Mathematical Study of a Parabolic System Describing the Evolution of the Solar Magnetic Field

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Summary. - *We study a system of two strongly coupled parabolic equations describing a solar dynamo wave. We investigate the existence and uniqueness of a classical solution and the existence of a periodic in time solution. In the Appendix, an existence result of periodic solutions for an auxiliary quasilinear parabolic equation is provided, together with a C^0 estimate of such solution.*

1. Introduction

In this paper we deal with the mathematical study of a system of two strongly coupled parabolic equations describing the evolution of the magnetic field of a solar dynamo wave.

It is nowadays accepted that the origin of the solar magnetic field could be ascribed to the existence of a dynamo mechanism operating in the convective region of the Sun. The dynamo process basically consists in the conversion of kinetic energy of plasma motions into

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magnetic field energy, as the result of the interplay between differential rotation and turbulent convective motions. Many different models have been proposed [9, 3], but particular attention has been payed to the so called $\alpha^2\omega$ dynamo model ([12, 2] and the references therein). The properties of this model make it attractive to explain some of the features of the solar and stellar magnetic activity.

Besides all these results, the mathematical properties of this model remain still largely unexplored, with respect to the existence and uniqueness of the solutions and in particular with respect to the periodic solutions corresponding to periodic boundary data which is of particular interest due to the well known periodic behavior of the solar activity.

In this paper we want to study from the mathematical point of view the evolution of the magnetic field B as described by an $\alpha^2\omega$ dynamo model.

The magnetic field vector $\mathbf{B}$ is usually represented as the sum of toroidal and poloidal components, $\mathbf{B} = \mathbf{B}_P + \mathbf{B}_T$, where

$$\mathbf{B}_P = \nabla \times (0, 0, A), \quad \mathbf{B}_T = (0, 0, B). \tag{1}$$

We follow the model described by Meunier [14] which has been developped by A. Bianchini and L. Zangrilli [1].

We adopt a system of spherical coordinates (r, θ, φ), which is suitable to study a dynamo operating inside a star. Taking into account the axisymmetry of the problem (so that the dependence on the φ coordinate can be accordingly neglected), setting $x = r$, $y = \theta$ the system to be studied is [12, 13]:

$$\frac{\partial A}{\partial t} = \frac{c_1 B}{1 + \mu B^2} + \frac{1}{\lambda^2}\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2}, \tag{2}$$

$$\frac{\partial B}{\partial t} = \frac{c_2}{1 + kB^2}\frac{\partial A}{\partial y} - \frac{c_3}{1 + \mu B^2}\frac{\partial^2 A}{\partial x^2} + \frac{1}{\lambda^2}\frac{\partial^2 B}{\partial x^2} + $$
$$+ \frac{\partial^2 B}{\partial y^2} - c_4 B^3, \tag{3}$$

where λ, μ, c_1, c_2, c_3, c_4 are suitable constants depending on the specific properties of the model.

The dissipation term $-c_4 B^3$ in equation (3) is motivated in [13] and it is due to the bouyancy of magnetic flux tubes, which escape from the region where the dynamo operates.

The system is studied in $Q_T = Q \times (0,T)$, $Q := \{(x,y) :\ r_0 < x < r_s,\ 0 < y < \pi\}$, where r_0, r_s correspond to the radius of the convective zone and the total solar radius. Equations (2)-(3) represent an uniformly nonlinear parabolic system whose equations for A and B are strongly coupled because of the presence of the term $\partial^2 A/\partial x^2$ in equation (3). For these reason we can not apply the classical results on systems of uniformly parabolic equations. On the parabolic boundary of Q_T we consider Dirichlet conditions [8]. This type of conditions are just the first case among some others that can be considered. An other type of conditions which will be studied in a next paper is a coupled system of equations for A and B, satisfied on the parabolic boundary of Q_T.

In this paper we deal with a proof of the existence and uniqueness of a classical solution for (2)-(3) with Dirichlet conditions on the parabolic boundary. We also prove the existence of a periodic in time classical solution, having prescribed periodic boundary data. Our results remain true even if the constant c_4 in equation (3) is zero, i.e. also for the original model problem of Meunier [12], hence our analysis can be applied to the simplified model described by several authors.

The paper is organized as follows: in Section 2 the existence and uniqueness of a global in time classical solution is proved. In Section 3 we present a result about the existence of a periodic in time solution. To obtain this result, we use an existence theorem for periodic solutions of an auxiliary quasilinear parabolic equation which is proved in the Appendix. This is a general result on the existence of periodic solutions for quasilinear equations in $W_q^{2,1}$ which provide also a C^0 estimate for such solution. We remark that such existence result can be obtained combining different methods presented in the papers of [4], [5], [7] and [11].

2. Existence of solutions of $\alpha^2 \omega$ dynamo equations

We will study the following system of parabolic equations, obtained from system (2)-(3) by adding into the first and the second equation respectively the terms $-cA$ and $-cB$, where c is a nonnegative

constant:

$$\frac{\partial A}{\partial t} = \frac{c_1 B}{1 + \mu B^2} + \frac{1}{\lambda^2}\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} - cA, \tag{4}$$

$$\frac{\partial B}{\partial t} = \frac{c_2}{1 + kB^2}\frac{\partial A}{\partial y} - \frac{c_3}{1 + \mu B^2}\frac{\partial^2 A}{\partial x^2} + \frac{1}{\lambda^2}\frac{\partial^2 B}{\partial x^2} +$$

$$+ \frac{\partial^2 B}{\partial y^2} - c_4 B^3 - cB, \tag{5}$$

with initial and boundary conditions:

$$
\begin{aligned}
A(x,y,0) &= A_0(x,y), \text{ on Q,}\\
B(x,y,0) &= B_0(x,y), \text{ on Q,} \qquad\qquad (6)\\
A(x,y,t) &= A_L(x,y,t), \text{ on } \partial Q \times (0,T),\\
B(x,y,t) &= B_L(x,y,t), \text{ on } \partial Q \times (0,T), \qquad (7)
\end{aligned}
$$

where A_0, B_0, A_L, B_L are such that

$$
\begin{aligned}
&A_0(x,y), \ B_0(x,y) \in C^{2,1}(\overline{Q}),\\
&A_L(x,y,t), \ B_L(x,y,t) \in C^{2,1}(\overline{Q} \times [0,T)),\\
&\text{the zero order compatibility conditions hold,} \qquad (8)\\
&\text{equations } (4,5) \text{ are fulfilled on the hedges of } \overline{Q} \times [0,T].
\end{aligned}
$$

The following inequality holds for any B:

$$\left|\frac{c_1 B}{1 + \mu B^2}\right| + \left|\frac{c_2}{1 + kB^2}\right| + \left|\frac{c_3}{1 + \mu B^2}\right| \leq \frac{c_1}{2\sqrt{\mu}} + c_2 + c_3 \equiv: K. \tag{9}$$

We give an existence and uniqueness result for a global in time solution of system (4)-(7) using Schauder's fixed point theorem.

THEOREM 2.1. *Under assumptions (8), taking c_i, $i = 1,...4$, k, c nonnegative constants, $\mu > 0$, $\lambda \neq 0$, the parabolic system (4)-(5), with initial and boundary conditions (6), (7), has an unique classical solution $(A, B) \in C^{2,1}(\overline{Q}_T)$.*

Proof. Consider

$$\Sigma = \{B(x,y,t) \in C^0(Q_T), |B|_{Q_T}^{(0)} \leq N\},$$

where N will be determined below.

Clearly the set Σ is a closed convex set.

We define an operator $\mathcal{T}$ on Σ in the following: $\forall \overline{B} \in \Sigma$, $\mathcal{T}\overline{B} = \tilde{B}$, where $(\tilde{A}, \tilde{B})$ is the solution of the following system:

$$\frac{\partial \tilde{A}}{\partial t} = \frac{c_1 \overline{B}}{1 + \mu \overline{B}^2} + \frac{1}{\lambda^2} \frac{\partial^2 \tilde{A}}{\partial x^2} + \frac{\partial^2 \tilde{A}}{\partial y^2} - c\tilde{A}, \tag{10}$$

$$\frac{\partial \tilde{B}}{\partial t} = \frac{c_2}{1 + k\overline{B}^2} \frac{\partial \tilde{A}}{\partial y} - \frac{c_3}{1 + \mu \overline{B}^2} \frac{\partial^2 \tilde{A}}{\partial x^2} + \frac{1}{\lambda^2} \frac{\partial^2 \tilde{B}}{\partial x^2} +$$

$$+ \frac{\partial^2 \tilde{B}}{\partial y^2} - c_4 \tilde{B}^3 - c\tilde{B}, \tag{11}$$

$$\tilde{A}(x,y,0) = A_0(x,y), \text{ on Q},$$
$$\tilde{B}(x,y,0) = B_0(x,y), \text{ on Q}, \tag{12}$$
$$\tilde{A}(x,y,t) = A_L(x,y,t), \text{ on } \partial Q \times (0,\text{T}),$$
$$\tilde{B}(x,y,t) = B_L(x,y,t), \text{ on } \partial Q \times (0,\text{T}). \tag{13}$$

The system (10)-(11) is no longer strongly coupled since equation (10) does not depend on $\tilde{B}$.

From Theorem 9.1 p.341 of [10], taking into account (9), we obtain that

$$\tilde{A}(x,y,t) \in W_q^{2,1}(Q_T), \quad \forall q > 1,$$

$$\|\tilde{A}\|_{q,Q_T}^{(2)} \le C(q), \quad \forall q > 1, \tag{14}$$

with $C(q)$ independent of N.

We want to remark that the regularity results in $W_q^{2,1}$ are obtained here and in the following even if ∂Q has corners. In fact, by subtracting to the solution $\tilde{A}$ the boundary data and by means of suitable reflections, we can reduce to a case to which the results of Section IV of [10] can be applied (see estimate 10.12 p.355), thus obtaining that Theorem 9.1 p.341 of [10] still holds. In particular we take advantage of the fact that the reflection technique allows us to get estimates near the boundary ∂Q, using only interior estimates.

We now pass to equation (11). First of all we prove that

$$|\tilde{B}|_{Q_T}^{(0)} \le C(\|\tilde{A}\|_{q,Q_T}^{(2)}) \le C(q), \tag{15}$$

where C is independent of N.

In fact, let us consider the solution $\hat{B}$ of the following problem

$$
\frac{\partial \hat{B}}{\partial t} = K\left|\frac{\partial \tilde{A}}{\partial y}\right| + K\left|\frac{\partial^2 \tilde{A}}{\partial x^2}\right| + \frac{1}{\lambda^2}\frac{\partial^2 \hat{B}}{\partial x^2} +
$$

$$
+ \frac{\partial^2 \hat{B}}{\partial y^2} - c_4 \hat{B}^3 - c\hat{B}, \tag{16}
$$

$$
\hat{B}(x,y,t)|_{\partial_p Q_T} = \max_{\partial_p Q_T} |\tilde{B}| =: M, \tag{17}
$$

where K in (16) is defined in (9).

Note that, from the maximum principle, $\hat{B} > 0$ and from the comparison principle $\hat{B} \geq \tilde{B}$. At this point, we consider the solution $\breve{B}$ of the linear problem

$$
\frac{\partial \breve{B}}{\partial t} = K\left|\frac{\partial \tilde{A}}{\partial y}\right| + K\left|\frac{\partial^2 \tilde{A}}{\partial x^2}\right| + \frac{1}{\lambda^2}\frac{\partial^2 \breve{B}}{\partial x^2} + \frac{\partial^2 \breve{B}}{\partial y^2}, \tag{18}
$$

$$
\breve{B}(x,y,t)|_{\partial_p Q_T} = M. \tag{19}
$$

From Theorem 7.1 p.181 of [10] we obtain

$$
|\breve{B}|_{Q_T}^{(0)} \leq C(\|\tilde{A}\|_{q,Q_T}^{(2)}) \leq C(q), \tag{20}
$$

where $C(q)$ is independent of N.

Moreover, from the comparison principle, we have

$$
\tilde{B} \leq \hat{B} \leq \breve{B},
$$

thus we obtain an upper bound for $\tilde{B}$, independent of N.

Analogously we find a lower bound for $\tilde{B}$, independent of N, i.e. estimate (15) is proved.

Regarding now $-c_4 \tilde{B}^3$ as a known bounded term because of (15), taking into account estimate (14), we apply to equation (11), Theorem 9.1 p. 341 and the Corollary at p.342 of [10]. Taking $q > 4$, we obtain an Hölder estimate for $\tilde{B}$, i.e.:

$$
|\tilde{B}|_{Q_T}^{(1+\alpha)} \leq C(1 + \|\tilde{A}\|_{q,Q_T}^{(2)}) \leq C(\alpha), \ \alpha = 1 - \frac{4}{q}, \tag{21}
$$

with C is independent of N.

If we take N, in the definition of Σ, such that $N > C(\alpha)$, ($C(\alpha)$ as in (21)), we obtain that $\mathcal{T}$ maps Σ into itself.

Moreover, from (21), we also get that $\mathcal{T}\Sigma$ is a precompact subset of Σ.

Next we show that $\mathcal{T}$ is a continuous mapping. Let be $\tilde{B}_1 = \mathcal{T}\overline{B}_1$ and $\tilde{B}_2 = \mathcal{T}\overline{B}_2$, we denote by $\tilde{A}_1$ and $\tilde{A}_2$ the solutions of (10) corresponding respectively to $\overline{B}_1$ and $\overline{B}_2$.

From Theorem 9.1 p.341 of [10], we have

$$\|\tilde{A}_1 - \tilde{A}_2\|_{q,Q_T}^{(2)} \leq C(N)(|\overline{B}_1 - \overline{B}_2|_{Q_T}^{(0)}), \ \forall q > 1. \qquad (22)$$

The equation satisfied by $\tilde{B}_1 - \tilde{B}_2$ is

$$\left(\frac{\partial(\tilde{B}_1 - \tilde{B}_2)}{\partial t} - \frac{1}{\lambda^2}\frac{\partial^2(\tilde{B}_1 - \tilde{B}_2)}{\partial x^2} - \frac{\partial^2(\tilde{B}_1 - \tilde{B}_2)}{\partial y^2} + \right.$$
$$a_1(\tilde{B}_1, \tilde{B}_2)(\tilde{B}_1 - \tilde{B}_2) \Big) = \left(a_2(\overline{B}_1, \overline{B}_2)\frac{\partial(\tilde{A}_1 - \tilde{A}_2)}{\partial y} + \right.$$
$$a_3(\overline{B}_1, \overline{B}_2)\frac{\partial^2(\tilde{A}_1 - \tilde{A}_2)}{\partial x^2} +$$
$$\left. a_4(\overline{B}_1, \overline{B}_2, \tilde{A}_{2y}, \tilde{A}_{2xx})(\overline{B}_1 - \overline{B}_2) \right). \qquad (23)$$

The coefficients a_i, $i = 1, \ldots 4$, in (23), are not explicitely written, for the sake of brevity and are, however, bounded in L_q, for any $q > 1$, because of (14)-(15). Regarding the terms inside the square brackets as known we can apply Theorem 9.1 p. 341 of [10], thus obtaining

$$\begin{aligned} \|\tilde{B}_1 - \tilde{B}_2\|_{q,Q_T}^{(2)} &\leq C(N)\left(|\overline{B}_1 - \overline{B}_2|_{Q_T}^{(0)} + \|\tilde{A}_1 - \tilde{A}_2\|_{q,Q_T}^{(2)} \right) \\ &\leq C(N)\left(|\overline{B}_1 - \overline{B}_2|_{Q_T}^{(0)} \right), \ \forall q > 1, \end{aligned} \qquad (24)$$

where the last inequality is obtained from (22).

From Corollary at p.342 of [10] and applying Schauder's fixed point theorem we conclude the existence of a function B in Σ for which $\mathcal{T}B = B$; $(A, B) \in C^0(Q_T)$.

At this point with a standard bootstrap argument we obtain further regularity for the solution (A, B) of problem (4), (7), more precisely we prove that $(A, B) \in C^{2,1}(\overline{Q}_T)$.

Moreover, from (24), we obtain that

$$|\tilde{B}_1 - \tilde{B}_2|_{Q_T}^{(0)} \le t^\gamma\, C(N) \left(|\overline{B}_1 - \overline{B}_2|_{Q_T}^{(0)} \right),\ \gamma < 1. \qquad (25)$$

Hence, taking $t \in [0, \tilde{T}]$, with $\tilde{T}$ sufficiently small, we get that the operator $\mathcal{T}$ is contractive and then in $[0, \tilde{T}]$ we also get uniqueness of the solution. Iterating step by step this procedure we obtain the uniqueness of the solution in whole Q_T. $\qquad\square$

3. Periodic solutions

In this section we investigate the existence of a periodic solution for our problem when periodic boundary data are assigned.

More precisely, we look for an existence theorem of a τ-periodic solution of system (4)-(5), with boundary data

$$A(x, y, t) = A_L(x, y, t),\ B(x, y, t) = B_L(x, y, t),\ \text{ on } \partial Q \times \mathbf{R}. \quad (26)$$

under the following assumptions

$$A_L(x, y, t),\ B_L(x, y, t) \in C^{2,1}(Q \times \mathbf{R}), \qquad (27)$$
$$A_L(x, y, t + \tau) = A_L(x, y, t),$$
$$B_L(x, y, t + \tau) = B_L(x, y, t),\ \tau > 0. \qquad (28)$$

As a first step we study problem (4)-(5) in the case in which $c > 0$.

We prove the following:

THEOREM 3.1. *Under assumptions (27), (28), if c_i, $i = 1,...4$, k nonnegative constants, $c > 0$, $\mu > 0$, $\lambda \ne 0$, the parabolic system (4)-(5), with boundary conditions (26), has at least a τ-periodic solution $(A, B) \in C^{2,1}(\overline{Q} \times \mathbf{R})$.*

Proof. Consider

$$\Sigma = \{B \in C^0(Q \times \mathbf{R}), |B|_{Q \times [0,\tau]}^{(0)} \le N,\ B(x, y, t + \tau) = B(x, y, t)\},$$

where N will be determined below.

We consider the operator $\mathcal{T}$ defined in the following way: if $\overline{B} \in \Sigma$, $\mathcal{T}\overline{B} = \tilde{B}$, where $(\tilde{A}, \tilde{B})$ is the solution of the following system in $Q \times \mathbf{R}$:

$$\frac{\partial \tilde{A}}{\partial t} = \frac{c_1}{1+\mu\overline{B}^2}\overline{B} + \frac{1}{\lambda^2}\frac{\partial^2 \tilde{A}}{\partial x^2} + \frac{\partial^2 \tilde{A}}{\partial y^2} - c\tilde{A}, \qquad (29)$$

$$\frac{\partial \tilde{B}}{\partial t} = \frac{c_2}{1+k\overline{B}^2}\frac{\partial \tilde{A}}{\partial y} - \frac{c_3}{1+\mu\overline{B}^2}\frac{\partial^2 \tilde{A}}{\partial x^2} + \frac{1}{\lambda^2}\frac{\partial^2 \tilde{B}}{\partial x^2} +$$

$$+\frac{\partial^2 \tilde{B}}{\partial y^2} - c_4\tilde{B}^3 - c\tilde{B}, \qquad (30)$$

$$\tilde{A}(x,y,t) = A_L(x,y,t), \quad \text{on } \partial Q \times \mathbf{R}, \qquad (31)$$

$$\tilde{B}(x,y,t) = B_L(x,y,t), \quad \text{on } \partial Q \times \mathbf{R}. \qquad (32)$$

Applying Theorem 4.1 of the Appendix to equation (29), we get that a τ-periodic solution of (29), (31), $\tilde{A} \in H^{1+\beta,\frac{1+\beta}{2}}(\overline{Q} \times \mathbf{R})$ exists such that

$$|\tilde{A}|_{Q\times\mathbf{R}}^{(1+\beta)} \leq M(\beta), \quad \forall\beta \in (0,1), \qquad (33)$$

$$\|\tilde{A}\|_{p,\Omega\times[0,\tau]}^{(2)} \leq M(p), \quad \forall p > 1. \qquad (34)$$

The quantity $M(p)$ in (34) is independent of N because of (9).

Regarding the term $\frac{c_2}{1+k\overline{B}^2}\frac{\partial \tilde{A}}{\partial y} - \frac{c_3}{1+\mu\overline{B}^2}\frac{\partial^2 \tilde{A}}{\partial x^2}$ as a known source term belonging to L_p, problem (30), (32) satisfies assumptions of Theorem 4.1 of the Appendix, hence a periodic solution $\tilde{B}$ exists, $\tilde{B}(x,t) \in H^{1+\beta,\frac{1+\beta}{2}}(\overline{Q} \times \mathbf{R})$.

Following a procedure similar to the one employed in the previous section, taking into account estimates (34) and (68), (67) of the Appendix, we obtain:

$$\|\tilde{B}\|_{p,Q\times[0,\tau]}^{(2)} \leq M(p), \qquad (35)$$

$$|\tilde{B}|_{Q\times\mathbf{R}}^{(1+\beta)} \leq M(\beta), \qquad (36)$$

where $M(p)$ and $M(\beta)$ are independent of N.

Taking $N > M(\beta)$ we obtain that $\mathcal{T}$ maps Σ into itself.
Finally, working as in Section 3, we prove the continuity of the operator $\mathcal{T}$, which implies, by means of the Schauder fixed point Theorem,

the existence of a fixed point for $\mathcal{T}$ and then a periodic solution of problem (4), (5), (26), $(A, B) \in C^{2,1}(\overline{Q} \times \mathbf{R})$.

Note that estimates (34) and (35) hold on $Q \times [0, \tau]$, where τ is the period of the boundary data, hence we can not prove uniqueness as done in Section 3. $\qquad\square$

We now treat the case in which $c = 0$. We obtain an existence result of a periodic solution via a compactness argument, passing to the limit when c tends to zero, making also use of (70).

THEOREM 3.2. *Under assumptions (27) (28), if $c = 0$ and c_i, $i = 1,...4$, k nonnegative constants, $\mu > 0$, $\lambda \neq 0$, the parabolic system (4)-(5), with boundary data (26), has at least a τ-periodic solution $(A, B) \in C^{2,1}(\overline{Q} \times \mathbf{R})$.*

Proof. From Theorem 4.1, there exists a periodic solution (A_n, B_n) of the system (4)-(5) with $c = \frac{1}{n}$, $n \in \mathbf{N}$ and boundary data (26). From estimate (70), taking into account (9), we have

$$|A_n|_{Q \times \mathbf{R}}^{(0)} \leq C(K), \tag{37}$$

where C is independent of n. Using estimate (10.12) p.355 of [10] and (37), we get that

$$\|A_n\|_{q, Q \times [t, t+\tau]}^{(2)} \leq C(K, q), \ \forall t \in \mathbf{R}, \ q > 1, \tag{38}$$

where C is independent of n.

At this point, applying estimate (70) to equation (5) and taking into account (38) which allows us to consider the terms containing A_n as known terms, we get that

$$|B_n|_{Q \times \mathbf{R}}^{(0)} \leq C(K), \tag{39}$$

and then, as done in (38):

$$\|B_n\|_{q, Q \times [t, t+\tau]}^{(2)} \leq C(K, q), \forall t \in \mathbf{R}, \quad q > 1. \tag{40}$$

where C is independent of n.

By means of an embedding theorem we obtain:

$$|A_n|_{Q \times \mathbf{R}}^{(1+\beta)} + |B_n|_{Q \times \mathbf{R}}^{(1+\beta)} \leq C(K, \beta), \ \beta \in (0, 1), \tag{41}$$

where C, as usual, is independent of n.

We can now extract a subsequence, which we still denote (A_n, B_n), converging, if $n \to \infty$, to a solution (A, B) of system (4)-(5) with $c = 0$ and boundary data (26). The functions A and B are τ-periodic functions such that

$$|A|_{Q \times \mathbf{R}}^{(1+\beta)} \leq C(K, \beta), \ \beta \in (0, 1), \tag{42}$$

$$|B|_{Q \times \mathbf{R}}^{(1+\beta)} \leq C(K, \beta), \ \beta \in (0, 1). \tag{43}$$

Finally, a bootstrap argument proves that $A, B \in C^{2,1}$, Theorem 3.2 is thus proved. $\qquad \square$

4. Appendix

In this Appendix, we consider the following parabolic equation

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= a_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} + b_i(x, t) \frac{\partial u}{\partial x_i} - bu^3 + \\
&\quad - c(x, t)u + f(x, t) \equiv L(u), \text{ on } \Omega \times \mathbf{R}, \tag{44} \\
u(x, t) &= h(x, t), \text{ on } \partial\Omega \times \mathbf{R}, \tag{45}
\end{aligned}
$$

where

$$
\begin{aligned}
&\Omega \subset \mathbf{R}^N, \\
&\Omega \text{ is a bounded smooth domain or} \\
&a \text{ cartesian product of intervals,} \\
&a_{ij}, b_i, c \in H^{\alpha, \frac{\alpha}{2}}(\Omega \times \mathbf{R}), \ \alpha \in (0, 1), \\
&|a_{ij}|_{\Omega \times \mathbf{R}}^{(\alpha)} + |b_i|_{\Omega \times \mathbf{R}}^{(\alpha)} + |c|_{\Omega \times \mathbf{R}}^{(\alpha)} \leq H, \\
&f \in L_{q,loc}(\Omega \times \mathbf{R}), \ q > N + 2, \\
&h \in C^{2,1}(\overline{\Omega} \times \mathbf{R}), \\
&a_{ij}(x, t) \geq \delta > 0, \ c(x, t) \geq \delta > 0, \ b \geq 0, \\
&a_{i,j}, b_i, c, f, h \text{ are } \tau - periodic.
\end{aligned} \tag{46}
$$

We can make more general assumptions on Ω but the previous ones are sufficient to our purpose.

THEOREM 4.1. *Under assumptions (46), the parabolic problem (44)-(45) has one and only one τ-periodic solution $\overline{u}(x,t)$ such that $\overline{u} \in H^{1+\beta,\frac{1+\beta}{2}}(\overline{\Omega} \times \mathbf{R}) \cap W_q^{2,1}(\overline{\Omega} \times \mathbf{R})$, $\forall q > N+2$, $\beta = 1 - \frac{N+2}{q}$.*

Proof. We introduce a suitable initial datum $u_0(x)$ satisfying the zero order compatibility conditions and the estimate

$$|u_0|_\Omega^{(2)} \leq M |h|_{\partial\Omega \times \mathbf{R}}^{(2)}. \qquad (47)$$

We now consider the problem (44)-(45) for $x \in \Omega$, $t \in [0,+\infty)$ with the initial condition

$$u(x,0) = u_0(x), \ x \in \Omega. \qquad (48)$$

Clearly this problem admits an unique solution $u^*(x,t)$.

Let us consider the following problem

$$\frac{\partial u}{\partial t} = a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + b_i\frac{\partial u}{\partial x_i} - bu^3 - cu + |f(x,t)|, \quad (49)$$
$$\text{on } \Omega \times (0,+\infty)$$
$$u(x,t) = \max\{\max_{\overline{\Omega}\times\mathbf{R}}|h|, \max_\Omega |u_0|\}$$
$$\equiv: \ B, \ \text{on } \partial_p(\Omega \times (0,+\infty)) \qquad (50)$$

We denote by $\tilde{u}$ the solution of problem (49), (50). We have that

$$\tilde{u}(x,t) \geq 0, \ \text{in } \Omega \times (0,+\infty)$$

and $\tilde{u} \geq u^*$.

Since the term $-b\tilde{u}^3$ is nonpositive, denoted by $\hat{u}$ the solution of the following problem:

$$\frac{\partial u}{\partial t} = a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + b_i\frac{\partial u}{\partial x_i} - cu + |f(x,t)|,$$
$$\text{on } \Omega \times (0,+\infty), \qquad (51)$$
$$u(x,t) = B, \ \text{on } \partial_p(\Omega \times (0,+\infty)), \qquad (52)$$

we have that $\hat{u} \geq \tilde{u}$. Applying Theorem 9.1 p.341 of [10] to problem (51), (52) and the embedding theorem of p.342 with $q > 1 + \frac{N}{2}$, we

obtain a C^0 estimate for $\hat{u}$, hence a upper bound for u^*. Analogously we can find a lower bound for u^*, hence we have that

$$|u^*|_{\Omega_T}^{(0)} \leq C(T, H, \|f\|_{q, \Omega \times (0,\tau)}, |h|_{\Omega \times \mathbf{R}}^{(2)}), \tag{53}$$

where $T \in (0, +\infty)$ is fixed.

We now prove an estimate for $u^*(x,t)$ which will be crucial in the following:

$$|u^*|_{\Omega \times [0,+\infty)}^{(0)} \leq C(H, \delta, \|f\|_{q, \Omega \times (0,\tau)}, |h|_{\Omega \times \mathbf{R}}^{(2)}). \tag{54}$$

Let us consider the sequence $u_n(x,t) :\equiv u^*(x, t + n\tau)$, $n \in \mathbf{N}$, $x \in \Omega$, $t \geq 0$.

We define $V_{n,m} :\equiv u_n - u_m = u^*(x, t + n\tau) - u^*(x, t + m\tau)$. Recalling that $a_{i,j}$, c, f, h are τ- periodic, $V_{n,m}$ satisfies:

$$\begin{aligned}
\mathcal{L}V_{n,m} \quad :\equiv \quad & \frac{\partial V_{n,m}}{\partial t} - a_{ij}\frac{\partial^2 V_{n,m}}{\partial x_i \partial x_j} - b_i \frac{\partial V_{n,m}}{\partial x_i} + \\
& + [c + b(u_n^2 + u_m^2 + u_n u_m)]V_{n,m} \\
= \quad & 0, \text{ on } \Omega \times (0, +\infty), \tag{55}\\
V_{n,m}(x,t) \quad = \quad & 0, \text{ on } \partial\Omega \times [0, +\infty), \tag{56}\\
V_{n,m}(x,0) \quad = \quad & u_n(x,0) - u_m(x,0), \; x \in \Omega. \tag{57}
\end{aligned}$$

As a first step, let us concentrate ourselves on $V_{1,0}(x,t) = u^*(x, t + \tau) - u^*(x,t)$, $t \in [0, \tau]$.

From (47) and (53), the initial datum $V_{1,0}(x,0) = u^*(x,\tau) - u_0(x)$ is bounded:

$$|V_{1,0}(x,0)|_{\Omega}^{(0)} \leq C + M|h|_{\partial\Omega \times \mathbf{R}}^{(2)} \equiv: C_1. \tag{58}$$

The function $z(x,t) :\equiv C_1 e^{-\delta t}$ is a supersolution of $V_{1,0}(x,t)$: in fact from (58), $z(x,0) \geq V_{1,0}(x,0)$, for any $x \in \Omega$ and $z(x,t) \geq V_{1,0}(x,t)$, for any $(x,t) \in \partial\Omega \times (0, +\infty)$. Moreover from the definition (55) of $\mathcal{L}$, $\mathcal{L}z = \left(b(u_1^2 + u_0^2 + u_1 u_0) + (c - \delta)\right)z$. From assumptions (46), we have that $\mathcal{L}z > 0 = \mathcal{L}V_{1,0}$ hence $z(x,t) = C_1 e^{-\delta t} \geq V_{1,0}(x,t)$. Analogously we can show that $-C_1 e^{-\delta t}$ is a subsolution of $V_{1,0}(x,t)$, then

$$\begin{aligned}
\max_{\Omega} |V_{1,0}(x,t)| \quad &\equiv \quad \max_{\Omega} |u(x, t + \tau) - u(x,t)| \\
&\leq \quad C_1 e^{-\delta t}, \; \forall t > 0. \tag{59}
\end{aligned}$$

At this point we observe that, for any $\overline{t} > 0$, there exists an n and a $t' \in [0, \tau]$ such that $\overline{t} = t' + n\tau$. Hence

$$
\begin{aligned}
|u^*(x, \overline{t})| & \leq |u^*(x, t' + n\tau) - u^*(x, t' + (n-1)\tau)| + \\
& \quad + |u^*(x, t' + (n-1)\tau)| \\
& \leq C_1 \exp\left(-\delta(t' + (n-1)\tau)\right) + \\
& \quad + |u^*(x, t' + (n-1)\tau)|, \quad (60)
\end{aligned}
$$

where the last inequality has been obtained from (59). Applying the previous procedure to $u^*(x, t' + (n-1)\tau)$ and iterating step by step, we get

$$
\begin{aligned}
|u^*(x, \overline{t})| & \leq C_1 \exp\left(-\delta t'\right)[\exp\left(-\delta(n-1)\tau\right) + \\
& \quad + \exp\left(-\delta(n-2)\tau\right) + \cdots + 1] + |u^*(x, t')|. \quad (61)
\end{aligned}
$$

Since $t' \in [0, \tau]$, the term $u^*(x, t')$ is bounded because of (53), hence

$$
|u^*(x, \overline{t})| \leq C_1 S + C \equiv K_1,
$$

where S is the sum of the geometrical series of ratio $\exp(-\delta\tau)$ and K_1 is independent of $\overline{t}$. Thus (54) is proved.

Now we prove that $\{u_n\}$ is a Cauchy sequence.

From (54), we get that

$$
|V_{n,m}(x, 0)|_{\Omega}^{(0)} \leq 2K_1,
$$

hence, proceeding as done to get (59), we obtain that $\pm 2K_1 e^{-\delta t}$ are super and sub solutions of $V_{n,m}$, i.e. :

$$
\max_{\Omega} |V_{n,m}(x, t)| = |u(x, t + n\tau) - u(x, t + m\tau)| \leq 2K_1 e^{-\delta t}, \ \forall t > 0.
$$

$$(62)$$

From estimate (62), we have that, for every $\overline{n}, \overline{m} \in \mathbf{N}$:

$$
|u_{\overline{n}}(x, t) - u_{\overline{m}}(x, t)| \leq 2K_1 e^{-\delta t}. \quad (63)
$$

Hence, for any $\epsilon > 0$, it is possible to choose a $\overline{T}$ such that, for any $t > \overline{T}$, $|u_{\overline{n}}(x, t) - u_{\overline{m}}(x, t)| \leq \epsilon$. Consequently if $n', m' \geq \max(\overline{n}, \overline{m}) + k$ (where k is such that $k\tau > \overline{T}$) we have

$$
|u_{n'}(x, t) - u_{m'}(x, t)| \leq \epsilon \ , \ \text{in } \Omega \times (0, +\infty)
$$

and then it is proved that $\{u_n\}$ is a Cauchy sequence, which implies that

$$\lim_n u_n = \overline{u}. \qquad (64)$$

Moreover

$$\begin{aligned}
\overline{u}(x, t+\tau) &= \lim_n u_n(x, t+\tau) \\
&= \lim_n u(x, t+(n+1)\tau) \\
&= \lim_n u_{n+1}(x, t) \\
&= \overline{u}(x, t).
\end{aligned}$$

We now find some estimates for $u^*(x, t)$ that hold in the whole $\Omega \times (0, +\infty)$.

We regard the term $-b(u^*)^3$ as a known equibounded term because of estimate (54). Applying estimate (10.12) p.355 of [10] to problem (44), (45), (48), we obtain

$$sup_{t\geq 0}\|u^*\|_{q,\Omega\times[t,t+\tau]}^{(2)} \leq M(q, H, \delta, \|f\|_{q,\Omega\times(0,\tau)}, |h|_{\Omega\times\mathbf{R}}^{(2)}), \qquad (65)$$

where $q > 1 + \frac{N}{2}$.

At this point, by means of an embedding theorem, see Corollary at p.342 of [10], taking $q > N + 2$ we find

$$\begin{aligned}
|u^*|_{\Omega\times[0,+\infty)}^{(1+\beta)} &\leq M(\beta, H, \delta, \|f\|_{q,\Omega\times(0,\tau)}, |h|_{\Omega\times\mathbf{R}}^{(2)}), \\
&\qquad \text{where } \beta = 1 - \frac{N+2}{q}.
\end{aligned} \qquad (66)$$

Obviously the previous estimates hold also for $u_n(x, t)$. Hence, passing to the limit, we obtain that estimates (65)-(66) hold also for $\overline{u}(x, t)$ and then $\overline{u}$ is a solution of (44)-(45) in $\Omega \times (0, +\infty)$.

The function $\overline{u}(x, t)$ can be extended to negative t, thus obtaining a τ-periodic solution of (44), (45) in $\Omega \times \mathbf{R}$ such that

$$\begin{aligned}
\|\overline{u}\|_{q,\Omega\times[t,t+\tau]}^{(2)} &\leq M(q, H, \delta, \|f\|_{q,\Omega\times(0,\tau)}, |h|_{\Omega\times\mathbf{R}}^{(2)}), \\
&\qquad \forall t \in \mathbf{R}, \ \forall q > N + 2, \\
|\overline{u}|_{\Omega\times\mathbf{R}}^{(1+\beta)} &\leq M(\beta, H, \delta, \|f\|_{q,\Omega\times(0,\tau)}, |h|_{\Omega\times\mathbf{R}}^{(2)}), \\
&\qquad \text{where } \beta = 1 - \frac{N+2}{q},
\end{aligned}$$

$$\qquad (67)$$
$$\qquad (68)$$

As far as uniqueness is concerned it suffices to remark that if $\overline{u}_1(x,t)$ and $\overline{u}_2(x,t)$ are two periodic solutions of (44)-(45), then $v(x,t) := \overline{u}_1(x,t) - \overline{u}_2(x,t)$ is τ-periodic. On the other hand $v(x,t)$ satisfies an equation similar to equation (55) hence an estimate like (59) holds, i.e.

$$|v| \leq 2Ce^{-\delta t},$$

which contradicts the periodicity of v.

Theorem 4.1 is thus completely proved. $\qquad\qquad\qquad\qquad\square$

We now prove an a priori estimate similar to (54) which is independent of δ.

The difficulty of this estimate lies in the fact that f belongs to $L_{q,loc}$. If this assumption is replaced by f bounded, the C^0 estimate can be straightforwardly obtained by means of explicit sub and super solutions of the problem (44)-(45) of the following form: $S_{up}(x,t) = -\gamma x_1^2 + \beta$, $s_{ub}(x,t) = \gamma x_1^2 - \beta$, where x_1 is an arbitrary space direction, and γ and β are suitable constants.

PROPOSITION 4.2. *Assume that*

$$\Omega \subset \mathbf{R}^N,$$

Ω *is a bounded smooth domain or*

  a cartesian product of intervals,

a_{ij}, b_i *are constant with* $a_{ij} > 0,$ $\qquad\qquad\qquad$ (69)

$c \in H^{\alpha, \frac{\alpha}{2}}(\Omega \times \mathbf{R}),\ \alpha \in (0,1), |c|_{\Omega \times \mathbf{R}}^{(\alpha)} \leq H,$

$f \in L_{q,loc}(\Omega \times \mathbf{R}),\ q > N + 2,\ h \in C^{2,1}(\overline{\Omega} \times \mathbf{R}),$

$c(x,t) \geq \delta > 0,\ b \geq 0,$

$c, f, h\ are\ \tau - periodic\ with\ respect\ to\ the\ time\ variable.$

Then the periodic solution $\overline{u}$ *of problem (44)-(45) is such that*

$$|\overline{u}|_{\Omega \times \mathbf{R}}^{(0)} \leq C(H_1, |h|_{\Omega \times \mathbf{R}}^{(2)}, \|f\|_{q, \Omega \times (0, \tau)}), \qquad (70)$$

where C *is independent of* δ.

Proof. Let u^* be the solution of problem (44), (45), (48).

Let $\hat{u}$ the solution of problem (51), (52) defined in the proof of Theorem 4.1.

We know that $\hat{u} \geq u^*$, $\hat{u} \geq 0$ in $\Omega \times (0, +\infty)$. Moreover $-\hat{u}$ is subsolution of u^*, hence:

$$-\hat{u}(x,t) \leq u^*(x,t) \leq \hat{u}(x,t), \text{ in } \Omega \times (0, +\infty). \tag{71}$$

We define now the following sequence:

$$\hat{u}_n(x,t) :\equiv \hat{u}(x, t + n\tau), n \in \mathbf{N}, \ x \in \Omega, \ t \geq 0.$$

Following the same procedure as in the proof of Theorem 4.1, we can show that $\hat{u}_n$ is a Cauchy sequence converging to a τ-periodic function $\overline{\hat{u}}$ defined in $\Omega \times \mathbf{R}$

$$\lim_n \hat{u}_n = \overline{\hat{u}}.$$

Recalling the definition (64) of $\overline{u}$, from (71) we have that

$$-\overline{\hat{u}}(x,t) \leq \overline{u}(x,t) \leq \overline{\hat{u}}(x,t), \text{ in } \Omega \times (0, +\infty). \tag{72}$$

The periodic function $v \equiv \overline{\hat{u}}(x,t) - B$ (B is defined in (52)) solves

$$\frac{\partial v}{\partial t} = a_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j} + b_i \frac{\partial v}{\partial x_i} +$$
$$-c(v + B) + |f|, \text{ on } \Omega \times \mathbf{R}, \tag{73}$$
$$v(x,t) = 0 \text{ on } \partial\Omega \times \mathbf{R}. \tag{74}$$

We now find an L_2 estimate for v in $\Omega \times (0, \tau)$ independent of δ.

We multiply by v equation (73) and we integrate in Ω. Taking into account that $v = 0$ on $\partial\Omega \times \mathbf{R}$, we obtain:

$$\frac{1}{2} \int_\Omega \frac{\partial(v^2)}{\partial t} dx + a_{ij} \int_\Omega |\nabla v|^2 dx =$$
$$= -\int_\Omega c(v + B) v \, dx + \int_\Omega |f| \, v \, dx \tag{75}$$
$$\leq \int_\Omega (|f| - cB) v \, dx$$
$$\leq \epsilon \int_\Omega v^2 \, dx + C(\epsilon) \int_\Omega (|f| - cB)^2 \, dx.$$

The last inequality is obtained by Young's inequality where ϵ and $C(\epsilon)$ are positive constants.

At this point we integrate with respect to the time variable in the interval $[0, \tau]$, taking into account the τ-periodicity of v, we obtain:

$$
\begin{aligned}
a_{ij} \int_0^\tau \int_\Omega |\nabla v|^2 \, dxdt \;&\leq\; \epsilon \int_0^\tau \int_\Omega v^2 \, dxdt + \\
&\qquad + C(\epsilon) \int_0^\tau \int_\Omega (|f| - cB)^2 \, dxdt \quad (76) \\
&\leq\; \epsilon C_1 \int_0^\tau \int_\Omega |\nabla v|^2 \, dxdt + C_2,
\end{aligned}
$$

the last inequality is obtained by Poincaré inequality, from assumption $f(x,t) \in L_{2,loc}(\Omega \times \mathbf{R})$ and the boundedness of cB in $\overline{\Omega} \times [0, \tau]$. At this point, taking ϵ suitably small, we obtain

$$
\int_0^\tau \int_\Omega |\nabla v(x,t)|^2 \, dxdt = \int_0^\tau \int_\Omega |\nabla \overline{\tilde{u}}(x,t)|^2 \, dxdt \leq C. \qquad (77)
$$

Note that the constant C in (77) is independent of δ.

From Poincaré inequality, we have

$$
\int_0^\tau \int_\Omega \overline{\tilde{u}}^2(x,t) \, dxdt \leq C, \qquad (78)
$$

where C is independent of δ.

Since $\overline{\tilde{u}}$ is τ-periodic, from (78), we have that

$$
\|\overline{\tilde{u}}\|_{2,\Omega \times [-\tau, 2\tau]} \leq 3C.
$$

At this point we can use estimate (10.12) p.355 of [10], thus obtaining

$$
\|\overline{\tilde{u}}\|_{q,\Omega \times [0,\tau]}^{(2)} \leq C, \;\; q \geq 2, \qquad (79)
$$

where C is independent of δ.

Hence, by the embedding theorem at p.342 of [10], we have

$$
|\overline{\tilde{u}}|_{\Omega \times \mathbf{R}}^{(0)} \leq C, \qquad (80)
$$

where C is independent of δ. From (72) and (80), (70) follows. $\qquad \square$

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