Representations of Additive Categories and Direct-Sum Decompositions of Objects

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ABSTRACT. The aim of this paper is to embed additive categories in which direct-sum decompositions into indecomposables are not unique but have a regular geometric behavior into categories in which the Krull-Schmidt Theorem holds, that is, to give a representation of additive categories into categories with unique direct-sum decompositions into indecomposables. Cf. Theorems 4.8, 6.1, 6.2, 7.2, and 8.2.

1. INTRODUCTION

During the past fifteen years, a number of classes of right R-modules with semilocal endomorphism rings have been discovered [4, 7, 10] (see the last part of this Introduction for the undefined terms). These classes $\mathcal{C}$ are usually closed under isomorphism, finite direct sums and direct summands, so that, viewed as full subcategories of the category Mod-R of all right R-modules, they turn out to be additive categories in which idempotents split. The monoid $V(\mathcal{C})$ of all isomorphism classes of modules in $\mathcal{C}$ is a reduced Krull monoid for these categories $\mathcal{C}$ [5], which implies a geometric regularity of direct-sum decompositions of modules belonging to $\mathcal{C}$. One of the things we do in this paper is to reinterpret the results obtained for right modules in the setting of additive categories in which idempotents split.

Additive categories in which idempotents split yield the right setting in which it is convenient to study problems of direct-sum decompositions. This was first realized by H. Bass [2, p. 20], who showed that the Krull-Schmidt Theorem holds in these categories (Theorem 2.2). Our aim is to characterize the additive categories $\mathcal{A}$ in which idempotents split and for which direct-sum decompositions have a reasonably good behavior via additive functors that enjoy the suitable properties we introduce in Section 3. For instance, in Theorem 4.8, we give a complete description of *weakly direct-summand reflecting functors* $F\colon \mathcal{A} \to \mathcal{B}$, where $\mathcal{A}$ is an additive category in which idempotents split and $\mathcal{B}$ is a directly finite, additive category.

In this paper we consider four cases. The first one is the case of a skeletally small additive category $\mathcal{A}$ in which idempotents split and in which endomorphism rings of objects are semiperfect. This is exactly the case considered by Bass, the case of the classical Krull-Schmidt Theorem. For such a category $\mathcal{A}$, the factor category of $\mathcal{A}$ modulo its Jacobson radical is an amenable semisimple category, cf. Theorem 7.1.

Let $\mathcal{A}$ be an additive category and let $\operatorname{Ob}\mathcal{A}$ be its class of objects. For every object $A \in \operatorname{Ob}\mathcal{A}$, let $\langle A\rangle := \{B \mid B \in \operatorname{Ob}\mathcal{A},\ A \cong B\}$ denote the *isomorphism class* of A. Set $V(\mathcal{A}) := \{\langle A\rangle \mid A \in \operatorname{Ob}\mathcal{A}\}$, so that $V(\mathcal{A})$ is a set if and only if $\mathcal{A}$ is skeletally small, and is a proper class otherwise. Define an addition on the class $V(\mathcal{A})$ by $\langle A\rangle + \langle B\rangle := \langle A \oplus B\rangle$ for every $A_R, B_R \in \operatorname{Ob}\mathcal{A}$. With this operation, the class $V(\mathcal{A})$ becomes an "additive commutative monoid." Here and in the rest of the paper, when we write a term referring to an algebraic structure between inverted commas, we mean that it can be not a set but a proper class. For instance, $V(\mathcal{A})$ is a "monoid" for every additive category $\mathcal{A}$, but it is a monoid if and only if $\mathcal{A}$ is skeletally small. We try to represent the category $\mathcal{A}$ we are studying into categories $\mathcal{B}$ with a better behavior as far as direct-sum decompositions are concerned via suitable functors: isomorphism reflecting functors, direct-summand reflecting functors, weakly direct-summand reflecting functors, and so on (Sections 3 and 4).

The second case we consider is the case of the skeletally small categories $\mathcal{A}$ whose objects decompose uniquely as a direct sum of indecomposables, that is, the case of the categories $\mathcal{A}$ for which $V(\mathcal{A})$ is a free monoid, that is, isomorphic to $\mathbb{N}_0^{(I)}$ for some set I (Sections 5 and 6). Here $\mathbb{N}_0$ denotes the additive monoid of non-negative integers. For example, let R be an arbitrary ring and set $C := \{$all finitely generated free right R-modules$\}$. Recall that a ring R has IBN (*invariant basis number*) if $R_R^n \cong R_R^m$ implies $n = m$. The monoid $V(C)$ is isomorphic to $\mathbb{N}_0$ if and only if R has IBN, otherwise it is isomorphic to a proper factor of $\mathbb{N}_0$.

The third case we consider is that of the categories whose endomorphism rings are semilocal (Sections 7 and 8), and the fourth case is that of the categories $\mathcal{A}$ for which $V(\mathcal{A})$ is a Krull monoid. If the endomorphism rings of all objects of a category $\mathcal{A}$ are semilocal, then $V(\mathcal{A})$ turns out to be a Krull monoid, cf. Theorems 8.2 and 8.3. In this setting a prominent role is played by the following notion of local functor, which is the categorical analogue of the notion of local ring morphism: an additive functor $F\colon \mathcal{A} \to \mathcal{B}$ is called a *local* functor if, for every pair A, A' of objects of $\mathcal{A}$ and every morphism $f\colon A \to A'$, if $F(f)\colon F(A) \to F(A')$ is an isomorphism then f is an isomorphism.

Categories $\mathcal{A}$ with $V(\mathcal{A})$ a Krull monoid are characterized in Theorem 6.1, and categories $\mathcal{A}$ with $V(\mathcal{A})$ a free commutative monoid are characterized in Theorem 6.2. In Theorem 8.2, we give the proper categorical setting to the main result of [5].

Recall that a ring R is *semilocal* if R modulo its Jacobson radical is semisimple artinian. A ring morphism $\varphi\colon R \to S$ is *local* if, for every $x \in R$, $\varphi(x)$ invertible in S implies x invertible in R. A ring R is semilocal if and only if there exists a local morphism of R into a semisimple artinian ring [4].

All the monoids in this paper are commutative and additively written. Let M be a (commutative additive) monoid. Denote by $U(M)$ the subgroup of all elements $a \in M$ with an additive inverse $-a$ in M. A monoid M is *reduced* if $U(M) = \{0\}$. For every monoid M, the factor monoid $M_{\mathrm{red}} := M/U(M)$ is reduced. The monoid $V(\mathcal{A})$ is reduced for every skeletally small additive category $\mathcal{A}$. For any monoid M, there is a natural pre-order $\leq$ on the set M defined by $x \leq y$ if there exists $z \in M$ such that $x + z = y$. It is called the *algebraic pre-order* on M. A monoid homomorphism $f\colon M \to M'$ is called a *divisor homomorphism* if, for every $x, y \in M$, $f(x) \leq f(y)$ implies $x \leq y$. A monoid M is a *Krull monoid* if there exists a divisor homomorphism of M into a free monoid. Equivalently, a monoid M is a Krull monoid if and only if there exists a set $\{v_i \mid i \in I\}$ of monoid homomorphisms $v_i\colon M \to \mathbb{N}_0$ such that:

(1) if $x, y \in M$ and $v_i(x) \leq v_i(y)$ for every $i \in I$, then $x \leq y$;
(2) for every $x \in M$, the set $\{i \in I \mid v_i(x) \neq 0\}$ is finite.

Since we shall mainly deal with reduced Krull monoids, we shall need the following elementary lemma, of which we give a proof for completeness. Recall that a monoid M is *directly finite* if, for every $x, y \in M$, $x = x + y$ implies $y = 0$.

Lemma 1.1. *Let $f\colon M \to F$ be a divisor homomorphism of a monoid M into a free monoid F. The following conditions are equivalent:*

(1) *The monoid M is reduced and cancellative.*
(2) *The monoid M is reduced and directly finite.*
(3) *The monoid M is reduced.*
(4) *The homomorphism f is injective.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are trivial implications.

(3) $\Rightarrow$ (4) Assume M reduced. If $x, y \in M$ and $f(x) = f(y)$, then $x \leq y$, so that there exists $z \in M$ with $x + z = y$. Then $f(x) = f(y) = f(x + z) = f(x) + f(z)$, from which $f(z) = 0$. Thus $z \leq 0$, that is, there exists $t \in M$ with $z + t = 0$. But M is reduced, so that $z = 0$, and $x = y$.

(4) $\Rightarrow$ (1) Every submonoid of a free monoid is reduced and cancellative. ☐

It is easily seen that if C, C' are two classes of R-modules closed under isomorphism and finite direct sums and $V(C)$, $V(C')$ are Krull monoids, then $V(C \cap C')$ also is a Krull monoid. For the undefined terms of Category Theory, we refer the reader to [12].

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2. BASIC NOTIONS AND NOTATION

Let $\mathcal{A}$ be an additive category, that is, a preadditive category with a zero object and finite products, and let $\operatorname{Ob}\mathcal{A}$ be its class of objects. If A, $B \in \operatorname{Ob}\mathcal{A}$, then $A \prod B \cong A \coprod B$ will be denoted $A \oplus B$ and called the *direct sum* of A and B. We give a proof of the following elementary lemma for the sake of completeness.

Lemma 2.1. *The following conditions are equivalent for an additive category $\mathcal{A}$:*

(1) *Idempotents have kernels in $\mathcal{A}$, that is, for every object B, every morphism $e\colon B \to B$ with $e^2 = e$ has a kernel.*
(2) *Idempotents split in $\mathcal{A}$, that is, for every object B and every idempotent $e\colon B \to B$ in $\mathcal{A}$ there exist an object A and morphisms $f\colon A \to B$ and $g\colon B \to A$ such that $e = fg$ and $gf = 1_A$.*

If these equivalent conditions hold and $f\colon A \to B$ and $g\colon B \to A$ are morphisms such that $gf = 1_A$, then $fg\colon B \to B$ is an idempotent whose kernel $k\colon K \to B$ is also a kernel for g, and $(f,k)\colon A \oplus K \to B$ is an isomorphism.

Proof. (1) $\Rightarrow$ (2). If $e\colon B \to B$ is an idempotent in $\mathcal{A}$ and $f\colon A \to B$ is a kernel of the idempotent $1_B - e$, then $(1_B - e)e = 0$ implies that there exists a unique morphism $g\colon B \to A$ such that $e = fg$. From $(1_B - e)f = 0$ it follows that $f = ef = fgf$. As kernels are monomorphisms, we obtain that $1_A = gf$.

(2) $\Rightarrow$ (1). Assume that (2) holds. Let $e\colon B \to B$ be an idempotent. Let $f\colon A \to B$ and $g\colon B \to A$ be such that $fg = 1_B - e$ and $gf = 1_A$. Then f is a kernel of e, because $ef = (1_B - fg)f = f - fgf = 0$ and, for every $t\colon D \to B$ such that $et = 0$, one has that $f(gt) = (1_B - e)t = t$. If t' is another morphism with $ft' = t$, then $t' = gft' = gt$. Thus (1) holds.

Assume now that (1) holds. Let $f\colon A \to B$ and $g\colon B \to A$ be morphisms such that $gf = 1_A$, so that fg is idempotent. Let $k\colon K \to B$ be a kernel of fg. Then $fgk = 0$, so that $gk = gfgk = 0$. In order to prove that k is also a kernel for g, let k' be a morphism such that $gk' = 0$. Then $fgk' = 0$, so that there is a unique morphism k'' with $kk'' = k'$. Hence k is also a kernel for g.

Consider the morphism $(f,k)\colon A \oplus K \to B$. Let ι_A, ι_K, π_A, π_K be such that $\pi_A\iota_A = 1_A$, $\pi_K\iota_K = 1_K$, $\pi_K\iota_A = 0$, $\pi_A\iota_K = 0$, and $\iota_A\pi_A + \iota_K\pi_K$ is the identity of $A \oplus K$. As $g(1_B - fg) = 0$, there exists a unique $h\colon B \to K$ such that $1_B - fg = kh$. We will check that $\iota_A g + \iota_K h\colon B \to A \oplus K$ is an inverse of $(f,k) = f\pi_A + k\pi_K$.

Let us compute $\beta := (\iota_A g + \iota_K h)(f\pi_A + k\pi_K)$. One has $\beta = \iota_A gf\pi_A + \iota_A gk\pi_K + \iota_K hf\pi_A + \iota_K hk\pi_K = \iota_A\pi_A + \iota_K hf\pi_A + \iota_K hk\pi_K$. Now $khf = (1_B - fg)f = 0$. As the kernel k is necessarily a monomorphism, it follows that $hf = 0$. Thus $\beta = \iota_A\pi_A + \iota_K hk\pi_K$. Moreover, $khk = (1_B - fg)k = k$, so that $hk = 1_K$ because k is a monomorphism. Therefore $\beta = 1$.

Finally, $(f\pi_A + k\pi_K)(\iota_A g + \iota_K h) = fg + kh = 1_B$. This proves that $\iota_A g + \iota_K h$ is an inverse of (f,k). ◻

Categories satisfying the equivalent conditions of Lemma 2.1 are sometimes called *idempotent complete* [11, p. 11], or *amenable* (Freyd), or categories *with split idempotents*. Notice that if $\mathcal{A}$ is an additive category, then any subclass of $\operatorname{Ob}\mathcal{A}$ closed under finite direct sums is the class of objects of an additive full subcategory of $\mathcal{A}$. Here and in the rest of the paper, a class C of objects is said to be closed under finite sums if it contains the zero object and $A, B \in C$ implies $A \oplus B \in C$. If $\mathcal{A}$ is an additive category in which idempotents split and S is a subclass of $\operatorname{Ob}\mathcal{A}$ closed under finite direct sums, then the additive full subcategory of $\mathcal{A}$ whose class of objects is S is a category in which idempotents split if and only if S is closed under direct summands. Additive categories in which idempotents split yield the proper setting to study direct-sum decompositions. For instance, as Bass observed in [2, p. 20], the following theorem holds:

Theorem 2.2 (Krull-Schmidt)**.** *Let $\mathcal{A}$ be an additive category in which idempotents split. Let $A_1, \dots, A_n$ be a finite family of nonzero objects of $\mathcal{A}$ with local endomorphism rings. Then:*

(1) *Every direct summand of $A_1 \oplus \cdots \oplus A_n$ is a finite direct sum of indecomposable objects.*
(2) *If $A_1 \oplus \cdots \oplus A_n \cong B_1 \oplus \cdots \oplus B_m$ with the B_j's indecomposable objects, then $n = m$ and there is a permutation σ of $\{1, 2, \dots, n\}$ such that $A_i \cong B_{\sigma(i)}$ for every $i = 1, 2, \dots, n$.*

3. Additive Functors

Since we want to study representations of an additive category $\mathcal{A}$, that is, additive functors $\mathcal{A} \to \mathcal{B}$, where $\mathcal{B}$ is a category with sufficiently good properties, we must introduce some terminology concerning additive functors. The notions we introduce now are weak forms of the notion of category equivalence $F\colon \mathcal{A} \to \mathcal{B}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories, and $F\colon \mathcal{A} \to \mathcal{B}$ an additive functor. We say that F is:

(1) *isomorphism reflecting* if, for every pair A, A' of objects of $\mathcal{A}$, $F(A) \cong F(A')$ implies $A \cong A'$;
(2) *direct-summand reflecting* if, for every pair A, A' of objects of $\mathcal{A}$ with $F(A)$ isomorphic to a direct summand of $F(B)$, A is isomorphic to a direct summand of B;
(3) *weakly direct-summand reflecting* if, for every pair A, A' of objects of $\mathcal{A}$ with $F(A)$ isomorphic to a direct summand of $F(B)$, there exists an object C of $\mathcal{A}$ with $F(C) = 0$ and A isomorphic to a direct summand of $B \oplus C$;
(4) *local* if, for every pair A, A' of objects of $\mathcal{A}$ and every morphism $f\colon A \to A'$ such that $F(f)\colon F(A) \to F(A')$ is an isomorphism, f is an isomorphism.

Every additive functor $F\colon \mathcal{A} \to \mathcal{B}$ induces a "monoid homomorphism" $V(F)\colon V(\mathcal{A}) \to V(\mathcal{B})$. The functor F is direct-summand reflecting if and only if the "monoid homomorphism" $V(F)$ is a "divisor homomorphism." The functor F is isomorphism reflecting if and only if the $V(F)$ is an "injective mapping." Every full, faithful, additive functor is isomorphism reflecting and local. If $\mathcal{A}$ is an additive category in which idempotents split, $\mathcal{B}$ is additive and $F\colon \mathcal{A} \to \mathcal{B}$ is a full, faithful, additive functor, then F is also direct-summand reflecting. We shall need the concept of local functor only in Section 8, but we have introduced it here for a better presentation and to have the possibility of giving a number of examples immediately.

Obviously, direct-summand reflecting implies weakly direct-summand reflecting. There is no relation, that is, no direct implication, between being direct-summand reflecting, isomorphism reflecting and local, as the following examples show.

Example 3.1. Let $\mathcal{B}$ be the additive category of all finite dimensional vector spaces over a fixed field k, and let $\mathcal{A}$ be its full subcategory whose objects are the vector spaces of dimension $\neq 1$. The embedding functor $\mathcal{A} \to \mathcal{B}$ is isomorphism reflecting and local, but not weakly direct-summand reflecting.

Example 3.2. Let $\mathcal{A}$ be the category of all finitely generated free abelian groups and $\mathcal{B}$ the category of all finite dimensional vector spaces over the field $\mathbf{Q}$. The functor $-\otimes \mathbf{Q}\colon \mathcal{A} \to \mathcal{B}$ is isomorphism reflecting and direct-summand reflecting, but not local.

Example 3.3 (Leavitt algebras)**.** Let k be a field, let n be a positive integer, let x_i, y_i ($i = 1, 2, \dots, n$) be $2n$ distinct noncommutative indeterminates, and let $k\langle x_i, y_i \mid i = 1, 2, \dots, n\rangle$ be the free k-algebra. If X is the $1\times n$ matrix with $(1, i)$ entry x_i and Y is the $n\times 1$ matrix with $(i, 1)$ entry y_i, then the 1×1 matrix $XY-1$ has its entry in the k-algebra $k\langle x_i, y_i \mid i = 1, 2, \dots, n\rangle$, and the $n \times n$ matrix $YX - 1_n$ has n^2 entries. Let I be the two-sided ideal of $k\langle x_i, y_i \mid i = 1, 2, \dots, n\rangle$ generated by these $1 + n^2$ elements, and set $R_n = k\langle x_i, y_i \mid i = 1, 2, \dots, n\rangle/I$. It is easy to see that the right R_n-modules R_n^n and R_n are isomorphic. It is possible to prove that all projective R_n-modules are free [3, Theorem 6.1] and that a complete set of non-isomorphic finitely generated free R_n-modules is $\{0, R_n, R_n^2, \dots, R_n^{n-1}\}$. It is easily seen that the position

$$\begin{array}{ll} x_1 \mapsto x_1 & y_1 \mapsto y_1 \\ x_2 \mapsto x_2x_1 & y_2 \mapsto y_1y_2 \\ x_3 \mapsto x_2^2 & y_2 \mapsto y_2^2 \end{array}$$

defines a ring homomorphism $R_3 \to R_2$. The functor $-\otimes_{R_3} R_2\colon \{0, R_3, R_3^2\} \to \{0, R_2\}$ is direct-summand reflecting, but not isomorphism reflecting.

Example 3.4. Let $\mathcal{A}$ be a Grothendieck category and let $\operatorname{Spec}\mathcal{A}$ be the *spectral category* of $\mathcal{A}$, that is, the category with the same objects as $\mathcal{A}$ and, for objects A and B of $\mathcal{A}$, with $\operatorname{Spec}\mathcal{A}(A, B) = \varinjlim \mathcal{A}(A', B)$, where the direct limit is taken over the downwards directed family of essential subobjects A' of A [8]. Let $P\colon \mathcal{A} \to \operatorname{Spec}\mathcal{A}$ be the canonical functor, that is, the functor that is the identity on objects and takes $f \in \mathcal{A}(A, B)$ to its canonical image in $\operatorname{Spec}\mathcal{A}(A, B)$. Dually, let $\mathcal{A}'$ be the category with the same objects as $\mathcal{A}$ and, for objects A and B of $\mathcal{A}$, with $\mathcal{A}'(A,B) = \varinjlim \mathcal{A}(A, B/B')$, where the direct limit is taken over the upwards directed family of superfluous subobjects B' of B [7, Section 6]. Let $F\colon \mathcal{A} \to \mathcal{A}'$ be the canonical functor. Then the functor $P \times F\colon \mathcal{A} \to \operatorname{Spec}\mathcal{A} \times \mathcal{A}'$ is a local functor (same proof as [7, Proposition 6.4]).

In general, $P \times F$ is neither weakly direct-summand reflecting nor isomorphism reflecting. For instance, let $\mathcal{A} = \mathrm{Ab}$ be the category of all abelian groups. Let A and B be the cyclic groups of order p and p^2 respectively, where p denotes a prime. As A is isomorphic to an essential subgroup of B, $P(A) \cong P(B)$ in $\operatorname{Spec}\mathcal{A}$. Similarly, there are superfluous epimorphisms $B \to A$, so that $F(A) \cong F(B)$ in $\mathcal{A}'$ [7, Lemma 6.1]. Therefore $(P \times F)(A) \cong (P \times F)(B)$, but A is not isomorphic to a direct summand of B. As for every object C, $P(C) = 0$ implies $C = 0$, it follows that there is no object C with $(P \times F)(C) = 0$ and A isomorphic to a direct summand of $B \oplus C$. Therefore $P \times F$ is neither weakly direct-summand reflecting nor isomorphism reflecting.

Example 3.5. Let $\mathcal{A}$ be an abelian category, $I\colon \mathcal{A} \to \mathcal{A}$ the identity functor, and $F\colon \mathcal{A} \to \mathcal{A}$ a subfunctor of I. Let I/F be the functor $\mathcal{A} \to \mathcal{A}$ defined on objects as $(I/F)(A) = A/F(A)$ for every $A \in \operatorname{Ob}\mathcal{A}$. Thus for every morphism $f\colon A \to B$ in $\mathcal{A}$ there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F(A) & \longrightarrow & A & \longrightarrow & A/F(A) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle F(f)} & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle (I/F)(f)} & & \\
0 & \longrightarrow & F(B) & \longrightarrow & B & \longrightarrow & B/F(B) & \longrightarrow & 0
\end{array} .$$

By the Snake Lemma, f is an isomorphism if and only if both $F(f)$ and $(I/F)(f)$ are isomorphisms. Therefore the functor $F \times I/F\colon \mathcal{A} \to \mathcal{A} \times \mathcal{A}$, defined on objects by $A \mapsto (F(A), A/F(A))$, is a local functor.

In the next lemma, we have collected some immediate consequences of the definitions given in this section. Recall that an additive category $\mathcal{A}$ is *directly finite* if, for every $A, B \in \operatorname{Ob}\mathcal{A}$, $A \cong A \oplus B$ implies $B = 0$, that is, if the "monoid" $V(\mathcal{A})$ is directly finite.

Lemma 3.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories, and $F\colon \mathcal{A} \to \mathcal{B}$ an additive functor.*

(a) *If F is direct-summand reflecting and $\mathcal{A}$ is directly finite, then F is isomorphism reflecting.*
(b) *If F is full and local, then F is isomorphism reflecting.*
(c) *If F is full and local and idempotents split in $\mathcal{A}$, then F is direct-summand reflecting.*
(d) *If F is local, then the ring homomorphism $F_A\colon \operatorname{End}_{\mathcal{A}}(A) \to \operatorname{End}_{\mathcal{B}}(F(A))$ induced by F is a local morphism for every object A of $\mathcal{A}$.*

(e) *If F is either full or isomorphism reflecting, then F is a local functor if and only if the ring homomorphism $F_A\colon \mathrm{End}_{\mathcal{A}}(A) \to \mathrm{End}_{\mathcal{B}}(F(A))$ is a local morphism for every object A of $\mathcal{A}$.*

Proof. The proofs are easy. Let us prove (c) and part of (e).

Assume that F is a full and local functor, $\mathcal{A}$ is an additive category in which idempotents split, A, B are objects of $\mathcal{A}$ and $F(A)$ is isomorphic to a direct summand of $F(B)$. As F is full, there exist morphisms $f\colon A \to B$ and $g\colon B \to A$ such that $F(g) \circ F(f)$ is the identity of $F(A)$. Since F is local, $gf\colon A \to A$ must be an isomorphism. Therefore there exists $h\colon A \to A$ such that $hgf = 1_A$. By Lemma 2.1, A is isomorphic to a direct summand of B. This proves (c).

Now let F be a full functor and assume that, for every object A of $\mathcal{A}$, the ring homomorphism $F_A\colon \mathrm{End}_{\mathcal{A}}(A) \to \mathrm{End}_{\mathcal{B}}(F(A))$ is a local morphism. Let A, A' be objects of $\mathcal{A}$ and $f\colon A \to A'$ a morphism such that $F(f)$ is an isomorphism. As F is full, there exists $g\colon A' \to A$ such that $F(g)$ is the inverse of $F(f)$. Then gf and fg are automorphisms because F_A and $F_{A'}$ are local morphisms, so that f is an isomorphism. □

Let us show with an example that (d) cannot be inverted, that is, that the hypothesis that either F is full or F is isomorphism reflecting is necessary in (e). Let $F\colon \mathcal{A} \to \mathrm{Spec}\,\mathrm{Ab}$ be the restriction of the canonical functor $P\colon \mathrm{Ab} \to \mathrm{Spec}\,\mathrm{Ab}$ to the full subcategory $\mathcal{A}$ of Ab whose objects are all artinian abelian groups, cf. Example 3.4. Then the ring morphism $F_A\colon \mathrm{End}_{\mathcal{A}}(A) \to \mathrm{End}_{\mathcal{B}}(F(A))$ is local for every artinian group A, but F is not a local functor, because if f is any essential monomorphism, for instance the inclusion of the cyclic group of order p into the cyclic group of order p^2, then $F(f) = P(f)$ is an isomorphism.

4. Weakly Direct-summand Reflecting Functors

This section is devoted to the study of weakly direct-summand reflecting functors of an additive category $\mathcal{A}$ into a directly finite, additive category $\mathcal{B}$. Recall that a *two-sided ideal* of an additive category $\mathcal{A}$ is a subfunctor of the two variable functor $\mathcal{A}(-,-)$ [11, p. 18]. That is, $\mathcal{I}$ is a two-sided ideal of $\mathcal{A}$ if for every pair of objects A, $B \in \mathrm{Ob}\,\mathcal{A}$, $\mathcal{I}(A,B)$ is a subgroup of $\mathcal{A}(A,B)$ such that for every morphism $\varphi\colon C \to A$, $\psi\colon A \to B$ and $\omega\colon B \to D$ with $\psi \in \mathcal{I}(A,B)$ one has that $\omega\psi\varphi \in \mathcal{I}(C,D)$.

Example 4.1 (Jacobson radical)**.** Let $\mathcal{A}$ be an additive category. If A, B are objects of $\mathcal{A}$ and $f\colon A \to B$, $g\colon B \to A$ are morphisms, then $1_A - gf$ has a left inverse if and only if $1_B - fg$ has a left inverse, because if $h(1_A - gf) = 1_A$, then $(1_B + fhg)(1_B - fg) = 1_B$. If one defines

$$J(A,B) := \left\{ f \in \mathcal{A}(A,B) \mid 1_A - gf \text{ has a left inverse for all } g \in \mathcal{A}(B,A) \right\},$$

then

$$J(A,B) = \Big\{f \in \mathcal{A}(A,B) \mid 1_A - gf \text{ has a two-sided inverse for all } g \in \mathcal{A}(B,A)\Big\},$$

and J turns out to be a two-sided ideal of the category $\mathcal{A}$, called the *Jacobson radical* of $\mathcal{A}$ [11, p. 21]. The quotient category $\mathcal{A}/J$ has zero Jacobson radical.

Proposition 4.2. *Let $\mathcal{A}$ be an additive category and J its Jacobson radical. Then:*

(a) *The canonical functor $G\colon \mathcal{A} \to \mathcal{A}/J$ is a full, isomorphism reflecting, local functor.*
(b) *If idempotents split in $\mathcal{A}$, then G is also direct-summand reflecting.*
(c) *Let $f\colon A \to A'$ be a morphism in $\mathcal{A}$. There exists a local, additive functor $F\colon \mathcal{A} \to \mathcal{B}$ of $\mathcal{A}$ into an arbitrary additive category $\mathcal{B}$ such that $F(f) = 0$ if and only if $f \in J(A, A')$.*

Proof.

(a) The proof is straightforward.
(b) Assume that $\mathcal{A}$ is an additive category in which idempotents split. In order to show that G is direct-summand reflecting, take two objects A, B of $\mathcal{A}$ with $G(A)$ isomorphic to a direct summand of $G(B)$. As G is obviously full, there exist morphisms $f\colon A \to B$ and $g\colon B \to A$ such that $G(g)G(f) = 1_{G(A)}$, that is, $1_A - gf \in J(A,A)$. Thus $gf\colon A \to A$ has a two-sided inverse $(gf)^{-1}$, so that A is isomorphic to a direct summand of B by Lemma 2.1.
(c) Every $f \in J(A,A')$ is annihilated by the local functor $G\colon \mathcal{A} \to \mathcal{A}/J$. Conversely, assume that $F(f) = 0$ for some local functor $F\colon \mathcal{A} \to \mathcal{B}$. Let $g \in \mathcal{A}(A',A)$. Then $F(1_A - gf)$ is the identity of $F(A)$. As F is local, the morphism $1_A - gf\colon A \to A$ is an isomorphism. Therefore $f \in J(A,A')$. □

Lemma 4.3. *Let $\mathcal{A}$ be an additive category and let S be a subclass of* $\operatorname{Ob}\mathcal{A}$ *closed under finite direct sums. For every A, $B \in \operatorname{Ob}\mathcal{A}$, set $\mathcal{I}_S(A,B) := \{f \in \mathcal{A}(A,B) \mid$ there exist $C \in S$, $g \in \mathcal{A}(A,C)$ and $h \in \mathcal{A}(C,B)$ with $f = hg\}$. Then $\mathcal{I}_S$ is a two-sided ideal of $\mathcal{A}$.*

Proof. It suffices to show that $\mathcal{I}_S(A,B)$ is a subgroup of $\mathcal{A}(A,B)$. If $f, f' \in \mathcal{I}_S(A,B)$, then there exist $C, C' \in S$, $g \in \mathcal{A}(A,C)$, $g' \in \mathcal{A}(A,C')$, $h \in \mathcal{A}(C,B)$ and $h' \in \mathcal{A}(C',B)$ with $f = hg$ and $f' = h'g'$. Let $\Delta_A\colon A \to A \oplus A$ be the diagonal morphism and $\nabla_B\colon B \oplus B \to B$ be the codiagonal morphism [12, p. 21]. Then $f + g$ is the composite morphism

$$A \xrightarrow{\Delta_A} A \oplus A \xrightarrow{g \oplus g'} C \oplus C' \xrightarrow{h \oplus h'} B \oplus B \xrightarrow{\nabla_B} B.$$

The rest is obvious. □

If M is a directly finite, reduced monoid, then the algebraic pre-order on M is a partial order. Conversely, if M is a monoid on which the algebraic pre-order is a partial order, then M is reduced, but not necessarily directly finite as Example 4.4 will show.

Example 4.4. Let M be the quotient monoid $\mathbb{N}_0^2/\sim$, where $\sim$ is the congruence on $\mathbb{N}_0^2$ generated by $(1,1) \sim (1,0)$. It is easily seen that $(m,n) \sim (m,0)$ for every $m \geq 1$ and every $n \geq 0$. It follows that a system of representatives of the elements of M is given by the elements $(m,0)$, $m \geq 1$, and the elements $(0,n)$, $n \geq 0$. Clearly, M is not directly finite. The monoid M with the algebraic pre-order is a well ordered set, because it is order-isomorphic to the ordinal $\omega 2$, where ω is the first infinite ordinal.

Let M, N be monoids. We will say that a monoid homomorphism $\varphi\colon M \to N$ is *essential* if, for every $x, y \in M$ with $\varphi(x) \leq \varphi(y)$, there exist $z, t \in M$ with $\varphi(z) = 0$ and $x + t = y + z$. Thus an additive functor $F\colon \mathcal{A} \to \mathcal{B}$ is weakly direct-summand reflecting if and only if the "monoid homomorphism" $V(F)$ is an "essential homomorphism." (We use the term essential monoid homomorphism in analogy with essential valutations of commutative rings and commutative monoids.)

We want to study weakly direct-summand reflecting functors of an additive category $\mathcal{A}$ into a directly finite, additive category $\mathcal{B}$. If $F\colon \mathcal{A} \to \mathcal{B}$ is such a functor, then F induces a "monoid homomorphism" $V(F)\colon V(\mathcal{A}) \to V(\mathcal{B})$, where $V(\mathcal{B})$ is a "directly finite, reduced monoid." Therefore we are interested in "essential monoid homomorphisms" $V(\mathcal{A}) \to N$, where $\mathcal{A}$ is an additive category and N is a "directly finite, reduced monoid." If $\mathcal{A}$ is an additive category, N is a "monoid," and $\varphi\colon V(\mathcal{A}) \to N$ is a "monoid homomorphism," we shall denote by $\mathcal{I}_\varphi$ the two-sided ideal $\mathcal{I}_S$ of $\mathcal{A}$ where $S = \{A \in \operatorname{Ob}\mathcal{A} \mid \varphi(\langle A \rangle) = 0\}$.

Theorem 4.5. *Let $\mathcal{A}$ be an additive category in which idempotents split, let N be a "directly finite, reduced monoid," and let $\varphi\colon V(\mathcal{A}) \to N$ be an "essential monoid homomorphism." Then $V(\mathcal{A}/\mathcal{I}_\varphi) \cong \varphi(V(\mathcal{A}))$ via the "monoid isomorphism" defined by $\langle A \rangle \mapsto \varphi(\langle A \rangle)$ for every object A of $\mathcal{A}/\mathcal{I}_\varphi$.*

Proof. Set $S = \{A \in \operatorname{Ob}\mathcal{A} \mid \varphi(\langle A \rangle) = 0\}$. It suffices to prove that if A and B are objects of $\mathcal{A}$, then A is isomorphic to B in $\mathcal{A}/\mathcal{I}_\varphi$ if and only if $\varphi(\langle A \rangle) = \varphi(\langle B \rangle)$.

We claim that if $A \leq B$ in $\mathcal{A}/\mathcal{I}_\varphi$, then $\varphi(\langle A \rangle) \leq \varphi(\langle B \rangle)$. Assume that $A \leq B$ in $\mathcal{A}/\mathcal{I}_\varphi$. Then there exist $f\colon A \to B$ and $f'\colon B \to A$ in $\mathcal{A}$ such that $f'f - 1_A \in \mathcal{I}_\varphi(A,A)$. Thus there exist $C \in S$, $g \in \mathcal{A}(A,C)$ and $h \in \mathcal{A}(C,A)$ with $f'f - 1_A = hg$. Therefore 1_A is the composite mapping of

$$A \xrightarrow{\Delta_A} A \oplus A \xrightarrow{f \oplus (-g)} B \oplus C \xrightarrow{f' \oplus h} A \oplus A \xrightarrow{\nabla_A} A,$$

where Δ_A and ∇_A denote the diagonal morphism and the codiagonal morphism respectively [12, p. 21]. By Lemma 2.1, the retraction $\nabla_A(f' \oplus h)$ has a kernel K

in $\mathcal{A}$ and $K \oplus A \cong B \oplus C$. This is an isomorphism in $\mathcal{A}$, so that, applying $\varphi(\langle - \rangle)$, we see that $\varphi(\langle A \rangle) \le \varphi(\langle K \oplus A \rangle) = \varphi(\langle B \rangle) + \varphi(\langle C \rangle) = \varphi(\langle B \rangle)$. This proves the claim.

Now assume that $A \cong B$ in $\mathcal{A}/\mathcal{I}_\varphi$. Then $A \le B$ and $B \le A$, so that, by the claim, both $\varphi(\langle A \rangle) \le \varphi(\langle B \rangle)$ and $\varphi(\langle B \rangle) \le \varphi(\langle A \rangle)$. Hence $\varphi(\langle A \rangle) = \varphi(\langle B \rangle)$ because $\le$ is a partial order on N.

Notice that, as a consequence, the category $\mathcal{A}/\mathcal{I}_\varphi$ is directly finite, because if X, Y are objects of $\mathcal{A}$, and $X \cong X \oplus Y$ in $\mathcal{A}/\mathcal{I}_\varphi$, then $\varphi(\langle X \rangle) = \varphi(\langle X \rangle) + \varphi(\langle Y \rangle)$, from which $\varphi(\langle Y \rangle) = 0$, hence $Y = 0$ in $\mathcal{A}/\mathcal{I}_\varphi$.

Conversely, suppose that A and B are objects of $\mathcal{A}$ such that $\varphi(\langle A \rangle) = \varphi(\langle B \rangle)$. As φ is an "essential homomorphism," from $\varphi(\langle A \rangle) \le \varphi(\langle B \rangle)$ it follows that there exists $C, D \in \operatorname{Ob} \mathcal{A}$ with $\varphi(\langle C \rangle) = 0$ and $A \oplus D \cong B \oplus C$ in $\mathcal{A}$. Thus $A \oplus D \cong B$ in $\mathcal{A}/\mathcal{I}_\varphi$. Interchanging the roles of A and B, one finds that B is isomorphic to a direct summand of A in $\mathcal{A}/\mathcal{I}_\varphi$. Since $\mathcal{A}/\mathcal{I}_\varphi$ is directly finite, A and B are isomorphic in the factor category $\mathcal{A}/\mathcal{I}_\varphi$. □

If N is a "directly finite, reduced monoid" and $\varphi \colon V(\mathcal{A}) \to N$ is an "essential monoid homomorphism," then we can consider the restriction $\varphi' \colon V(\mathcal{A}) \to \varphi(V(\mathcal{A}))$. As $\varphi(V(\mathcal{A}))$ is a "directly finite, reduced submonoid" of N and φ' is a fortiori "an essential monoid homomorphism," we could have supposed φ "surjective" in Theorem 4.5 without loss of information.

Let us determine all possible surjective essential monoid homomorphisms of a monoid M into a directly finite, reduced monoid N. This is equivalent to determining all possible congruences $\sim$ of the monoid M such that the canonical projection $M \to M/\sim$ is an essential monoid homomorphism and $M/\sim$ is directly finite and reduced. Surprisingly, there is a one-to-one correspondence between the set of all these congruences and the set of all directed convex subgroups of the Grothendieck group $G(M)$ of the monoid M. To present this result in detail, we need some terminology on prime ideals of commutative monoids and directed convex subgroups of pre-ordered abelian groups. Recall that a *prime ideal* of a commutative monoid M is a proper subset P of M such that for any $x, y \in M$ one has $x + y \in P$ if and only if either $x \in P$ or $y \in P$. The set $\operatorname{Spec}(M)$ of all prime ideals of M, partially ordered by set inclusion, is a complete lattice whose greatest element is the prime ideal $M \setminus U(M)$ and whose least element is the empty ideal $\emptyset$. If P is a prime ideal of M, then the *localization* M_P of M at P is the monoid whose elements are all formal differences $x - s$ with $x \in M$ and $s \in M \setminus P$, and in which, for all $x, x' \in M$ and $s, s' \in M \setminus P$, $x - s = x' - s'$ if and only if there exists $t \in M \setminus P$ such that $x + s' + t = x' + s + t$. The monoid $(M_P)_{\mathrm{red}} = M_P/U(M_P)$ is called the *reduced localization* of M at P. The canonical homomorphism $\varphi \colon M \to (M_P)_{\mathrm{red}}$, defined by $x \mapsto x - 0 + U(M_P)$, is surjective. Its kernel is the congruence $\sim_P$ on M defined, for every $x, y \in M$, by $x \sim_P y$ if there exist $z, t \in M \setminus P$ such that $x + z = y + t$.

The localization $M_\varnothing$ of M at its empty prime ideal $\varnothing$ is an abelian group, which is usually called the *Grothendieck group* of M, and denoted $G(M)$. It becomes a pre-ordered group taking the image of the canonical homomorphism $M \to G(M)$ as positive cone $G(M)_+$. A *convex subgroup* of a pre-ordered group G is any subgroup H of G such that $0 \le a \le b$ with $b \in H$ and $a \in G$ implies $a \in H$. A *directed* subgroup H of G is one such that for every $h \in H$ there exists h', $h'' \in G_+ \cap H$ with $h = h' - h''$. Let $\mathcal{L}(G)$ denote the set of all directed convex subgroups of the pre-ordered group G, and $\operatorname{Spec}'(M)$ denote the set of all $P \in \operatorname{Spec}(M)$ with $(M_P)_{\text{red}}$ directly finite.

Proposition 4.6 ([1, Proposition 6.2])**.** *Let M be a commutative monoid. Then there is an inclusion reversing one-to-one correspondence* $\operatorname{Spec}'(M) \to \mathcal{L}(G(M))$.

Proposition 4.7. *Let M be a commutative monoid. Then the assignement*

$$P \in \operatorname{Spec}'(M) \mapsto \sim_P$$

is a one-to-one correspondence of $\operatorname{Spec}'(M)$ *onto the set of all the congruences $\sim$ on M such that the canonical projection $M \to M/\sim$ is an essential monoid homomorphism and $M/\sim$ is directly finite and reduced.*

Proof. Let $P \in \operatorname{Spec}'(M)$, so that $M/\sim_P \cong (M_P)_{\text{red}}$ is directly finite and reduced. Let us show that the canonical projection $\pi\colon M \to M/\sim_P$ is an essential homomorphism. If $x, y \in M$ and $\pi(x) \le \pi(y)$, then there exists $t' \in M$ with $x+t' \sim_P y$. Therefore there exist t'', $z \in M \setminus P$ such that $x+t'+t'' = y+z$. Thus π is an essential monoid homomorphism. This shows that the correspondence is well defined.

In order to prove that the correspondence is injective, assume that P, P' are two distinct primes belonging to $\operatorname{Spec}'(M)$. We can suppose $P \not\subseteq P'$, so that there exists $p \in P$, $p \notin P'$. Then $p \sim_{P'} 0$ and $p \not\sim_P 0$. Hence $\sim_P \neq \sim_{P'}$.

We will now show that the correspondence is surjective. Let $\sim$ be a congruence on M such that the canonical projection $M \to M/\sim$ is essential and $M/\sim$ is directly finite and reduced. Let $P := \{x \in M \mid x \not\sim 0\}$. It is easily seen that P is a prime ideal. Let us show that $\sim_P = \sim$. If $x, y \in M$ and $x \sim_P y$, then there exist $z, t \in M \setminus P$ with $x + z = y + t$. Then $z \sim 0$ and $t \sim 0$, so that $x \sim x + z = y + t \sim y$. Conversely, assume $x \sim y$. Then $\pi(x) = \pi(y)$. As π is essential, there exist $z, t \in M$ with $\pi(z) = 0$ and $x + t = y + z$. Thus $\pi(x)+\pi(t) = \pi(y) = \pi(x)$. Since $M/\sim$ is directly finite, we get that $\pi(t) = 0$. Thus $z, t \notin P$, hence $x \sim_P y$. This concludes the proof that $\sim_P = \sim$.

Finally, $(M_P)_{\text{red}} \cong M/\sim_P = M/\sim$ is directly finite. Thus $P \in \operatorname{Spec}'(M)$. □

It is easily checked that, for a prime ideal P of a commutative monoid M, $(M_P)_{\text{red}}$ is directly finite if and only if $x + y = x + z$ and $y \notin P$ implies $z \notin P$ for every $x, y, z \in M$.

We are ready to apply the ideas developed in this section to weakly direct-summand reflecting functors.

Theorem 4.8 (Isomorphism theorem). *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a weakly direct summand reflecting functor of an additive category $\mathcal{A}$ in which idempotents split into a directly finite, additive category $\mathcal{B}$. Then:*

(a) *The class $S := \{A \in \operatorname{Ob}\mathcal{A} \mid F(A) = 0\}$ is closed under isomorphism, finite direct sums and direct summands, and, for every $A, B, C \in \operatorname{Ob}\mathcal{A}$, $A \oplus B \cong A \oplus C$ and $B \in S$ implies $C \in S$.*
(b) *For every $A, B \in \operatorname{Ob}\mathcal{A}$, $F(A) \cong F(B)$ if and only if there exist $C, D \in S$ such that $A \oplus C \cong B \oplus D$.*
(c) *The functor F induces an isomorphism reflecting, direct-summand reflecting functor $\mathcal{A}/\mathcal{I}_S \to \mathcal{B}$, and $\mathcal{A}/\mathcal{I}_S$ is a directly finite category.*

Conversely, let $\mathcal{A}$ be an additive category in which idempotents split and let S be a subclass of $\operatorname{Ob}\mathcal{A}$ closed under isomorphism, finite direct sums and direct summands, and such that, for every $A, B, C \in \operatorname{Ob}\mathcal{A}$, $A \oplus B \cong A \oplus C$ and $B \in S$ implies $C \in S$. Then:

(a) *The canonical projection $P\colon \mathcal{A} \to \mathcal{A}/\mathcal{I}_S$ is a weakly direct-summand reflecting functor.*
(b) *The additive category $\mathcal{A}/\mathcal{I}_S$ is directly finite.*
(c) *$S = \{A \in \operatorname{Ob}\mathcal{A} \mid P(A) = 0\}$.*

The proof is now elementary and direct.

5. IBN Categories

A *valuation* of an additive category $\mathcal{A}$ is an "onto mapping" $v\colon \operatorname{Ob}\mathcal{A} \to G_+$, where G is a totally ordered abelian group and $G_+ = \{g \in G \mid g \geq 0\}$ is its positive cone, such that:

(1) if A, B are isomorphic objects of $\mathcal{A}$, then $v(A) = v(B)$;
(2) $v(A \oplus B) = v(A) + v(B)$ for every pair A, B of objects of $\mathcal{A}$. It follows that if 0 is the zero object, then $v(0) = 0$.

A *discrete valuation* is a valuation $v\colon \operatorname{Ob}\mathcal{A} \to G_+$ for which $G \cong \mathbb{Z}$, so that $G_+ = \mathbb{N}_0$ without loss of generality. An IBN *category* is an additive category $\mathcal{A}$ with a discrete valuation $v\colon \operatorname{Ob}\mathcal{A} \to \mathbb{N}_0$ such that for every pair A, B of objects of $\mathcal{A}$, $v(A) = v(B)$ implies $A \cong B$. Let $\mathcal{A}$ be an IBN category and let G be an object of $\mathcal{A}$ with $v(G) = 1$. Then every other object A of $\mathcal{A}$ is isomorphic to G^n, where $n = v(A)$. It follows that G is the unique indecomposable object of $\mathcal{A}$ up to isomorphism. Thus an object of $\mathcal{A}$ is indecomposable if and only if its valuation is 1. It follows that a discrete valuation $v\colon \operatorname{Ob}\mathcal{A} \to \mathbb{N}_0$ such that $v(A) = v(B)$ implies $A \cong B$ is unique, when it exists.

A trival example of an IBN category is the category of all finite dimensional right vector spaces over a fixed division ring.

Proposition 5.1. *If a ring R has* IBN *and $\mathcal{F}_R$ is the full subcategory of* Mod-*R whose objects are all finitely generated free right R-modules, then $\mathcal{F}_R$ is an* IBN *category. Conversely, for every* IBN *category $\mathcal{A}$, there exists a ring R with* IBN *and $\mathcal{F}_R$ equivalent to $\mathcal{A}$.*

Proof. For the first part of the statement, it is sufficient to take as v the rank of the free module. Conversely, let $\mathcal{A}$ be an IBN category and G an object of $\mathcal{A}$ with $v(G) = 1$. Consider the functor $\mathcal{A}(G,-)\colon \mathcal{A} \to \mathrm{Ab}$. For every object A of $\mathcal{A}$, the abelian group $\mathcal{A}(G,A)$ is a finitely generated free right module over the ring $R = \mathcal{A}(G,G)$. Thus we have a functor $\mathcal{A}(G,-)\colon \mathcal{A} \to \mathcal{F}_R$, which is an equivalence of categories. In particular, $\mathcal{F}_R$ is an IBN category, so that R has IBN. □

For an arbitrary additive category $\mathcal{A}$, the valuations of $\mathcal{A}$ correspond exactly to the "surjective monoid homomorphisms" of $V(\mathcal{A})$ into the positive cone of a totally ordered abelian group, and the essential discrete valuations of $\mathcal{A}$ correspond to the "essential valuations" of the "monoid" $V(\mathcal{A})$ that are "surjective" (equivalently, of index 1; cf. [6]).

6. Additive Categories and Krull Monoids

In this section, we will characterize the categories $\mathcal{A}$ with $V(\mathcal{A})$ a Krull monoid. We will say that a category $\mathcal{A}$ is a *Krull category* if it is an additive category in which idempotents split and for which there exists a family $v_i\colon \mathrm{Ob}\,\mathcal{A} \to \mathbb{N}_0$, $i \in I$, of discrete valuations of $\mathcal{A}$ such that:

(1) if $A, B \in \mathrm{Ob}\,\mathcal{A}$ and $v_i(A) \le v_i(B)$ for every $i \in I$, then there exists a section $A \to B$;

(2) for every $A \in \mathrm{Ob}\,\mathcal{A}$, the set $\{i \in I \mid v_i(A) \neq 0\}$ is finite.

If $\{\mathcal{A}_j \mid j \in J\}$ is a family of additive categories indexed in a set J, let $\prod_{j\in J} \mathcal{A}_j$ be the product category, whose objects are the sequences $S = (A_j)_{j\in J}$ with $A_j \in \mathcal{A}_j$, and whose morphisms are given by $(\prod_{j\in J} \mathcal{A}_j)((A_j)_{j\in J}, (A'_j)_{j\in J}) = \prod_{j\in J} \mathcal{A}_j(A_j, A'_j)$. Let $\coprod_{j\in J} \mathcal{A}_j$ be the full subcategory of $\prod_{j\in J} \mathcal{A}_j$ whose objects are the sequences $S = (A_j)_{j\in J}$ with almost all $A_j = 0$, that is, the sequences $S = (A_j)_{j\in J}$ for which there exists a finite subset J_S of J with $A_j = 0$ for every $j \in J \setminus J_S$.

Theorem 6.1. *Let $\mathcal{A}$ be a skeletally small additive category in which idempotents split. The following conditions are equivalent:*

(a) *The monoid $V(\mathcal{A})$ is a Krull monoid.*

(b) *There exist a family $\{\mathcal{A}_j \mid j \in J\}$ of* IBN *categories and a direct-summand reflecting functor $\mathcal{A} \to \coprod_{j\in J} \mathcal{A}_j$.*

(c) *The category $\mathcal{A}$ is a Krull category.*

Proof. (a) ⇒ (b). If $V(\mathcal{A})$ is a Krull monoid, the set $\{v_j \mid j \in J\}$ of all surjective essential valuations of the monoid $V(\mathcal{A})$ defines a divisor theory $V(\mathcal{A}) \to \mathbb{N}_0^{(J)}$ [6, Proposition 4.3]. Let F_j be the canonical functor of $\mathcal{A}$ into $\mathcal{A}_j := \mathcal{A}/\mathcal{I}_{v_j}$. As the divisor theory $V(\mathcal{A}) \to \mathbb{N}_0^{(J)}$ has its values in $\mathbb{N}_0^{(J)}$ and is a divisor homomorphism, one sees that the product functor $F = \prod_{j\in J} F_j\colon \mathcal{A} \to \prod_{j\in J} \mathcal{A}_j$ sends objects of $\mathcal{A}$ to objects of $\coprod_{j\in J} \mathcal{A}_j$ and that its restriction $\mathcal{A} \to \coprod_{j\in J} \mathcal{A}_j$ is direct-summand reflecting.

(b) $\Rightarrow$ (c) Let $G\colon \mathcal{A} \to \coprod_{j\in J} \mathcal{A}_j$ be a direct-summand reflecting functor with the $\mathcal{A}_j$'s IBN categories. The functor $G\colon \mathcal{A} \to \coprod_{j\in J} \mathcal{A}_j$ induces a monoid homomorphism $V(G)\colon V(\mathcal{A}) \to V(\coprod_{j\in J} \mathcal{A}_j) \cong \mathbb{N}_0^{(J)}$. As G is direct-summand reflecting, the monoid homomorphism $V(G)$ is a divisor homomorphism. Hence $V(\mathcal{A})$ is a Krull monoid.

(c) $\Rightarrow$ (a) is obvious. □

Notice that the commutative monoid $V(\mathcal{A})$ is free if and only if the Krull-Schmidt property holds in $\mathcal{A}$, i.e., every object of $\mathcal{A}$ has a unique direct-sum decomposition into indecomposable objects up to isomorphism and a permutation of direct summands.

Theorem 6.2. *Let $\mathcal{A}$ be a skeletally small additive category in which idempotents split. The following conditions are equivalent:*

(a) *The monoid $V(\mathcal{A})$ is a free monoid.*
(b) *There exist a family $\{\mathcal{A}_j \mid j \in J\}$ of* IBN *categories $\mathcal{A}_j$ and an isomorphism reflecting functor $F\colon \mathcal{A} \to \coprod_{j\in J} \mathcal{A}_j$ such that every object of $\coprod_{j\in J} \mathcal{A}_j$ is isomorphic to $F(A)$ for some object A of $\mathcal{A}$.*

Proof. (a) $\Rightarrow$ (b). Assume $V(\mathcal{A})$ free, so that there is an isomorphism $\varphi : V(\mathcal{A}) \to \mathbb{N}_0^{(J)}$ for some set J. For every $j \in J$, the canonical projection $\pi_j\colon \mathbb{N}_0^{(J)} \to \mathbb{N}_0$ is an essential monoid homomorphism, so that the composite mapping $v_j := \pi_j\varphi\colon V(\mathcal{A}) \to \mathbb{N}_0$ is an essential monoid homomorphism. Apply Theorem 4.5, so that the factor category $\mathcal{A}_j := \mathcal{A}/\mathcal{I}_{v_j}$ is an IBN category. Let $F_j\colon \mathcal{A} \to \mathcal{A}_j$ be the canonical functor and $F = \prod_{j\in J} F_j$. Then $V(F) = \prod_{j\in J} \pi_j\varphi\colon V(\mathcal{A}) \to N_0^{(J)}$ coincides with φ. Statement (b) is now obvious.

(b) $\Rightarrow$ (a) The conditions on $F\colon \mathcal{A} \to \coprod_{j\in J} \mathcal{A}_j$ in (b) say that the monoid homomorphism $V(F)\colon V(\mathcal{A}) \to V(\coprod_{j\in J} \mathcal{A}_j)$ is both injective and surjective. As the $\mathcal{A}_j$'s are IBN categories, one gets that $V(\coprod_{j\in J} \mathcal{A}_j) \cong \bigoplus_{j\in J} V(\mathcal{A}_j) \cong \mathbb{N}_0^{(J)}$. □

7. Categories Whose Endomorphism Rings Are Semisimple Artinian

For every division ring k, we shall denote by vect-k the category of all finite dimensional right vector spaces over k.

Theorem 7.1. *The following conditions are equivalent for a skeletally small additive category $\mathcal{A}$:*

(a) *Idempotents split in $\mathcal{A}$, and the endomorphism rings* $\mathrm{End}_{\mathcal{A}}(A)$ *of all objects A of $\mathcal{A}$ are semisimple artinian.*
(b) *There exists a set $\{k_j \mid j \in J\}$ of division rings such that $\mathcal{A}$ is equivalent to* $\coprod_{j\in J}$ vect-k_j.

Proof. (a) ⇒ (b). Let $\mathcal{A}$ be a skeletally small additive category in which idempotents split and in which every endomorphism ring is semisimple artinian. As $\mathcal{A}$ is skeletally small, there exists a set $\{A_j \mid j \in J\}$ of representatives of the indecomposable objects of $\mathcal{A}$ up to isomorphism. Since idempotents split in $\mathcal{A}$, the endomorphism ring of every A_j is a division ring k_j. Every object A of $\mathcal{A}$ decomposes as a direct sum of finitely many objects whose endomorphism rings are division rings, hence they are necessarily indecomposable objects. Assume that A_j and $A_{j'}$ are indecomposable objects and that $\mathcal{A}(A_j, A_{j'}) \neq 0$. We have already remarked that the endomorphism rings of A_j and $A_{j'}$ are division rings. In the additive category $\mathcal{A}$, the endomorphism ring of $A_j \oplus A_{j'}$ is the matrix ring

$$E = \begin{pmatrix} \mathcal{A}(A_j, A_j) & \mathcal{A}(A_{j'}, A_j) \\ \mathcal{A}(A_j, A_{j'}) & \mathcal{A}(A_{j'}, A_{j'}) \end{pmatrix},$$

which is a semisimple artinian ring. Let $f\colon A_j \to A_{j'}$ be a non-zero morphism. Then the element $\left(\begin{smallmatrix} 0 & 0 \\ f & 0 \end{smallmatrix}\right) \in E$ is a non-zero element of E that induces by left multiplication a non-zero morphism of right E-modules from the indecomposable right ideal

$$\begin{pmatrix} 1_{A_j} & 0 \\ 0 & 0 \end{pmatrix} E$$

into the indecomposable right ideal

$$\begin{pmatrix} 0 & 0 \\ 0 & 1_{A_{j'}} \end{pmatrix} E$$

As indecomposable right ideals are simple E-modules because E is semisimple artinian, the non-zero morphism induced by left multiplication is an isomorphism, and thus it has an inverse isomorphism. This inverse isomorphism

$$\begin{pmatrix} 0 & 0 \\ 0 & 1_{A_{j'}} \end{pmatrix} E \to \begin{pmatrix} 1_{A_j} & 0 \\ 0 & 0 \end{pmatrix} E$$

is given by left multiplication by an element $\alpha \in E$, which is necessarily of the form $\alpha = \left(\begin{smallmatrix} 0 & g \\ 0 & 0 \end{smallmatrix}\right)$. So $g\colon A_{j'} \to A_j$ is a morphisms in $\mathcal{A}$ such that $gf = 1_{A_j}$ and $fg = 1_{A_{j'}}$. We have thus proved that if A_j and $A_{j'}$ are two objects in $\mathcal{A}$ whose endomorphism rings are division rings and there is a non-zero morphism $A_j \to A_{j'}$, then $A_j \cong A_{j'}$.

As every object A of $\mathcal{A}$ decomposes as a direct sum of finitely many objects with local endomorphism rings, by Theorem 2.2 there are only finitely many j's such that $\mathcal{A}(A_j, A) \neq 0$. Thus $F = \prod_{j \in J} \mathcal{A}(A_j, -)\colon \mathcal{A} \to \coprod_{j \in J} \text{vect-}k_j$ is an equivalence.

(b) ⇒ (a) is obvious. □

The skeletally small additive categories satisfying the equivalent conditions of Theorem 7.1 are called *amenable semisimple* [11, Section 4]. They are necessarily abelian.

Remarks. (1) Let $\mathcal{A}$ be an additive category, let A be an object of $\mathcal{A}$ and let $E := \mathrm{End}_{\mathcal{A}}(A)$ be its endomorphism ring. Let Mod-E denote the category of all right E-modules and consider the functor $\mathcal{A}(A, -)\colon \mathcal{A} \to$ Mod-E. Let $\mathrm{add}(A)$ denote the full subcategory of $\mathcal{A}$ whose objects are all objects of $\mathcal{A}$ that are isomorphic to direct summands of A^n for some $n \in \mathbb{N}_0$. Then $V(\mathrm{add}(A))$ is a submonoid of $V(\mathcal{A})$. Let proj-E be the full subcategory of Mod-E whose objects are all finitely generated projective right E-modules. Thus proj-$E = \mathrm{add}(E_E)$ in the category Mod-E. The restriction of the functor $\mathcal{A}(A, -)\colon \mathcal{A} \to$ Mod-E to the full subcategories $\mathrm{add}(A)$ of $\mathcal{A}$ and proj-E of Mod-E is a full and faithful functor $\mathcal{A}(A, -)\colon \mathrm{add}(A) \to$ proj-E. If idempotents split in $\mathcal{A}$, the functor $\mathcal{A}(A, -)\colon \mathrm{add}(A) \to$ proj-E is an equivalence of categories. In particular, $V(\mathrm{add}(A))$ and $V(\text{proj-}E)$ are canonically isomorphic monoids if idempotents split in $\mathcal{A}$. Under this canonical homomorphism, the class $\langle A \rangle$ corresponds to $\langle E_E \rangle$.

(2) Let $v\colon \mathrm{Ob}\,\mathcal{A} \to G_+$ be a valuation of an additive category $\mathcal{A}$ in which idempotents split, where G is a rank one group, that is, a group order-isomorphic to a subgroup of the additive group $\mathbb{D}$ of all real numbers, and let A be an object of $\mathcal{A}$. Then v induces a "surjective monoid homomorphisms" $V(\mathcal{A}) \to G_+$, which induces, by restriction, a monoid homomorphism $V(\mathrm{add}(A)) \to \mathbb{D}_+$. As we have just seen in (1), this corresponds to a monoid homomorphism $V(\text{proj-}\mathrm{End}_{\mathcal{A}}(A)) \to \mathbb{D}_+$ that sends $\langle \mathrm{End}_{\mathcal{A}}(A) \rangle$ to $v(A)$. There are two cases. Either $v(A) = 0$, in which case $v(B) = 0$ for all $B \in \mathrm{add}(A)$. Or $v(A) \neq 0$, so that by dividing by $v(A)$ and passing to the Grothendieck group

$$K_0(\mathrm{End}_{\mathcal{A}}(A)) := G(V(\text{proj-}\mathrm{End}_{\mathcal{A}}(A))),$$

we obtain a group homomorphism $K_0(\mathrm{End}_{\mathcal{A}}(A)) \to \mathbb{D}$ that is a projective rank function on the ring $\mathrm{End}_{\mathcal{A}}(A)$ [13, p. 5]. We have thus shown that a valuation of an additive category in which idempotents split into a totally ordered abelian group of rank one is either zero on an object A of $\mathcal{A}$ or induces a projective rank function on $\mathrm{End}_{\mathcal{A}}(A)$.

(3) There is a strong relation between the topic of this paper and the topic of [13]. Apart from what we have just seen in (2), let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor of an additive category $\mathcal{A}$ into an amenable semisimple category $\mathcal{B}$. For every object A of $\mathcal{A}$, the functor F induces a ring homomorphism $F_A\colon \mathrm{End}_{\mathcal{A}}(A) \to \mathrm{End}_{\mathcal{B}}(F(A))$ into the semisimple artinian ring $\mathrm{End}_{\mathcal{B}}(F(A))$. Homomorphisms of rings into semisimple artinian rings are the object of study of [13]. Every homomorphism of a ring R into a semisimple artinian ring induces a projective rank function on R.

(4) For every semisimple artinian ring S, let $c(S)$ denote the composition length of the semisimple right S-module S_S, which is equal to the composition

length of the semisimple left S-module ${}_SS$. For instance, for every finite dimensional right vector space V_k over a division ring k, $c(\mathrm{End}(V_k))$ is equal to the dimension of the vector space V_k. In particular, if V_k, W_k are finite dimensional right vector spaces over k, $c(\mathrm{End}(V_k \oplus W_k)) = c(\mathrm{End}(V_k)) + c(\mathrm{End}(W_k))$. Therefore, if B, B' are objects of an amenable semisimple category $\mathcal{B}$, then $c(\mathrm{End}_{\mathcal{B}}(B \oplus B')) = c(\mathrm{End}_{\mathcal{B}}(B)) + c(\mathrm{End}_{\mathcal{B}}(B'))$. It follows if $F\colon \mathcal{A} \to \mathcal{B}$ is an additive functor of an additive category $\mathcal{A}$ into an amenable semisimple category $\mathcal{B}$ and every object of $\mathcal{B}$ is isomorphic to $F(A)$ for some object A of $\mathcal{A}$, then F canonically induces a discrete valuation $v\colon \mathrm{Ob}\,\mathcal{A} \to \mathbb{N}_0$, defined by $v(A) = c(\mathrm{End}_{\mathcal{B}}(F(A)))$ for every object A of $\mathcal{A}$.

For every additive category $\mathcal{A}$, there exists a functor $F\colon \mathcal{A} \to \hat{\mathcal{A}}$ into an additive category $\hat{\mathcal{A}}$ in which idempotents split, uniquely determined up to a categorical equivalence, with the following universal property: for every functor $G\colon \mathcal{A} \to \mathcal{A}'$ of $\mathcal{A}$ into an additive category $\mathcal{A}'$ in which idempotents split, there exists a unique functor $H\colon \hat{\mathcal{A}} \to \mathcal{A}'$ such that $G = HF$. The category $\widehat{\mathcal{A}}$ is called an *idempotent completion* of $\mathcal{A}$. To prove the existence of the idempotent completion of $\mathcal{A}$, take as objects of $\hat{\mathcal{A}}$ the pairs (A, e), where A is an object of $\mathcal{A}$ and e is an idempotent of $\mathrm{End}_{\mathcal{A}}(A)$, and as morphisms $(A, e) \to (B, f)$ the morphisms $\varphi\colon A \to B$ in $\mathcal{A}$ such that $f\varphi e = \varphi$. Thus $\widehat{\mathcal{A}}((A, e), (B, f))$ is a subgroup of $\mathcal{A}(A, B)$. Define the functor $F\colon \mathcal{A} \to \hat{\mathcal{A}}$ by $F(A) = (A, 1_A)$ for every object A of $\mathcal{A}$. Notice that the functor F is full and faithful, hence, a fortiori, isomorphism reflecting and local.

Theorem 7.2. *The following conditions are equivalent for a skeletally small additive category $\mathcal{A}$:*

(a) *The endomorphism ring* $\mathrm{End}_{\mathcal{A}}(A)$ *of every object of $\mathcal{A}$ is a semisimple artinian ring.*
(b) *There exist a set $\{k_j \mid j \in J\}$ of division rings and a full and faithful functor $H\colon \mathcal{A} \to \coprod_{j \in J} \text{vect-}k_j$.*

Proof. (a) $\Rightarrow$ (b). Assume that (a) holds for the category $\mathcal{A}$. Then (a) also holds for the idempotent completion $\hat{\mathcal{A}}$. Apply Theorem 7.1 to the skeletally small additive category $\widehat{\mathcal{A}}$, so that there exist a set $\{k_j \mid j \in J\}$ of division rings and an equivalence $G\colon \widehat{\mathcal{A}} \to \coprod_{j \in J} \text{vect-}k_j$. The composite functor $H = GF$ of G and the canonical functor $F\colon \mathcal{A} \to \hat{\mathcal{A}}$ is full and faithful.

(b) $\Rightarrow$ (a) is obvious. □

8. Categories Whose Endomorphism Rings Are Semilocal

As we have already remarked in the Introduction, the main source of examples of classes C of modules with $V(C)$ a Krull monoid is given by the classes C of modules whose endomorphism rings are semilocal. In this section, we shall determine the categorical interpretation of this fact.

Theorem 8.1. *Let $\mathcal{A}$ be an additive category with Jacobson radical J. Let $G\colon \mathcal{A} \to \widehat{\mathcal{A}/J}$ be the canonical functor of $\mathcal{A}$ into the idempotent completion $\widehat{\mathcal{A}/J}$ of the factor category $\mathcal{A}/J$. Then G is a full, isomorphism reflecting, local functor. If, moreover, idempotents split in $\mathcal{A}$, then G is also direct-summand reflecting.*

Proof. The objects of $\widehat{\mathcal{A}/J}$ are the pairs $(A, \bar{\varphi})$, where A is an object of $\mathcal{A}$, $\varphi\colon A \to A$ is an endomorphism of A in $\mathcal{A}$ and $\bar{\varphi} = \varphi + J(A,A)$ is an idempotent of $\mathcal{A}(A,A)/J(A,A) = \operatorname{End}_{\mathcal{A}}(A)/J(\operatorname{End}_{\mathcal{A}}(A))$. The morphisms $(A, \bar{\varphi}) \to (B, \bar{\psi})$ in $\widehat{\mathcal{A}/J}$ are the cosets $\bar{f} = f + J(A,B)$, where $f\colon A \to B$ is a morphisms in $\mathcal{A}$ such that $\psi f \varphi - f \in J(A,B)$.

The canonical functor $G\colon \mathcal{A} \to \widehat{\mathcal{A}/J}$ is the composite functor of

(1) the functor $\mathcal{A} \to \mathcal{A}/J$, which is full, isomorphism reflecting, local and, when idempotents split in $\mathcal{A}$, also direct-summand reflecting (Proposition 4.2), and
(2) the functor $\mathcal{A}/J \to \widehat{\mathcal{A}/J}$, which is full, faithful, isomorphism reflecting and local (last sentence before the statement of Theorem 7.2).

Thus G is full, isomorphism reflecting and local.

Now assume that idempotents split in $\mathcal{A}$. Let A, A' be a pair of objects of $\mathcal{A}$ with $G(A)$ isomorphic to a direct summand of $G(A')$. Let $(K, \bar{\omega}) \in \operatorname{Ob} \widehat{\mathcal{A}/J}$ be such that $G(A) \oplus (K, \bar{\omega}) \cong G(A')$. Then there are morphisms $\bar{f} = f + J(A,A')\colon (A, \overline{1_A}) \to (A', \overline{1_{A'}})$ and $\bar{g} = g + J(A',A)\colon (A', \overline{1_{A'}}) \to (A, \overline{1_A})$ with $\overline{gf} = \overline{1_A}$ and $\ker \bar{g} = \ker \overline{fg} = (K, \bar{\omega})$; cf. Lemma 2.1. Then $1_A - gf \in J(\operatorname{End}_{\mathcal{A}}(A))$, so that gf is invertible in the ring $\operatorname{End}_{\mathcal{A}}(A)$. Thus $f(gf)^{-1}g\colon A' \to A'$ is idempotent. In the proof of Lemma 2.1, we have seen that to get the kernel of an idempotent e it is sufficient to write the idempotent $1 - e$ as $k\ell$ with $\ell k = 1$, because then k is necessarily the kernel of e. Apply this remark to the idempotent $f(gf)^{-1}g$. Thus write $1_{A'} - f(gf)^{-1}g = k\ell$ for some $\ell\colon A' \to B$ and some $k\colon B \to A'$ with $\ell k = 1_B$, so that k is the kernel of $f(gf)^{-1}g$. Applying the functor G we get that, for the idempotent $\overline{f(gf)^{-1}g} = \overline{fg}$, one has $\overline{1_{A'}} - \overline{fg} = \overline{k\ell}$ with $\overline{\ell k} = \overline{1_B}$, so that $\bar{k}\colon G(B) \to G(A')$ is the kernel of $\overline{fg}$. As kernels are unique up to isomorphism, we conclude that $G(B) \cong (K, \bar{\omega})$. In particular, this proves that G is direct-summand reflecting, because k kernel of the idempotent $f(gf)^{-1}g$ implies $A \oplus B \cong A'$ by Lemma 2.1. □

Notice that in the proof of Theorem 8.1 we have proved that, when idempotents split in $\mathcal{A}$, if A, A' are objects in $\mathcal{A}$ such that there exists $(K, \bar{\omega}) \in \operatorname{Ob} \widehat{\mathcal{A}/J}$ with $G(A) \oplus (K, \bar{\omega}) \cong G(A')$, there exists $B \in \operatorname{Ob} \mathcal{A}$ such that $A \oplus B \cong A'$ and $G(B) \cong (K, \bar{\omega})$. This is stronger than the last sentence in the statement of Theorem 8.1.

When the endomorphism rings $\operatorname{End}_{\mathcal{A}}(A)$ are all semilocal, the functor $G\colon \mathcal{A} \to \widehat{\mathcal{A}/J}$ considered in Theorem 8.1 maps $\mathcal{A}$ to the particularly good category $\widehat{\mathcal{A}/J}$, as the next results show.

Theorem 8.2. *Let $\mathcal{A}$ be a skeletally small additive category with Jacobson radical J and with the property that* $\mathrm{End}_{\mathcal{A}}(A)$ *is a semilocal ring for every object A of $\mathcal{A}$. Then the idempotent completion $\widehat{\mathcal{A}/J}$ of the factor category $\mathcal{A}/J$ is an amenable semisimple category.*

Proof. As the endomorphism ring of every object in $\mathcal{A}$ is semilocal, the endomorphism ring of every object in $\mathcal{A}/J$ is semisimple artinian, so that $\widehat{\mathcal{A}/J}$ is an amenable semisimple category by Theorem 7.1. □

For example, let R be an arbitrary ring and let $\mathcal{A}$ be the full subcategory of Mod-R whose objects are all modules of finite composition length, so that endomorphism rings of objects of $\mathcal{A}$ are semiperfect and the Krull-Schmidt Theorem holds for the modules in $\mathcal{A}$. Let $\{M_i \mid i \in I\}$ be a set of representatives of the indecomposable objects of $\mathcal{A}$, so that $V(\mathcal{A}) \cong \mathbb{N}_0^{(I)}$. Notice that the fact that $V(\mathcal{A}) \cong \mathbb{N}_0^{(I)}$ does not correspond to a decompostion of $\mathcal{A}$ as a product or a coproduct of categories $\mathcal{A}_i$'s, essentially because there can be non-zero morphisms $M_i \to M_j$ for $i \neq j$. An isomorphism reflecting functor $F\colon \mathcal{A} \to \coprod_{i\in I} \mathcal{A}_i$ satisfying condition (b) in the statement of Theorem 6.2 is given by the canonical projection $\mathcal{A} \to \mathcal{A}/J(\mathcal{A})$. Notice that $\mathcal{A}/J(\mathcal{A})$ is the coproduct of the categories vect-k_i, where k_i is the division ring obtained factoring out the Jacobson radical of the local ring $\mathrm{End}_R(M_i)$. Thus $\mathcal{A}$ is not a product or a coproduct of categories, but the existence of the isomorphism reflecting functor $F\colon \mathcal{A} \to \coprod_{i\in I}$ vect-k_i and the fact that every object of $\coprod_{i\in I}$ vect-k_i is isomorphic to an object in the image of the functor F are sufficient for the Krull-Schmidt Theorem to hold.

Theorem 8.3. *Let $\mathcal{A}$ be a skeletally small additive category. Let F be an additive functor of $\mathcal{A}$ into an amenable semisimple category $\mathcal{B}$. If either*

(1) *F is direct-summand reflecting, or*
(2) *idempotents split in $\mathcal{A}$, and F is local,*

then $V(\mathcal{A})$ is a Krull monoid.

Proof. The functor $F\colon \mathcal{A} \to \mathcal{B}$ induces a monoid homomorphism $V(F)\colon V(\mathcal{A}) \to V(\mathcal{B})$. The monoid $V(\mathcal{B})$ is free because $\mathcal{B}$ is amenable semisimple.

The functor F is direct-summand reflecting if and only if $V(F)$ is a divisor homomorphism. This concludes the proof in case (1).

If idempotents split in $\mathcal{A}$ and F is local, then F induces a local ring homomorphism $\mathrm{End}_{\mathcal{A}}(A) \to \mathrm{End}_{\mathcal{B}}(F(A))$ for every object A of $\mathcal{A}$. Thus the endomorphism rings of all objects of $\mathcal{A}$ are semilocal rings [4]. By Theorem 8.2, there is a direct-summand reflecting functor G of $\mathcal{A}$ into the amenable semisimple category $\widehat{\mathcal{A}/J}$. Thus $V(\mathcal{A})$ is a Krull monoid by case (1). □

Remark. We have already seen in Remark (1) of Section 7 that if $\mathcal{A}$ is an additive category, A is an object of $\mathcal{A}$ and $E := \mathrm{End}_{\mathcal{A}}(A)$, then the functor $\mathcal{A}(A, -)$ is a full and faithful functor of the category $\mathrm{add}(A)$ into the category proj-E of all finitely generated projective right E-modules. If, moreover, idempotents split in

$\mathcal{A}$, then this functor is an equivalence $\mathrm{add}(A) \cong \text{proj-}E$ of categories. In other words, the decompositions of an object A of an additive category $\mathcal{A}$ in which idempotents split correspond to the decompositions of the finitely generated projective right E-module E_E. This fact can be generalized from one object A to the whole category $\mathcal{A}$ as follows.

Let $\mathcal{A}$ be a skeletally small preadditive category. Let Mod-$\mathcal{A}$ be the category of all additive contravariant functors of $\mathcal{A}$ to the category Ab of all abelian groups, so that Mod-$\mathcal{A}$ is a Grothendieck category. The Yoneda functor $\mathcal{A} \to \text{Mod-}\mathcal{A}$, which maps an object A of $\mathcal{A}$ to $\mathcal{A}(-, A)$ is a covariant functor, and the image $\mathcal{A}(-, A)$ of each object A of $\mathcal{A}$ is a finitely generated projective object of Mod-$\mathcal{A}$. Thus every skeletally small preadditive category $\mathcal{A}$ can be viewed as a full subcategory of the category proj-$\mathcal{A}$ of all finitely generated projective objects of the Grothendieck category Mod-$\mathcal{A}$, and, after this identification, $\mathrm{add}(\mathcal{A}) = \text{proj-}\mathcal{A}$, in the sense that every finitely generated projective object of Mod-$\mathcal{A}$ is isomorphic to a direct summand of a finite coproduct $\coprod_{i=1}^{n} \mathcal{A}(-, B_i)$ for suitable $B_i \in \mathrm{Ob}\,\mathcal{A}$. Thus, if $\mathcal{A}$ is a skeletally small additive category, then the idempotent completion $\hat{\mathcal{A}}$ of $\mathcal{A}$ is equivalent to the category proj-$\mathcal{A}$ of all finitely generated projective objects of a Grothendieck category. In particular, every skeletally small additive category $\mathcal{A}$ in which idempotents split is equivalent to the category proj-$\mathcal{A}$ of all finitely generated projective objects of the Grothendieck category Mod-$\mathcal{A}$.

We conclude noticing that other techniques allow us to construct Krull monoids whose elements are isomorphism classes of objects of an abelian category. For instance, it is well known that if A, B are objects of an abelian category $\mathcal{A}$, and A and $A \oplus B$ have an injective envelope, then B has an injective envelope. Therefore, let $\mathcal{A}$ be an abelian category, and let $\mathrm{Ob}\,\mathcal{A}$ be its class of objects. Let C be a subclass of $\mathrm{Ob}\,\mathcal{A}$ closed under isomorphism and finite direct sums. Let C' be the class of all objects in C with an injective envelope in $\mathcal{A}$. Then the embedding $V(C') \to V(C)$ is a "divisor homomorphism." Therefore, if $V(C)$ is a "Krull monoid," then so is $V(C')$.

For instance, let R be any ring and C be the class of all finitely generated semisimple right R-modules. Then $V(C)$ is a free monoid, so that if C' is the class of all finitely generated semisimple right R-modules with a projective cover, the monoid $V(C')$ turns out to be a reduced Krull monoid.

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