J. reine angew. Math. **603** (2007), 113—131
DOI 10.1515/CRELLE.2007.013

Journal für die reine und
angewandte Mathematik

Levi umbilical surfaces in complex space

By *Roberto Monti* at Padova and *Daniele Morbidelli* at Bologna

Abstract. We define a complex connection on a real hypersurface of $\mathbb{C}^{n+1}$ which is naturally inherited from the ambient space. Using a system of Codazzi-type equations, we classify connected real hypersurfaces in $\mathbb{C}^{n+1}$, $n \geqq 2$, which are Levi umbilical and have non zero constant Levi curvature. It turns out that such surfaces are contained either in a sphere or in the boundary of a complex tube domain with spherical section.

1. Introduction

Let M be a $(2n+1)$-dimensional real surface embedded in $\mathbb{C}^{n+1}$, denote by h the $\mathbb{C}$-linear extension of the second fundamental form of M and by g be the restriction to the complexified tangent bundle $\mathbb{C}TM$ of the standard hermitian product of $\mathbb{C}^{n+1}$. The surface M is Levi umbilical if $h(Z, \overline{W}) = Hg(Z, \overline{W})$ for some scalar function H (the Levi curvature) and for all holomorphic tangent vector fields Z and W. Levi umbilicality is weaker than Euclidean umbilicality because it contains no information on terms of the form $h(Z, W)$ with holomorphic Z and W. In particular, it is easy to construct Levi umbilical surfaces which are neither spheres nor hyperplanes. Indeed, any surface which is the zero set $F = 0$ of a smooth defining function $F(z, \bar{z}) = |z|^2 + \Phi(z, \bar{z})$, where Φ is any polyharmonic function in $\mathbb{C}^{n+1}$, is Levi umbilical (see Example 2.3).

In view of these examples, a natural question is whether there is any version of the classical Darboux theorem for usual umbilical surfaces. In this paper we classify Levi umbilical surfaces with *constant* non zero Levi curvature. An example of such surfaces are, of course, the spheres $\{z \in \mathbb{C}^{n+1} : |z| = r\}$, $r > 0$. A less trivial example is the boundary of spherical tubes, i.e. surfaces of the form (see Example 2.2)

$$\left\{ z \in \mathbb{C}^{n+1} : \sum_{h=1}^{n+1} (z_h + \bar{z}_h)^2 = r^2 \right\}, \quad r > 0. \tag{1.1}$$

Our main result states that there are no other examples. More precisely, we prove that any $(2n+1)$-dimensional oriented connected surface embedded in $\mathbb{C}^{n+1}$, $n \geqq 2$, which is Levi umbilical and has non zero constant Levi curvature is necessarily contained either in a sphere or, up to complex isometries of $\mathbb{C}^{n+1}$, in a spherical cylinder of the form (1.1). This is proved in Theorem 5.1. It is interesting to observe the appearance of tube domains, which are relevant objects in several complex variables, see [Kr].

This classification follows from the analysis of a system of Codazzi equations for h, where covariant derivatives are computed with respect to a suitable complex connection ∇ on M. Though very natural, this connection and the corresponding Codazzi equations do not seem to be studied in the literature. The main features of ∇ are:

(a) both the holomorphic and the antiholomorphic bundles are parallel;

(b) the restriction g to $\mathbb{C}TM$ of the hermitian product in $\mathbb{C}^{n+1}$ satisfies $\nabla g = 0$.

Briefly, the connection is constructed in the following way. Let v be a real unit normal to M and consider $N = 2^{-1/2}(v - iT)$, the holomorphic unit normal to M. Here, $T = J(v)$ where J is the standard complex structure of $\mathbb{C}^{n+1}$. Then, given a holomorphic tangent vector field Z and a tangent vector U, we define

$$\nabla_U Z = D_U Z - g(D_U Z, \overline{N})N,$$

where D is the standard connection in $\mathbb{C}^{n+1}$. Then, this definition, along with $\nabla T = 0$, is extended to the whole tangent bundle, giving rise to a connection satisfying (a) and (b) (see Section 3).

Properties (a) and (b) are similar to the ones of the Tanaka-Webster connection on strictly pseudoconvex Cauchy-Riemann manifolds (see [T] and [W]). Whereas for this connection the Levi form $-id\vartheta$ associated with a contact form ϑ plays the role of the metric and is required to be parallel, in our case the metric inherited from $\mathbb{C}^{n+1}$ is required to be parallel. See also the discussion in Remark 3.2. This produces a connection which seems to be more suitable for our purposes. A different connection is introduced by Klingenberg in [Kl]. It arises as orthogonal projection of the standard connection in the space and, in general, does not satisfy property (a).

A typical example of Codazzi equation for h, written in components with respect to a holomorphic frame $Z_1, \dots, Z_n$, is (see Remark 4.2)

$$\nabla_\alpha h_{\beta\bar{\gamma}} - \nabla_\beta h_{\alpha\bar{\gamma}} = i h_{\beta\bar{\gamma}} h_{\alpha 0} - i h_{\alpha\bar{\gamma}} h_{\beta 0}, \tag{1.2}$$

where $h_{\alpha\bar{\beta}} = h(Z_\alpha, \overline{Z}_\beta)$ for $\alpha, \beta = 1, \dots, n$ and index 0 refers to T. In Theorem 4.1, we compute the system of equations needed in the classification theorem. In these equations, as in (1.2), there is a non vanishing right-hand side, reflecting both the non vanishing of Tor_∇ and the non vanishing of $g(D_Z N, \overline{N})$.

Concerning the restriction $n \geqq 2$ in the classification theorem, note that for $n = 1$ the umbilicality property is satisfied by any hypersurface of $\mathbb{C}^2$. Moreover, by the existence and regularity results proved by Slodkowski and Tomassini [ST] and Citti, Lanconelli and Montanari [CLM] for the Levi equation, there are smooth graphs in $\mathbb{C}^2$ with prescribed boundary and with constant Levi curvature which do not belong to the classes described above. Then, the natural question is whether a *compact* surface in $\mathbb{C}^2$ having constant Levi curvature is necessarily a sphere. This question has been recently addressed in [HL] by

Hounie and Lanconelli, who give an affirmative answer in the class of Reinhardt domains.

Another result implied by our Codazzi equations is the classification of connected pseudoconvex surfaces with non zero constant Levi curvature and vanishing $h_{\alpha\beta}$ (the symmetric part of the second fundamental form). Up to complex isometry, such surfaces are contained in a sphere or in a spherical cylinder of the form

$$\left\{z \in \mathbb{C}^{n+1} : \sum_{i=m}^{n+1} |z_i|^2 = r^2\right\}, \quad r > 0,\ 1 \leqq m \leqq n.$$

This is established in Theorem 5.2, which improves [Kl], Theorem 5.2, where the result is proved by a global argument under compactness and strict pseudoconvexity assumptions (see Remark 5.3).

The notion of Levi curvature was introduced by Bedford and Gaveau in [BG] and it has been recently generalized by Montanari and Lanconelli in [ML]. There is an increasing interest on problems concerning this curvature, mainly from the point of view of partial differential equations. Other significant references are Citti and Montanari [CM], Huisken and Klingenberg [HK] and Montanari and Lascialfari [MLa]. The tools developed in this work could be useful in the study also of other problems concerning real hypersurfaces in complex space.

Concerning terminology, we call "Levi form" the hermitian map $(Z, W) \mapsto h(Z, \overline{W})$, with holomorphic Z and W. This is justified by the fact that $h(Z, \overline{W})$ coincides with the Levi form associated with a natural pseudohermitian structure (see [W] or [JL] for this notion) inherited by M from the ambient (see the discussion in Section 2).

Notation. Greek indices α, β etc. run from 1 to n, Latin indices h, k run from 1 to $n+1$. We let $\partial_h = \dfrac{\partial}{\partial z_h}$, $\partial_{\bar{h}} = \dfrac{\partial}{\partial \bar{z}_h}$ and $F_h = \partial_h F$. J is the standard complex structure and D is the usual connection in $\mathbb{C}^{n+1}$. The standard hermitian product g in $\mathbb{C}^{n+1}$ is normalized by $g(\partial_h, \partial_{\bar{k}}) = g(\partial_{\bar{h}}, \partial_k) = \frac{1}{2}\delta_{hk}$, $g(\partial_h, \partial_k) = g(\partial_{\bar{h}}, \partial_{\bar{k}}) = 0$, where δ_{hk} is the Kronecker symbol. The metric tensors $g_{\alpha\bar{\beta}}$ and $g^{\alpha\bar{\beta}}$, which are related by $g^{\alpha\bar{\beta}} g_{\gamma\bar{\beta}} = \delta_{\alpha\gamma}$, are used to lower and raise indices, e.g. $h_\alpha{}^\beta = g^{\beta\bar{\gamma}} h_{\alpha\bar{\gamma}}$. If h is symmetric, we write h_α^β instead of $h_\alpha{}^\beta$. We adopt the summation convention. If E is a bundle we denote by $\Gamma(E)$ the sections of E. Finally, $[U, V]$ denotes the Lie bracket of vector fields and $\mathrm{Tor}_\nabla(U, V) = \nabla_U V - \nabla_V U - [U, V]$ is the torsion of the connection ∇.

Acknowledgments. We are indebted to Ermanno Lanconelli and Annamaria Montanari for several fruitful conversations on Levi curvature.

2. Levi form and examples

Let $M \subset \mathbb{C}^{n+1}$ be a real hypersurface oriented by a real unit normal ν. We denote by $\mathscr{H} = T^{1,0}M$ (resp. $\overline{\mathscr{H}} = T^{0,1}M$) the holomorphic (resp. antiholomorphic) tangent bundle of M. We restrict the complex structure J to $\mathscr{H} \oplus \overline{\mathscr{H}}$ and the metric g to $\mathbb{C}TM$. The vector field $T = J(v)$ is tangent to M. Then, the complexified tangent bundle $\mathbb{C}TM$ can be decomposed as a direct sum $\mathscr{H} \oplus \overline{\mathscr{H}} \oplus \mathbb{C}T$ and the decomposition is orthogonal with respect to g. The *holomorphic unit normal* to M is the holomorphic vector field

$$N = \frac{1}{\sqrt{2}}(v - iT). \tag{2.1}$$

Up to orientation, N is defined uniquely on M by $|N| = 1$ and $g(N, U) = 0$ for all $U \in \mathscr{H} \oplus \overline{\mathscr{H}}$. Here and in the following, $|V|^2 = g(V, \overline{V})$. We have the relations

$$T = \frac{i}{\sqrt{2}}(N - \overline{N}), \quad v = \frac{1}{\sqrt{2}}(N + \overline{N}). \tag{2.2}$$

There is a unique real 1-form η on M such that

$$\eta(T) = 1 \quad \text{and} \quad \eta(Z) = 0 \quad \text{for all } Z \in \mathscr{H} \oplus \overline{\mathscr{H}}. \tag{2.3}$$

Precisely, $\eta(Z) = g(Z, T)$ for any $Z \in \mathbb{C}TM$. The *Levi form* on M associated with η is the hermitian form on $\mathscr{H}$ defined by

$$L_\eta(Z, \overline{W}) = \frac{1}{2i} d\eta(Z, \overline{W}), \quad Z, W \in \mathscr{H}. \tag{2.4}$$

Denote by h the $\mathbb{C}$-linear extension to $\mathbb{C}TM \times \mathbb{C}TM$ of the second fundamental form of M. For $Z, W \in \mathbb{C}TM$ let

$$h(Z, W) = g(Z, D_W v). \tag{2.5}$$

Note that $h(Z, W) = h(W, Z)$ and $\overline{h(Z, W)} = h(\overline{Z}, \overline{W})$.

The Levi form associated with η coincides with the hermitian part of the second fundamental form, i.e. $L_\eta(Z, \overline{W}) = h(Z, \overline{W})$ for all $Z, W \in \mathscr{H}$. Indeed, by (2.3) and (2.2),

$$\begin{aligned} d\eta(Z, \overline{W}) &= Z\eta(\overline{W}) - \overline{W}\eta(Z) - \eta([Z, \overline{W}]) = -\eta([Z, \overline{W}]) \\ &= g([\overline{W}, Z], T) = \frac{i}{\sqrt{2}} g(D_{\overline{W}} Z - D_Z \overline{W}, N - \overline{N}). \end{aligned} \tag{2.6}$$

Since $g(D_{\overline{W}} Z, N) = g(D_Z \overline{W}, \overline{N}) = 0$, we find

$$d\eta(Z, \overline{W}) = -\frac{i}{\sqrt{2}}\big(g(D_{\overline{W}} Z, N + \overline{N}) + g(D_Z \overline{W}, N + \overline{N})\big) = -2ig(D_{\overline{W}} Z, v). \tag{2.7}$$

The claim follows.

The *Levi curvature* H of M is the trace of the Levi form. The surface M is *Levi umbilical* if $h(Z, \overline{W}) = Hg(Z, \overline{W})$ for all $Z, W \in \mathscr{H}$. In order to express these definitions in components, fix a frame $Z_1, \dots, Z_n$ of holomorphic tangent vector fields. Let $h_{\alpha\bar\beta} = h(Z_\alpha, Z_{\bar\beta})$ and $g_{\alpha\bar\beta} = g(Z_\alpha, Z_{\bar\beta})$. The Levi curvature of M is

$$H = \frac{1}{n} h_\alpha^\alpha. \tag{2.8}$$

The surface M is Levi umbilical if $h_{\alpha\bar\beta} = H g_{\alpha\bar\beta}$. Observe that the relation between the Levi curvature $H = H_{\mathbb{C}}$ and the standard mean curvature $H_{\mathbb{R}}$ is $(2n+1)H_{\mathbb{R}} = 2nH_{\mathbb{C}} + h(T,T)$.

It is useful to compute the Levi curvature by means of a defining function. Let $M = \{z \in \mathbb{C}^{n+1} : F(z) = 0\}$ for some smooth function $F : \mathbb{C}^{n+1} \to \mathbb{R}$. The holomorphic unit normal is

$$N = \sqrt{2}\frac{F_{\bar h}}{|\partial F|}\partial_h, \quad \text{where } |\partial F|^2 = F_h F_{\bar h}. \tag{2.9}$$

The complex Hessian D^2F induces a hermitian form on holomorphic vector fields of $\mathbb{C}^{n+1}$ by letting $D^2F(U, \overline{V}) = U^h V^{\bar k} F_{h\bar k}$, where $U = U^h \partial_h$ and $\overline{V} = V^{\bar k}\partial_{\bar k}$. As observed in [ML], the Levi form can be written as

$$h(U, \overline{V}) = \frac{1}{2|\partial F|} D^2F(U, \overline{V}), \quad U, V \in \mathscr{H}. \tag{2.10}$$

Moreover, the Levi curvature of M is

$$H = \frac{1}{n|\partial F|}\left(F_{h\bar h} - \frac{F_k F_{\bar h} F_{h\bar k}}{|\partial F|^2}\right). \tag{2.11}$$

We briefly check (2.10). By (2.2) and (2.9), we have

$$h(U, \overline{V}) = g(U, D_{\overline{V}} v) = \frac{1}{\sqrt{2}} g\big(U, D_{\overline{V}}(N + \overline{N})\big) = g\left(U, D_{\overline{V}}\left(\frac{F_h}{|\partial F|}\partial_{\bar h}\right)\right).$$

As $g(F_h \partial_{\bar h}, U) = 0$, we get

$$g\left(U, D_{\overline{V}}\left(\frac{F_h}{|\partial F|}\partial_{\bar h}\right)\right) = \frac{1}{|\partial F|} g\big(U, D_{\overline{V}}(F_h \partial_{\bar h})\big) = \frac{1}{2|\partial F|} D^2F(U, \overline{V}).$$

In order to prove (2.11), assume, for instance, $F_{n+1} \neq 0$ near a point $P \in M$ and consider the local holomorphic frame near P

$$Z_\alpha = \partial_\alpha - \frac{F_\alpha}{F_{n+1}}\partial_{n+1}, \quad \alpha = 1, \dots, n. \tag{2.12}$$

The application of (2.10) to the Z_α's gives

$$h_{\alpha\bar\beta} = \frac{1}{2|\partial F|}\left\{F_{\alpha\bar\beta} - \frac{F_{\bar\beta}}{F_{\overline{n+1}}} F_{\alpha\overline{n+1}} - \frac{F_\alpha}{F_{n+1}} F_{n+1,\bar\beta} + \frac{F_\alpha F_{\bar\beta}}{|F_{n+1}|^2} F_{n+1,\overline{n+1}}\right\}.$$

The metric tensor and its inverse are respectively

$$g_{\alpha\bar{\beta}} = \frac{1}{2}\left(\delta_{\alpha\beta} + \frac{F_\alpha F_{\bar{\beta}}}{|F_{n+1}|^2}\right) \quad \text{and} \quad g^{\alpha\bar{\beta}} = 2\left(\delta_{\alpha\beta} - \frac{F_\beta F_{\bar{\alpha}}}{|\partial F|^2}\right).$$

Then, a short computation gives

$$H = \frac{1}{n} g^{\alpha\bar{\beta}} h_{\alpha\bar{\beta}} = \frac{1}{n|\partial F|}\left(F_{h\bar{h}} - \frac{F_k F_{\bar{h}} F_{h\bar{k}}}{|\partial F|^2}\right).$$

In the next proposition we collect some useful identities.

Proposition 2.1. *Let $M \subset \mathbb{C}^{n+1}$ be an oriented surface with real unit normal v, $T = J(v)$ and holomorphic unit normal N. Then*:

(i) $g(D_Z N, \overline{N}) = g([T, Z], T)$ *for all* $Z \in \Gamma(\mathscr{H})$.

(ii) $g(D_Z N, \overline{N}) = ih(T, Z)$ *for all* $Z \in \Gamma(\mathbb{C}TM)$.

(iii) $g([Z, \overline{W}], T) = -2ih(Z, \overline{W})$ *for all* $Z, W \in \Gamma(\mathscr{H})$.

Proof. Note that $g(D_Z T, T) = 0$, because T is real. Moreover, by (2.2), we have for any $Z \in \Gamma(\mathscr{H})$

$$g([T, Z], T) = g(D_T Z - D_Z T, T) = \frac{1}{2} g(D_{N-\overline{N}} Z, \overline{N}).$$

We used the orthogonality $g(D_{N-\overline{N}} Z, \overline{N}) = 0$, which holds because Z is holomorphic. Thus

$$\begin{aligned} 2g([T, Z], T) &= g(D_Z N + [N, Z], \overline{N}) - g(D_Z \overline{N} + [\overline{N}, Z], \overline{N}) \\ &= g([N - \overline{N}, Z], \overline{N}) + g(D_Z N, \overline{N}) \\ &= \frac{1}{i} g([T, Z], v + iT) + g(D_Z N, \overline{N}) = g([T, Z], T) + g(D_Z N, \overline{N}). \end{aligned}$$

We used again (2.2) and $g([T, Z], v) = 0$. This proves (i).

In order to check (ii), note that

$$\begin{aligned} g(D_Z N, \overline{N}) &= g(D_Z N, N + \overline{N}) = \sqrt{2} g(D_Z N, v) = g(D_Z v, v) - ig(D_Z T, v) \\ &= -ig(D_Z T, v) = ih(T, Z). \end{aligned}$$

Identity (iii) is proved in (2.6)–(2.7). □

Now we discuss a couple of examples showing the existence of non trivial Levi umbilical surfaces.

Example 2.2 (Boundary of spherical tubes). The surface $M = \{z \in \mathbb{C}^{n+1} : F(z) = 0\}$, where

$$F(z) = \frac{1}{2}\sum_{h=1}^{n+1}(z_h + \bar{z}_h)^2 - 1,$$

is a Levi umbilical cylinder with spherical section having constant Levi curvature. Indeed, the complex derivatives of F are $F_h = F_{\bar{h}} = z_h + \bar{z}_h$ and $F_{h\bar{k}} = \delta_{hk}$. Then $|\partial F| = \sqrt{2}$ and, by (2.11), the Levi curvature is $H = 1/\sqrt{2}$. The complex Hessian of F is the identity and, by (2.10), the condition $h_{\alpha\bar{\beta}} = \frac{1}{\sqrt{2}} g_{\alpha\bar{\beta}}$ is identically satisfied on M.

Example 2.3. It is possible to construct compact Levi umbilical surfaces by polyharmonic perturbations of the sphere. Consider

$$M = \{z \in \mathbb{C}^{n+1} : |z|^2 + \lambda\Phi(z) = 1\}, \tag{2.13}$$

where λ is a real parameter and

$$\Phi(z) = \frac{1}{2}\sum_{h=1}^{n+1}(z_h^2 + \bar{z}_h^2).$$

The derivative of the defining function $F(z) = |z|^2 + \lambda\Phi(z) - 1$ are $F_h = \bar{z}_h + \lambda z_h$ and $F_{h\bar{k}} = \delta_{hk}$. On the set M we have $|\partial F(z)|^2 = 2 - (1 - \lambda^2)|z|^2$. Then, $|\partial F|$ is constant on M if and only if $\lambda = 0, 1, -1$. If $|\lambda| < 1$, M is a smooth compact surface bounding the region $\{z \in \mathbb{C}^{n+1} : F(z) < 0\}$. Indeed, M is an ellipsoid: letting $z = x + iy$, we have $F(z) = (1 + \lambda)|x|^2 + (1 - \lambda)|y|^2 - 1$. Moreover, on M

$$|\partial F(z)|^2 = 2 - (1 + \lambda)(1 - 2\lambda|x|^2) \geqq 1 - \lambda > 0.$$

By formula (2.11), the Levi curvature of M is $H = |\partial F|^{-1}$. The complex Hessian of F is the identity and, by (2.10), the surface M is Levi umbilical and

$$h_{\alpha\bar{\beta}} = \frac{1}{|\partial F|} g_{\alpha\bar{\beta}}. \tag{2.14}$$

Many other examples of compact Levi umbilical surfaces can be constructed, taking as Φ in (2.13) any polyharmonic function, i.e. any smooth function satisfying $\Phi_{h\bar{k}} = 0$. In fact, the complex Hessian of the corresponding defining function is the identity. Therefore condition (2.14) is satisfied.

3. The connection and its properties

In this section, we define the covariant derivative ∇ on an oriented, smooth hypersurface $M \subset \mathbb{C}^{n+1}$ starting from the standard connection D in $\mathbb{C}^{n+1}$. A vector field $V \in \Gamma(\mathbb{C}TM)$ can be uniquely decomposed as

$$V = Z + \overline{W} + fT, \tag{3.1}$$

where $Z, W \in \Gamma(\mathscr{H})$ and $f \in C^\infty(M)$ is a complex valued function. We define $\nabla : \Gamma(\mathbb{C}TM) \times \Gamma(\mathbb{C}TM) \to \Gamma(\mathbb{C}TM)$ by letting, for $U, V \in \Gamma(\mathbb{C}TM)$ with V as in (3.1),

$$\nabla_U V = D_U Z - g(D_U Z, \overline{N})N + D_U \overline{W} - g(D_U \overline{W}, N)\overline{N} + (Uf)T. \tag{3.2}$$

Here, N is the holomorphic unit normal. Equivalently, let for $U \in \Gamma(\mathbb{C}TM)$ and $Z, W \in \Gamma(\mathscr{H})$

$$\begin{aligned} \nabla_U Z &= D_U Z - g(D_U Z, \overline{N})N, \\ \nabla_U \overline{W} &= D_U \overline{W} - g(D_U \overline{W}, N)\overline{N}, \\ \nabla_U T &= 0. \end{aligned} \tag{3.3}$$

We have the following

Theorem 3.1. *∇ is a complex connection on M and satisfies the following properties*:

(C1) $\overline{\nabla_U V} = \nabla_{\overline{U}} \overline{V}$ *for all* $U, V \in \Gamma(\mathbb{C}TM)$.

(C2) $\nabla_U\big(J(V)\big) = J(\nabla_U V)$ *for all* $U, V \in \Gamma(\mathbb{C}TM)$.

(C3) *The bundles $\mathscr{H}$ and $\overline{\mathscr{H}}$ are parallel.*

(C4) $\nabla g = 0$.

(C5) $\mathrm{Tor}_\nabla(U, V) = 0$ *for all* $U, V \in \Gamma(\mathscr{H})$.

(C6) $\mathrm{Tor}_\nabla(U, \overline{V}) = -g([U, \overline{V}], T)T$ *for all* $U, V \in \Gamma(\mathscr{H})$.

Proof. Properties (C1), (C2) and the fact that ∇ is a connection are easy and we omit their proof.

Property (C3) amounts to say that the covariant derivative of a holomorphic (resp. antiholomorphic) vector field is still a holomorphic (resp. antiholomorphic) vector field. But this is an immediate consequence of definition (3.2) and of the orthogonal decomposition $T_P^{1,0}\mathbb{C}^{n+1} = \mathscr{H}_P \oplus \mathbb{C}N_P$, at any point $P \in M$.

In order to prove property (C4), let

$$V_1 = Z_1 + \overline{W}_1 + f_1 T, \quad V_2 = Z_2 + \overline{W}_2 + f_2 T,$$

where $Z_1, Z_2, W_1, W_2 \in \Gamma(\mathscr{H})$ and f_1, f_2 are complex valued functions. By the metric property of the standard connection D in $\mathbb{C}^{n+1}$, we have

$$\begin{aligned} Ug(V_1, \overline{V}_2) &= Ug(Z_1, \overline{Z}_2) + Ug(\overline{W}_1, W_2) + Ug(f_1 T, \bar{f}_2 T) \\ &= g(D_U Z_1, \overline{Z}_2) + g(Z_1, D_U \overline{Z}_2) + g(D_U \overline{W}_1, W_2) \\ &\quad + g(\overline{W}_1, D_U W_2) + g\big(D_U(f_1 T), \bar{f}_2 T\big) + g\big(f_1 T, D_U(\bar{f}_2 T)\big). \end{aligned} \tag{3.4}$$

We claim that the following identities hold:

$$(3.5)\qquad \begin{aligned} g(D_U Z_1, \overline{Z}_2) &= g(\nabla_U Z_1, \overline{V}_2), \quad g(D_U \overline{W}_1, W_2) = g(\nabla_U \overline{W}_1, \overline{V}_2),\\ g(Z_1, D_U \overline{Z}_2) &= g(V_1, \nabla_U \overline{Z}_2), \quad g(\overline{W}_1, D_U W_2) = g(V_1, \nabla_U W_2). \end{aligned}$$

We check the first one only. Since $g(N, \overline{Z}_2) = 0$, we have $g(D_U Z_1, \overline{Z}_2) = g(\nabla_U Z_1, \overline{Z}_2)$ and property (C3) gives $g(\nabla_U Z_1, \overline{Z}_2) = g(\nabla_U Z_1, \overline{V}_2)$. The following identities also hold:

$$(3.6)\qquad g\big(D_U(f_1 T), \bar{f}_2 T\big) = g\big((Uf_1)T, \overline{V}_2\big), \quad g\big(f_1 T, D_U(\bar{f}_2 T)\big) = g\big(V_1, (U\bar{f}_2)T\big).$$

We check the first one. Since $g(D_U T, T) = 0$, then

$$g\big(D_U(f_1 T), \bar{f}_2 T\big) = g\big((Uf_1)T, \bar{f}_2 T\big) + g(f_1 D_U T, \bar{f}_2 T) = g\big((Uf_1)T, \bar{f}_2 T\big).$$

But T is orthogonal to Z_2 and $\overline{W}_2$. Thus we get the claim. Replacing (3.5) and (3.6) into (3.4) we get $Ug(V_1, \overline{V}_2) = g(\nabla_U V_1, \overline{V}_2) + g(V_1, \nabla_U \overline{V}_2)$, which means $\nabla g = 0$.

Statement (C5), $\mathrm{Tor}_\nabla(U, V) = 0$ for $U, V \in \Gamma(\mathscr{H})$, follows from $\mathrm{Tor}_D(U, V) = 0$ and $[U, V] \in \Gamma(\mathscr{H})$. Concerning property (C6), observe that a connection leaving $\mathscr{H}$ and $\overline{\mathscr{H}}$ parallel cannot be, in general, torsion free, because the horizontal distribution needs not be integrable (in other words, it may be $[\mathscr{H}, \overline{\mathscr{H}}] \subsetneqq \mathscr{H} \oplus \overline{\mathscr{H}}$). Take $W \in \Gamma(\mathscr{H} \oplus \overline{\mathscr{H}})$. Then

$$g\big(\mathrm{Tor}_\nabla(U, \overline{V}), \overline{W}\big) = g\big(\mathrm{Tor}_D(U, \overline{V}) + g(D_{\overline{V}} U, \overline{N})N - g(D_U \overline{V}, N)\overline{N}, \overline{W}\big) = 0,$$

because $\mathrm{Tor}_D(U, \overline{V}) = 0$ and $g(N, \overline{W}) = g(\overline{N}, \overline{W}) = 0$. Then $\mathrm{Tor}_\nabla(U, \overline{V}) = \lambda T$ for some function λ and, by (C3), $\lambda = g\big(\mathrm{Tor}_\nabla(U, \overline{V}), T\big) = -g([U, \overline{V}], T)$. □

Recall that the restriction of the hermitian product in $\mathbb{C}^{n+1}$ to $\mathbb{C}TM$ induces the orthogonal decomposition

$$(3.7)\qquad \mathbb{C}TM = \mathscr{H} \oplus \overline{\mathscr{H}} \oplus \mathbb{C}T.$$

Denote by $\Pi_{\mathscr{H}} : \mathbb{C}TM \to \mathscr{H}$ the projection onto $\mathscr{H}$ and by $\Pi_{\overline{\mathscr{H}}}$ the projection onto $\overline{\mathscr{H}}$. Then it is easy to check that for $U, V \in \Gamma(\mathscr{H})$, we have

$$(3.8)\qquad \nabla_{\overline{U}} V = \Pi_{\mathscr{H}}([\overline{U}, V]) \quad \text{and} \quad \nabla_U \overline{V} = \Pi_{\overline{\mathscr{H}}}([U, \overline{V}]).$$

This follows from (C3) and (C6).

Remark 3.2. If M is a strictly pseudoconvex CR manifold, there is a natural connection associated with a given contact form ϑ, which was introduced by Tanaka and Webster in [T] and [W]. Although it was designed for different scopes from ours, we highlight some analogies and differences between our connection ∇ and the Tanaka-Webster one.

The Levi form $(Z, W) \mapsto -\mathrm{i}d\vartheta(Z, \overline{W})$ is a non degenerate hermitian form on $\mathscr{H}$. Then, ϑ induces a decomposition of $\mathbb{C}TM$ similar to (3.7). The vector field T is replaced in this construction by the *characteristic vector field* T', defined by $\vartheta(T') = 1$ and $d\vartheta(T, Z) = 0$ for all $Z \in \mathscr{H}$. In the Tanaka-Webster connection, the Levi form $d\vartheta$ essentially plays the role of the metric and is required to be parallel. Covariant derivatives of holomorphic vector fields along antiholomorphic ones are defined by relations analogous to (3.8) (see [T], Lemma 3.2, p. 31), but with the projections $\Pi'_{\mathscr{H}}$ and $\Pi'_{\overline{\mathscr{H}}}$ induced by $d\vartheta$. The characteristic vector field T' of ϑ is in general different from T for any choice of the contact form ϑ.

Similarly to the Tanaka-Webster connection, the property $\nabla T = 0$ is forced by (C4) and (C5). Indeed, the one dimensional bundle generated by T is the orthogonal complement with respect to the parallel metric g of the parallel bundle $\mathscr{H} \oplus \mathscr{H}$. Then $\nabla_U T = \lambda T$ for some function λ and $U \in \mathbb{C}TM$. But, since T is real, $0 = Ug(T,T) = 2g(\nabla_U T, T) = 2\lambda$. Therefore $\nabla T = 0$.

Remark 3.3. The connection ∇ is not uniquely determined on the whole tangent bundle $\Gamma(\mathbb{C}TM)$ by properties (C1)–(C6). In particular, $\nabla_T U$ with $U \in \Gamma(\mathscr{H})$ is not uniquely determined. In (3.3), we let $\nabla_T U = D_T U - g(D_T U, \overline{N})N$. An alternative possibility, consistent with (3.8), is to set

$$\nabla'_T U = \Pi_{\mathscr{H}}([T,U]) \quad \text{and} \quad \nabla'_T \overline{U} = \Pi_{\overline{\mathscr{H}}}([T,\overline{U}]).$$

The resulting connection ∇' still satisfies (C1)–(C6). Our choice ∇, however, seems to be more suitable than ∇' to work with Codazzi equations.

Remark 3.4. The real tangent bundle has the orthogonal decomposition $TM = \mathrm{Re}(\mathscr{H} \oplus \overline{\mathscr{H}}) \oplus \mathbb{R}T$. Then, for $Y \in \Gamma\big(\mathrm{Re}(\mathscr{H} \oplus \overline{\mathscr{H}})\big)$, $V \in \Gamma(TM)$ and f real function, we have

$$\nabla_V(Y + fT) = D_V Y - g(D_V Y, v)v - g(D_V Y, T)T + (Vf)T. \tag{3.9}$$

Indeed, taking $X = Z + \overline{Z}$ with holomorphic Z, we have

$$\begin{aligned}
\nabla_V(Z + \overline{Z}) &= \nabla_V Z + \nabla_V \overline{Z} = D_V Z - g(D_V Z, \overline{N})N + D_V \overline{Z} - g(D_V \overline{Z}, N)\overline{N} \\
&= D_V X - g(D_V Z, v)(v - iT) - g(D_V \overline{Z}, v)(v + iT) \\
&= D_V X - g(D_V X, v)v + ig\big(D_V(Z - \overline{Z}), v\big)T \\
&= D_V X - g(D_V X, v)v + g\big(D_V\big(J(X)\big), -J(T)\big)T \\
&= D_V X - g(D_V X, v)v - g(D_V X, T)T.
\end{aligned}$$

We used $v = -J(T)$ and the property $D_V \circ J = J \circ D_V$.

4. Codazzi equations

In this section we compute the system of Codazzi equations.

Theorem 4.1. *The Levi form h on a hypersurface $M \subset \mathbb{C}^{n+1}$ satisfies the following Codazzi equations*:

$$\nabla_\beta h_{\alpha\bar{\gamma}} - \nabla_{\bar{\gamma}} h_{\alpha\beta} = i h_{\alpha\beta} h_{\bar{\gamma}0} - i h_{\alpha\bar{\gamma}} h_{\beta 0} - 2i h_{\beta\bar{\gamma}} h_{\alpha 0}, \tag{4.1a}$$

$$\nabla_{\bar\beta} h_{\alpha 0} - \nabla_0 h_{\alpha\bar\beta} = i h_{\alpha\lambda} h_{\bar\beta}^{\lambda} - i h_{\alpha\bar\lambda} h_{\bar\beta}^{\bar\lambda} + i h_{\alpha\bar\beta} h_{00}, \tag{4.1b}$$

$$\nabla_{\beta} h_{\alpha 0} - \nabla_0 h_{\alpha\beta} = i h_{\beta\bar\lambda} h_{\alpha}^{\bar\lambda} - i h_{\beta\lambda} h_{\alpha}^{\lambda} + i h_{\alpha\beta} h_{00} - 2i h_{\alpha 0} h_{\beta 0}, \tag{4.1c}$$

$$\nabla_{\alpha} h_{00} - \nabla_0 h_{\alpha 0} = 2i h_{\alpha}^{\lambda} h_{\lambda 0} - 2i h_{\alpha}^{\bar\lambda} h_{\bar\lambda 0} - i h_{\alpha\lambda} h_{0}^{\lambda} + i h_{\alpha\bar\lambda} h_{0}^{\bar\lambda} - i h_{\alpha 0} h_{00}. \tag{4.1d}$$

Proof. The proof relies on the fact that the standard connection D in $\mathbb{C}^{n+1}$ has vanishing curvature. We shall also use several times the formula

$$D_Z U = \nabla_Z U - \sqrt{2} h(U, Z) N, \quad U \in \Gamma(\mathscr{H}),\ Z \in \Gamma(\mathbb{C}TM). \tag{4.2}$$

Let $Z, W \in \Gamma(\mathbb{C}TM)$ and $U \in \Gamma(\mathscr{H})$. Denote by R_D and R_∇ the standard curvature endomorphisms of D and ∇. Using (4.2), we have

$$\begin{aligned} 0 = R_D(Z, W)U &= D_Z D_W U - D_W D_Z U - D_{[Z,W]} U \\ &= D_Z\big(\nabla_W U - \sqrt{2} h(U, W)N\big) - D_W\big(\nabla_Z U - \sqrt{2} h(U, Z)N\big) - D_{[Z,W]} U \\ &= R_\nabla(Z, W)U - \sqrt{2} h(U, W) D_Z N + \sqrt{2} h(U, Z) D_W N \\ &\quad - \sqrt{2}\big(Z h(U, W) - h(\nabla_Z U, W) - W h(U, Z) \\ &\qquad + h(\nabla_W U, Z) - h(U, [Z, W])\big) N. \end{aligned} \tag{4.3}$$

Multiplying by $\overline{N}$ and using $g\big(R_\nabla(Z, W)U, \overline{N}\big) = 0$, we get the equation for h

$$\begin{aligned} \nabla_Z h(U, W) - \nabla_W h(U, Z) &= h\big(U, \mathrm{Tor}_\nabla(W, Z)\big) - h(U, W) g(D_Z N, \overline{N}) \\ &\quad + h(U, Z) g(D_W N, \overline{N}), \end{aligned} \tag{4.4}$$

where $\nabla_Z h(U, W) = Z h(U, W) - h(\nabla_Z U, W) - h(U, \nabla_Z W)$ is the covariant derivative of h. Note that by Proposition 2.1, we have $g(D_Z N, \overline{N}) = i h(Z, T)$ for any $Z \in \Gamma(\mathbb{C}TM)$.

In order to prove (4.1a), take $Z, W, U \in \Gamma(\mathscr{H})$ and write (4.4) with $\overline{W}$ instead of W. By Theorem 3.1, the torsion satisfies $\mathrm{Tor}_\nabla(Z, \overline{W}) = -g([Z, \overline{W}], T)T$. Moreover, by Proposition 2.1 we have $g([Z, \overline{W}], T) = -2i h(Z, \overline{W})$. Thus, equation (4.4) becomes

$$\begin{aligned} \nabla_Z h(U, \overline{W}) - \nabla_{\overline{W}} h(U, Z) &= i h(U, Z) h(\overline{W}, T) - i h(U, \overline{W}) h(Z, T) \\ &\quad - 2i h(Z, \overline{W}) h(U, T). \end{aligned} \tag{4.5}$$

This is formula (4.1a).

In order to prove (4.1b), we take $Z, U \in \Gamma(\mathscr{H})$. By (4.4), we have

$$\begin{aligned} \nabla_{\bar Z} h(U, T) - \nabla_T h(U, \bar Z) &= h(U, \nabla_T \bar Z + [\bar Z, T]) - i h(U, T) h(\bar Z, T) \\ &\quad + i h(U, \bar Z) h(T, T), \end{aligned} \tag{4.6}$$

because $\nabla T = 0$.

We analyze the right-hand side of (4.6). By $\mathrm{Tor}_D(\bar Z, T) = 0$ and the second equation of (3.3), we have

$$\begin{aligned}(4.7)\qquad \nabla_T\overline{Z}+[\overline{Z},T] &= D_{\overline{Z}}T-g(D_T\overline{Z},N)\overline{N}\\ &= D_{\overline{Z}}T-g(D_{\overline{Z}}T,v)v-ig(D_T\overline{Z},v)T.\end{aligned}$$

We also used $g([\overline{Z},T],v)=0$, which implies $g(D_{\overline{Z}}T,v)=g(D_T\overline{Z},v)$. The vector field $V=D_{\overline{Z}}T-g(D_{\overline{Z}}T,v)v$ is tangent to M and $g(V,T)=0$. Therefore, for any holomorphic frame $Z_1,\dots,Z_n$, we have

$$\begin{aligned}(4.8)\qquad V &= g^{\lambda\bar{\mu}}g(D_{\overline{Z}}T,Z_{\bar{\mu}})Z_\lambda+g^{\mu\bar{\lambda}}g(D_{\overline{Z}}T,Z_\mu)Z_{\bar{\lambda}}\\ &= ig^{\lambda\bar{\mu}}g(D_{\overline{Z}}v,Z_{\bar{\mu}})Z_\lambda-ig^{\mu\bar{\lambda}}g(D_{\overline{Z}}v,Z_\mu)Z_{\bar{\lambda}}\\ &= ig^{\lambda\bar{\mu}}h(Z_{\bar{\mu}},\overline{Z})Z_\lambda-ig^{\mu\bar{\lambda}}h(Z_\mu,\overline{Z})Z_{\bar{\lambda}}.\end{aligned}$$

In order to get the second equality in (4.8), we used the isometry J and the relations $T=J(v)$, $J(D_{\overline{Z}}T)=D_{\overline{Z}}\big(J(T)\big)$, $J(Z_\mu)=iZ_\mu$ and $J(Z_{\bar{\mu}})=-iZ_{\bar{\mu}}$. Thus, (4.7)–(4.8) give

$$\begin{aligned}(4.9)\qquad h(U,\nabla_T\overline{Z}+[\overline{Z},T]) &= ih(T,\overline{Z})h(U,T)+ig^{\lambda\bar{\mu}}h(Z_{\bar{\mu}},\overline{Z})h(U,Z_\lambda)\\ &\quad -ig^{\mu\bar{\lambda}}h(Z_\mu,\overline{Z})h(U,Z_{\bar{\lambda}}).\end{aligned}$$

Replacing (4.9) into (4.6), we finally find

$$\begin{aligned}\nabla_{\overline{Z}}h(U,T)-\nabla_Th(U,\overline{Z}) &= ih(U,\overline{Z})h(T,T)+ig^{\lambda\bar{\mu}}h(Z_{\bar{\mu}},\overline{Z})h(U,Z_\lambda)\\ &\quad -ig^{\mu\bar{\lambda}}h(Z_\mu,\overline{Z})h(U,Z_{\bar{\lambda}}),\end{aligned}$$

which is identity (4.1b).

In order to prove (4.1c), take $Z,U\in\Gamma(\mathscr{H})$. By (4.4), we have

$$\begin{aligned}(4.10)\qquad \nabla_Zh(U,T)-\nabla_Th(U,Z) &= ih(U,Z)h(T,T)-ih(U,T)h(Z,T)\\ &\quad +h(U,\nabla_TZ+[Z,T]).\end{aligned}$$

On conjugating (4.7), we find $\nabla_TZ+[Z,T]=\overline{V}-ih(Z,T)T$, where the vector field $\overline{V}=D_ZT-g(D_ZT,v)v$ is, by (4.8),

$$\overline{V}=ig^{\lambda\bar{\mu}}h(Z,Z_{\bar{\mu}})Z_\lambda-ig^{\mu\bar{\lambda}}h(Z,Z_\mu)Z_{\bar{\lambda}}.$$

Thus, equation (4.10) reads

$$\begin{aligned}\nabla_Zh(U,T)-\nabla_Th(U,Z) &= ih(U,Z)h(T,T)-2ih(U,T)h(Z,T)\\ &\quad +ig^{\lambda\bar{\mu}}h(U,Z_\lambda)h(Z,Z_{\bar{\mu}})-ig^{\mu\bar{\lambda}}h(Z,Z_\mu)h(U,Z_{\bar{\lambda}}).\end{aligned}$$

The proof of identity (4.1c) is accomplished.

In order to prove (4.1d), take $Z\in\Gamma(\mathscr{H})$ and start from the identity

$$D_T D_Z T - D_Z D_T T - D_{[T,Z]} T = 0. \tag{4.11}$$

Observe that $D_Z T = U - h(Z,T)v$ for some $U \in \Gamma(\mathscr{H} \oplus \overline{\mathscr{H}})$, because $g(D_Z T, T) = 0$. Precisely, as in (4.8), we have

$$U = ig^{\lambda\bar{\mu}} h(Z, Z_{\bar{\mu}}) Z_\lambda - ig^{\mu\bar{\lambda}} h(Z, Z_\mu) Z_{\bar{\lambda}}. \tag{4.12}$$

Then $D_T D_Z T = D_T U - Th(Z,T)v - h(Z,T) D_T v$, and multiplying by v,

$$g(D_T D_Z T, v) = -h(U,T) - Th(Z,T), \tag{4.13}$$

because $g(D_T v, v) = 0$.

We analyze the second term in the left-hand side of (4.11). A computation similar to (4.8) furnishes

$$D_T T - g(D_T T, v)v = ig^{\lambda\bar{\mu}} h(Z_{\bar{\mu}}, T) Z_\lambda - ig^{\mu\bar{\lambda}} h(Z_\mu, T) Z_{\bar{\lambda}} = W, \tag{4.14}$$

where $W \in \Gamma(\mathscr{H} \oplus \overline{\mathscr{H}})$ is defined by the last equality. Thus,

$$g(D_Z D_T T, v) = -h(Z, W) - Zh(T,T). \tag{4.15}$$

Finally, we study the third term in the left-hand side of (4.11). We have

$$\begin{aligned} [T,Z] &= D_T Z - D_Z T = \nabla_T Z + g(D_T Z, \overline{N}) N - D_Z T \\ &= \nabla_T Z - h(Z,T)(v - iT) - D_Z T = \nabla_T Z - U + ih(Z,T)T, \end{aligned}$$

where U is defined after (4.11). This yields

$$\begin{aligned} g(D_{[T,Z]} T, v) &= -h([T,Z], T) \\ &= -h(\nabla_T Z, T) + h(U,T) - ih(Z,T)h(T,T). \end{aligned} \tag{4.16}$$

Multiplying (4.11) by v and using (4.13), (4.15) and (4.16), we obtain

$$\nabla_Z h(T,T) - \nabla_T h(Z,T) = 2h(U,T) - h(Z,W) - ih(Z,T)h(T,T).$$

Replacing the expressions for U and W in (4.12) and (4.14), we get formula (4.1d). □

Remark 4.2. The second fundamental form h satisfies also other Codazzi equations. For instance, we have

$$\nabla_\alpha h_{\beta\bar{\gamma}} - \nabla_\beta h_{\alpha\bar{\gamma}} = ih_{\beta\bar{\gamma}} h_{\alpha 0} - ih_{\alpha\bar{\gamma}} h_{\beta 0}, \tag{4.17a}$$

$$\nabla_\alpha h_{\beta\gamma} - \nabla_\gamma h_{\beta\alpha} = ih_{\beta\alpha} h_{\gamma 0} - ih_{\beta\gamma} h_{\alpha 0}. \tag{4.17b}$$

Identity (4.17a) can be obtained interchanging α and β in identity (4.1a) and taking the difference of the two equations. Identity (4.17b) follows from (4.4) on choosing $Z, U, W \in \Gamma(\mathscr{H})$ and using $\mathrm{Tor}_\nabla(W, Z) = 0$.

Notice also that, letting $Z, U, V, W \in \Gamma(\mathscr{H})$ and multiplying identity (4.3) by V, we get the Gauss-type equation

$$g\big(R_\nabla(Z, \overline{W})U, \overline{V}\big) = 2\{h(U, \overline{W})h(\overline{V}, Z) - h(U, Z)h(\overline{V}, \overline{W})\}.$$

5. Classification results

In this section we prove the following results:

Theorem 5.1. *Let $M \subset \mathbb{C}^{n+1}$, $n \geqq 2$, be a $(2n+1)$-dimensional, connected Levi umbilical surface with constant Levi curvature $H \neq 0$. Then M is contained either in a sphere or in the boundary of a spherical tube.*

Theorem 5.2. *Let M be a connected pseudovonvex hypersurface in $\mathbb{C}^{n+1}$, $n \geqq 1$, with constant Levi curvature $H \neq 0$ and $h_{\alpha\beta} = 0$. Then, up to a complex isometry, M is contained in a sphere or in a cylinder of the form*

$$\left\{z \in \mathbb{C}^{n+1} : \sum_{i=m}^{n+1} |z_i|^2 = r^2\right\}, \quad r > 0,\ 1 \leqq m \leqq n. \tag{5.1}$$

Remark 5.3. The only compact surface among the ones defined in (5.1) is the sphere. Theorem 5.2 improves [Kl], Theorem 5.2, because we assume neither compactness nor strict pseudoconvexity of M.

A slight modification of the argument also shows that if strict pseudoconvexity (but not compactness) is added as hypothesis in Theorem 5.2, then the surface M must be contained in a sphere.

Proof of Theorem 5.1. Possibly changing the orientation of M, assume $H > 0$. Observe preliminarily that, given an orthonormal frame Z_α, by Proposition 2.1 part (iii), we have

$$\sum_{\alpha=1}^{n} g([Z_\alpha, Z_{\bar\alpha}], T) = -2inH \neq 0, \tag{5.2}$$

provided that $H \neq 0$. Then at least one term in the sum is non zero and the distribution $\operatorname{Re}(\mathscr{H} \oplus \overline{\mathscr{H}})$ is bracket generating.

We accomplish the proof in several steps.

Step 1. We claim that

$$h_{\alpha 0} = 0. \tag{5.3}$$

Indeed, contracting the indices α and $\bar\gamma$ in the Codazzi equation (4.17a), we get

$$\nabla_\alpha h_\beta^\alpha - \nabla_\beta h_\alpha^\alpha = i h_\beta^\alpha h_{\alpha 0} - i h_\alpha^\alpha h_{\beta 0}. \tag{5.4}$$

The fundamental form satisfies $h_{\alpha\bar{\beta}} = Hg_{\alpha\bar{\beta}}$ and thus $h_\alpha^\beta = H\delta_\alpha^\beta$ and $h_\alpha^\alpha = nH$. Then the left-hand side in (5.4) vanishes. Therefore $(n-1)Hh_{\beta 0} = 0$. The claim follows.

As a consequence of (5.3), it turns out that h satisfies the identities

(5.5a) $$h_\alpha^{\bar{\beta}} h_{\bar{\beta}}^\mu = (H^2 - h_{00}H)\delta_\alpha^\mu,$$

(5.5b) $$\nabla_0 h_{\alpha\beta} + i h_{00} h_{\alpha\beta} = 0,$$

(5.5c) $$\nabla_\alpha h_{00} = 0.$$

To show (5.5a), observe that, since (5.3) holds and $h_{\alpha\bar{\beta}} = Hg_{\alpha\bar{\beta}}$, the left-hand side of identity (4.1b) vanishes. Thus, using again $h_{\alpha\bar{\beta}} = Hg_{\alpha\bar{\beta}}$ in the right-hand side, we find the equation

$$h_{\alpha\lambda} h_{\bar{\beta}}^\lambda = (H^2 - h_{00}H)g_{\alpha\bar{\beta}}.$$

Contracting with $g^{\mu\bar{\beta}}$ yields (5.5a). Equations (5.5b) and (5.5c) follow from (4.1c) and (4.1d), letting $h_{\alpha 0} = 0$ and $h_{\alpha\bar{\beta}} = Hg_{\alpha\bar{\beta}}$.

Notice also that equation (5.5a) gives

(5.6) $$|h_{\alpha\beta}|^2 := h_\alpha^{\bar{\beta}} h_{\bar{\beta}}^\alpha = nH(H - h_{00}),$$

which implies $h_{00} \leqq H$. Moreover, equation (5.5c) and $\nabla T = 0$ give

$$Zh(T,T) = \nabla_Z h(T,T) = 0 \quad \text{on } M \text{ for any } Z \in \mathscr{H}.$$

On conjugating, the equation is satisfied also for all $Z \in \overline{\mathscr{H}}$. Since M is connected, from (5.2) it follows that

(5.7) $$h(T,T) = \text{constant} = h_{00} \quad \text{on } M.$$

Take $P \in M$ and denote by L the shape operator, $L(X) = D_X \nu$, $X \in T_P M$.

Step 2. If $X \in T_P M$ is an eigenvector of L with $|X| = 1$ and $g(X,T) = 0$, then $Y = J(X)$ is an eigenvector of L with $|Y| = 1$.

Indeed, assume that $L(X) = \lambda X$ for some $\lambda \in \mathbb{R}$ and let

$$Z = X - iJ(X) = X - iY \in \mathscr{H}_P.$$

By (5.3), since $L(X)$ is orthogonal to T,

$$0 = h(Z,T) = g\big(L(X) - iL(Y), T\big) = -ig\big(L(Y), T\big).$$

Therefore $L(Y)$ is orthogonal to T. Moreover, by the symmetry of L, $g\big(L(Y), X\big) = g\big(L(X), Y\big) = \lambda g(X,Y) = \lambda g\big(X, J(X)\big) = 0$. Finally, if $W \in \mathscr{H}_P$ satisfies $g(Z, \overline{W}) = 0$, it must be also $g(X, \overline{W}) = 0$ and thus $g\big(L(X), \overline{W}\big) = \lambda g(X, \overline{W}) = 0$. Since M is Levi umbilical, we also have $g\big(L(Z), \overline{W}\big) = Hg(Z, \overline{W})$. Eventually, we get

$$g(L(Y), \overline{W}) = ig(L(X) - iL(Y), \overline{W}) = ig(L(Z), \overline{W}) = iHg(Z, \overline{W}) = 0.$$

Taking the conjugate we also find $g(L(Y), W) = 0$. Ultimately, we showed that $L(Y)$ has no component orthogonal to Y and our claim is proved.

Step 3. At any point $P \in M$ there exists an orthonormal basis

$$\{X_\alpha, Y_\alpha = J(X_\alpha), T : \alpha = 1, \dots, n\} \quad \text{of } T_P M$$

such that

$$(5.8) \qquad \begin{aligned} L(X_\alpha) &= (H + \sqrt{H^2 - h_{00}H})X_\alpha, \\ L(Y_\alpha) &= (H - \sqrt{H^2 - h_{00}H})Y_\alpha, \\ L(T) &= h_{00}T. \end{aligned}$$

Note first that, by (5.3), $h(T, X) = 0$ for any $X \in \mathscr{H}_P \oplus \overline{\mathscr{H}}_P$. Then we have $L(T) = h_{00}T$, by (5.7), and the orthogonal complement of T at any point $P \in M$ is an invariant subspace for L. We diagonalize L restricted to this invariant subspace. By Step 1, for any eigenvector X_α with eigenvalue λ_α, there is an eigenvector $Y_\alpha = J(X_\alpha)$ with eigenvalue μ_α. Thus we get an orthonormal basis $\{T, X_\alpha, Y_\alpha, \alpha = 1, \dots, n\}$ of $T_P M$. We may assume $\lambda_\alpha \geqq \mu_\alpha$.

The values of λ_α and μ_α are determined by (5.5a) and by Levi umbilicality. Indeed, letting $Z_\alpha = X_\alpha - iY_\alpha$, we have $g_{\alpha\bar{\beta}} = 2\delta_{\alpha\beta}$. Since M is Levi umbilical,

$$(5.9) \qquad 2H = Hg(Z_\alpha, Z_{\bar{\alpha}}) = g(L(Z_\alpha), Z_{\bar{\alpha}}) = g(\lambda_\alpha X_\alpha - i\mu_\alpha Y_\alpha, X_\alpha + iY_\alpha) = \lambda_\alpha + \mu_\alpha.$$

Moreover, since $h_{\alpha\beta} = g(Z_\alpha, L(Z_\beta)) = (\lambda_\alpha - \mu_\alpha)\delta_{\alpha\beta}$, it is $h_\alpha^{\bar{\beta}} = \frac{1}{2}(\lambda_\alpha - \mu_\alpha)\delta_\alpha^\beta$. Thus,

$$(5.10) \qquad (H^2 - h_{00}H)\delta_\alpha^\gamma = h_\alpha^{\bar{\beta}} h_{\bar{\beta}}^\gamma = \frac{1}{4}(\lambda_\alpha - \mu_\alpha)^2 \delta_\alpha^\lambda.$$

The solutions to equations (5.9) and (5.10) are $\lambda_\alpha = H + \sqrt{H^2 - h_{00}H}$ and $\mu_\alpha = H - \sqrt{H^2 - h_{00}H}$. The proof of Step 3 is concluded.

In Step 3, we established that the principal curvatures of M are the constant numbers (5.8). By a classical result going back to Segre [S], if a connected hypersurface in $\mathbb{R}^{N+1}$ has constant principal curvatures, then it must be a plane, a sphere or a cylinder, i.e. a Cartesian product $\mathbb{S}^p \times \mathbb{R}^{N-p}$, where $\mathbb{S}^p$ is a p-dimensional sphere and $0 \leqq p \leqq N$. In particular, a surface with constant curvatures can have at most two different ones. The numbers in (5.8) are not pairwise different only in the following two cases:

Case A: $h_{00} = H$

and

Case B: $h_{00} = 0$.

In Case A, all the principal curvatures are equal to H and the surface M must be contained in a sphere of radius $1/H$. In Case B the surface must be a cylinder. In the latter case, equations (5.8) become

$$L(X_\alpha) = 2HX_\alpha, \quad L(Y_\alpha) = 0 \quad \text{and} \quad L(T) = 0. \tag{5.11}$$

Fix a point P. After a complex rotation, we may assume that the vectors at P satisfying (5.11) are $X_\alpha = \partial_{x_\alpha}$, $Y_\alpha = \partial_{y_\alpha}$ and $T = \partial_{y_{n+1}}$. This means that

$$\ker(L) = \operatorname{span}\{\partial_{y_h} : h = 1, \ldots, n+1\}.$$

For a cylinder, $\ker(L)$ is the same at any point (after the trivial identification between different tangent spaces of $\mathbb{R}^{2n}$). Moreover, the remaining n principal curvatures are all equal to $2H$. Then the surface is contained in a cylinder of equation

$$\sum_{k=1}^{n+1} (x_k - b_k)^2 = \frac{1}{4H^2},$$

for suitable constants b_k. The proof is concluded. □

Proof of Theorem 5.2. Without loss of generality we can assume $H > 0$.

Step A. First we prove that $h_{\alpha 0} = 0$. Since $h_{\alpha\beta} = 0$, (4.1a) becomes $\nabla_\beta h_{\alpha\bar{\gamma}} + ih_{\alpha\bar{\gamma}}h_{\beta 0} + 2ih_{\alpha 0}h_{\beta\bar{\gamma}} = 0$. Contracting with $g^{\bar{\gamma}\alpha}$ gives $\nabla_\beta h_\alpha^\alpha + ih_\alpha^\alpha h_{\beta 0} + 2ih_{\alpha 0}h_\beta^\alpha = 0$. The Levi curvature is constant and then

$$nHh_{\beta 0} + 2h_{\alpha 0}h_\beta^\alpha = 0. \tag{5.12}$$

Denote by $k_{(\lambda)}$, $\lambda = 1, \ldots, n$, the principal Levi curvatures of M at a point P. This means that there is an orthonormal family of holomorphic vectors $V_{(\lambda)} = V_{(\lambda)}^\beta Z_\beta \in \mathscr{H}_P$, $\lambda = 1, \ldots, n$, such that $h_\beta^\alpha V_{(\lambda)}^\beta = k_{(\lambda)} V_{(\lambda)}^\alpha$. Contracting (5.12) with $V_{(\lambda)}^\beta$ yields

$$(nH + 2k_{(\lambda)})h(T, V_{(\lambda)}) = 0.$$

By pseudoconvexity, it is $k_{(\lambda)} \geqq 0$ for all $\lambda = 1, \ldots, n$. Since $H > 0$, this implies $h(T, V_{(\lambda)}) = 0$ for any $\lambda = 1, \ldots, n$, which ensures $h_{\alpha 0} = 0$.

Inserting $h_{\alpha\beta} = 0$, $h_{\alpha 0} = 0$ and $h_\alpha^\alpha = nH = \text{constant}$ in equations (4.1a), (4.1b) and (4.1d), we find

$$\nabla_\beta h_{\alpha\bar{\gamma}} = 0, \tag{5.13a}$$

$$\nabla_0 h_{\alpha\bar{\beta}} = ih_{\alpha\bar{\lambda}}h_{\bar{\beta}}^{\bar{\lambda}} - ih_{\alpha\bar{\beta}}h_{00}, \tag{5.13b}$$

$$\nabla_\alpha h_{00} = 0. \tag{5.13c}$$

Equation (5.13c) and $\nabla T = 0$ imply that $Zh_{00} = 0$ for all holomorphic Z. Since the horizontal distribution is bracket generating, we conclude that h_{00} is constant on M. Contracting α and $\bar{\beta}$ in (5.13b) and using $H =$constant, we find $h_{\alpha\bar{\lambda}}h^{\alpha\bar{\lambda}} = nHh_{00}$. If $h_{00} = 0$, it follows that $h_{\alpha\bar{\lambda}} = 0$ and thus $H = 0$. This is not possible and h_{00} must be a non zero constant. Since $h_{\alpha 0} = 0$, by Step A we also have $L(T) = h_{00}T$.

Step B. If $X \in T_P M$ is a real tangent vector orthogonal to T and such that $L(X) = \lambda X$, then the vector $Y = J(X)$ satisfies $L(Y) = \mu Y$. This follows from $h_{\alpha\beta} = 0$ and can be proved as in Step 2 of the proof of Theorem 5.1. Moreover, letting $Z = X - iY$ we have

$$0 = h(Z, Z) = g\big(L(Z), Z\big) = g(\lambda X - i\mu Y, X - iY) = \lambda - \mu.$$

Therefore $\lambda = \mu$.

Iterating this process n times, we find an orthonormal basis

$$\{X_\alpha,\ Y_\alpha = J(X_\alpha), T : \alpha = 1, \dots, n\} \quad \text{of } T_P M$$

such that

$$L(X_\alpha) = \lambda_\alpha X_\alpha, \quad L(Y_\alpha) = \lambda_\alpha Y_\alpha. \tag{5.14}$$

Notice that L sends $\mathscr{H}$ into $\mathscr{H}$, because $h_{\alpha\beta} = 0$ and $h_{\alpha 0} = 0$. Moreover, letting $Z_\alpha = X_\alpha - iY_\alpha$ we have $L(Z_\alpha) = \lambda_\alpha Z_\alpha$. The numbers $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the Levi form at the point P, i.e. $h(Z_\alpha, Z_{\bar{\beta}}) = \lambda_\alpha g(Z_\alpha, Z_{\bar{\beta}})$.

Step C. We claim that the eigenvalues of L are constant. First observe that any pair of points in M can be connected by a horizontal path $\gamma : [0, 1] \to M$, i.e. a piecewise C^1 curve such that $g(\dot{\gamma}, T) = 0$. This follows from the rank condition (5.2). Take $P, Q \in M$ and connect them by a horizontal curve γ with $\gamma(0) = P$ and $\gamma(1) = Q$. Let $\{X_\alpha^P,\ Y_\alpha^P = J(X_\alpha^P), T : \alpha = 1, \dots, n\}$ be an orthonormal basis of $T_P M$ satisfying (5.14). Let $Z_\alpha^P = X_\alpha^P - iY_\alpha^P$ and let Z_α be the parallel extension of Z_α^P along γ, that is

$$\nabla_{\dot{\gamma}} Z_\alpha = 0 \quad \text{along } \gamma \quad \text{and} \quad Z_\alpha(P) = Z_\alpha^P. \tag{5.15}$$

The vector field Z_α is holomorphic.

Equation (5.13a) and its conjugate imply

$$\nabla_{\dot{\gamma}} h(Z, \overline{W}) = 0, \quad \text{for all holomorphic } Z, W. \tag{5.16}$$

Then, from (5.16) and (5.15) it follows that

$$\frac{d}{dt} h(Z_\alpha, Z_{\bar{\beta}}) = \nabla_{\dot{\gamma}} h(Z_\alpha, Z_{\bar{\beta}}) + h(\nabla_{\dot{\gamma}} Z_\alpha, Z_{\bar{\beta}}) + h(Z_\alpha, \nabla_{\dot{\gamma}} Z_{\bar{\beta}}) = 0.$$

Thus $h(Z_\alpha, Z_{\bar{\beta}})$ is constant along γ and

$$h(Z_\alpha, Z_{\bar{\beta}}) = h(Z_\alpha^P, Z_{\bar{\beta}}^P) = 2\lambda_\alpha \delta_{\alpha\beta},$$

where the λ_α's are the Levi eigenvalues at P. Since g is parallel, we also have $g(Z_\alpha, Z_{\bar{\beta}}) = g(Z_\alpha^P, Z_{\bar{\beta}}^P) = 2\delta_{\alpha\beta}$. Eventually, we get $h(Z_\alpha, Z_{\bar{\beta}}) = \lambda_\alpha g(Z_\alpha, Z_{\bar{\beta}})$, where the λ_α's are again the eigenvalues at P. This means that also at the point $Q = \gamma(1)$ the eigenvalues of L are $\lambda_1, \lambda_2, \ldots, \lambda_n$ and h_{00}.

Step D. The shape operator L has constant eigenvalues $\lambda_1, \ldots, \lambda_n$, h_{00}. Each eigenvalue λ_α has multiplicity 2 and the corresponding eigenspace is a complex subspace of $\mathbb{C}^{n+1}$. By Segre's theorem on hypersurfaces with constant curvatures, M can have not more than two different constant curvatures and it is contained either in a sphere or in a cylinder with spherical section. We may assume $\lambda_1 = \cdots = \lambda_m = 0$ and $\lambda_{m+1} = \cdots = \lambda_n = h_{00}$ for some $0 \leqq m \leqq n-1$. In case $m = 0$ we have a sphere. In case $1 \leqq m \leqq n-1$ we have a cylinder of the form (5.1). The case $m = n$ is excluded, because we have a cylinder of the form $\mathbb{C}^n \times \mathbb{S}^1$ which has $H = 0$.

The proof is concluded. □

References

[BG] *E. Bedford*, *B. Gaveau*, Hypersurfaces with bounded Levi form, Indiana Univ. Math. J. **27** (1978), 867–873.

[CLM] *G. Citti*, *E. Lanconelli*, *A. Montanari*, Smoothness of Lipschitz-continuous graphs with nonvanishing Levi curvature, Acta Math. **188** (2002), no. 1, 87–128.

[CM] *G. Citti*, *A. Montanari*, Strong solutions for the Levi curvature equation, Adv. Diff. Equ. **5** (2000), no. 1–3, 323–342.

[HL] *J. Hounie*, *E. Lanconelli*, An Alexandrov type theorem for Reinhardt domains in $\mathbb{C}^2$, Contemp. Math. **400** (2006), 129–146.

[HK] *G. Huisken*, *W. Klingenberg*, Flow of real hypersurfaces by the trace of the Levi form, Math. Res. Lett. **6** (1999), no. 5–6, 645–661.

[JL] *D. Jerison*, *J. M. Lee*, The Yamabe problem on CR manifolds, J. Diff. Geom. **25** (1987), no. 2, 167–197.

[Kl] *W. Klingenberg*, Real hypersurfaces in Kähler manifolds, Asian J. Math. **5** (2001), no. 1, 1–17.

[Kr] *S. Krantz*, Several complex variables, 2nd edition, AMS Chelsea Publishing, 2001.

[ML] *A. Montanari*, *E. Lanconelli*, Pseudoconvex fully nonlinear partial differential operators: strong comparison theorem, J. Diff. Equ. **202** (2004), 306–331.

[MLa] *A. Montanari*, *F. Lascialfari*, The Levi Monge-Ampère equation: smooth regularity of strictly Levi convex solutions, J. Geom. Anal. **14** (2004), no. 2, 331–353.

[S] *B. Segre*, B. Famiglie di superficie isoparametriche negli spazi euclidei ad un numero qualunque di dimensioni, Atti Accad. Naz. Lincei Rend. (VI) **27** (1938), 203–207.

[ST] *Z. Slodkowski*, *G. Tomassini*, Geometric properties of solutions of the Levi curvature equation in C^2, J. Funct. Anal. **138** (1996), no. 1, 188–212.

[T] *N. Tanaka*, A differential geometric study of strongly pseudo-convex manifolds, Lectures in Mathematics, Department of Mathematichs, Kyoto University, Kinokuniya Book-store, Co., Ltd. Tokyo, 1975.

[W] *S. Webster*, Pseudohermitian structures on a real hypersurface, J. Diff. Geom. **13** (1978), 25–41.

Dipartimento di Matematica Pura ed Applicata, Università di Padova, Via Belzoni, 7, 35100 Padova, Italy
e-mail: monti@math.unipd.it

Dipartimento di Matematica, Università di Bologna, Piazza di Porta San Donato, 5, 40127 Bologna, Italy
e-mail: morbidel@dm.unibo.it

Eingegangen 7. November 2005