - b/m_{k_0}$ converges to $(a-b)/m_{k_0} \neq 0$.

(ii) Assume there exists a strictly increasing sequence k_i, $1 \leqslant i \in \mathbb{N}$, such that $y_{k_i} = a_i/m_{k_i}$ with $0 < a_i < m_{k_i}$ and $a_1 < a_2 < a_3 < \cdots$. If a_i/m_{k_i} does not converge to zero in $\mathbb{T}$ for $i \to \infty$, we set $u_n := \pi_{k_n}$ and $u_n(z)$ converges to zero, while $u_n(y)$ does not. Then we can suppose that a_i/m_{k_i} converges to zero in $\mathbb{T}$. Now we distinguish three cases:

(a) Suppose a_i/m_{k_i} converges to zero in $\mathbb{R}$; hence it is not restrictive to assume that $2a_i < m_{k_i}$ for $1 \leqslant i \in \mathbb{N}$. Let $u_n = [m_{k_n}/2a_n]\pi_{k_n}$; then $u_n(z) = [m_{k_n}/2a_n]/m_{k_n}$ converges to zero, while $u_n(y) = [m_{k_n}/2a_n]a_n/m_{k_n}$ converges to $\frac{1}{2}$.

(b) Suppose a_i/m_{k_i} converges to one in $\mathbb{R}$. We have $a_i/m_{k_i} = (a_i - m_{k_i})/m_{k_i}$ in $\mathbb{T}$; we can assume without loss of generality that $a_i - m_{k_i}$ is a strictly decreasing sequence of negative integers, otherwise (i) can be applied. Then, analogously to the previous case, let $u_n = [m_{k_n}/2(m_{k_n} - a_n)]\pi_{k_n}$; then $u_n(z) = [m_{k_n}/2(m_{k_n} - a_n)]/m_{k_n}$ converges to zero, while $u_n(y) = [m_{k_n}/2(m_{k_n} - a_n)](a_n - m_{k_n})/m_{k_n}$ converges to $-\frac{1}{2} = \frac{1}{2}$ in $\mathbb{T}$.

(c) Suppose a_i/m_{k_i} does not converge to zero or to one in $\mathbb{R}$. Then there exist a subsequence k_{i_r} of k_i such that $a_{i_r}/m_{k_{i_r}}$ converges to zero or to one in $\mathbb{R}$. Then for $a_{i_r}/m_{k_{i_r}}$ (a) or (b) can be applied. □

Corollary 4.5. *The totally disconnected compact groups have the* $\mathfrak{g}$*-closure property*.

Proof. Let G be a totally disconnected compact group, i.e. a profinite group. As we have observed, it suffices to assume that G is monothetic. Then the group G is isomorphic to a product $N \times H$, where $N = \prod_{p \in P} \mathbb{Z}_p$, $H = \prod_{p \in P'} \mathbb{Z}(p^{n_p})$ and P, P' are disjoint sets of prime numbers. According to Lemmas 4.4, 4.3 and Proposition 2.14, the cyclic subgroups of $G = N \times H$ are $\mathfrak{g}$-closed. □

Corollary 4.6. *The* $\mathfrak{g}$*-closure property is preserved by arbitrary products of precompact groups*.

Proof. Let $\{G_i : i \in I\}$ be a family of precompact abelian groups satisfying the $\mathfrak{g}$-closure property. We have to prove that also $G := \prod_i G_i$ satisfies the $\mathfrak{g}$-closure property. To check it pick an element $x = (x_i) \in G$ and $y \in \mathfrak{g}(x)$.

Case 1: There exists $i \in I$ such that x_i is non-torsion. Now for every $j \in I \setminus \{i\}$ consider the projection $p_j : G \to G_i \times G_j$. Since $G_i \times G_j$ satisfies the $\mathfrak{g}$-closure property by Proposition 2.14, and $p_j(y) \in \mathfrak{g}_{G_i \times G_j}(p_j(x))$, by Proposition 2.6, we conclude that

$$p_j(y) \in \langle p_j(x) \rangle. \tag{9}$$

Let $p_j(y) = k_j p_j(x)$ with $k_j \in \mathbb{Z}$. Now consider the projection $p : G \to G_i$. Since every G_i satisfies the $\mathfrak{g}$-closure property and $p(y) \in \mathfrak{g}_{G_i}(p(x))$ we conclude that $p(y) \in \langle p(x) \rangle$, hence $p(y) = kp(x) = kx_i$ for some $k \in \mathbb{Z}$. Projecting (9) on G_i we get $kx_i = p(y) = k_j x_i$. Since x_i is non-torsion, we conclude that $k_j = k$. This proves that $y_j = kx_j$ for every *j*, i.e., $y = kx$. Hence $y \in \langle x \rangle$.

Case 2: Every x_i is torsion. Hence x is contained in the totally disconnected compact subgroup $H = \prod_{i \in I} \langle x_i \rangle$. By Corollary 4.5 H satisfies the $\mathfrak{g}$-closure property, so $y \in \langle x \rangle$. □

[7, Theorem 1] (see also [10, Theorem 4.7]) and Corollary 4.6 yield

Corollary 4.7. $\mathbb{T}^\lambda$ *has the* $\mathfrak{g}$*-closure property for every infinite cardinal* λ.

We are ready now to prove that every cyclic subgroup of a compact abelian group is $\mathfrak{g}$-closed.

Theorem 4.8. *The compact abelian groups have the* $\mathfrak{g}$*-closure property*.

Proof. Let G be a compact abelian group. Denote by X the discrete Pontryagin dual of G. Fix a free subgroup F of X such that X/F is torsion. For every $n \in \mathbb{N}$ let

$$F_n = \{x \in X : n!x \in F\},$$

(note that $F_1 = F$ and F_n form an ascending chain). Then F_n has a bounded torsion part $T_n = F_n[n!]$ that necessarily splits by Baer's theorem, hence $F_n = L_n \oplus T_n$, where L_n is torsion free. Since the multiplication by $n!$ defines a homomorphism $f : F_n \to F$ whose restriction on L_n is a monomorphism, the group L_n is free. Identifying $\widehat{\widehat{G}}$ with G, we consider the annihilator N_n of F_n as a subgroup of G. Then N_n is a closed subgroup of G such that

$$G/N_n \cong \widehat{F_n} \cong \widehat{L_n} \times \widehat{T_n} \quad \text{and} \quad \widehat{N_1} \cong X/F, \tag{10}$$

and the subgroups N_n form a descending chain. Here $L_n \cong \bigoplus_\lambda \mathbb{Z}$ and $T_n \cong \bigoplus_{i\in I} \mathbb{Z}(k_i)$ (as T_n is a bounded torsion group, the second isomorphism follows from Prüfer's theorem). Therefore, (10) gives

$$G/N_n \cong \mathbb{T}^\lambda \times \prod_{i\in I} \mathbb{Z}(k_i). \tag{11}$$

Denote by h_n the canonical homomorphism $G \to G/N_n$.

In order to prove that G has the $\mathfrak{g}$-closure property, fix an element $x \in G$.

It is not restrictive to assume that $\langle x\rangle$ is infinite and dense in G. Here two cases are possible.

Case 1: *$\langle x\rangle$ non-trivially meets the closed subgroup N_1*. Then N_1 is open in G since $N_1 \cap \langle x\rangle$ has finite index in $\langle x\rangle$ and consequently it is open in $\langle x\rangle$. Since by (10) N_1 is totally disconnected (as the quotient group X/F is torsion), also the group G is totally disconnected. Therefore, Corollary 4.5 applies.

Case 2: *$\langle x\rangle$ trivially meets the subgroup N_1* (*hence, every subgroup N_n*). Pick a $y \in \mathfrak{g}(x)$. Then $h_n(y) \in h_n(\mathfrak{g}(x)) \subseteq \mathfrak{g}_{G/N_n}(h_n(x))$ for every $n \in \mathbb{N}$. Since G/N_n has the $\mathfrak{g}$-closure property by (11), Corollaries 4.7 and 4.6, we conclude that $h_n(y) \in \langle h_n(x)\rangle = h_n(\langle x\rangle)$. Consequently, $y \in \langle x\rangle + N_n$ for every $n \in \mathbb{N}$. By our assumption each one of the sums $\langle x\rangle + N_n = \langle x\rangle \oplus N_n$ is direct. Therefore, $y \in \bigcap_n \langle x\rangle \oplus N_n$ yields $y \in \langle x\rangle \oplus \bigcap_n N_n$. Indeed, let $y = k_n x + a_n$, where $k_n \in \mathbb{Z}$ and $a_n \in N_n$ for every n. Then for $n < m$ one gets $k_n x + a_n = k_m x + a_m$. Since $N_n \geqslant N_m$ and $\langle x\rangle \oplus N_n$ is direct, we conclude, by uniqueness, that $k_n x = k_m x$ and $a_n = a_m$. This proves that $y = kx + a$, where $k \in \mathbb{Z}$ and $a \in \bigcap_n N_n$. On the other hand, since $\bigcup_n F_n = X$, one has $\bigcap_n N_n = 0$ and consequently $y \in \langle x\rangle \oplus \bigcap_n N_n = \langle x\rangle$. $\square$

Here comes the proof of our main result.

Proof of Theorem 1.2. One direction follows immediately by Proposition 2.9 (b). Next, as we have observed, by Proposition 2.9 we can assume that G is a precompact abelian group. Then G embeds as dense topological subgroup of a compact group $\widetilde{G}$. Denote by $j : G \hookrightarrow \widetilde{G}$ this topological embedding. According to Theorem 4.8 $\widetilde{G}$ has the $\mathfrak{g}$-closure property, hence $j(\mathfrak{g}_G(\langle x\rangle)) \subseteq \mathfrak{g}_{\widetilde{G}}(\langle j(x)\rangle) = \langle j(x)\rangle$ for every $x \in G$ by Proposition 2.6. $\square$

Corollary 4.9. *All non-zero cyclic subgroups of a topological abelian group G are $\mathfrak{g}$-closed if and only if G is MAP or G is isomorphic to the Prüfer group $\mathbb{Z}(p^\infty)$ endowed with a Hausdorff group topology τ such that $n(\mathbb{Z}(p^\infty), \tau) = \mathbb{Z}(p)$.*

Proof. Let G be a topological abelian group such that all non-zero cyclic subgroups are $\mathfrak{g}$-closed. Assume that G is not MAP. Then G is infinite and $n(G) \neq 0$. If $n(G)$ is not of the form $\mathbb{Z}(p)$ for some $p \in \mathbb{P}$, then for every non-zero $x \in n(G)$ such that $\langle x \rangle$ is a proper subgroup of $n(G)$, the cyclic subgroup $\langle x \rangle$ is not $\mathfrak{g}$-closed. Hence we are left with the case $n(G) \cong \mathbb{Z}(p)$ for some $p \in \mathbb{P}$. It suffices to observe that for every non-zero $y \in G$ one has $\langle y \rangle \subseteq n(G) + \langle y \rangle \subseteq \mathfrak{g}_G(y) = \langle y \rangle$. Hence $\langle y \rangle \supseteq n(G) \cong \mathbb{Z}(p)$. Then $G \cong \mathbb{Z}(p^\infty)$.

Conversely, assume that $n(\mathbb{Z}(p^\infty)) = \mathbb{Z}(p)$ so that $\mathbb{Z}(p^\infty)/\mathbb{Z}(p)$ is MAP and let $\phi : \mathbb{Z}(p^\infty) \to \mathbb{Z}(p^\infty)/\mathbb{Z}(p)$ be the canonical homomorphism. Then for every $0 \neq y \in \mathbb{Z}(p^\infty)$, $\langle y \rangle = \phi^{-1}(\phi(\langle y \rangle))$ is $\mathfrak{g}$-closed by Theorem 1.2 and Proposition 2.6. $\square$

Observe that the group topology τ considered in the above corollary can be viewed as the group topology of $\mathbb{Z}(p^\infty)$ such that $\mathbb{Z}(p^\infty)^\sharp = (\mathbb{Z}(p^\infty), T_H)$ where H is a dense subgroup of $p\mathbb{Z}_p$.

5. Open questions

It was proved in [7] (see also [11]) that every countable subgroup of $\mathbb{T}$ is (basic) $\mathfrak{g}$-closed. So it will be natural to extend the question also to compact abelian groups:

Problem 5.1. *Characterize the class $\mathscr{C}$ of those compact abelian groups G such that every countable subgroup of G is $\mathfrak{g}$-closed.*

Clearly, $\mathbb{T} \in \mathscr{C}$. One may start with the larger class $\mathscr{C}'$ of those compact abelian groups G such that every *finitely generated* subgroup of G is $\mathfrak{g}$-closed.

Question 5.2. Does $\mathbb{Z}_p \in \mathscr{C}'$ or $\mathscr{C}$ for some (all) prime(s) p?

Theorem 4.8 shows that every cyclic subgroup of a compact abelian group is $\mathfrak{g}$-closed. On the other hand, [7, Theorem 1] yields that every cyclic subgroup of $\mathbb{T}$ is a basic $\mathfrak{g}$-closed subgroup. We do not know if this is true for *every* compact abelian group.

Problem 5.3. *Characterize the class $\mathscr{C}_b$ of those compact abelian groups G such that every cyclic subgroup of G is a basic $\mathfrak{g}$-closed subgroup of G.*

Notice that by Remarks 2.11 (b) and 2.15 the class $\mathscr{C}_b$ is closed under taking finite product. Since by [7, Theorem 1] $\mathbb{T} \in \mathscr{C}_b$ and by Corollary 4.2 finite products of p-adic integers are in $\mathscr{C}_b$, every compact group of the form $\mathbb{T}^n \times \prod_\pi \mathbb{Z}_p \times B$, where π is a finite subset of $\mathbb{P}$ and B is a bounded group, belongs to $\mathscr{C}_b$.

One can easily prove that $\mathscr{C}_b$ is contained in the class of compact metrizable abelian groups. Moreover, for every compact metrizable abelian group G and every element $x \in G$ such that $\langle x \rangle$ is a basic $\mathfrak{g}$-closed subgroup of $\overline{\langle x \rangle}$, $\langle x \rangle$ is a basic $\mathfrak{g}$-closed subgroup of G.

This immediately yields that:

(a) $\mathbb{Z}_p^{\mathbb{N}} \in \mathscr{C}_b$ since for every $x \in \mathbb{Z}_p^{\mathbb{N}}$ one has $\overline{\langle x \rangle} \cong \mathbb{Z}_p$ and the class $\mathscr{C}_b$ is closed under topological isomorphisms.
(b) Every compact, totally disconnected, topologically p-torsion group G belongs to the class $\mathscr{C}_b$. Indeed, for every $x \in G$ one has $\overline{\langle x \rangle} \cong \mathbb{Z}_p$ or $\overline{\langle x \rangle} \cong \mathbb{Z}(p)$.
(c) $G \in \mathscr{C}_b$ for every compact totally disconnected group $G = \prod_{p \in \mathbb{P}} G_p$ such that $G_p = 0$ for all but finitely many $p \in \mathbb{P}$.

It can be deduced from recent unpublished results of A. Biró that the class $\mathscr{C}_b$ coincides with the class of *all* compact metrizable abelian groups.

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