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TILTING OBJECTS IN ABELIAN CATEGORIES AND QUASITILTED RINGS

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ABSTRACT. D. Happel, I. Reiten and S. Smalø initiated an investigation of quasitilted artin K-algebras that are the endomorphism rings of tilting objects in hereditary abelian categories whose Hom and Ext groups are all finitely generated over a commutative artinian ring K. Here, employing a notion of $*$-objects, tilting objects in arbitrary abelian categories are defined and are shown to yield a version of the classical tilting theorem between the category and the category of modules over their endomorphism rings. This leads to a module theoretic notion of quasitilted rings and their characterization as endomorphism rings of tilting objects in hereditary cocomplete abelian categories.

Tilting modules for finite-dimensional and artin algebras A and the resulting tilting theorem between mod-A and the finitely generated modules over the endomorphism ring of a tilting A-module were introduced by Brenner and Butler [3] and Happel and Ringel [15] as a generalization of the Morita equivalence theorem between categories of modules over a pair of algebras. A particularly tractable account was given by Bongartz in [2]. Subsequently, Miyashita [19] and Colby and Fuller [4] showed that if A is an arbitrary ring and V_A is a tilting module, then the tilting theorem holds between Mod-A and Mod-R, where $R = \mathrm{End}(V_A)$. The tilting theorem is basically a pair of equivalences $\mathcal{T} \rightleftarrows \mathcal{Y}$ and $\mathcal{F} \rightleftarrows \mathcal{X}$ between the members of torsion pairs $(\mathcal{T}, \mathcal{F})$ of A-modules and $(\mathcal{X}, \mathcal{Y})$ of R-modules. Particularly useful, from a representation theory point of view, is the case in which A is hereditary, for then $(\mathcal{X}, \mathcal{Y})$ splits. In this case R is said to be *tilted*.

Given a commutative artinian ring K, a *locally finite abelian K-category* $\mathcal{A}$ is an abelian category in which the Hom and Ext groups are K-modules of finite length and composition of morphisms is K-bilinear. Happel, Reiten and Smalø [14] defined a *quasitilted (artin) algebra* as the endomorphism algebra of a tilting object in a hereditary locally finite abelian K-category. They characterized quasitilted algebras as those with a split torsion pair $(\mathcal{X}, \mathcal{Y})$ in mod-R such that $R_R \in \mathcal{Y}$ and $\operatorname{proj\,dim} \mathcal{Y} \leq 1$, and showed that then $\operatorname{inj\,dim} \mathcal{X} \leq 1$ and $\operatorname{gl\,dim} R \leq 2$. They also characterized these algebras as those with global dimension ≤ 2 such that every finitely generated indecomposable module has either injective or projective dimension ≤ 1.

Here, following [13], we say that R is a *(right) quasitilted ring* if it has split torsion pair $(\mathcal{X}, \mathcal{Y})$ in Mod-R such that $R_R \in \mathcal{Y}$ and $\operatorname{proj\,dim} \mathcal{Y} \leq 1$. As we shall show, quasitilted rings turn out to be precisely the endomorphism rings of tilting objects

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in hereditary cocomplete (i.e., with arbitrary coproducts) abelian categories, and they satisfy $\operatorname{inj\,dim} \mathcal{X} \leq 1$ and $\operatorname{rt\,gl\,dim} R \leq 2$.

When they introduced quasitilted algebras, Happel, Reiten and Smalø [14] showed that a tilting object V in an abelian K-category $\mathcal{A}$ induces a pair of equivalences between torsion theories in $\mathcal{A}$ and mod-R for the artin algebra $R = \operatorname{End}_{\mathcal{A}}(V)$; and, conversely, they proved that if R is an artin algebra and $(\mathcal{X}, \mathcal{Y})$ is a torsion theory in mod-R such that $R_R \in \mathcal{Y}$, then there is a locally finite abelian K-category $\mathcal{A}$ with a tilting object V such that $R = \operatorname{End}_{\mathcal{A}}(V)$ and $(\mathcal{X}, \mathcal{Y})$ is given by V.

In [7] Colpi, employing the notion of a $*$-object (a version of the $*$-modules of Menini and Orsatti [17]), proved that a tilting object V in a Grothendieck category $\mathcal{G}$ induces a tilting theorem between $\mathcal{G}$ and Mod-R, for $R = \operatorname{End}_{\mathcal{G}}(V)$. In order to prove our characterization of quasitilted rings, we employ a similar approach to show that a tilting object in a cocomplete abelian category $\mathcal{A}$ induces a tilting theorem between $\mathcal{A}$ and the category of right modules over its endomorphism ring; and using an argument similar to one in [14], we show conversely that if $(\mathcal{X}, \mathcal{Y})$ is a torsion theory in Mod-R with $R_R \in \mathcal{Y}$, then there is a tilting object V in a cocomplete abelian category $\mathcal{A}$ such that $R = \operatorname{End}_{\mathcal{A}}(V)$ and $(\mathcal{X}, \mathcal{Y})$ is given by V.

Our concluding sections contain an example of a non-noetherian quasitilted ring that is not tilted, some open questions, and an appendix providing needed results on the behavior of coproducts under the functor $\operatorname{Ext}^1_{\mathcal{A}}(-, L)$ for an abelian category $\mathcal{A}$.

1. Maximal equivalences

In the sequel, $\mathcal{A}$ denotes a fixed abelian category and V an object of $\mathcal{A}$ such that $V^{(\alpha)}$ exists in $\mathcal{A}$ for any cardinal α.

Proposition 1.1. *Let $R = \operatorname{End}_{\mathcal{A}}(V)$, $H_V = \operatorname{Hom}_{\mathcal{A}}(V, -) : \mathcal{A} \to \text{Mod-}R$. Then H_V has a left adjoint additive functor $T_V : \text{Mod-}R \to \mathcal{A}$ such that $T_V(R) = V$. Let $\sigma : 1_{\text{Mod-}R} \to H_V T_V$ and $\rho : T_V H_V \to 1_{\mathcal{A}}$ be respectively the unit and the counit of the adjunction $\langle T_V, H_V \rangle$. Let us define*

$$\operatorname{Tr}_V : \mathcal{A} \to \mathcal{A} \text{ by } \operatorname{Tr}_V(M) = \sum \{\operatorname{Im} f \mid f \in \operatorname{Hom}_{\mathcal{A}}(V, M)\},$$

$$\operatorname{Ann}_V : \text{Mod-}R \to \text{Mod-}R \text{ by } \operatorname{Ann}_V(N) = \sum \{L \mid L \overset{i}{\hookrightarrow} N,\ T_V(i) = 0\}$$

and

$$\operatorname{Gen} V = \{M \in \mathcal{A} \mid \operatorname{Tr}_V(M) = M\}, \quad \operatorname{Faith} V = \{N \in \text{Mod-}R \mid \operatorname{Ann}_V(N) = 0\}.$$

Then:

a) *The canonical inclusion $\operatorname{Tr}_V(M) \hookrightarrow M$ induces a natural isomorphism $H_V(\operatorname{Tr}_V(M)) \cong M$, and the canonical projection $N \twoheadrightarrow N/\operatorname{Ann}_V(N)$ induces a natural isomorphism $T_V(N) \cong T_V(N/\operatorname{Ann}_V(N))$.*
b) *Tr_V is an idempotent preradical, and Ann_V is a radical.*
c) *$\operatorname{Tr}_V(M) = \operatorname{Im} \rho_M$, and $\operatorname{Ann}_V(N) = \operatorname{Ker} \sigma_N$.*
d) *$T_V(\text{Mod-}R) \subseteq \operatorname{Gen} V$, and $H_V(\mathcal{A}) \subseteq \operatorname{Faith} V$.*
e) *$\operatorname{Gen} V$ is closed under (existing) coproducts and factors in $\mathcal{A}$, and $\operatorname{Faith} V$ is closed under products and submodules in Mod-R.*

Proof. The first part of the statement was proved by Popescu in [20], Corollary 7.3 and Note 1 on page 109.

a) The first part is clear, since $\operatorname{Im}(f) \subseteq \operatorname{Tr}_V(M)$ for any $f \in \operatorname{Hom}_{\mathcal{A}}(V, M)$. Now let $\operatorname{Ann}_V(N) = \sum\{L_\lambda \mid \lambda \in \Lambda\}$, with $j_\lambda : L_\lambda \hookrightarrow \operatorname{Ann}_V(N)$, for each $i_\lambda : L_\lambda \hookrightarrow N$ with $T_V(i_\lambda) = 0$. Applying the functor T_V to the commutative diagram

$$\begin{array}{ccc} & \oplus L_\lambda & \\ {\scriptstyle \oplus j_\lambda}\swarrow & & \searrow{\scriptstyle \oplus i_\lambda} \\ \operatorname{Ann}_V(N) & \xrightarrow{\ \ i\ \ } & N \end{array}$$

we immediately obtain $T_V(i) = 0$, since $\oplus j_\lambda$ is an epimorphism and T_V is right exact and commutes with direct sums. Therefore, if we apply T_V to the exact sequence

$$0 \to \operatorname{Ann}_V(N) \xrightarrow{i} N \xrightarrow{\pi} N/\operatorname{Ann}_V(N) \to 0$$

we obtain the exact sequence

$$T_V(\operatorname{Ann}_V(N)) \xrightarrow{0} T_V(N) \xrightarrow{T_V(\pi)} T_V(N/\operatorname{Ann}_V(N)) \to 0$$

which shows that $T_V(\pi)$ is an isomorphism.

b) The first part is clear. Moreover, using the right exactness of T_V and, for $i_\lambda : L_\lambda \hookrightarrow M$ and $f : M \to N$, using the commutative diagram

$$\begin{array}{ccc} L_\lambda & \xrightarrow{\ \ f i_\lambda\ \ } & N \\ & \searrow \quad \nearrow & \\ & f(L_\lambda) & \end{array}$$

it becomes clear that Ann_V is a preradical. Let us prove that Ann_V is a radical, i.e., $\operatorname{Ann}_V(N/\operatorname{Ann}_V(N)) = 0$. Let $\operatorname{Ann}_V(N) \leq L \leq N$ such that $T_V(i) = 0$ for $i : L/\operatorname{Ann}_V(N) \hookrightarrow N/\operatorname{Ann}_V(N)$. Applying T_V to the commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{\ \ j\ \ } & N \\ \downarrow{\scriptstyle \pi_L} & & \downarrow{\scriptstyle \pi_N} \\ L/\operatorname{Ann}_V(N) & \xrightarrow{\ \ i\ \ } & N/\operatorname{Ann}_V(N) \end{array}$$

we have $0 = T_V(i)T_V(\pi_L) = T_V(i\pi_L) = T_V(\pi_N j) = T_V(\pi_N)T_V(j)$, so that $T_V(j) = 0$, since $T_V(\pi_N)$ is an isomorphism by a). This gives $L \leq \operatorname{Ann}_V(N)$, as desired.

c) $\operatorname{Im}\rho_M$ is a factor of $T_V H_V(M)$, and $T_V H_V(M) \in T_V(\text{Mod-}R) = T_V(\operatorname{Gen} R) \subseteq \operatorname{Gen} T_V(R) = \operatorname{Gen} V$ since T_V is right exact and preserves coproducts. Therefore $\operatorname{Im}\rho_M \in \operatorname{Gen} V$, i.e., $\operatorname{Im}\rho_M \subseteq \operatorname{Tr}_V(M)$. Conversely, let $V^{(\alpha)} \xrightarrow{\varphi} M$ be a morphism such that $\operatorname{Im}\varphi = \operatorname{Tr}_V(M)$. In the commutative diagram

$$\begin{array}{ccc} V^{(\alpha)} & \xrightarrow{\ \ \varphi\ \ } & M \\ \uparrow{\scriptstyle \rho_{V^{(\alpha)}}} & & \uparrow{\scriptstyle \rho_M} \\ T_V H_V(V^{(\alpha)}) & \xrightarrow{\ T_V H_V(\varphi)\ } & T_V H_V(M) \end{array}$$

$\rho_{V^{(\alpha)}}$ is epi-split by adjointness, since $V^{(\alpha)} = T_V(R^{(\alpha)})$. Thus $\operatorname{Tr}_V(M) = \operatorname{Im}\varphi \leq \operatorname{Im}\rho_M$, and so they are equal.

Now let $N \in \text{Mod-}R$. From the commutative diagram

$$\begin{array}{ccc} \operatorname{Ann}_V(N) & \xrightarrow{\quad i \quad} & N \\ \big\downarrow{\scriptstyle \sigma_{\operatorname{Ann}_V(N)}} & & \big\downarrow{\scriptstyle \sigma_N} \\ H_V T_V(\operatorname{Ann}_V(N)) & \xrightarrow{H_V T_V(i)} & H_V T_V(N) \end{array}$$

since $T_V(i) = 0$ (as in the proof of a)), we see that $\sigma_N i = 0$, i.e., $\operatorname{Ann}_V(N) \leq \operatorname{Ker}\sigma_N$. Conversely, if $i : \operatorname{Ker}\sigma_N \hookrightarrow N$ is the canonical inclusion, then $\sigma_N i = 0$, so that $T_V(\sigma_N)T_V(i) = 0$, and so $T_V(i) = 0$, since $T_V(\sigma_N)$ is mono-split by adjointness. This proves that $\operatorname{Ker}\sigma_N \leq \operatorname{Ann}_V(N)$, and so they are equal.

d) By c), it follows that $M \in \operatorname{Gen} V$ if and only if ρ_M is epic, and $N \in \operatorname{Faith} V$ if and only if σ_N is monic. Since by adjointness ρ_M is epi-split for any $M \in T_V(\text{Mod-}R)$, and σ_N is mono-split for any $N \in H_V(\mathcal{A})$, d) follows.

e) follows from b), thanks to [22], Ch. VI, Proposition 1.4. □

Remark 1.2. From the statements a), b), c) and d) in Proposition 1.1 it follows that $\operatorname{Gen} V \subseteq \mathcal{A}$ and $\operatorname{Faith} V \subseteq \text{Mod-}R$ are the largest full subcategories between which the adjunction $\langle T_V, H_V \rangle$ can induce an equivalence.

This suggests the following.

Definition 1.3. $V \in \mathcal{A}$ is called a $*$-object if $\langle T_V, H_V \rangle$ induces an equivalence

$$H_V : \operatorname{Gen} V \rightleftarrows \operatorname{Faith} V : T_V.$$

Note that $\operatorname{Gen} V$ is closed under factors and coproducts, and $\operatorname{Faith} V$ is closed under submodules and direct products, thanks to Proposition 1.1e). These properties, together with the equivalence, characterize $*$-objects, in view of the following version of Menini and Orsatti's theorem [17] (see also [5], section 2).

Theorem 1.4. *Let $\mathcal{A}$ be a cocomplete abelian category, and let R be a ring. Let $\mathcal{G} \subseteq \mathcal{A}$ be a full subcategory closed under factors and coproducts, and let $\mathcal{F} \subseteq \text{Mod-}R$ be a full subcategory closed under submodules and direct products, and suppose that there is a category equivalence*

$$H : \mathcal{G} \rightleftarrows \mathcal{F} : T.$$

Let $\overline{R} = R/\mathbf{r}_R(\mathcal{F})$. Then $\overline{R}_R$ is in $\mathcal{F}$, and setting $V = T(\overline{R})$ we have natural isomorphisms $H \cong H_V$ and $T \cong T_V$, and equalities $\mathcal{G} = \operatorname{Gen} V$ and $\mathcal{F} = \operatorname{Faith} V$. In particular, V is a $*$-object in $\mathcal{A}$ and $\overline{R} \cong \operatorname{End}_{\mathcal{A}}(V)$.

Proof. Since $\mathcal{F}$ is closed under submodules and products, $\overline{R}_R$ is in $\mathcal{F}$. For any $M \in \mathcal{G}$ we have $H(M) \cong \operatorname{Hom}_R(\overline{R}, H(M)) \cong \operatorname{Hom}_{\mathcal{A}}(V, M)$ canonically in Mod-R. Moreover $\operatorname{End}_{\mathcal{A}}(V) \cong \operatorname{End}_R(\overline{R}) \cong \overline{R}$ canonically. Given any $N \in \mathcal{F} \subseteq \text{Mod-}\overline{R}$, from the exact sequence $\overline{R}^{(\alpha)} \to N \to 0$ we obtain the exact sequence $V^{(\alpha)} \to T_V(N) \to 0$ which gives $T_V(N) \in \mathcal{G}$, since $\mathcal{G}$ is closed under coproducts and factors. Therefore $T \cong T_V$, as both functors are left adjoint to $H \cong H_V$. From statement c) in Proposition 1.1 we derive the inclusions $\mathcal{G} \subseteq \operatorname{Gen} V$ and $\mathcal{F} \subseteq \operatorname{Faith} V$. On the other hand, $V \in \mathcal{G}$ and the closure properties of $\mathcal{G}$ immediately give $\operatorname{Gen} V \subseteq \mathcal{G}$. Moreover, if $N \in \operatorname{Faith} V$, then from statements b) and c) in Proposition 1.1 we derive $N \overset{\sigma_N}{\hookrightarrow} H_V T_V(N) \in H_V(\operatorname{Gen} V) = H_V(\mathcal{G}) = H(\mathcal{G}) \subseteq \mathcal{F}$, hence $N \in \mathcal{F}$ by the closure properties of $\mathcal{F}$. This shows that $\operatorname{Faith} V \subseteq \mathcal{F}$. □

For convenience, we restate [7], Lemma 1.5.

Lemma 1.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, and let $\mathcal{G} \subseteq \mathcal{A}$ and $\mathcal{F} \subseteq \mathcal{B}$ be full subcategories, each one of which is either closed under subobjects or factor objects. Let $\langle T, H\rangle$ be an adjoint pair of additive functors $\mathcal{G} \underset{T}{\overset{H}{\rightleftarrows}} \mathcal{F}$, with unit $\sigma : 1 \to HT$ and counit $\rho : TH \to 1$. Then:*

a) *If ρ_M is an isomorphism for all $M \in \mathcal{G}$, then T preserves the exactness of short exact sequences with objects in $H(\mathcal{G})$.*
b) *If σ_N is an isomorphism for all $N \in \mathcal{F}$, then H preserves the exactness of short exact sequences with objects in $T(\mathcal{F})$.*

2. Tilting objects

Let $\mathcal{A}$ be an abelian category. Following Dickson [9], a *torsion theory* in $\mathcal{A}$ is a pair of classes of objects $(\mathcal{T}, \mathcal{F})$ of $\mathcal{A}$ such that

(1) $\mathcal{T} = \{T \in \mathcal{A} \mid \mathrm{Hom}_{\mathcal{A}}(T, F) = 0 \ \forall F \in \mathcal{F}\}$,
(2) $\mathcal{F} = \{F \in \mathcal{A} \mid \mathrm{Hom}_{\mathcal{A}}(T, F) = 0 \ \forall T \in \mathcal{T}\}$,
(3) for each $X \in \mathcal{A}$ there is a short exact sequence $0 \to T \to X \to F \to 0$, with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

Now let V be an object of $\mathcal{A}$ such that $V^{(\alpha)} \in \mathcal{A}$ for any cardinal α. We shall denote by $\mathrm{Gen}\, V$ the full subcategory of $\mathcal{A}$ generated by V and by $\overline{\mathrm{Gen}}\, V$ the closure of $\mathrm{Gen}\, V$ under subobjects: $\overline{\mathrm{Gen}}\, V$ is the smallest exact abelian subcategory of $\mathcal{A}$ containing $\mathrm{Gen}\, V$. Moreover we let $\mathrm{Pres}\, V$ denote the full subcategory of $\mathrm{Gen}\, V$ which consists of the objects in $\mathcal{A}$ presented by V, i.e., $\mathrm{Pres}\, V = \{M \in \mathcal{A} \mid \exists$ an exact sequence $V^{(\beta)} \to V^{(\alpha)} \to M \to 0\}$. Finally, let $R = \mathrm{End}_{\mathcal{A}}(V)$ and

$$V^{\perp} = \mathrm{Ker}\,\mathrm{Ext}^1_{\mathcal{A}}(V, -), \quad V_{\perp} = \mathrm{Ker}\,\mathrm{Hom}_{\mathcal{A}}(V, -).$$

In this setting we have analogues of results regarding non-finitely generated tilting modules from [8] (see also [6], section 3.1).

Proposition 2.1. *Let $V \in \mathcal{A}$.*

a) *If $\mathrm{Gen}\, V \subseteq V^{\perp}$, then Tr_V is a radical. In particular $(\mathrm{Gen}\, V, V_{\perp})$ is a torsion theory in $\mathcal{A}$.*
b) *If $\mathrm{Gen}\, V = V^{\perp}$, then $\mathrm{Gen}\, V = \mathrm{Pres}\, V$.*
c) *If $\overline{\mathrm{Gen}}\, V = \mathcal{A}$, then the equality $\mathrm{Gen}\, V = V^{\perp}$ is equivalent to the following conditions:*
   i) $\mathrm{proj\,dim}\, V \leq 1$,
   ii) $\mathrm{Ext}^1_{\mathcal{A}}(V, V^{(\alpha)}) = 0$ *for any cardinal α,*
   iii) *if $M \in \mathcal{A}$ and $\mathrm{Hom}_{\mathcal{A}}(V, M) = 0 = \mathrm{Ext}^1_{\mathcal{A}}(V, M)$, then $M = 0$.*

Proof. a) Let $M \in \mathcal{A}$ and consider the canonical exact sequence

$$0 \to \mathrm{Tr}_V(M) \to M \to M/\mathrm{Tr}_V(M) \to 0.$$

We obtain the exact sequence

$$0 \to H_V(\mathrm{Tr}_V(M)) \xrightarrow{\cong} H_V(M) \to H_V(M/\mathrm{Tr}_V(M)) \to \mathrm{Ext}^1_{\mathcal{A}}(V, \mathrm{Tr}_V(M)) = 0$$

which shows that $H_V(M/\mathrm{Tr}_V(M)) = 0$, i.e., $\mathrm{Tr}_V(M/\mathrm{Tr}_V(M)) = 0$. This and Proposition 1.1b) prove that Tr_V is an idempotent radical. This shows that for any $M \in \mathcal{A}$, $\mathrm{Tr}_V(M)$ is the unique subobject of M such that $\mathrm{Tr}_V(M) \in \mathrm{Gen}\, V$ and $M/\mathrm{Tr}_V(M) \in V_{\perp}$, and so $(\mathrm{Gen}\, V, V_{\perp})$ is a torsion theory in $\mathcal{A}$.

b) Let $M \in \operatorname{Gen} V$ and $\alpha = \operatorname{Hom}_{\mathcal{A}}(V, M)$. Then we have the exact sequences

$$0 \to K \to V^{(\alpha)} \xrightarrow{\varphi} M \to 0$$

and

$$H_V(V^{(\alpha)}) \xrightarrow{H_V(\varphi)} H_V(M) \to \operatorname{Ext}^1_{\mathcal{A}}(V, K) \to 0$$

where the morphism $H_V(\varphi)$ is an epimorphism by construction. Therefore $\operatorname{Ext}^1_{\mathcal{A}}(V, K) = 0$, so by assumption $K \in \operatorname{Gen} V$. This proves that $M \in \operatorname{Pres} V$.

c) Let $\overline{\operatorname{Gen}}\, V = \mathcal{A}$ and $\operatorname{Gen} V = V^{\perp}$. Let us prove i), showing that $\operatorname{Ext}^2_{\mathcal{A}}(V, M) = 0$ for any $M \in \mathcal{A}$. Indeed, given a representative of an element $\epsilon \in \operatorname{Ext}^2_{\mathcal{A}}(V, M)$, say

$$(\epsilon) \qquad 0 \to M \to E_1 \xrightarrow{f} E_2 \to V \to 0,$$

let $I = \operatorname{Im} f$. Embedding E_1 in a suitable object $X \in \operatorname{Gen} V$, we first have a push-out diagram (dual to [22], Proposition 5.1, page 90)

$$(1) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & E_1 & \longrightarrow & I & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & X & \longrightarrow & P' & \longrightarrow & 0 \end{array}$$

where X, and so P', are in $\operatorname{Gen} V$. Then we have a second push-out diagram

$$(2) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & E_2 & \longrightarrow & V & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & P' & \longrightarrow & P'' & \longrightarrow & V & \longrightarrow & 0. \end{array}$$

By glueing (1) and (2) together, we derive a commutative diagram with exact rows

$$(3) \qquad \begin{array}{ccccccccccc} 0 & \longrightarrow & M & \longrightarrow & E_1 & \xrightarrow{f} & E_2 & \longrightarrow & V & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & M & \longrightarrow & X & \xrightarrow{g} & P'' & \xrightarrow{\pi} & V & \longrightarrow & 0 \end{array}$$

where $\operatorname{Im} g = P' \in V^{\perp}$. Then π is epi-split, and so $\epsilon \sim 0$. This proves i). Condition ii) is contained in the hypothesis, and condition iii) follows from a).

Conversely, let us assume that conditions i), ii) and iii) hold. The first condition assures that $V^{\perp}$ is closed under factors. Therefore, using the second condition we immediately see that $\operatorname{Gen} V \subseteq V^{\perp}$. In order to prove the opposite inclusion, given any $M \in V^{\perp}$, from the exact sequence $0 \to \operatorname{Tr}_V(M) \to M \to M/\operatorname{Tr}_V(M) \to 0$ and using condition i) we obtain the exact sequence

$$\begin{aligned} 0 \to \operatorname{Hom}_{\mathcal{A}}(V, \operatorname{Tr}_V(M)) &\xrightarrow{\cong} \operatorname{Hom}_{\mathcal{A}}(V, M) \to \operatorname{Hom}_{\mathcal{A}}(V, M/\operatorname{Tr}_V(M)) \\ &\to \operatorname{Ext}^1_{\mathcal{A}}(V, \operatorname{Tr}_V(M)) = 0 = \operatorname{Ext}^1_{\mathcal{A}}(V, M) \to \operatorname{Ext}^1_{\mathcal{A}}(V, M/\operatorname{Tr}_V(M)) \to 0. \end{aligned}$$

Hence $\operatorname{Hom}_{\mathcal{A}}(V, M/\operatorname{Tr}_V(M)) = 0 = \operatorname{Ext}^1_{\mathcal{A}}(V, M/\operatorname{Tr}_V(M))$. Now condition iii) gives $M/\operatorname{Tr}_V(M) = 0$, i.e., $M = \operatorname{Tr}_V(M) \in \operatorname{Gen} V$. This proves that $V^{\perp} \subseteq \operatorname{Gen} V$. $\square$

Remark 2.2. If $\mathcal{A}$ is cocomplete with exact coproducts, or $\mathcal{A}$ has enough injectives, then $\overline{\operatorname{Gen}}\, V = \mathcal{A}$ whenever $\operatorname{Gen} V = V^{\perp}$.

Proof. If $\mathcal{A}$ has enough injectives, then every object of $\mathcal{A}$ embeds in an injective object which, by definition, belongs to $V^{\perp} = \operatorname{Gen} V$. Let us assume, now, that $\mathcal{A}$ is cocomplete with exact coproducts. Let $M \in \mathcal{A}$ and α be the cardinality of a spanning set for $\operatorname{Ext}^1_{\mathcal{A}}(V, M)$ as a right R-module. Then, arguing as in [6], Lemma 3.4.4, we can find an exact sequence

$$0 \to M \to X \to V^{(\alpha)} \to 0$$

such that the connecting homomorphism $\operatorname{Hom}_{\mathcal{A}}(V, V^{(\alpha)}) \to \operatorname{Ext}^1_{\mathcal{A}}(V, M)$ is onto. This gives $\operatorname{Ext}^1_{\mathcal{A}}(V, X) = 0$, i.e., $X \in V^{\perp} = \operatorname{Gen} V$, and so it proves that $\overline{\operatorname{Gen}}\, V = \mathcal{A}$. □

In view of this last remark, we add a third condition to the Definition 2.3 of [7] to obtain

Definition 2.3. An object V in an abelian category $\mathcal{A}$ that contains arbitrary coproducts of copies of V is called a tilting object if:

i) V is selfsmall (i.e., $\operatorname{Hom}_{\mathcal{A}}(V, V^{(\alpha)}) \cong R^{(\alpha)}$ for any cardinal α);
ii) $\operatorname{Gen} V = V^{\perp}$;
iii) $\overline{\operatorname{Gen}}\, V = \mathcal{A}$.

So, to any tilting object $V \in \mathcal{A}$ is naturally associated a torsion theory $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$, namely $\mathcal{T} = V^{\perp}$ and $\mathcal{F} = V_{\perp}$.

Now we can extend [7], Theorem 3.2, as follows.

Theorem 2.4. *Let $\mathcal{A}$ be an abelian category such that $V^{(\alpha)} \in \mathcal{A}$ for any cardinal α. Then the following are equivalent:*

(a) *V is a $*$-object;*
(b) *V is a tilting object in $\overline{\operatorname{Gen}}\, V$;*
(c) *ρ is monic in $\mathcal{A}$ and σ is epic in* Mod-R*;*
(d) *V is selfsmall,* $\operatorname{Gen} V = \operatorname{Pres} V$ *and H_V preserves short exact sequences in $\mathcal{A}$ with all terms in* $\operatorname{Gen} V$*;*
(e) *V is selfsmall, and for any short exact sequence $0 \to L \to M \to N \to 0$ in $\mathcal{A}$ with M (and N) in* $\operatorname{Gen} V$*, the sequence $0 \to H_V(L) \to H_V(M) \to H_V(N) \to 0$ is exact if and only if* $L \in \operatorname{Gen} V$*.*

Proof. (a) $\Rightarrow$ (b) We see that V is selfsmall, since $H_V(V^{(\alpha)}) = H_V T_V(R^{(\alpha)}) \cong R^{(\alpha)} = H_V(V)^{(\alpha)}$ canonically. We can assume that $\mathcal{A} = \overline{\operatorname{Gen}}\, V$. In order to prove that $\operatorname{Gen} V \subseteq V^{\perp}$, given any $M \in \operatorname{Gen} V$ we show that any short exact sequence in $\mathcal{A}$,

$$0 \to M \to X \xrightarrow{\pi} V \to 0, \tag{1}$$

splits. Let $X \overset{i}{\hookrightarrow} L$ be a fixed embedding with $L \in \operatorname{Gen} V$, and let us consider the push-out diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & X & \xrightarrow{\pi} & V & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} & & \\
0 & \longrightarrow & M & \longrightarrow & L & \xrightarrow{p} & P & \longrightarrow & 0
\end{array} \tag{2}$$

where the second row is in $\operatorname{Gen} V$. From (2) we obtain the commutative diagram with exact rows

$$(3)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & H_V(M) & \longrightarrow & H_V(X) & \xrightarrow{H_V(\pi)} & H_V(V) & \xrightarrow{\delta} & \operatorname{Ext}^1_{\mathcal{A}}(V,M) \\ & & \Vert & & \downarrow & & \downarrow & & \Vert \\ 0 & \longrightarrow & H_V(M) & \longrightarrow & H_V(L) & \xrightarrow{H_V(p)} & H_V(P) & \xrightarrow{\gamma} & \operatorname{Ext}^1_{\mathcal{A}}(V,M). \end{array}$$

Since, from statement b) in Lemma 1.5, the morphism $H_V(p)$ in (3) is epic, we see that $\gamma = 0$, so that $\delta = 0$, too. This shows that $H_V(\pi)$ is epic, so that (1) splits.

Conversely, let us prove that $V^\perp \subseteq \operatorname{Gen} V$. Given any $M \in V^\perp$, let

$$0 \to M \to X_0 \xrightarrow{\varphi} X_1 \to 0$$

be a fixed exact sequence with X_0 (and X_1) in $\operatorname{Gen} V$. Since $\operatorname{Ext}^1_{\mathcal{A}}(V,M) = 0$ by assumption, $H_V(\varphi)$ is epic. Therefore we have the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & X_0 & \xrightarrow{\varphi} & X_1 & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \rho_M} & & \cong\uparrow{\scriptstyle \rho_{X_0}} & & \cong\uparrow{\scriptstyle \rho_{X_1}} & & \\ \cdots & \longrightarrow & T_V H_V(M) & \longrightarrow & T_V H_V(X_0) & \xrightarrow{T_V H_V(\varphi)} & T_V H_V(X_1) & \longrightarrow & 0 \end{array}$$

which shows that ρ_M is epic, i.e., $M \in \operatorname{Gen} V$.

(b) $\Rightarrow$ (e) Assume that $0 \to L \to M \to N \to 0$ is an exact sequence in $\mathcal{A}$ with M (and N) in $\operatorname{Gen} V$. Then, since by assumption $\operatorname{Gen} V = V^\perp$, the sequence $0 \to H_V(L) \to H_V(M) \to H_V(N) \to \operatorname{Ext}^1_{\mathcal{A}}(V,L) \to 0$ is exact, and so $\operatorname{Ext}^1_{\mathcal{A}}(V,L) = 0$ if and only if $L \in \operatorname{Gen} V$.

(e) $\Rightarrow$ (d) Let $M \in \operatorname{Gen} V$, and let $\alpha = H_V(M)$. Then there is a short exact sequence $0 \to K \to V^{(\alpha)} \xrightarrow{\varphi} M \to 0$ such that $H_V(\varphi)$ is epic. By hypothesis, we must have $K \in \operatorname{Gen} V$. This shows that $M \in \operatorname{Pres} V$.

(d) $\Rightarrow$ (c) Let $N \in$ Mod-R and let $R^{(\beta)} \to R^{(\alpha)} \xrightarrow{\varphi} N \to 0$ be exact. Since H_V is exact on $\operatorname{Gen} V$ by assumption, it preserves the exactness of the sequence $0 \to K \to T_V(R^{(\alpha)}) \xrightarrow{T_V(\varphi)} T_V(N) \to 0$. Thus we have a commutative diagram with exact rows

$$\begin{array}{ccccc} R^{(\alpha)} & \xrightarrow{\varphi} & N & \longrightarrow & 0 \\ \cong\downarrow{\scriptstyle \sigma_{R^{(\alpha)}}} & & \downarrow{\scriptstyle \sigma_N} & & \\ H_V T_V(R^{(\alpha)}) & \xrightarrow{H_V T_V(\varphi)} & H_V T_V(N) & \longrightarrow & 0 \end{array}$$

where $\sigma_{R^{(\alpha)}}$ is an isomorphism, since V is selfsmall. This proves that σ_N is epic, for any $N \in$ Mod-R. In order to prove that ρ is monic in $\mathcal{A}$, thanks to statement a) in Proposition 1.1, it is sufficient to prove that ρ is monic in $\operatorname{Gen} V = \operatorname{Pres} V$. Moreover, we see that ρ is monic in $T_V(\text{Mod-}R)$, since by adjunction $\rho_{T_V(-)} \circ T_V(\sigma_-) = 1_{T_V(-)}$, and $T_V(\sigma_-)$ is an isomorphism, since we have already proved that σ_-, and so $T_V(\sigma_-)$, is an epimorphism in Mod-R. Therefore, it remains to be proved that $\operatorname{Pres} V \subseteq T_V(\text{Mod-}R)$. Let $M \in \operatorname{Pres} V$ and let $V^{(\beta)} \to V^{(\alpha)} \xrightarrow{\varphi} M \to 0$ be exact. Applying T_V to the exact sequence

$$H_V(V^{(\beta)}) \to H_V(V^{(\alpha)}) \xrightarrow{\operatorname{Coker} H_V(\varphi)} C \to 0$$

we obtain the commutative diagram with exact rows

$$\begin{array}{ccccccc} T_V H_V(V^{(\beta)}) & \longrightarrow & T_V H_V(V^{(\alpha)}) & \longrightarrow & T_V(C) & \longrightarrow & 0 \\ \cong \big\downarrow \rho_{V^{(\beta)}} & & \cong \big\downarrow \rho_{V^{(\alpha)}} & & & & \\ V^{(\beta)} & \longrightarrow & V^{(\alpha)} & \xrightarrow{\varphi} & M & \longrightarrow & 0 \end{array}$$

which proves that $M \cong T_V(C) \in T_V(\text{Mod-}R)$.

(c) $\Rightarrow$ (a) This is an immediate consequence of Proposition 1.1. □

3. The tilting theorem

Here we shall obtain a tilting theorem in our present setting with the aid of

Lemma 3.1. *Let $V \in \mathcal{A}$ be a tilting object, $R = \operatorname{End}_{\mathcal{A}}(V)$, and let $T_V^{(i)}$, $i \geq 1$, be the i-th left derived functor of T_V. Then:*

a) $\operatorname{Faith} V = \operatorname{Ker} T'_V$*;*
b) $T_V^{(i)} = 0$ *for all* $i \geq 2$*;*
c) Ann_V *is an idempotent radical;*
d) $(\operatorname{Ker} T_V, \operatorname{Ker} T'_V)$ *is a torsion theory in* Mod-R*;*
e) *for any* $N \in$ Mod-R *the canonical inclusion* $\operatorname{Ann}_V(N) \hookrightarrow N$ *induces a natural isomorphism* $T'_V(\operatorname{Ann}_V(N)) \cong T'_V(N)$.

Proof. a) If $N \in \operatorname{Faith} V$, then by d) and e) of Proposition 1.1 there is an exact sequence in $\operatorname{Faith} V$

$$0 \to K \to R^{(\alpha)} \to N \to 0.$$

On the one hand we have the exact sequence

$$0 \to T'_V(N) \to T_V(K) \to T_V(R^{(\alpha)}) \to T_V(N) \to 0,$$

on the other hand, thanks to Theorem 2.4, we know that V is a $*$-object, and so by Lemma 1.5 a) the functor T_V preserves the exactness of sequences in $\operatorname{Faith} V$. Thus $T'_V(N) = 0$, and the inclusion $\operatorname{Faith} V \subseteq \operatorname{Ker} T'_V$ is proved. Conversely, for any $N \in \operatorname{Ker} T'_V$ we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & R^{(\alpha)} & \longrightarrow & N & \longrightarrow & 0 \\ & & \big\downarrow \cong & & \big\downarrow \cong & & \big\downarrow \sigma_N & & \\ 0 & \longrightarrow & H_V T_V(K) & \longrightarrow & H_V T_V(R^{(\alpha)}) & \longrightarrow & H_V T_V(N) & & \end{array}$$

where the first two vertical canonical maps are isomorphisms thanks to Theorem 2.4. This shows that σ_N is monic, so that $N \in \operatorname{Faith} V$ by statement c) in Proposition 1.1.

b) Given any $N \in$ Mod-R and short exact sequence

$$0 \to K \to R^{(\alpha)} \to N \to 0,$$

since $K \in \operatorname{Faith} V = \operatorname{Ker} T'_V$, we see by induction that $T^{(i+1)}(N) \cong T^{(i)}(K)$ is zero for any $i \geq 1$.

c) We have already remarked in b) of Proposition 1.1 that Ann_V is a radical. Since by a) $\operatorname{Faith} V = \operatorname{Ker} T'_V$ is obviously closed under extensions, we can conclude that the associate radical Ann_V is idempotent.

d) Thanks to c) we see that $(\mathcal{T}, \operatorname{Ker} T'_V)$ is a torsion theory, where $\mathcal{T} = \{N \in \text{Mod-}R \mid \operatorname{Ann}_V(N) = N\}$. It remains to be proved that $\mathcal{T} = \operatorname{Ker} T_V$. First, let

$N \in \mathcal{T}$. Then by a) in Proposition 1.1 we have $T_V(N) \cong T_V(N/\operatorname{Ann}_V(N)) = T_V(0) = 0$. Conversely, if $N \in \operatorname{Ker} T_V$, then for any embedding $L \hookrightarrow N$ we have $T_V(i) = 0$, which proves that $\operatorname{Ann}_V(N) = N$, i.e., $N \in \mathcal{T}$.

e) Given any $N \in \text{Mod-}R$ and the associated canonical exact sequence

$$0 \to \operatorname{Ann}_V(N) \to N \to N/\operatorname{Ann}_V(N) \to 0,$$

employing a), b) and d) we see that $T'_V(\operatorname{Ann}_V(N)) \cong T'_V(N)$ canonically. □

Our Tilting Theorem follows. We note that several of the arguments are closely related to those in the proofs of various less general versions, but we include them for the sake of completeness.

Theorem 3.2. *Let V be a tilting object in an abelian category $\mathcal{A}$, $R = \operatorname{End}_{\mathcal{A}}(V)$, $H_V = \operatorname{Hom}_{\mathcal{A}}(V, -)$, $H'_V = \operatorname{Ext}^1_{\mathcal{A}}(V, -)$, T_V the left adjoint to H_V, and T'_V the first left derived functor of T_V. Set*

$$\mathcal{T} = \operatorname{Ker} H'_V,\ \mathcal{F} = \operatorname{Ker} H_V,\ \mathcal{X} = \operatorname{Ker} T_V,\ \mathcal{Y} = \operatorname{Ker} T'_V.$$

Then:

- a) *$(\mathcal{T}, \mathcal{F})$ is a torsion theory in $\mathcal{A}$ with $\mathcal{T} = \operatorname{Gen} V$, and $(\mathcal{X}, \mathcal{Y})$ is a torsion theory in* Mod-R *with $\mathcal{Y} = \operatorname{Faith} V$;*
- b) *the functors $H_V\restriction_{\mathcal{T}}$, $T_V\restriction_{\mathcal{Y}}$, $H'_V\restriction_{\mathcal{F}}$, $T'_V\restriction_{\mathcal{X}}$ are exact, and they induce a pair of category equivalences $\mathcal{T} \underset{T_V}{\overset{H_V}{\rightleftarrows}} \mathcal{Y}$ and $\mathcal{F} \underset{T'_V}{\overset{H'_V}{\rightleftarrows}} \mathcal{X}$;*
- c) *$T_V H'_V = 0 = T'_V H_V$ and $H_V T'_V = 0 = H'_V T_V$;*
- d) *there are natural transformations θ and η that, together with the adjoint transformations ρ and σ, yield exact sequences*

  $$0 \to T_V H_V(M) \xrightarrow{\rho_M} M \xrightarrow{\eta_M} T'_V H'_V(M) \to 0$$

  and

  $$0 \to H'_V T'_V(N) \xrightarrow{\theta_N} N \xrightarrow{\sigma_N} H_V T_V(N) \to 0$$

  for each $M \in \mathcal{A}$ and for each $N \in$ Mod-R.

Proof. Statement a) is contained in Proposition 2.1 and Lemma 3.1. The first part of b) regarding the exactness of the four restricted functors and the existence of the first equivalence is an immediate consequence of Theorem 2.4, Lemma 1.5, Proposition 2.1c) and Lemma 3.1b). Moreover, part of d) is contained in Theorem 2.4 and Proposition 1.1.

In order to prove c), we start with an arbitrary object $M \in \mathcal{A}$ and a fixed associated short exact sequence

$$(*) \qquad 0 \to M \to X_0 \to X_1 \to 0$$

with X_0 and X_1 objects of $\operatorname{Gen}(V) = \mathcal{T}$. Applying $\operatorname{Hom}_{\mathcal{A}}(V, -)$, we obtain the exact sequence $H_V(X_0) \to H_V(X_1) \to H'_V(M) \to H'_V(X_0) = 0$. Applying T_V we obtain the commutative diagram with exact rows

$$\begin{array}{ccccccc} X_0 & \longrightarrow & X_1 & \longrightarrow & 0 & & \\ {\scriptstyle \rho_{X_0}}\uparrow{\scriptstyle \cong} & & {\scriptstyle \rho_{X_1}}\uparrow{\scriptstyle \cong} & & & & \\ T_V H_V(X_0) & \longrightarrow & T_V H_V(X_1) & \longrightarrow & T_V H'_V(M) & \longrightarrow & 0 \end{array}$$

which shows that $T_V H'_V(M) = 0$. Moreover, thanks to Proposition 1.1d) and Lemma 3.1a), we have $H_V(M) \in \operatorname{Faith} V = \operatorname{Ker} T'_V$, and so $T'_V H_V(M) = 0$.

On the other hand, for any $N \in \text{Mod-}R$ let us consider an exact sequence of the form

$$(**) \qquad 0 \to K \to R^{(\alpha)} \to N \to 0.$$

Note that both $R^{(\alpha)}$ and the submodule K belong to $\operatorname{Faith} V$. Applying H_V to the exact sequence $0 = T'_V(R^{(\alpha)}) \to T'_V(N) \to T_V(K) \to_V (R^{(\alpha)})$, we obtain the commutative diagram with exact rows

$$\begin{array}{ccccccc} & & 0 & \longrightarrow & K & \longrightarrow & R^{(\alpha)} \\ & & & & \sigma_K \downarrow \cong & & \sigma_{R^{(\alpha)}} \downarrow \cong \\ 0 & \longrightarrow & H_V T'_V(N) & \longrightarrow & H_V T_V(K) & \longrightarrow & H_V T_V(R^{(\alpha)}) \end{array}$$

which shows that $H_V T'_V(N) = 0$. Finally, by Proposition 1.1d) and hypothesis, we have $T(N) \in \operatorname{Gen}(V) = \mathcal{T} = \operatorname{Ker} H'_V$, therefore $H'_V T_V(N) = 0$. This completes the proof of c).

In order to prove the second half of b), first we remark that the inclusion $\operatorname{Im} H'_V \subseteq \mathcal{X}$ follows from $T_V H'_V = 0$ and, similarly, the inclusion $\operatorname{Im} T'_V \subseteq \mathcal{F}$ follows from $H_V T'_V = 0$.

Next, let $M \in \mathcal{F}$. Applying $\operatorname{Hom}_{\mathcal{A}}(V, -)$ to the exact sequence (*), we obtain the exact sequence $0 \to H_V(X_0) \to H_V(X_1) \to H'_V(M) \to 0$, and applying T_V to this, we obtain the diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & X_0 & \longrightarrow & X_1 & \longrightarrow & 0 \\ & & \downarrow \eta_M & & \rho_{X_0} \uparrow \cong & & \rho_{X_1} \uparrow \cong & & \\ 0 & \longrightarrow & T'_V H'_V(M) & \longrightarrow & T_V H_V(X_0) & \longrightarrow & T_V H_V(X_1) & \longrightarrow & 0 \end{array}$$

where η_M is the unique isomorphism making the diagram commutative. Similarly, given any $N \in \mathcal{X}$ and any exact sequence of the form (**), we define $\theta_N : H'_V T'_V(N) \to N$ as the unique isomorphism making commutative the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & R^{(\alpha)} & \longrightarrow & N & \longrightarrow & 0 \\ & & \sigma_K \downarrow \cong & & \sigma_{R^{(\alpha)}} \downarrow \cong & & \uparrow \theta_N & & \\ 0 & \longrightarrow & H_V T_V(K) & \longrightarrow & H_V T_V(R^{(\alpha)}) & \longrightarrow & H'_V T'_V(N) & \longrightarrow & 0 \end{array}$$

It can be shown that θ_N does not depend on the choice of (**), and that $(\eta_M)_{M \in \mathcal{F}}$ and $(\theta_N)_{N \in \mathcal{X}}$ are natural maps.

This proves that $\mathcal{F} \underset{T'_V}{\overset{H'_V}{\rightleftarrows}} \mathcal{X}$ is an equivalence.

To complete the proof of d), we first recall that Lemma 3.1e) says that for any $N \in \text{Mod-}R$ the canonical inclusion $\operatorname{Ann}_V(N) \hookrightarrow N$ induces a natural isomorphism $T'_V(\operatorname{Ann}_V(N)) \cong T'_V(N)$. Second, since from Proposition 2.1c) we have $\operatorname{proj\,dim} V \leq 1$, we can similarly prove that for any $M \in \mathcal{A}$ the canonical projection $M \twoheadrightarrow M/\operatorname{Tr}_V(M)$ induces a natural isomorphism $H'_V(M) \cong H'_V(M/\operatorname{Tr}_V(M))$.

Because of this, we can extend the definitions of η and θ to a pair of natural homomorphisms defined in $\mathcal{A}$ and in Mod-R respectively, making the diagrams

$$\begin{array}{ccccc} M & \longrightarrow & M/\operatorname{Tr}_V(M) & \longrightarrow & 0 \\ \Big\downarrow \eta_M & & \cong \Big\downarrow \eta_{M/\operatorname{Tr}_V(M)} & & \\ T'_V H'_V(M) & \xrightarrow{\cong} & T'_V H'_V(M/\operatorname{Tr}_V(M)) & & \end{array}$$

and

$$\begin{array}{ccccc} 0 & \longrightarrow & \operatorname{Ann}_V(N) & \longrightarrow & N \\ & & \cong \Big\uparrow \theta_{\operatorname{Ann}_V(N)} & & \Big\uparrow \theta_N \\ & & H'_V T'_V(\operatorname{Ann}_V(N)) & \xrightarrow{\cong} & H'_V T'_V(N) \end{array}$$

commutative for any $M \in \mathcal{A}$ and any $N \in$ Mod-R. Thus we see that η_M is epic, $\operatorname{Ker}(\eta_M) = \operatorname{Tr}_V(M)$, θ_N is monic and $\operatorname{Im}(\theta_N) = \operatorname{Ann}_V(N)$. Applying Proposition 1.1c), we complete the proof of d). □

4. Representing faithful torsion theories

Given any abelian category $\mathcal{M}$, let us denote by $\mathcal{D}^b(\mathcal{M})$ the bounded derived category of $\mathcal{M}$. If $(\mathcal{X}, \mathcal{Y})$ is a torsion theory in $\mathcal{M}$, then $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ is the full subcategory of $\mathcal{D}^b(\mathcal{M})$ defined as

$$\mathcal{H}(\mathcal{X}, \mathcal{Y}) = \{X \in \mathcal{D}^b(\mathcal{M}) \mid H^{-1}(X) \in \mathcal{Y},\ H^0(X) \in \mathcal{X},\ H^i(X) = 0\ \forall i \neq -1, 0\}.$$

$\mathcal{H}(\mathcal{X}, \mathcal{Y})$ is called the heart of the t-structure in $\mathcal{D}^b(\mathcal{M})$ associated with $(\mathcal{X}, \mathcal{Y})$.

Regarding a map $X^{-1} \xrightarrow{x} X^0$ as a complex $\dots 0 \to X^{-1} \xrightarrow{x} X^0 \to 0 \dots$, the objects of $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ are represented, up to isomorphism, by complexes of the form

$$X:\ X^{-1} \xrightarrow{x} X^0 \text{ with } \operatorname{Ker} x \in \mathcal{Y} \text{ and } \operatorname{Coker} x \in \mathcal{X}.$$

A morphism in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ is a formal fraction $\varphi = s^{-1} f$, where:

(1) $X \xrightarrow{f} Y$ is a representative of a homotopy class of complex maps

$$\begin{array}{cccc} X & X^{-1} & \xrightarrow{x} & X^0 \\ f\Big\downarrow & f^{-1}\Big\downarrow & & f^0\Big\downarrow \\ Y & Y^{-1} & \xrightarrow{y} & Y^0 \end{array}$$

where $X \xrightarrow{f} Y$ is null-homotopic if there is a map $r^0 : X^0 \to Y^{-1}$ such that

$$f^0 = y r^0 \quad \text{and} \quad f^{-1} = r^0 x;$$

(2) $X \xrightarrow{s} Y$ is a quasi-isomorphism, i.e., there are isomorphisms making the diagrams

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \operatorname{Ker} x & \longrightarrow & X^{-1} & \xrightarrow{x} & X^0 & \longrightarrow & \operatorname{Coker} x & \longrightarrow & 0 \\ & & \cong\Big\downarrow & & s^{-1}\Big\downarrow & & s^0\Big\downarrow & & \cong\Big\downarrow & & \\ 0 & \longrightarrow & \operatorname{Ker} y & \longrightarrow & Y^{-1} & \xrightarrow{y} & Y^0 & \longrightarrow & \operatorname{Coker} y & \longrightarrow & 0 \end{array}$$

commute: quasi-isomorphisms are invertible in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$.

It turns out that $\mathcal{H}(\mathcal{X},\mathcal{Y})$ is an abelian category, and setting

$$\mathcal{T} = \mathcal{Y}[1] = \{Y \to 0 \mid Y \in \mathcal{Y}\} \text{ and } \mathcal{F} = \mathcal{X} = \{0 \to X \mid X \in \mathcal{X}\}$$

the pair $(\mathcal{T},\mathcal{F})$ is a torsion theory in $\mathcal{H}(\mathcal{X},\mathcal{Y})$ with category equivalences $\mathcal{T} \cong \mathcal{Y}$ and $\mathcal{F} \cong \mathcal{X}$.

An exhaustive description of $\mathcal{H}(\mathcal{X},\mathcal{Y})$ is contained in [14], Chapter 1.

If $\mathcal{M}$ has products and coproducts with good behaviour, as in the case of Mod-R, then $\mathcal{H}(\mathcal{X},\mathcal{Y})$ is cocomplete.

Lemma 4.1. *Let $\mathcal{M}$ be a complete and cocomplete abelian category with exact coproducts, such that for any family of objects the canonical map from their coproduct to their product is monic. Then for any torsion theory $(\mathcal{X},\mathcal{Y})$ in $\mathcal{M}$ the associated heart $\mathcal{H}(\mathcal{X},\mathcal{Y})$ is cocomplete.*

Proof. Let α be any cardinal. By hypothesis, the diagram

$$\amalg : \prod_{\alpha} \mathcal{M} \rightleftarrows \mathcal{M} : \Delta,$$

where $\amalg$ is the coproduct functor and Δ is the diagonal functor, defines an adjoint pair $\langle \amalg, \Delta \rangle$. This adjunction naturally extends componentwise to the corresponding homotopy categories. Moreover, since both $\amalg$ and Δ are exact, they extend to a pair of functors $\widehat{\amalg}$ and $\widehat{\Delta}$ between the corresponding derived categories. Moreover, thanks to [16], Section 3, the diagram

$$\widehat{\amalg} : \mathcal{D}^b(\prod_{\alpha} \mathcal{M}) \cong \prod_{\alpha} \mathcal{D}^b(\mathcal{M}) \rightleftarrows \mathcal{D}^b(\mathcal{M}) : \widehat{\Delta}$$

still defines an adjoint pair $\langle \widehat{\amalg}, \widehat{\Delta} \rangle$. This shows that $\mathcal{D}^b(\mathcal{M})$ admits arbitrary coproducts, and that they are defined componentwise. Moreover, since the assumptions on $\mathcal{M}$ guarantee that both $\mathcal{X}$ and $\mathcal{Y}$ are closed under arbitrary coproducts, we see that $\mathcal{H}(\mathcal{X},\mathcal{Y})$ is closed under coproducts in $\mathcal{D}^b(\mathcal{M})$. □

Thus by Theorem 1.4 we immediately have:

Proposition 4.2. *If $(\mathcal{X},\mathcal{Y})$ is a torsion theory in* Mod-R *there is a $*$-object $V = (R/\mathbf{r}_R(\mathcal{Y}))[1]$ in $\mathcal{H}(\mathcal{X},\mathcal{Y})$ that induces an equivalence*

$$H_V : \mathcal{T} \rightleftarrows \mathcal{Y} : T_V.$$

Definition 4.3. A torsion theory $(\mathcal{X},\mathcal{Y})$ in Mod-R is *faithful* if $\mathbf{r}_R(\mathcal{Y}) = 0$.

Note that $(\mathcal{X},\mathcal{Y})$ is faithful if and only if $R_R \in \mathcal{Y}$ or, equivalently, if $\mathcal{Y}$ *generates* Mod-R.

We shall show that when $(\mathcal{X},\mathcal{Y})$ is faithful in Mod-R, the equivalence $H_V : \mathcal{T} \rightleftarrows \mathcal{Y} : T_V$ in Proposition 4.2 is actually induced by a tilting object V with $\mathrm{End}_{\mathcal{H}}(V) = R$. To do so we need

Lemma 4.4. *If $\mathcal{Y}$ generates $\mathcal{M}$, then every object of $\mathcal{H}(\mathcal{X},\mathcal{Y})$ is isomorphic to a complex of the form $Y^{-1} \to Y^0$, with $Y^{-1}, Y^0 \in \mathcal{Y}$.*

Proof. Let $Z^{-1} \xrightarrow{z} Z^0 \in \mathcal{H}(\mathcal{X},\mathcal{Y})$ to obtain exact sequences

$$0 \to Y \to Z^{-1} \to I \to 0 \text{ and } 0 \to I \to Z^0 \to X \to 0$$

with $I = \operatorname{Im} z$, $Y \in \mathcal{Y}$ and $X \in \mathcal{X}$. Then there are an object $Y^0 \in \mathcal{Y}$, an epimorphism $Y^0 \to Z^0$ and a pullback diagram

$$
(1)\qquad \begin{array}{ccccccccc}
0 & \longrightarrow & P & \longrightarrow & Y^0 & \longrightarrow & X & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & I & \longrightarrow & Z^0 & \longrightarrow & X & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

where P is in $\mathcal{Y}$, since $\mathcal{Y}$ is closed under subobjects. Then we obtain a further pullback diagram

$$
(2)\qquad \begin{array}{ccccccccc}
0 & \longrightarrow & Y & \longrightarrow & Y^{-1} & \longrightarrow & P & \longrightarrow & 0\\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & Z^{-1} & \longrightarrow & I & \longrightarrow & 0\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

where Y^{-1} is in $\mathcal{Y}$, since $\mathcal{Y}$ is closed under extensions. Now (1) and (2) combine to give a commutative diagram with exact rows

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & Y & \longrightarrow & Y^{-1} & \longrightarrow & Y^0 & \longrightarrow & X & \longrightarrow & 0\\
 & & \| & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & Y & \longrightarrow & Z^{-1} & \longrightarrow & Z^0 & \longrightarrow & X & \longrightarrow & 0
\end{array}
$$

and so the desired quasi-isomorphism. □

This allows us to prove the following version of Proposition 3.2(ii) on page 17 of [14].

Proposition 4.5. *If $\mathcal{Y}$ generates $\mathcal{M}$, then $\mathcal{T} = \mathcal{Y}[1]$ cogenerates $\mathcal{H}(\mathcal{X}, \mathcal{Y})$.*

Proof. By the last lemma, we know that every object in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ is isomorphic to a complex of the form $Y^{-1} \xrightarrow{y} Y^0$ with $Y^{-1}, Y^0 \in \mathcal{Y}$. We shall show that

$$
\begin{array}{ccc}
Y^{-1} & \xrightarrow{\;y\;} & Y^0\\
\| & & \downarrow\\
Y^{-1} & \longrightarrow & 0
\end{array}
$$

is a monomorphism. So suppose that the commutative diagram

$$
\begin{array}{ccc}
Z^{-1} & \xrightarrow{\;z\;} & Z^0\\
{\scriptstyle \varphi^{-1}}\downarrow & & \downarrow{\scriptstyle \varphi^{0}}\\
Y^{-1} & \xrightarrow{\;y\;} & Y^0\\
\| & & \downarrow\\
Y^{-1} & \longrightarrow & 0
\end{array}
$$

yields a null-homotopic map, i.e., that there is a map $r^0 : Z^0 \to Y^{-1}$ such that

$$\varphi^{-1} = r^0 z.$$

Let $\gamma = yr^0 - \varphi^0 : Z^0 \to Y^0$ so that

$$\gamma z = yr^0 z - \varphi^0 z = y\varphi^{-1} - y\varphi^{-1} = 0$$

and hence $\operatorname{Im} z \subseteq \operatorname{Ker} \gamma$. But $Z^0 / \operatorname{Im} z \in \mathcal{X}$ and $Z^0 / \operatorname{Ker} \gamma \in \mathcal{Y}$. Thus $\gamma = 0$ and so

$$\varphi^0 = yr^0.$$

In other words the map

$$\begin{array}{ccc} Z^{-1} & \xrightarrow{z} & Z^0 \\ {\scriptstyle \varphi^{-1}} \downarrow & & \downarrow {\scriptstyle \varphi^0} \\ Y^{-1} & \xrightarrow{y} & Y^0 \end{array}$$

is zero in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$, which proves our assertion. □

Now we have the needed results to prove

Theorem 4.6. *A torsion theory $(\mathcal{X}, \mathcal{Y})$ in* Mod-R *is faithful if and only if there is a cocomplete abelian category $\mathcal{H}$ and a tilting object V of $\mathcal{H}$ such that $R = \operatorname{End}_{\mathcal{H}}(V)$ and $\mathcal{Y} = \operatorname{Faith} V$.*

Proof. The condition is sufficient by Theorem 3.2. Necessity follows from Propositions 4.2 and 4.5, and from Theorem 2.4. □

5. The hereditary case

Throughout this section $\mathcal{A}$ is a fixed *hereditary* cocomplete abelian category, V is a tilting object in $\mathcal{A}$ with $R = \operatorname{End}_{\mathcal{A}}(V)$, and $(\mathcal{T}, \mathcal{F})$, $(\mathcal{X}, \mathcal{Y})$ are the induced torsion theories in $\mathcal{A}$ and Mod-R, respectively. Here we shall show that R is quasitilted, and verify that it satisfies key properties of quasitilted algebras.

Lemma 5.1. $\operatorname{proj\,dim} N \leq 1$ *for any $N \in \mathcal{Y}$.*

Proof. Let $N \in \mathcal{Y}$ and consider an exact sequence in Mod-R,

$$0 \to K \to P \to N \to 0,$$

with P projective. Since this sequence is exact in $\mathcal{Y}$ we have

$$(*) \qquad 0 \to T_V(K) \to T_V(P) \to T_V(N) \to 0$$

exact in $\mathcal{T}$. Since $\mathcal{T} = \operatorname{Pres} V$, there is an exact sequence

$$(**) \qquad 0 \to L \to V^{(\alpha)} \to T_V(K) \to 0$$

in $\mathcal{T}$. Now apply $\operatorname{Hom}_{\mathcal{A}}(-, L)$ to (*) to obtain

$$\operatorname{Ext}^1_{\mathcal{A}}(T_V(P), L) \to \operatorname{Ext}^1_{\mathcal{A}}(T_V(K), L) \to 0 = \operatorname{Ext}^2_{\mathcal{A}}(T_V(N), L).$$

But, since $T_V(P) \in \operatorname{Add} V$ and $L \in \mathcal{T}$, $\operatorname{Ext}^1_{\mathcal{A}}(T_V(P), L) = 0$ (see Corollary 8.3), and hence $\operatorname{Ext}^1_{\mathcal{A}}(T_V(K), L) = 0$. Thus (**) splits, so $T_V(K) \in \operatorname{Add} V$ and $K \cong H_V T_V(K)$ is projective. □

Proposition 5.2. *The torsion theory $(\mathcal{X}, \mathcal{Y})$ splits in* Mod-R.

Proof. Let $A \in \mathcal{T}$ and $B \in \mathcal{F}$. Then, since $\overline{\mathrm{Gen}}\, V = \mathcal{A}$, there is an exact sequence

$$0 \to B \xrightarrow{f} A_1 \xrightarrow{g} A_2 \to 0$$

with $A_1,\ A_2 \in \mathcal{T}$. Applying $H = H_V$ we obtain an exact sequence

$$0 \to HA_1 \xrightarrow{Hg} HA_2 \xrightarrow{\partial} H'B \to 0$$

since $H'A_1 = 0$. Now, from these two exact sequences we obtain a commutative diagram

$$\begin{array}{ccccccc} \mathrm{Ext}^1_{\mathcal{A}}(A, A_1) & \xrightarrow{\mathrm{Ext}^1_{\mathcal{A}}(A,g)} & \mathrm{Ext}^1_{\mathcal{A}}(A, A_2) & \xrightarrow{\partial'} & 0 & & \\ \cong\downarrow & & \cong\downarrow & & & & \\ \mathrm{Ext}^1_R(HA, HA_1) & \xrightarrow{\mathrm{Ext}^1_R(HA,Hg)} & \mathrm{Ext}^1_R(HA, HA_2) & \xrightarrow{\mathrm{Ext}^1_R(HA,\partial)} & \mathrm{Ext}^1_R(HA, H'B) & \xrightarrow{\partial''} & 0 \end{array}$$

with exact rows, noting that $\partial' = 0$ since $\mathcal{A}$ is hereditary, and $\partial'' = 0$ by Lemma 5.1. Also, the vertical maps are isomorphisms, since $\mathcal{T}$ and $\mathcal{Y}$ are closed under extensions, and H_V is exact on $\mathcal{T}$ and T_V is exact on $\mathcal{Y}$. Thus $\mathrm{Ext}^1_R(HA, H'B) = 0$, and so the exact sequence

$$0 \to H'_V T'_V(N) \xrightarrow{\theta_N} N \xrightarrow{\sigma_N} H_V T_V(N) \to 0$$

splits for all $N \in$ Mod-R. □

Thus, using the Tilting Theorem, Theorem 3.2, we have shown that R is a quasitilted ring. We shall conclude this section by showing that R enjoys two further properties possessed by the quasitilted algebras of [14].

Proposition 5.3. $\mathrm{rt\,gl\,dim}\, R \leq 2$.

Proof. Suppose

$$0 \to K \to P_1 \xrightarrow{d} P_0 \to M \to 0$$

exact with P_0 and P_1 projective in Mod-R. Let $I = \mathrm{Im}\, d$, so that

$$0 \to K \to P_1 \to I \to 0$$

is exact in $\mathcal{Y}$. Now apply Lemma 5.1. □

Note that this last argument can be modified to show that an arbitrary ring R with faithful torsion theory $(\mathcal{X}, \mathcal{Y})$ in Mod-R satisfies

$$\mathrm{rt\,gl\,dim}\, R \leq \mathrm{proj\,dim}\, \mathcal{Y} + 1.$$

Proposition 5.4. $\mathrm{inj\,dim}\, N \leq 1$ *for any* $N \in \mathcal{X}$.

Proof. If $N \in \mathcal{X}$, then so is $\mathrm{E}(N)$, since from Proposition 5.2 we know that the torsion theory $(\mathcal{X}, \mathcal{Y})$ splits. Thus there is an exact sequence in $\mathcal{X}$

$$(K) \qquad 0 \to N \xrightarrow{d_0} E_0 \xrightarrow{d_1} E_1 \xrightarrow{d} C \to 0$$

in which each $\mathrm{Im}\, d_i$ is essential in the injective module E_i and

$$(T'_V K) \qquad 0 \to T'_V(N) \xrightarrow{T'_V(d_0)} T'_V(E_0) \xrightarrow{T'_V(d_1)} T'_V(E_1) \xrightarrow{T'_V(d)} T'_V(C) \to 0$$

represents the zero element in $\mathrm{Ext}^2_{\mathcal{A}}(T'_V(C), T'_V(N))$. Let

$$\begin{array}{ccc} E_0 & \xrightarrow{d_1} & E_1 \\ & {\scriptstyle p}\searrow \quad \nearrow{\scriptstyle i} & \\ & I & \end{array}$$

be the epi-monic factorization through $I = \operatorname{Im} d_1 \in \mathcal{X}$, to obtain short exact sequences

$$(T'_V E) \qquad 0 \to T'_V(N) \xrightarrow{T'_V(d_0)} T'_V(E_0) \xrightarrow{T'_V(p)} T'_V(I) \to 0$$

and

$$(T'_V F) \qquad 0 \to T'_V(I) \xrightarrow{T'_V(i)} T'_V(E_1) \xrightarrow{T'_V(d)} T'_V(C) \to 0$$

then, according to [18], page 175, Lemma 4.1, $T'_V K \sim 0$ if and only if there is a short exact sequence L in $\mathcal{A}$ such that $T'_V F \sim T'_V(p)L$ and $T'_V E T'_V(p) \sim 0$. In particular the first condition gives a commutative diagram with exact rows

$$\begin{array}{lccccccccc} (L) & 0 & \longrightarrow & T'_V(E_0) & \longrightarrow & D & \longrightarrow & T'_V(C) & \longrightarrow & 0 \\ & & & {\scriptstyle T'_V(p)}\downarrow & & \downarrow & & \| & & \\ (T'_V F) & 0 & \longrightarrow & T'_V(I) & \xrightarrow{T'_V(i)} & T'_V(E_1) & \longrightarrow & T'_V(C) & \longrightarrow & 0 \end{array}$$

Now, since $\mathcal{F}$ is closed under extensions, all the previous diagrams belong to $\mathcal{F}$, and so we can apply H'_V to see that there is a commutative diagram with exact rows in Mod-R

$$\begin{array}{lccccccccc} (H'_V L) & 0 & \longrightarrow & E_0 & \longrightarrow & H'_V(D) & \xrightarrow{\delta} & C & \longrightarrow & 0 \\ & & & {\scriptstyle p}\downarrow & & \downarrow & & \| & & \\ (F) & 0 & \longrightarrow & I & \xrightarrow{T'_V(i)} & E_1 & \xrightarrow{d} & C & \longrightarrow & 0 \end{array}$$

in which, since E_0 is injective, δ is epi-split, and so d is such. Thus F splits, I is injective, and

$$0 \to N \xrightarrow{d_0} E_0 \xrightarrow{p} I \to 0$$

is an injective resolution of N. □

6. Quasitilted rings characterized

We reiterate from the Introduction:

Definition 6.1. A ring R is called a right quasitilted ring if there is a faithful splitting torsion theory $(\mathcal{X}, \mathcal{Y})$ in Mod-R such that $\operatorname{proj\,dim} \mathcal{Y} \leq 1$.

The results from Section 5 can be summarized in the following.

Proposition 6.2. *If V is a tilting object in a hereditary cocomplete abelian category $\mathcal{A}$, then $R = \mathrm{End}_{\mathcal{A}}(V)$ is a quasitilted ring with $\mathcal{X} = \operatorname{Ker} T_V$ and $\mathcal{Y} = \operatorname{Faith} V$. Moreover* $\operatorname{inj\,dim} \mathcal{X} \leq 1$ *and* $\operatorname{rt\,gl\,dim} R \leq 2$.

This section is devoted to proving the converse of Proposition 6.2. To do so, we need one more lemma.

Lemma 6.3. *Let V be a tilting object in an abelian category $\mathcal{A}$, and let $R = \operatorname{End}_{\mathcal{A}}(V)$. Then for all $L, M \in \mathcal{T} = \operatorname{Gen} V$ and for all $i \geq 0$,*

$$\operatorname{Ext}^i_R(H_V(L), H_V(M)) \cong \operatorname{Ext}^i_{\mathcal{A}}(L, M).$$

Proof. The same proof as in [6], Lemma 3.6.2, works in this general setting. □

Theorem 6.4. *A ring R is right quasitilted via the torsion theory $(\mathcal{X}, \mathcal{Y})$ if and only if there exist a hereditary cocomplete abelian category $\mathcal{A}$ and a tilting object V in $\mathcal{A}$ such that $R \cong \operatorname{End}_{\mathcal{A}}(V)$. Moreover in this case* $\operatorname{inj}\dim \mathcal{X} \leq 1$ *and* $\operatorname{rt}\operatorname{gl}\dim R \leq 2$.

Proof. Thanks to Proposition 6.2 it remains to be proved that any quasitilted ring R is isomorphic to $\operatorname{End}_{\mathcal{A}}(V)$ for a tilting object in a suitable hereditary cocomplete abelian category. Now let $\mathcal{H}$, V and $(\mathcal{T}, \mathcal{F})$ as in Theorem 4.6. To finish the proof, we have to show that $\mathcal{H}$ is hereditary. First, since by Proposition 4.5 any object M in $\mathcal{H}$ admits an exact sequence

$$(*) \qquad 0 \to M \to X_0 \to X_1 \to 0$$

with $X_0, X_1 \in \mathcal{T}$, for any $L \in \mathcal{H}$ we have an exact sequence

$$\operatorname{Ext}^2_{\mathcal{H}}(X_0, L) \to \operatorname{Ext}^2_{\mathcal{H}}(M, L) \to \operatorname{Ext}^3_{\mathcal{H}}(X_1, L).$$

Therefore it is enough to prove that $\operatorname{Ext}^2_{\mathcal{H}}(\mathcal{T}, \mathcal{H}) = 0$ (from which it follows easily from [18], Lemma 4.1, page 75, that even $\operatorname{Ext}^3_{\mathcal{H}}(\mathcal{T}, \mathcal{H}) = 0$) in order to see that $\operatorname{Ext}^2_{\mathcal{H}}(\mathcal{H}, \mathcal{H}) = 0$. Moreover, since $(\mathcal{T}, \mathcal{F})$ is a torsion theory in $\mathcal{H}$, we see that $\operatorname{Ext}^2_{\mathcal{H}}(\mathcal{T}, \mathcal{H}) = 0$ if and only if $\operatorname{Ext}^2_{\mathcal{H}}(\mathcal{T}, \mathcal{T}) = 0$ and $\operatorname{Ext}^2_{\mathcal{H}}(\mathcal{T}, \mathcal{F}) = 0$. The first Ext-vanishing is an immediate consequence of Lemma 6.3 in the case of $i = 2$, since $\operatorname{proj}\dim \mathcal{Y} \leq 1$ by assumption. In order to prove the second Ext-vanishing, let us consider $L \in \mathcal{T}$ and $M \in \mathcal{F}$. Given an exact sequence (*) for M, applying $H = H_V$, since $HM = 0$ and $H'X_0 = 0$, we obtain an exact sequence

$$(**) \qquad 0 \to HX_0 \to HX_1 \to H'M \to 0,$$

and from (*) and (**) we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc}
\operatorname{Ext}^1_{\mathcal{H}}(L, X_0) & \longrightarrow & \operatorname{Ext}^1_{\mathcal{H}}(L, X_1) & \longrightarrow & \operatorname{Ext}^2_{\mathcal{H}}(L, M) & \longrightarrow & \operatorname{Ext}^2_{\mathcal{H}}(L, X_0) \\
\downarrow \cong & & \downarrow \cong & & & & \\
\operatorname{Ext}^1_R(HL, HX_0) & \longrightarrow & \operatorname{Ext}^1_R(HL, HX_1) & \longrightarrow & \operatorname{Ext}^1_R(HL, H'M) & &
\end{array}$$

where $\operatorname{Ext}^2_{\mathcal{H}}(L, X_0) = 0$ since L and X_0 belong to $\mathcal{T}$, and $\operatorname{Ext}^1_R(HL, H'M) = 0$ since $HL \in \mathcal{Y}$, $H'M \in \mathcal{X}$ and $(\mathcal{X}, \mathcal{Y})$ splits by assumption. Thus $\operatorname{Ext}^2_{\mathcal{H}}(L, M) = 0$, and the proof is complete. □

7. An example and two questions

Following the artin algebra tradition, we say that a ring R is *right tilted* if there is a right hereditary ring S with a finitely generated tilting module V_S such that $R = \operatorname{End}(V_S)$ (see [4] for noetherian examples of such rings). Now, Theorem 6.4 shows that tilted rings are particular cases of quasitilted rings. In this section we will see that the class of (right) quasitilted rings properly extends the class of (right) tilted rings, and we shall discuss two problems that arise in connection with quasitilted algebras.

In the following,

$$R = \begin{bmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Z} \end{bmatrix}$$

denotes the ring of upper triangular 2×2 matrices over $\mathbb{Q}$ with $2,2$-entries in $\mathbb{Z}$. We let

$$e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } f = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

in R, and we note that if $J = J(R)$, then

$$fRe = 0, \quad fR = fRf \cong \mathbb{Z}, \quad eRe \cong \mathbb{Q}, \quad eRf = eJ = J \cong {}_{\mathbb{Q}}\mathbb{Q}_{\mathbb{Z}}.$$

We shall show that R is right quasitilted.

The ring R is left, but not right hereditary, as observed by L. Small in [21]. Indeed according to a well-known result from [12]

$$\operatorname{rt\,gl\,dim} R \leq \operatorname{rt\,gl\,dim} eRe + \operatorname{rt\,gl\,dim} fRf + 1 = 2.$$

However there is an exact sequence

$$0 \to K \longrightarrow fR^{(\alpha)} \longrightarrow eR \longrightarrow eR/eJ \to 0,$$

so we see that $\operatorname{proj\,dim} eR/eJ = 2$.

To prove that R is quasitilted, we shall employ the following lemmas.

Lemma 7.1. *All direct sums of copies of eR/eJ and of eR are injective.*

Proof. To see that eR/eJ is injective, let $I \leq R_R$ and $\gamma : I \to eR/eJ$. If $I \leq Rf$, then $\gamma = 0$. Otherwise $e \in I = eR + If$, and one can show that Baer's Criterion applies.

Next we will show that eR is injective relative to both fR and eR, so [1], Proposition 16.12, applies. The former follows since $J = eRf_{fRf} \cong \mathbb{Q}_{\mathbb{Z}}$ and $fR = fRf_{fRf} \cong \mathbb{Z}_{\mathbb{Z}}$. For the latter, suppose that $I < eR$, $\gamma : I \to eR$. Then $I = eIf$ and again $\gamma(I) \leq eRf$ which is injective over fRf. Thus there is a map $\overline{\gamma} : eRf \to eRf$ that extends γ. Identifying $eRf = \mathbb{Q}_{\mathbb{Z}}$ we see that there is a $x \in \mathbb{Q}$ such that $\overline{\gamma}(erf) = xerf$ for all $erf \in eRf$. Now multiplication by $xe \in eRe \cong \operatorname{End}(eR_R)$ extends γ.

Clearly ${}_{eRe}eR$ and eR/eJ have the d.c.c. on submodules, and in particular on annihilators of subsets of R. Thus (see [11], page 181) R has a.c.c. on annihilators of subsets of eR and eR/eJ. Now, since eR/eJ and eR are injective, the result follows from [11], Proposition 3, page 184. □

Let $\mathcal{C} = \{eR/K \mid 0 \neq K \leq eR\}$ and let $(\mathcal{X}, \mathcal{Y})$ be the torsion theory generated by $\mathcal{C}$. Thus, letting

$$\mathcal{Y} = \{Y_R \mid \operatorname{Hom}_R(C, Y) = 0 \text{ for all } C \in \mathcal{C}\}$$

we have

$$\mathcal{X} = \{X_R \mid \operatorname{Hom}_R(X, Y) = 0 \text{ for all } Y \in \mathcal{Y}\}.$$

Lemma 7.2. $\mathcal{Y} = \{eR^{(\alpha)} \oplus N \mid N = Nf\}$.

Proof. Let $Y \in \mathcal{Y}$. Since $\operatorname{Hom}_R(eR/K, Y) = 0$ whenever $0 \neq K \leq eR$, it follows for $x \in Y$ that $xe \neq 0$ implies $xeR \cong eR$. Thus,

$$Y = \sum_I w_\alpha eR + \sum_L b_\lambda fR$$

with each $w_\alpha eR \cong eR$. Now let $H \subseteq I$ be maximal with $\{w_\alpha eR \mid \alpha \in H\}$ independent, so that $P = \bigoplus_H w_\alpha eR \cong eR^{(H)}$ is an (injective by Lemma 7.1) projective direct summand of $\sum_I w_\alpha eR$. One easily checks that $Y \cong eR^{(H)} \oplus N$ with $N = Nf$.

Suppose that $M = eR^{(\alpha)} \oplus N$ with $N = Nf$. If $0 \neq \gamma \in \operatorname{Hom}_R(eR/K, M)$, then $\operatorname{Im}\gamma \subseteq eR^{(\alpha)}$ and $\operatorname{Im}\gamma \nsubseteq eJ^{(\alpha)} = eJ^{(\alpha)}f$, and so some $\pi_\alpha\gamma : eR/K \to eR$ is a split epimorphism. Thus $K = 0$ and $M \in \mathcal{Y}$. $\square$

Corollary 7.3. *$R \in \mathcal{Y}$ and* $\operatorname{proj\,dim}\mathcal{Y} \leq 1$.

Proof. Clearly $R \in \mathcal{Y}$, and $\operatorname{proj\,dim}(eR^{(\alpha)} \oplus Nf) \leq 1$ since $eR^{(\alpha)}$ is projective and $\operatorname{proj\,dim} Nf \leq 1$ as it is an $fR = fRf \cong \mathbb{Z}$-module. $\square$

It only remains to show that $(\mathcal{X}, \mathcal{Y})$ splits. To do so we need

Lemma 7.4. $\mathcal{X} = \operatorname{Gen}\mathcal{C}$.

Proof. Clearly $\operatorname{Gen}\mathcal{C} \subseteq \mathcal{X}$. So let $X \in \mathcal{X}$ and consider X/XJ. Since every direct sum of copies of eR/eJ is injective by Lemma 7.1, as in the proof of Lemma 7.2, $X/XJ \cong eR/eJ^{(\alpha)} \oplus N$ with $N = Nf$. But then $N \in \mathcal{X} \cap \mathcal{Y} = 0$. Thus, since J is nilpotent, there exist $t_\alpha \in X \setminus XJ$ such that $\sum t_\alpha eR = X$, and by Lemmas 7.1 and 7.2, each $t_\alpha eR \cong eR/K_\alpha$ with $K_\alpha \neq 0$. $\square$

Proposition 7.5. *R is right quasitilted with torsion theory $(\mathcal{X}, \mathcal{Y})$.*

Proof. If $X \in \mathcal{X}$ and (see Lemma 7.2) $Y = eR^{(\alpha)} \oplus Nf$, then $\operatorname{Ext}^1_R(Y, X) = \operatorname{Ext}^1_R(Nf, X)$. To show that the latter is 0, noting that by Lemma 7.4, $\mathcal{X} \subseteq \operatorname{Gen} eR$, we will actually show that $\operatorname{Ext}^1_R(N, G) = 0$ for any $G \in \operatorname{Gen} eR$ and $N = Nf$ in Mod $-R$. So suppose that

$$0 \to K \longrightarrow eR^{(\alpha)} \longrightarrow G \to 0$$

is exact, to obtain an exact sequence

$$0 = \operatorname{Ext}^1_R(N, eR^{(\alpha)}) \longrightarrow \operatorname{Ext}^1_R(N, G) \longrightarrow \operatorname{Ext}^2_R(N, K) = 0.$$

Here the first equality is by Lemma 7.1 and the second is because $N = Nf$ has projective dimension ≤ 1. $\square$

Now, if R is any right tilted ring with torsion theory $(\mathcal{X}, \mathcal{Y})$ in Mod-R, then there is a right hereditary ring S with a (finitely generated) tilting module V_S such that $R = \operatorname{End}(V_S)$ and $\mathcal{X} = \operatorname{Ker}(- \otimes_R V)$. In any case if V_S is a tilting module with $R = \operatorname{End}(V_S)$, then ${}_RV$ is a tilting module and so is finitely presented, so that $\operatorname{Ker}(- \otimes_R V)$ is closed under direct products. As pointed out to us by Enrico Gregorio, the torsion theory $(\mathcal{X}, \mathcal{Y})$ of Proposition 7.5 cannot result from any tilting module, because the torsion-free injective module eR embeds as a direct summand in $\prod_{0 \neq K \leq eR} eR/K$, so $\mathcal{X}$ is not closed under direct products. In a forthcoming article with Gregorio, we shall prove that R is actually not a tilted ring.

A quasitilted artin algebra in the sense of Happel, Reiten and Smalø [14] is one that has a split torsion theory $(\mathcal{X}_0, \mathcal{Y}_0)$ in mod-R such that $\operatorname{proj\,dim}\mathcal{Y}_0 \leq 1$ and $R \in \mathcal{Y}_0$, and, necessarily, $\operatorname{inj\,dim} X_0 \leq 1$. Clearly a quasitilted ring that happens to be an artin algebra is quasitilted in their sense. We wonder if the converse is true, and we shall next present some observations that suggest that it may be true.

Let R be a quasitilted artin algebra with torsion theory $(\mathcal{X}_0, \mathcal{Y}_0)$ in mod-R, and let $(\mathcal{X}, \mathcal{Y})$ be the torsion theory in Mod-R generated by $\mathcal{X}_0$. Then, according to [22], Proposition 2.5, page 140,

$$\mathcal{X} = \{X \in \text{Mod-}R \mid \text{non-zero factors of } X \text{ have non-zero submodules in } \mathcal{X}_0\}.$$

Claim 7.6. $\operatorname{inj\,dim}\mathcal{X} \leq 1$.

Proof. If $X \in \mathcal{X}$ there is a non-zero submodule $X_0 \leq X$ with $\operatorname{inj}\dim X_0 \leq 1$, so that given any simple module S_R we have $\operatorname{Ext}^2_R(S, X_0) = 0$. But then we have $X_1/X_0 \leq X/X_0$ with $\operatorname{Ext}^2_R(S, X_1/X_0) = 0$, and so $\operatorname{Ext}^2_R(S, X_1) = 0$. Continue this way transfinitely to see that X is a direct limit of modules of injective dimension ≤ 1, and use [10], Lemma 3.1.16, to get $\operatorname{Ext}^2_R(S, X) = 0$ for each simple S_R. But then for any $M \in \text{Mod-}R$, considering the Loewy series of M, we have $\operatorname{Ext}^2_R(M, X) = 0$. Thus $\operatorname{inj}\dim \mathcal{X} \leq 1$. □

Now

$$\mathcal{Y} = \{Y_R \mid \operatorname{Hom}_R(\mathcal{X}_0, Y) = 0\} = \{Y_R \mid \text{every fin. gen. submodule of } Y \text{ is in } \mathcal{Y}_0\},$$

and, of course, $R_R \in \mathcal{Y}$.

Claim 7.7. $\operatorname{proj}\dim \mathcal{Y} \leq 1$.

Proof. To show that $Y \in \mathcal{Y}$ has $\operatorname{proj}\dim Y \leq 1$, consider an exact sequence

$$0 \to K \to P \xrightarrow{f} Y \to 0$$

where P is projective. Since R is semiperfect, $P = \bigoplus_I P_\alpha$, where each P_α is finitely generated. Now, for each finite subset $F \subseteq I$, let $K_F = \operatorname{Ker} f\restriction_{\oplus_F P_\alpha}$, so that each K_F is projective, since $f(\bigoplus_F P_\alpha) \in \mathcal{Y}_0$. But then $K = \bigcup_{F \subseteq I} K_F$ is a direct limit of projective modules, and so is projective since R is perfect. Thus $\operatorname{proj}\dim \mathcal{Y} \leq 1$. □

We also note that a proof similar to the one for Claim 7.6 yields

Claim 7.8. $\operatorname{Ext}^1_R(\mathcal{Y}_0, \mathcal{X}) = 0$.

So our question becomes one of extending this to $\operatorname{Ext}^1_R(\mathcal{Y}, \mathcal{X}) = 0$.

As we mentioned in our introductory remarks, Happel, Reiten and Smalø [14] also characterized quasitilted artin algebras as those of global dimension ≤ 2 whose finitely generated indecomposable right modules each have either injective or projective dimension at most 1 (so, by duality, any right quasitilted artin algebra is also left quasitilted). Thus we are led to question whether a ring of right global dimension ≤ 2, each of whose right modules is a direct sum of a module of injective dimension ≤ 1 and a module of projective dimension ≤ 1, is a quasitilted ring.

8. Appendix: Ext and direct sums

We do not know if the analogue of the natural isomorphism $\operatorname{Ext}^1_R(\bigoplus_I M_\alpha, L) \cong \Pi_I \operatorname{Ext}^1_R(M_\alpha, L)$ for R-modules is valid for infinite sets I and cocomplete abelian categories. However, for the purpose of this paper it will suffice to show that there is an embedding $\operatorname{Ext}^1_{\mathcal{A}}(\bigoplus_I M_\alpha, L) \rightarrowtail \Pi_I \operatorname{Ext}^1_{\mathcal{A}}(M_\alpha, L)$. To this end, assume $\mathcal{A}$ is a cocomplete abelian category and consider an exact sequence

$$E: \quad 0 \to L \xrightarrow{f} N \xrightarrow{g} \bigoplus_I M_\alpha \to 0$$

with injections $\iota_a : M_a \to \bigoplus_I M_\alpha$, representing an element of $\mathrm{Ext}^1_{\mathcal{A}}(\bigoplus_I M_\alpha, L)$, also consider the pullback diagrams

$$(*)\qquad \begin{array}{ccccccccc}
0 & \longrightarrow & L & \xrightarrow{f} & N & \xrightarrow{g} & \bigoplus_I M_\alpha & \longrightarrow & 0 \\
 & & \| & & {\scriptstyle j_\alpha}\uparrow & & \uparrow{\scriptstyle \iota_\alpha} & & \\
0 & \longrightarrow & L & \xrightarrow{f_\alpha} & B_\alpha & \xrightarrow{g_\alpha} & M_\alpha & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

to obtain representatives

$$E_\alpha : \quad 0 \to L \xrightarrow{f_\alpha} B_\alpha \xrightarrow{g_\alpha} M_\alpha \to 0$$

of $\mathrm{Ext}^1_{\mathcal{A}}(M_\alpha, L)$, and let $\Theta(E) = (E_\alpha)_I \in \Pi_I \,\mathrm{Ext}^1_{\mathcal{A}}(M_\alpha, L)$.

To see that Θ is additive, consider the commutative diagram

$$\begin{array}{rccccccc}
 & 0\to & L\oplus L & \xrightarrow{f_\alpha\oplus f'_\alpha} & B_\alpha\oplus B'_\alpha & \xrightarrow{g_\alpha\oplus g'_\alpha} & M_\alpha\oplus M_\alpha & \to 0 \\
 & & \| & & {\scriptstyle j_\alpha\oplus j'_\alpha}\downarrow & & {\scriptstyle \iota_\alpha\oplus\iota'_\alpha}\downarrow & \\
 & 0\to & L\oplus L & \xrightarrow{f\oplus f'} & N\oplus N' & \xrightarrow{g\oplus g'} & (\bigoplus_I M_\alpha)\oplus(\bigoplus_I M_\alpha) & \to 0 \\
 & & {\scriptstyle \nabla_L}\downarrow & & \downarrow & & \| & \\
 & 0\to & L & \longrightarrow & P_o & \longrightarrow & (\bigoplus_I M_\alpha)\oplus(\bigoplus_I M_\alpha) & \to 0 \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle \Delta_{\bigoplus_I M_\alpha}} & \\
E+E': & 0\to & L & \longrightarrow & P_b & \longrightarrow & \bigoplus_I M_\alpha & \to 0 \\
 & & \| & & \uparrow & & {\scriptstyle \iota_\alpha}\uparrow & \\
\pi_\alpha\Theta(E+E'): & 0\to & L & \longrightarrow & C_\alpha & \longrightarrow & M_\alpha & \to 0
\end{array}$$

Since $\Delta_{\bigoplus_I M_\alpha} \circ \iota_\alpha = \Delta_{M_\alpha}$ and $\mathrm{Im}\, \iota_\alpha \oplus \iota'_\alpha \supseteq \mathrm{Im}\ \Delta_{M_\alpha}$, we see that $\pi_\alpha\Theta(E+E') \sim \pi_\alpha\Theta(E) + \pi_\alpha\Theta(E')$, and so Θ is indeed additive.

To show that Θ is well defined, suppose that E splits with $gi = 1_{\bigoplus_I M_\alpha}$. Then

$$i\iota_\alpha : M_\alpha \to N \quad \text{and} \quad 1_{M_\alpha} : M_\alpha \to M_\alpha$$

with

$$gi\iota_\alpha = \iota_\alpha 1_{M_\alpha}.$$

Thus there is a unique morphism $k_\alpha : M_\alpha \to B_\alpha$ with

$$\iota_\alpha k_\alpha = 1_{M_\alpha}$$

(and $j_\alpha k_\alpha = i\iota_\alpha$) and so every E_α splits. Thus Θ is well defined.

Now suppose that each

$$E_\alpha : \quad 0 \to L \xrightarrow{f_\alpha} B_\alpha \xrightarrow{g_\alpha} M_\alpha \to 0$$

splits with some $k_\alpha : M_\alpha \to B_\alpha$ such that

$$g_\alpha k_\alpha = 1_{M_\alpha}.$$

Then there is a unique morphism

$$i : \bigoplus_I M_\alpha \to N$$

with

$$i\iota_\alpha = j_\alpha k_\alpha$$

and hence

$$gii_\alpha = gj_\alpha k_\alpha = \iota_\alpha g_\alpha k_\alpha = \iota_a.$$

Thus $gi = 1_{\bigoplus_I M_\alpha}$, and so E splits.

Now we have proved

Proposition 8.1. $\Theta : \mathrm{Ext}^1_{\mathcal{A}}(\bigoplus_I M_\alpha, L) \to \Pi_I \mathrm{Ext}^1_{\mathcal{A}}(M_\alpha, L)$ *is a monomorphism of abelian groups.*

If I is finite then Θ is an isomorphism.

Proposition 8.2. *If F is a finite set and $\mathcal{A}$ is an arbitrary abelian category, then* $\Theta : \mathrm{Ext}^1_{\mathcal{A}}(\bigoplus_F M_\alpha, L) \to \Pi_F \mathrm{Ext}^1_{\mathcal{A}}(M_\alpha, L)$ *is a isomorphism of abelian groups.*

Proof. Given exact sequences

$$E_\alpha : \quad 0 \to L \xrightarrow{f_\alpha} B_\alpha \xrightarrow{g_\alpha} M_\alpha \to 0$$

for $\alpha = 1, 2$, consider the pushout diagram

$$(\#) \qquad \begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & L & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & M_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle \varphi_2} & & \| & & \\
0 & \longrightarrow & B_2 & \xrightarrow{\varphi_1} & B & \xrightarrow{p_1} & M_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle p_2} & & & & \\
 & & M_2 & = & M_2 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

and let p be the product morphism

$$\begin{array}{ccc}
B & \xrightarrow{p} & M_1 \Pi M_2 \\
 {\scriptstyle p_\alpha} \searrow & & \swarrow {\scriptstyle \pi_\alpha} \\
 & M_i &
\end{array}$$

Then $\pi_1 p \varphi_2 f_1 = p_1 \varphi_1 f_2 = 0$, and similarly $\pi_2 p \varphi_1 f_2 = 0$. Thus $\mathrm{Im}\, \varphi_2 f_1 \subseteq \mathrm{Ker}\, p$. On the other hand, if $K \xrightarrow{\varphi} B$ is the kernel of p, then, since $B_1 \xrightarrow{\varphi_2} B$ is the kernel of p_2 and $p_2 \varphi = 0$ there is a commutative diagram

$$\begin{array}{ccc}
B_1 & \xrightarrow{\varphi_2} & B \\
 {\scriptstyle \lambda} \nwarrow & & \nearrow {\scriptstyle \varphi} \\
 & M_i &
\end{array}$$

(with unique λ). Now

$$0 = p_1 \varphi = p_1 \varphi_2 \lambda = g_1 \lambda$$

so, since $L \xrightarrow{f_1} B_1$ is the kernel of g_1, there is a unique $\lambda' : K \to L$ with $f_1 \lambda' = \lambda$. Thus

$$\varphi_2 f_1 \lambda' = \varphi_2 \lambda = \varphi$$

and

$$\mathrm{Im}\, \varphi_2 f_1 \supseteq \mathrm{Im}\, \varphi_2 f_1 \lambda' = \mathrm{Im}\, \varphi = \mathrm{Ker}\, p.$$

So we have an exact sequence

$$E: \quad 0 \to L \xrightarrow{\varphi_2 f_1} B \xrightarrow{p} M_1 \oplus M_2 \to 0$$

with $E \in \mathrm{Ext}^1_{\mathcal{A}}(M_1 \oplus M_2)$. Finally, upon checking that the diagram

$$\begin{array}{lccccccccc} E_1: & 0 & \longrightarrow & L & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & M_1 & \longrightarrow & 0 \\ & & & \| & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \iota_1} & & \\ E_1: & 0 & \longrightarrow & L & \xrightarrow{\varphi_2 \circ f_1} & B & \xrightarrow{p} & M_1 \oplus M_1 & \longrightarrow & 0 \end{array}$$

commutes, we see that $\pi_1 \Theta E \sim E_1$, and similarly $\pi_2 \Theta E \sim E_2$. □

Corollary 8.3. *Let $\mathcal{A}$ be a cocomplete abelian category. If $\mathrm{Ext}^1_{\mathcal{A}}(V, L) = 0$ and $P \in \mathrm{Add}(V)$, then $\mathrm{Ext}^1_{\mathcal{A}}(P, L) = 0$.*

Added in proof

(1) The article with E. Gregorio mentioned in the paragraph following the proof of Proposition 7.5 has appeared in *Colloq. Math.*, 104, 151–156, 2006, MR2195804.

(2) In *Symposia Mathematica*, vol. XXIII, 321–412, Instituto Naz. Alta Mat., 1979, MR0565613 (81i:16032), C. M. Ringel proved that if R is a finite-dimensional hereditary algebra, the preinjective modules form a torsion class $\mathcal{X}_0$ in mod-R that generates a torsion theory $(\mathcal{X}, \mathcal{Y})$ in Mod-R that splits if and only if R is tame. This fact provides a negative answer to our question preceding Claim 7.6.

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