



ELSEVIER

Available online at www.sciencedirect.com

SCIENCE @ DIRECT®

NUCLEAR
PHYSICS B

Nuclear Physics B 720 (2005) 64–88

Tri-bimaximal neutrino mixing from discrete symmetry in extra dimensions

Guido Altarelli ^{a,b}, Ferruccio Feruglio ^c

^a *CERN, Department of Physics, Theory Division, CH-1211 Geneva 23, Switzerland*

^b *Dipartimento di Fisica 'E. Amaldi', Università di Roma Tre, INFN, Sezione di Roma Tre, I-00146 Rome, Italy*

^c *Dipartimento di Fisica 'G. Galilei', Università di Padova, INFN, Sezione di Padova, Via Marzolo 8, I-35131 Padua, Italy*

Received 4 May 2005; accepted 19 May 2005

Available online 2 June 2005

Abstract

We discuss a particularly symmetric model of neutrino mixings where, with good accuracy, the atmospheric mixing angle θ_{23} is maximal, $\theta_{13} = 0$ and the solar angle satisfies $\sin^2 \theta_{12} = 1/3$ (Harrison–Perkins–Scott (HPS) matrix). The discrete symmetry A_4 is a suitable symmetry group for the realization of this type of model. We construct a model where the HPS matrix is exactly obtained in a first approximation without imposing ad hoc relations among parameters. The crucial issue of the required VEV alignment in the scalar sector is discussed and we present a natural solution of this problem based on a formulation with extra dimensions. We study the corrections from higher dimensionality operators allowed by the symmetries of the model and discuss the conditions on the cut-off scales and the VEVs in order for these corrections to be completely under control. Finally, the observed hierarchy of charged lepton masses is obtained by assuming a larger flavour symmetry. We also show that, under general conditions, a maximal θ_{23} can never arise from an exact flavour symmetry.

© 2005 Elsevier B.V. All rights reserved.

E-mail addresses: guido.altarelli@cern.ch (G. Altarelli), feruglio@pd.infn.it (F. Feruglio).

0550-3213/\$ – see front matter © 2005 Elsevier B.V. All rights reserved.
doi:10.1016/j.nuclphysb.2005.05.005

1. Introduction

By now there is convincing evidence for solar and atmospheric neutrino oscillations. The Δm^2 values and mixing angles are known with fair accuracy [1]. For Δm^2 we have: $\Delta m_{\text{atm}}^2 \sim 2.5 \times 10^{-3} \text{ eV}^2$ and $\Delta m_{\text{sol}}^2 \sim 8 \times 10^{-5} \text{ eV}^2$. As for the mixing angles, two are large and one is small. The atmospheric angle θ_{23} is large, actually compatible with maximal but not necessarily so: at 3σ : $0.31 \leq \sin^2 \theta_{23} \leq 0.72$ with central value around 0.5. The solar angle θ_{12} is large, $\sin^2 \theta_{12} \sim 0.3$, but certainly not maximal (by about $5\text{--}6\sigma$ now [2]). The third angle θ_{13} is strongly limited, mainly by the CHOOZ experiment, and has at present a 3σ upper limit given by about $\sin^2 \theta_{13} \leq 0.08$.

In spite of this experimental progress there are still many alternative routes in constructing models of neutrino masses. This variety is mostly due to the considerable ambiguities that remain. First of all, it is essential to know whether the LSND signal [3], which has not been confirmed by KARMEN [4] and is currently being double-checked by MiniBoone [5], will be confirmed or will be excluded. If LSND is right we probably need at least four light neutrinos; if not we can do with only the three known ones, as we assume here in the following. As neutrino oscillations only determine mass squared differences a crucial missing input is the absolute scale of neutrino masses (within the existing limits from terrestrial experiments and cosmology [6,7]). Even for three neutrinos the pattern of the neutrino mass spectrum is still undetermined: it can be approximately degenerate, or of the inverse hierarchy type or normally hierarchical. Given for granted that neutrinos are Majorana particles, their masses can still arise either from the see-saw mechanism or from generic dimension-five non-renormalizable operators.

At a more direct level, we do not know how small the mixing angle θ_{13} is and how close to maximal is θ_{23} . One can make a distinction between “normal” and “special” models. For normal models θ_{23} is not too close to maximal and θ_{13} is not too small, typically a small power of the self-suggesting order parameter \sqrt{r} , with $r = \Delta m_{\text{sol}}^2 / \Delta m_{\text{atm}}^2 \sim 1/35$. Special models are those where some symmetry or dynamical feature assures in a natural way the near vanishing of θ_{13} and/or of $\theta_{23} - \pi/4$. Normal models are conceptually more economical and much simpler to construct. We expect that experiment will eventually find that θ_{13} is not too small and that θ_{23} is sizably not maximal. But if, on the contrary, either θ_{13} very small or θ_{23} very close to maximal will emerge from experiment, then theory will need to cope with this fact. Thus it is interesting to conceive and explore dynamical structures that could lead to special models in a natural way.

We want to discuss here some particularly special models where both θ_{13} and $\theta_{23} - \pi/4$ exactly vanish.¹ Then the neutrino mixing matrix U_{fi} ($f = e, \mu, \tau, i = 1, 2, 3$), in the basis of diagonal charged leptons, is given by, apart from sign convention redefinitions:

$$U_{fi} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12}/\sqrt{2} & c_{12}/\sqrt{2} & -1/\sqrt{2} \\ -s_{12}/\sqrt{2} & c_{12}/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}, \quad (1)$$

where c_{12} and s_{12} stand for $\cos \theta_{12}$ and $\sin \theta_{12}$, respectively. It is much simpler to write natural models of this type with s_{12} small and thus many such attempts are present in the

¹ More precisely, they vanish in a suitable limit, with correction terms that can be made negligibly small.

early literature. More recently, given the experimental value of θ_{12} , the more complicated case of s_{12} large was also attacked, using non-Abelian symmetries, either continuous or discrete [8–14]. In many examples the invoked symmetries are particularly ad hoc and/or no sufficient attention is devoted to corrections from higher-dimensional operators that can spoil the pattern arranged at tree level and to the highly non-trivial vacuum alignment problems that arise if naturalness is required also at the level of vacuum expectation values (VEVs).

An interesting special case of Eq. (1) is obtained for $s_{12} = 1/\sqrt{3}$, i.e. the so-called tri-bimaximal or Harrison–Perkins–Scott mixing pattern (HPS) [13], with the entries in the second column all equal to $1/\sqrt{3}$ in absolute value:

$$U_{\text{HPS}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (2)$$

This matrix is a good approximation to present data.² It would be interesting to find a natural and appealing scheme that leads to this matrix with good accuracy. In fact this is a most special model where not only θ_{13} and $\theta_{23} - \pi/4$ vanish but also θ_{12} assumes a particular value. Clearly, in a natural realization of this model, a very constraining and predictive dynamics must be underlying. We think it is interesting to explore particular structures giving rise to this very special set of models in a natural way. In this case we have a maximum of “order” implying special values for all mixing angles: at the other extreme, anarchical models have been proposed [15], where no structure at all is assumed in the lepton sector, so that, for example, θ_{13} and θ_{23} are predicted to be in no way special, except that there must be a smallest angle (probably near to the present bound) and a largest angle (expected sizably different from maximal).

Interesting ideas on how to obtain the HPS mixing matrix have been discussed in Ref. [13]. The most attractive models are based on the discrete symmetry A_4 , which appears as particularly suitable for the purpose, and were presented in Refs. [10,11]. In the present paper we start by discussing some general features of HPS models. We then present a new version of an A_4 model, with (moderate) normal hierarchy, and discuss in detail all aspects of naturalness in this model, also considering effects beyond tree level and the problem of vacuum alignment. There are a number of substantial improvements in our version with respect to Ma in Ref. [11]. First, the HPS matrix is exactly obtained in a first approximation when higher-dimensional operators are neglected, without imposing ad hoc relations among parameters (in Ref. [11] the equality of b and c is not guaranteed by the symmetry). The observed hierarchy of charged lepton masses is obtained by assuming a larger flavour symmetry. The crucial issue of the required VEV alignment in the scalar sector is considered with special attention and we present a natural solution of this problem. We also keep the flavour scalar fields distinct from the normal Higgs bosons (a proliferation of Higgs doublets is disfavoured by coupling unification) and singlets under the Standard

² In the HPS scheme $\tan^2 \theta_{12} = 0.5$, to be compared with the latest experimental determination [2]: $\tan^2 \theta_{12} = 0.45^{+0.09}_{-0.08}$.

Model gauge group. Last not least, we study the corrections from higher dimensionality operators allowed by the symmetries of the model and discuss the conditions on the cut-off scales and the VEVs in order for these corrections to be completely under control.

2. General considerations

The HPS mixing matrix implies that in a basis where charged lepton masses are diagonal the effective neutrino mass matrix is given by $m_\nu = U_{\text{HPS}} \text{diag}(m_1, m_2, m_3) U_{\text{HPS}}^T$:

$$m_\nu = \left[\frac{m_3}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} + \frac{m_2}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \frac{m_1}{6} \begin{pmatrix} 4 & -2 & -2 \\ -2 & 1 & 1 \\ -2 & 1 & 1 \end{pmatrix} \right]. \quad (3)$$

The eigenvalues of m_ν are m_1, m_2, m_3 with eigenvectors $(-2, 1, 1)/\sqrt{6}$, $(1, 1, 1)/\sqrt{3}$ and $(0, 1, -1)/\sqrt{2}$, respectively. In general, apart from phases, there are six parameters in a real symmetric matrix like m_ν : here only three are left after the values of the three mixing angles have been fixed à la HPS. For a hierarchical spectrum $m_3 \gg m_2 \gg m_1$, $m_3^2 \sim \Delta m_{\text{atm}}^2$, $m_2^2/m_3^2 \sim \Delta m_{\text{sol}}^2/\Delta m_{\text{atm}}^2$ and m_1 could be negligible. But also degenerate masses and inverse hierarchy can be reproduced: for example, by taking $m_3 = -m_2 = m_1$ we have a degenerate model, while for $m_1 = -m_2$ and $m_3 = 0$ an inverse hierarchy case (stability under renormalization group running strongly prefers opposite signs for the first and the second eigenvalue which are related to solar oscillations and have the smallest mass squared splitting). From the general expression of the eigenvectors one immediately sees that this mass matrix, independent of the values of m_i , leads to the HPS mixing matrix. It is a curiosity that the eigenvectors are the same as in the case of the Fritzsch–Xing (FX) matrix [16] but with the roles of the first and the third ones interchanged (so that for HPS θ_{23} is maximal while $\sin^2 2\theta_{12} = 8/9$, while for FX the two mixing angles keep the same values but are interchanged).

If the atmospheric mixing angle is really maximal as in the HPS ansatz or close to maximal, it seems quite natural to interpret this as the effect of a flavour symmetry. It would be tempting to think of an approximate flavour symmetry such that $\theta_{23} = \pi/4$ arises in the limit of exact symmetry, that is by neglecting all symmetry breaking effects. Here we will show that this is not the case and that, under quite general conditions, we can never obtain $\theta_{23} = \pi/4$ as a result of an *exact* flavour symmetry.³ We assume that this symmetry is a meaningful symmetry, that is it is only broken by small effects, in the real world. In other words here we exclude symmetries that need breaking terms of order one to describe the observed fermion masses and mixing angles. Apart from that the symmetry can be of whatever type, global or local, continuous or discrete. Being interested in the limit of exact symmetry, we can neglect the sector giving rise to flavour symmetry breaking. We assume that the fields on which such symmetry acts are the fields of the standard model, plus possibly the right-handed neutrinos, so that our results will also cover the see-saw case. Last, we assume canonical kinetic terms, so that the symmetry acts on the fields of the standard model through unitary transformations.

³ For related observations see Ref. [17].

Since the flavour symmetry is broken only by small effects, the mass matrices for charged leptons and neutrinos can be written as:

$$m_e = m_e^0 + \dots, \quad m_\nu = m_\nu^0 + \dots, \quad (4)$$

where dots denote symmetry breaking effects and m_e^0 has rank less or equal than one. Rank greater than one, as for instance, when both the tau and the muon have non-vanishing masses in the symmetry limit, is clearly an unacceptable starting point, since the difference between the two non-vanishing masses can only be explained by large breaking effects, which we have excluded, or by a fine-tuning, which we wish to avoid. If the rank of m_e vanishes, than all mixing angles in the charged lepton sector are undetermined in the symmetry limit and θ_{23} is also completely undetermined. Therefore we can focus on the case when m_e^0 has rank one. If m_e^0 has rank one, then by a unitary transformations we can always go to a field basis where

$$m_e^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & m_\tau^0 \end{pmatrix}. \quad (5)$$

As in the original basis, the action of the flavour symmetry on the new field basis is perfectly defined. If U_ν and U_e are the unitary matrices that diagonalize m_ν^0 and $m_e^{0\dagger} m_e^0$, it will be possible to adopt the parametrization [18]

$$U_\nu = K_\nu R_{23}(\theta_{23}^\nu) P^\dagger R_{13}(\theta_{13}^\nu) P R_{12}(\theta_{12}^\nu), \quad (6)$$

where R_{ij} is the orthogonal matrix representing a rotation in the ij sector, $P = \text{diag}(1, 1, \exp(i\delta))$ and $K = \text{diag}(\exp(i\alpha_1), \exp(i\alpha_2), \exp(i\alpha_3))$. Moreover:

$$U_e = R_{12}(\theta_{12}^e), \quad (7)$$

where the angle θ_{12}^e is completely undetermined. The physical mixing matrix is $U_{PMNS} = U_e^\dagger U_\nu$ and we find:

$$|\tan \theta_{23}| = \left| \cos \theta_{12}^e \tan \theta_{23}^\nu e^{i\alpha_2} + \sin \theta_{12}^e \frac{\tan \theta_{13}^\nu}{\cos \theta_{23}^\nu} e^{i(\delta+\alpha_1)} \right|. \quad (8)$$

Therefore, in general, the atmospheric mixing angle is always undetermined at the leading order. When small symmetry breaking terms are added to m_e^0 and m_ν^0 , it is possible to obtain $\theta_{23} = \pi/4$, provided these breaking terms have suitable orientations in the flavour space. If the breaking terms are produced by a spontaneous symmetry breaking through the minimization of the potential energy of the theory, in general two independent scalar sectors are needed. One of them communicates the breaking to charged fermions and the other one feeds the breaking to neutrinos. In such a framework a maximal atmospheric mixing angle is always the result of a special vacuum alignment.

In the literature there are symmetries predicting θ_{23} large, not necessarily maximal, in the limit of exact symmetry [19]. For instance, this is produced by U(1) flavour symmetries, when the U(1) charges of left-handed leptons and right-handed charged leptons are $(q_1, 0, 0)$ and $(p_1, p_2, 0)$, respectively, with q_1 and $p_{1,2}$ all non-vanishing and different. In

the symmetry limit, such an assignment implies ($m_e \sim \bar{R}L$, $m_\nu \sim L^T L$):

$$m_e^{0\dagger} m_e^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & |\alpha|^2 & \bar{\alpha}\beta \\ 0 & \alpha\bar{\beta} & |\beta|^2 \end{pmatrix}, \quad (9)$$

and:

$$m_\nu^{0\dagger} m_\nu^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & |\alpha'|^2 + |\gamma'|^2 & \bar{\alpha}'\gamma' + \bar{\gamma}'\beta' \\ 0 & \alpha'\bar{\gamma}' + \gamma'\bar{\beta}' & |\beta'|^2 + |\gamma'|^2 \end{pmatrix}, \quad (10)$$

with α , β , α' , β' and γ' independent parameters of the same order of magnitude. If there is no conspiracy among these parameters, the resulting θ_{23} mixing is generically large.

In conclusion, a large lepton mixing in the 23 sector is possible as the result of an exact flavour symmetry. But if we want to reproduce $\theta_{23} = \pi/4$ in some limit of our theory, necessarily this limit cannot correspond to an exact symmetry in flavour space. A maximal atmospheric mixing angle can only originate from breaking effects as a solution of a vacuum alignment problem.

3. Basic structure of the model

Our model is based on the discrete group A_4 following Refs. [10,11], where its structure and representations are described in detail. Here we simply recall that A_4 is the discrete symmetry group of the rotations that leave a tetrahedron invariant, or the group of the even permutations of 4 objects. It has 12 elements and 4 inequivalent irreducible representations denoted 1, 1', 1'' and 3 in terms of their respective dimensions. Introducing ω , the cubic root of unity, $\omega = \exp(i\frac{2\pi}{3})$, so that $1 + \omega + \omega^2 = 0$, the three one-dimensional representations are obtained by dividing the 12 elements of A_4 in three classes, which are determined by the multiplication rule, and assigning to (class 1, class 2, class 3) a factor (1, 1, 1) for 1, or (1, ω , ω^2) for 1' or (1, ω^2 , ω) for 1''. The product of two 3 gives $3 \times 3 = 1 + 1' + 1'' + 3 + 3$. Also $1' \times 1' = 1''$, $1' \times 1'' = 1$, $1'' \times 1'' = 1'$, etc. For $3 \sim (a_1, a_2, a_3)$, $3' \sim (b_1, b_2, b_3)$ the irreducible representations obtained from their product are:

$$1 = a_1 b_1 + a_2 b_2 + a_3 b_3, \quad (11)$$

$$1' = a_1 b_1 + \omega a_2 b_2 + \omega^2 a_3 b_3, \quad (12)$$

$$1'' = a_1 b_1 + \omega^2 a_2 b_2 + \omega a_3 b_3, \quad (13)$$

$$3 \sim (a_2 b_3, a_3 b_1, a_1 b_2), \quad (14)$$

$$3 \sim (a_3 b_2, a_1 b_3, a_2 b_1). \quad (15)$$

Following Ref. [11] we assigns leptons to the four inequivalent representations of A_4 : left-handed lepton doublets l transform as a triplet 3, while the right-handed charged leptons e^c , μ^c and τ^c transform as 1, 1' and 1'', respectively. The flavour symmetry is broken by two real triplets φ and φ' and by a real singlet ξ . At variance with the choice made by [11],

these fields are gauge singlets. Hence we only need two Higgs doublets $h_{u,d}$ (not three generations of them as in Ref. [11]), which we take invariant under A_4 . We assume that some mechanism produces and maintains the hierarchy $\langle h_{u,d} \rangle = v_{u,d} \ll \Lambda$ where Λ is the cut-off scale of the theory.⁴ The Yukawa interactions in the lepton sector read:

$$\mathcal{L}_Y = y_e e^c(\varphi l) + y_\mu \mu^c(\varphi l)'' + y_\tau \tau^c(\varphi l)' + x_a \xi(l l) + x_d(\varphi' l l) + \text{h.c.} + \dots \quad (16)$$

In our notation, (33) transforms as 1, (33)' transforms as 1' and (33)'' transforms as 1''. Also, to keep our notation compact, we use a two-component notation for the fermion fields and we set to 1 the Higgs fields $h_{u,d}$ and the cut-off scale Λ . For instance, $y_e e^c(\varphi l)$ stands for $y_e e^c(\varphi l) h_d / \Lambda$, $x_a \xi(l l)$ stands for $x_a \xi(l h_u l h_u) / \Lambda^2$ and so on. The Lagrangian \mathcal{L}_Y contains the lowest order operators in an expansion in powers of $1/\Lambda$. Dots stand for higher-dimensional operators that will be discussed in Section 6. Some terms allowed by the flavour symmetry, such as the terms obtained by the exchange $\varphi' \leftrightarrow \varphi$, or the term $(l l)$ are missing in \mathcal{L}_Y . Their absence is crucial and will be motivated later on.

As we will demonstrate in Section 5, the fields φ' , φ and ξ develop a VEV along the directions:

$$\begin{aligned} \langle \varphi' \rangle &= (v', 0, 0), \\ \langle \varphi \rangle &= (v, v, v), \\ \langle \xi \rangle &= u. \end{aligned} \quad (17)$$

Therefore, at the leading order of the $1/\Lambda$ expansion, the mass matrices m_l and m_ν for charged leptons and neutrinos are given by:

$$m_l = v_d \frac{v}{\Lambda} \begin{pmatrix} y_e & y_e & y_e \\ y_\mu & y_\mu \omega & y_\mu \omega^2 \\ y_\tau & y_\tau \omega^2 & y_\tau \omega \end{pmatrix}, \quad (18)$$

$$m_\nu = \frac{v_u^2}{\Lambda} \begin{pmatrix} a & 0 & 0 \\ 0 & a & d \\ 0 & d & a \end{pmatrix}, \quad (19)$$

where

$$a \equiv x_a \frac{u}{\Lambda}, \quad d \equiv x_d \frac{v'}{\Lambda}. \quad (20)$$

Charged leptons are diagonalized by

$$l \rightarrow \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} l, \quad (21)$$

and charged fermion masses are given by:

$$m_e = \sqrt{3} y_e v_d \frac{v}{\Lambda}, \quad m_\mu = \sqrt{3} y_\mu v_d \frac{v}{\Lambda}, \quad m_\tau = \sqrt{3} y_\tau v_d \frac{v}{\Lambda}. \quad (22)$$

⁴ This is the well-known hierarchy problem that can be solved, for instance, by realizing a supersymmetric version of this model.

We can easily obtain a natural hierarchy among m_e , m_μ and m_τ by introducing an additional $U(1)_F$ flavour symmetry under which only the right-handed lepton sector is charged. We assign F -charges 0, 2 and 3–4 to τ^c , μ^c and e^c , respectively. By assuming that a flavon θ , carrying a negative unit of F , acquires a VEV $\langle\theta\rangle/\Lambda \equiv \lambda < 1$, the Yukawa couplings become field dependent quantities $y_{e,\mu,\tau} = y_{e,\mu,\tau}(\theta)$ and we have

$$y_\tau \approx O(1), \quad y_\mu \approx O(\lambda^2), \quad y_e \approx O(\lambda^{3-4}). \quad (23)$$

In the flavour basis the neutrino mass matrix reads⁵:

$$m_\nu = \frac{v_u^2}{\Lambda} \begin{pmatrix} a + 2d/3 & -d/3 & -d/3 \\ -d/3 & 2d/3 & a - d/3 \\ -d/3 & a - d/3 & 2d/3 \end{pmatrix}, \quad (24)$$

and is diagonalized by the transformation:

$$U^T m_\nu U = \frac{v_u^2}{\Lambda} \text{diag}(a + d, a, -a + d), \quad (25)$$

with

$$U = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & +1/\sqrt{2} \end{pmatrix}. \quad (26)$$

The leading order predictions are $\tan^2 \theta_{23} = 1$, $\tan^2 \theta_{12} = 0.5$ and $\theta_{13} = 0$. The neutrino masses are $m_1 = a + d$, $m_2 = a$ and $m_3 = -a + d$, in units of v_u^2/Λ . We can express $|a|$, $|d|$ in terms of $r \equiv \Delta m_{\text{sol}}^2/\Delta m_{\text{atm}}^2 \equiv (|m_2|^2 - |m_1|^2)/(|m_3|^2 - |m_1|^2)$, $\Delta m_{\text{atm}}^2 \equiv |m_3|^2 - |m_1|^2$ and $\cos \Delta$, Δ being the phase difference between the complex numbers a and d :

$$\begin{aligned} \sqrt{2}|a| \frac{v_u^2}{\Lambda} &= \frac{-\sqrt{\Delta m_{\text{atm}}^2}}{2 \cos \Delta \sqrt{1 - 2r}}, \\ \sqrt{2}|d| \frac{v_u^2}{\Lambda} &= \sqrt{1 - 2r} \sqrt{\Delta m_{\text{atm}}^2}. \end{aligned} \quad (27)$$

To satisfy these relations a moderate tuning is needed in our model. Due to the absence of (ll) in Eq. (16) which we will motivate in the next section, a and d are of the same order in $1/\Lambda$, see Eq. (20). Therefore we expect that $|a|$ and $|d|$ are close to each other and, to satisfy Eq. (27), $\cos \Delta$ should be negative and of order one. We obtain:

$$\begin{aligned} |m_1|^2 &= \left[-r + \frac{1}{8 \cos^2 \Delta (1 - 2r)} \right] \Delta m_{\text{atm}}^2, \\ |m_2|^2 &= \frac{1}{8 \cos^2 \Delta (1 - 2r)} \Delta m_{\text{atm}}^2, \\ |m_3|^2 &= \left[1 - r + \frac{1}{8 \cos^2 \Delta (1 - 2r)} \right] \Delta m_{\text{atm}}^2. \end{aligned} \quad (28)$$

⁵ Notice that a unitary change of basis like the one in Eq. (21) will in general change the relative phases of the eigenvalues of m_ν .

If $\cos \Delta = -1$, we have a neutrino spectrum close to hierarchical:

$$|m_3| \approx 0.053 \text{ eV}, \quad |m_1| \approx |m_2| \approx 0.017 \text{ eV}. \quad (29)$$

In this case the sum of neutrino masses is about 0.087 eV. If $\cos \Delta$ is accidentally small, the neutrino spectrum becomes degenerate. The value of $|m_{ee}|$, the parameter characterizing the violation of total lepton number in neutrinoless double beta decay, is given by:

$$|m_{ee}|^2 = \left[-\frac{1+4r}{9} + \frac{1}{8 \cos^2 \Delta (1-2r)} \right] \Delta m_{\text{atm}}^2. \quad (30)$$

For $\cos \Delta = -1$ we get $|m_{ee}| \approx 0.005 \text{ eV}$, at the upper edge of the range allowed for normal hierarchy, but unfortunately too small to be detected in a near future. Independently from the value of the unknown phase Δ we get the relation:

$$|m_3|^2 = |m_{ee}|^2 + \frac{10}{9} \Delta m_{\text{atm}}^2 \left(1 - \frac{r}{2} \right), \quad (31)$$

which is a prediction of our model.

It is also important to get some constraint on the mass scales involved in our construction. From Eqs. (27) and (20), by assuming $x_d \approx 1v_u \approx 250 \text{ GeV}$, we have

$$\Lambda \approx 1.8 \times 10^{15} \left(\frac{v'}{\Lambda} \right) \text{ GeV}. \quad (32)$$

Since, to have a meaningful expansion, we expect $v' \leq \Lambda$, we have the upper bound

$$\Lambda < 1.8 \times 10^{15} \text{ GeV}. \quad (33)$$

Beyond this energy scale, new physics should come into play. The smaller the ratio v'/Λ , the smaller becomes the cut-off scale. For instance, when $v'/\Lambda = 0.03$, Λ should be close to 10^{14} GeV . A complementary information comes from the charged lepton sector, Eq. (22). A lower bound on v/Λ can be derived from the requirement that the Yukawa coupling y_τ remains in a perturbative regime. By asking $y_\tau v_d < 250 \text{ GeV}$, we get

$$\frac{v}{\Lambda} > 0.004. \quad (34)$$

Finally, by assuming that all the VEVs fall in approximately the same range, which will be shown in Section 5, we obtain the range

$$0.004 < \frac{v'}{\Lambda} \approx \frac{v}{\Lambda} \approx \frac{u}{\Lambda} < 1, \quad (35)$$

that will be useful to estimate the effects of higher-dimensional operators in Section 6. Correspondingly the cut-off scale will range between about 10^{13} and $1.8 \times 10^{15} \text{ GeV}$.

4. Vacuum alignment

In this section we investigate the problem of achieving the vacuum alignment of Eq. (17). At the same time we should prevent, at least at some level, the interchange between the fields φ and φ' to produce the desired mass matrices in the neutrino and charged

lepton sectors. As we will see, there are several difficulties to naturally accomplish these requirements. By minimizing the scalar potential of the theory with respect to φ and φ' we get six equations that we would like to satisfy in terms of the two unknown v and v' . Even though we expect that, due to the symmetry A_4 , the six minimum conditions are not necessarily independent, such an expectation turns out to be wrong in the specific case, unless some additional relation is enforced on the parameters of the scalar potential. These additional relations are in general not natural. For instance, even by imposing them at the tree level, they are expected to be violated at the one-loop order. Therefore, as we will now illustrate, the minimum conditions cannot be all satisfied by our vacuum configuration.

As an example here we analyze the most general renormalizable scalar potential invariant under A_4 and depending upon the triplets φ and φ' of the Lagrangian \mathcal{L} in Eq. (16). The term (ll) in \mathcal{L} can be forbidden by an additional symmetry, commuting with A_4 . One possibility is just the total lepton number L or a discrete subgroup of it. Here we consider a Z_4 symmetry under which f^c transform into $-if^c$ ($f = e, \mu, \tau$), l into il , φ is invariant and φ' changes sign. This symmetry also explains why φ and φ' cannot be interchanged. The scalar potential V contains bilinears B_i , trilinears T_i and quartic terms Q_i , invariant under the group $A_4 \times Z_4$. A choice of independent invariants is:

$$\begin{aligned}
 B_1 &= \varphi_1^2 + \varphi_2^2 + \varphi_3^2, \\
 B_2 &= \varphi_1'^2 + \varphi_2'^2 + \varphi_3'^2, \\
 T_1 &= \varphi_1 \varphi_2 \varphi_3, \\
 T_2 &= \varphi_1 \varphi_2' \varphi_3' + \varphi_2 \varphi_3' \varphi_1' + \varphi_3 \varphi_1' \varphi_2', \\
 Q_1 &= \varphi_1^2 \varphi_2^2 + \varphi_2^2 \varphi_3^2 + \varphi_3^2 \varphi_1^2, \\
 Q_2 &= |\varphi_1^2 + \omega^2 \varphi_2^2 + \omega \varphi_3^2|^2, \\
 Q_3 &= \varphi_1'^2 \varphi_2'^2 + \varphi_2'^2 \varphi_3'^2 + \varphi_3'^2 \varphi_1'^2, \\
 Q_4 &= |\varphi_1'^2 + \omega^2 \varphi_2'^2 + \omega \varphi_3'^2|^2, \\
 Q_5 &= \varphi_1 \varphi_2 \varphi_1' \varphi_2' + \varphi_2 \varphi_3 \varphi_2' \varphi_3' + \varphi_3 \varphi_1 \varphi_3' \varphi_1', \\
 Q_6 &= (\varphi_1^2 + \varphi_2^2 + \varphi_3^2)(\varphi_1'^2 + \varphi_2'^2 + \varphi_3'^2), \\
 Q_7 &= (\varphi_1^2 + \omega^2 \varphi_2^2 + \omega \varphi_3^2)(\varphi_1'^2 + \omega \varphi_2'^2 + \omega^2 \varphi_3'^2).
 \end{aligned} \tag{36}$$

The scalar potential reads:

$$\begin{aligned}
 V &= \frac{M_1^2}{2} B_1^2 + \frac{M_2^2}{2} B_2^2 + \mu_1 T_1 + \mu_2 T_2 + c_1 Q_1 + c_2 Q_2 + c_3 Q_3 + c_4 Q_4 \\
 &\quad + c_5 Q_5 + c_6 Q_6 + (c_7 Q_7 + \text{c.c.}).
 \end{aligned} \tag{37}$$

We start by analyzing the field configuration:

$$\langle \varphi \rangle = (v, v, v), \quad \langle \varphi' \rangle = (v', 0, 0). \tag{38}$$

The minimum conditions are:

$$\frac{\partial V}{\partial \varphi_1} = M_1^2 v + \mu_1 v^2 + 4c_1 v^3 + 2c_6 v v'^2 + 2(c_7 + \bar{c}_7) v v'^2 = 0,$$

$$\begin{aligned}
\frac{\partial V}{\partial \varphi_2} &= M_1^2 v + \mu_1 v^2 + 4c_1 v^3 + 2c_6 v v'^2 + 2(\omega^2 c_7 + \omega \bar{c}_7) v v'^2 = 0, \\
\frac{\partial V}{\partial \varphi_3} &= M_1^2 v + \mu_1 v^2 + 4c_1 v^3 + 2c_6 v v'^2 + 2(\omega c_7 + \omega^2 \bar{c}_7) v v'^2 = 0, \\
\frac{\partial V}{\partial \varphi'_1} &= M_2^2 v' + 4c_4 v'^3 + 6c_6 v^2 v' = 0, \\
\frac{\partial V}{\partial \varphi'_2} &= \mu_2 v v' + c_5 v^2 v' = 0, \\
\frac{\partial V}{\partial \varphi'_3} &= \mu_2 v v' + c_5 v^2 v' = 0.
\end{aligned} \tag{39}$$

The equations $\partial V / \partial \varphi_i = 0$ are clearly incompatible unless $c_7 = 0$. Even by forcing c_7 to vanish, we are left with three independent equations for the two unknown v and v' , which, for generic values of the coefficients, admit only the trivial solution $v = v' = 0$. This negative results cannot be modified by adding to V the terms depending on the singlet ξ . Also by investigating the problem in a slightly more general framework, with φ real and (φ', ξ) complex, we reach the same conclusion. Although we have not a no-go theorem, these examples show the difficulty to obtain the desired alignment.

The difficulty illustrated above is not common to all vacua. For instance, the other possible alignment:

$$\langle \varphi \rangle = (v, v, v), \quad \langle \varphi' \rangle = (v', v', v') \tag{40}$$

leads to the minimum conditions:

$$\begin{aligned}
\frac{\partial V}{\partial \varphi_i} &= M_1^2 v + \mu_1 v^2 + \mu_2 v'^2 + 4c_1 v^3 + (2c_5 + 6c_6) v v'^2 = 0, \\
\frac{\partial V}{\partial \varphi'_i} &= M_2^2 v' + 2\mu_2 v v' + 4c_3 v'^3 + (2c_5 + 6c_6) v^2 v' = 0.
\end{aligned} \tag{41}$$

In a non-vanishing portion of the parameter space, these equations have non-trivial solution with non-vanishing v and v' .

It is possible to show that, by sufficiently restricting the form of the most general scalar potential invariant under A_4 , the desired alignment can be obtained. Restrictions that are unnatural in a generic model becomes technically natural in a supersymmetric (SUSY) model. The well-known non-renormalization properties of the superpotential allow to accept, at least from a technical viewpoint, a restricted number of terms, compared to what the A_4 symmetry would permit. Undesired terms of the superpotential that are set to zero at the tree level are not generated at any order in perturbation theory. Indeed we have produced a SUSY example of this type, where the alignment problem is solved and this example is discussed in detail in [Appendix A](#). However, our real aim is to build a fully natural model, where all the terms allowed by the symmetries are present and where the only deviations from the symmetry limit are provided by higher-dimensional operators, rather than by small violations of ad-hoc imposed relations. As we will now see, there exist a simple and economic solution in the context of theories with one extra spatial dimension.

5. A_4 model in an extra dimension

One of the problems we should overcome in the search for the correct alignment is that of keeping neutrino and charged lepton sectors separate, including the respective symmetry breaking sectors. Here we show that such a separation can be achieved by means of an extra spatial dimension. The space–time is assumed to be five-dimensional, the product of the four-dimensional Minkowski space–time times an interval going from $y = 0$ to $y = L$. At $y = 0$ and $y = L$ the space–time has two four-dimensional boundaries, which we will call branes. Our idea is that matter $SU(2)$ singlets such as e^c, μ^c, τ^c are localized at $y = 0$, while $SU(2)$ doublets, such as l are localized at $y = L$ (see Fig. 1). Neutrino masses arise from local operators at $y = L$. Charged lepton masses are produced by non-local effects involving both branes. Later on we will see how such non-local effects can arise in this theory. The simplest possibility is to introduce a bulk fermion, depending on all space–time coordinates, that interacts with e^c, μ^c, τ^c at $y = 0$ and with l at $y = L$. The exchange of such a fermion can provide the desired non-local coupling between right- and left-handed ordinary fermions. Finally, assuming that φ and (φ', ξ) are localized respectively at $y = 0$ and $y = L$, we obtain a natural separation between the two sectors.

5.1. Alignment in an extra dimension

Such a separation also greatly simplify the vacuum alignment problem. We can determine the minima of two scalar potentials V_0 and V_L , depending only, respectively, on φ and (φ', ξ) . Indeed, as we shall see, there are whole regions of the parameter space where $V_0(\varphi)$ and $V_L(\varphi', \xi)$ have the minima given in Eq. (17). Notice that in the present setup dealing with a discrete symmetry such as A_4 provides a great advantage as far as the alignment problem is concerned. A continuous flavour symmetry such as, for instance, $SO(3)$ would need some extra structure to achieve the desired alignment. Indeed the potential energy $\int d^4x [V_0(\varphi) + V_L(\varphi', \xi)]$ would be invariant under a much bigger symmetry,

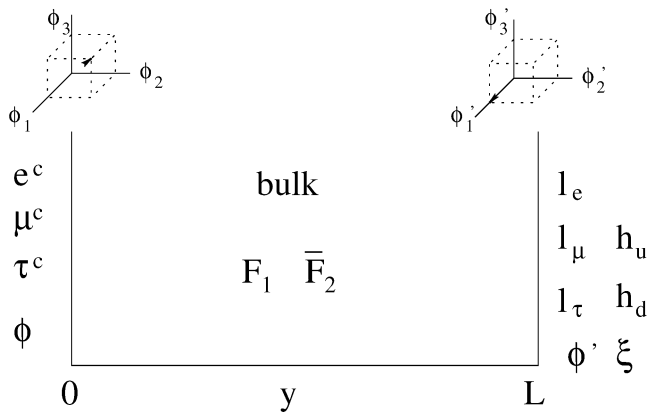


Fig. 1. Fifth dimension and localization of scalar and fermion fields. The symmetry breaking sector includes the A_4 triplets φ and φ' , localized at the opposite ends of the interval. Their VEVs are dynamically aligned along the directions shown at the top of the figure.

$SO(3)_0 \times SO(3)_L$, with the $SO(3)_0$ acting on φ and leaving (φ', ξ) invariant and vice-versa for $SO(3)_L$. This symmetry would remove any alignment between the VEVs of φ and those of (φ', ξ) . If, for instance, (17) is minimum of the potential energy, then any other configuration obtained by acting on (17) with $SO(3)_0 \times SO(3)_L$ would also be a minimum and the relative orientation between the two sets of VEVs would be completely undetermined. A discrete symmetry such as A_4 has not this problem, as we will show now.

Consider first the scalar potential $V_0(\varphi)$:

$$V_0(\varphi) = \frac{M_1^2}{2} B_1^2 + \mu_1 T_1 + c_1 Q_1 + c_2 Q_2, \quad (42)$$

where $B_1, T_1, Q_{1,2}$ are defined in Eq. (36). The minimum conditions at $\varphi = (v, v, v)$ are:

$$\frac{\partial V_0}{\partial \varphi_i} = v(M_1^2 + \mu_1 v + 4c_1 v^2) = 0 \quad (i = 1, 2, 3), \quad (43)$$

while the minimum condition at $\varphi = (v, 0, 0)$ is:

$$\frac{\partial V_0}{\partial \varphi_1} = v(M_1^2 + 4c_2 v^2) = 0, \quad (44)$$

since in this case $(\partial V_0 / \partial \varphi_{2,3}) = 0$ are automatically satisfied. Both $\varphi = (v, v, v)$ and $\varphi = (v, 0, 0)$ can be local minima of V_0 , depending on the parameters. The constants $c_{1,2}$ should be positive, to have V_0 bounded from below. We can look at the region where $|\mu_1| \ll |M_1|$. When $c_1 \gg c_2$ and $M_1^2 < 0$, the minimum at $\varphi = (v, 0, 0)$ is the absolute one, while for $c_2 \gg c_1$ and $M_1^2 < 0$ V_0 is minimized by $\varphi = (v, v, v)$. Therefore we have a large portion of the parameter space where the minimum is of the desired form: $\varphi = (v, v, v)$. To be precise, in this region, there are four degenerate minima: $\varphi = (v, v, v)$, $\varphi = (v, -v, -v)$, $\varphi = (-v, v, -v)$, $\varphi = (-v, -v, v)$, related by A_4 transformations.

Now we turn to $V_L(\varphi', \xi)$. As we did in Section 3, we assume both φ' and ξ real and odd under the action of a discrete Z_4 symmetry. The most general renormalizable invariant potential is a combination of $B_2, Q_{3,4}$ in Eq. (36) and the following invariants:

$$\begin{aligned} B_3 &= \xi^2, \\ Q_8 &= \xi^4, \\ Q_9 &= \xi \varphi'_1 \varphi'_2 \varphi'_3, \\ Q_{10} &= \xi^2 (\varphi_1'^2 + \varphi_2'^2 + \varphi_3'^2). \end{aligned} \quad (45)$$

We have:

$$V_L(\varphi', \xi) = \frac{M_2^2}{2} B_2 + \frac{M_3^2}{2} B_3 + c_3 Q_3 + c_4 Q_4 + c_8 Q_8 + c_9 Q_9 + c_{10} Q_{10}. \quad (46)$$

We search for a minimum at $\varphi' = (v', 0, 0)$ and $\xi = u$:

$$\begin{aligned} \frac{\partial V_L}{\partial \varphi'_1} &= v'(M_2^2 + 4c_4 v'^2 + 2c_{10} u^2) = 0, \\ \frac{\partial V_L}{\partial \xi} &= u(M_3^2 + 4c_8 u^2 + 2c_{10} v'^2) = 0, \end{aligned} \quad (47)$$

while $(\partial V_L / \partial \varphi'_{2,3}) = 0$ are always satisfied. There is a region of the parameter space where the absolute minimum is of this type. Taking into account the A_4 symmetry, in all this region we have six degenerate minima: $\varphi' = (\pm v', 0, 0)$, $\varphi' = (0, \pm v', 0)$ and $\varphi' = (0, 0, \pm v')$. Putting together the minima of $V_0(\varphi)$ and $V_L(\varphi', \xi)$ we have 24 degenerate minima of the potential energy, differing for signs or ordering. It can be shown that these 24 minima produce exactly the same mass pattern discussed in Section 3, up to field and parameter redefinitions. Therefore, it is not restrictive to choose one of them, for instance, $\varphi = (v, v, v)$ and $\varphi' = (v', 0, 0)$, to analyze the property of this model.

The observed hierarchy among lepton masses can be efficiently described by an additional $U(1)_F$ flavour symmetry, under which only right-handed charged leptons are charged: $F(e^c, \mu^c, \tau^c) = (4, 2, 0)$. To spontaneously break this symmetry and to produce the desired hierarchy, we need a scalar field θ , carrying a negative unit of F and developing a VEV $\langle \theta \rangle / \Lambda \approx 0.22$. In our framework θ is localized on the brane at $y = 0$ and the scalar potential V_0 of Eq. (42) is modified into:

$$V_0 \rightarrow V_0 + M_4^2 B_4 + c_{11} Q_{11} + c_{12} Q_{12}, \tag{48}$$

where

$$\begin{aligned} B_4 &= |\theta|^2, \\ Q_{11} &= |\theta|^4, \\ Q_{12} &= |\theta|^2 (\varphi_1^2 + \varphi_2^2 + \varphi_3^2). \end{aligned} \tag{49}$$

The minimum conditions at $\varphi = (v, v, v)$ and $|\theta| = t$ read:

$$\begin{aligned} \frac{\partial V_0}{\partial \varphi_i} &= v(M_1^2 + \mu_1 v + 4c_1 v^2 + 2c_{12} t^2) = 0 \quad (i = 1, 2, 3), \\ \frac{\partial V_0}{\partial |\theta|} &= 2t(M_4^2 + 2c_{11} t^2 + 3c_{12} v^2) = 0. \end{aligned} \tag{50}$$

These conditions are satisfied by non-vanishing (t, v) in a finite portion of the parameter space. Therefore the inclusion of an Abelian flavour symmetry is fully compatible with the mechanism for vacuum alignment discussed above.

5.2. Lepton masses and mixing angles

We now show how it is possible to take advantage of above results to obtain the desired lepton masses. To this purpose we introduce a bulk fermion field $F(x, y) = (F_1, \bar{F}_2)$, singlet under $SU(2)$ with hypercharge $Y = -1$ and transforming as a triplet of A_4 . We also impose the discrete Z_4 symmetry introduced in Section 4 under which $(f^c, l, F, \varphi, \varphi', \xi)$ transform into $(-if^c, il, iF, \varphi, -\varphi', -\xi)$. The action is

$$\begin{aligned} S &= \int d^4x dy \left\{ \left[iF_1 \sigma^\mu \partial_\mu \bar{F}_1 + iF_2 \sigma^\mu \partial_\mu \bar{F}_2 + \frac{1}{2}(F_2 \partial_y F_1 - \partial_y F_2 F_1 + \text{h.c.}) \right] \right. \\ &\quad - M(F_1 F_2 + \bar{F}_1 \bar{F}_2) \\ &\quad \left. + V_0(\varphi) \delta(y) + V_L(\varphi', \xi) \delta(y - L) \right\} \end{aligned}$$

$$\begin{aligned}
& + [Y_e e^c (\varphi F_1) + Y_\mu \mu^c (\varphi F_1)'' + Y_\tau \tau^c (\varphi F_1)' + \text{h.c.}] \delta(y) \\
& + \left[\frac{x_d}{\Lambda^2} \xi (ll) h_u h_u + \frac{x_d}{\Lambda^2} (\varphi' ll) h_u h_u + Y_L (F_2 l) h_d + \text{h.c.} \right] \delta(y - L) \Big\} + \dots,
\end{aligned} \tag{51}$$

where the constants Y have mass dimension $-1/2$. The first two lines represent the five-dimensional kinetic and mass terms of the bulk field F . The third line is the scalar potential and the remaining terms are the lowest order invariant operators localized at the two branes. Dots stand for the kinetic terms of $f^c, l, \varphi, \varphi', \xi$ and for higher-dimensional operators, which will be classified in Section 6.

The potential energy is given, at lowest order by:

$$U = \int d^4x [V_0(\varphi) + V_L(\varphi', \xi)], \tag{52}$$

and, under the conditions discussed above, is minimized by Eqs. (17). It is clear that φ and (φ', ξ) are strictly separated only at lowest order. Indeed higher-dimensional brane interactions like, for instance, $(\varphi \varphi F_1 F_2)/\Lambda^2$, $(\varphi' \varphi' F_1 F_2)/\Lambda^2$ are allowed. At the one-loop level, the exchange of the bulk fermion F will give rise to the structures $Q_{5,6,7}$ of Eq. (36) and this will necessarily deform the vacuum (17). Here we will assume that such a deformation is sufficiently small. Indeed, as we shall see in Section 6, the operators of the type $(\varphi \varphi \varphi' \varphi')$ arising from one-loop F -exchange, are suppressed by $1/\Lambda^4 L^4$.

We now discuss the effects of the tree-level exchange of F . To this purpose we consider the equations of motion for (F_1, F_2) :

$$\begin{aligned}
i\sigma^\mu \partial_\mu \bar{F}_2 + \partial_y F_1 - M F_1 &= 0, \\
i\sigma^\mu \partial_\mu \bar{F}_1 - \partial_y F_2 - M F_2 &= 0.
\end{aligned} \tag{53}$$

If M is large and positive, we can prove that all the modes contained in (F_1, F_2) become heavy, at a scale greater than or comparable to $1/L$, which we assume to be much higher than the electroweak scale. If we are only interested in energies much lower than $1/L$, we can solve the equations of motion in the static approximation, by neglecting the four-dimensional kinetic term:

$$\begin{aligned}
F_1(y) &= F_1(L) e^{M(y-L)}, \\
F_2(y) &= F_2(0) e^{-My}.
\end{aligned} \tag{54}$$

These equations must be supplemented with appropriate boundary conditions, which we can identify by varying the action S with respect the fields (F_1, F_2) . The boundary terms read

$$\begin{aligned}
(\delta S)_{\text{boundary}} &= \int d^4x \left\{ \delta F_1(L) \left[\frac{1}{2} F_2(L) \right] + \delta F_2(L) \left[-\frac{1}{2} F_1(L) + Y_L l h_d \right] \right. \\
&\quad + \delta F_1(0) \left[-\frac{1}{2} F_2(0) + Y_e e^c \varphi + Y_\mu \mu^c \varphi_\mu + Y_\tau \tau^c \varphi_\tau \right] \\
&\quad \left. + \delta F_2(0) \left[\frac{1}{2} F_1(0) \right] \right\},
\end{aligned} \tag{55}$$

where $\varphi_\mu = (\varphi_1, \omega\varphi_2, \omega^2\varphi_3)$ and $\varphi_\tau = (\varphi_1, \omega^2\varphi_2, \omega\varphi_3)$. We can chose as boundary conditions:

$$\begin{aligned} F_1(L) &= 2Y_L l h_d, \\ F_2(0) &= 2(Y_e e^c \varphi + Y_\mu \mu^c \varphi_\mu + Y_\tau \tau^c \varphi_\tau). \end{aligned} \quad (56)$$

Since $\delta F_1(L) = \delta F_2(0) = 0$, we have $(\delta S)_{\text{boundary}} = 0$, as desired. By substituting back Eqs. (54) and (56) into the action S we get

$$\begin{aligned} S = U + \int d^4x & \left[\frac{y_e}{\Lambda} e^c(\varphi l) h_d + \frac{y_\mu}{\Lambda} \mu^c(\varphi l)'' h_d + \frac{y_\tau}{\Lambda} \tau^c(\varphi l)' h_d \right. \\ & \left. + \frac{x_d}{\Lambda^2} \xi(l l) h_u h_u + \frac{x_d}{\Lambda^2} (\varphi' l l) h_u h_u \right] + \dots, \end{aligned} \quad (57)$$

with

$$\frac{y_f}{\Lambda} = 4Y_L Y_f e^{-ML} \quad (f = e, \mu, \tau). \quad (58)$$

Therefore, in lowest order approximation we have reproduced the Lagrangian \mathcal{L}_Y of Eq. (16) and the discussion of Section 3 applies.

We also recall that, to account for the observed hierarchy of the charged lepton masses, we have included an additional U(1) flavour symmetry. Therefore, in the present picture, the quantities $Y_{e,\mu,\tau}$ stand for:

$$Y_e = \tilde{Y}_e \left(\frac{\theta}{\Lambda} \right)^4, \quad Y_\mu = \tilde{Y}_\mu \left(\frac{\theta}{\Lambda} \right)^2, \quad Y_\tau = \tilde{Y}_\tau, \quad (59)$$

where $\tilde{Y}_{e,\mu,\tau}$ are field-independent constants having similar values. After spontaneous breaking of U(1), the Yukawa couplings y_f possess the desired hierarchy.

6. Higher-order corrections

The results of the previous section hold to first approximation. Higher-dimensional operators, suppressed by additional powers of the cut-off Λ , can be added to the leading terms in Eqs. (42), (46), (52), (57), (58). Here we will classify these terms and analyze their physical effects. In particular we will show that these corrections are completely under control in our model and that they can be made negligibly small without any fine-tuning. We can order higher-order operators into three groups.

6.1. Local corrections to m_ν

There are higher-order operators that are local in the five-dimensional theory and do not depend upon the heavy fermion sector (F_1, \bar{F}_2). As we have seen, at leading order, the neutrino mass matrix m_ν arises entirely from operators of this type that are localized at $y = L$. On this brane we only have scalar fields (ξ, φ'), odd under Z_4 . Therefore, higher-dimensional operators modifying m_ν and localized at $y = L$ are down by two powers of

the cut-off, compared to the leading ones. After A_4 breaking, the only two operators that cannot be absorbed by a redefinition of the parameters $x_{a,d}$ are:

$$\begin{aligned} & \frac{x_b}{\Lambda^4} \xi(\varphi' \varphi')(l l)'' h_u h_u, \\ & \frac{x_c}{\Lambda^4} \xi(\varphi' \varphi)''(l l)' h_u h_u. \end{aligned} \quad (60)$$

After adding these operators localized at $y = L$ to the five-dimensional action of Eq. (51), we get a neutrino mass matrix

$$m_\nu = \frac{v_u^2}{\Lambda} \begin{pmatrix} a + b + c & 0 & 0 \\ 0 & a + \omega b + \omega^2 c & d \\ 0 & d & a + \omega^2 b + \omega c \end{pmatrix}, \quad (61)$$

where

$$b \equiv x_b \frac{uv'^2}{\Lambda^3}, \quad c \equiv x_c \frac{uv'^2}{\Lambda^3}, \quad (62)$$

to be compared with a and d of Eq. (20).

6.2. Corrections from tree-level F -exchange

Another set of higher-dimensional operators arise from the exchange of the heavy fermion (F_1, \bar{F}_2) in the static limit and in the tree-level approximation. To classify them, we should list all operators localized at the two branes that are linear in the bulk fermion (F_1, \bar{F}_2) . At $y = 0$ such operators have the generic structure

$$Y_f f^c \varphi F_1, \quad \frac{Y_f^{(1)}}{\Lambda} f^c \varphi^2 F_1, \quad \frac{Y_f^{(2)}}{\Lambda^2} f^c \varphi^3 F_1, \quad \dots \quad (f = e, \mu, \tau). \quad (63)$$

After spontaneous A_4 breaking, the effect of these operators can be absorbed by redefining the coupling constants Y_f , ($f = e, \mu, \tau$), at least up to order φ^3 . Thus the leading interactions between f^c and F_1

$$[Y_e e^c(\varphi F_1) + Y_\mu \mu^c(\varphi F_1)'' + Y_\tau \tau^c(\varphi F_1)' + \text{h.c.}] \delta(y) \quad (64)$$

are unchanged up to relative order $1/\Lambda^2$. We are left with the couplings of F_2 at the brane $y = L$. Neglecting all operators that, after A_4 breaking, only lead to a renormalization of the parameter Y_L , we find four new terms:

$$\begin{aligned} & \frac{Z_1}{\Lambda^2} (\varphi' \varphi)'(F_2 l)'' h_d, \\ & \frac{Z_2}{\Lambda^2} (\varphi' \varphi)''(F_2 l)' h_d, \\ & \frac{Z_3}{\Lambda^2} \xi [\varphi'_1(F_2)_2 l_3 + \varphi'_2(F_2)_3 l_1 + \varphi'_3(F_2)_1 l_2] h_d, \\ & \frac{Z_4}{\Lambda^2} \xi [\varphi'_1(F_2)_3 l_2 + \varphi'_2(F_2)_1 l_3 + \varphi'_3(F_2)_2 l_1] h_d. \end{aligned} \quad (65)$$

After the breaking of A_4 , the leading order interaction of F_2 at $y = L$ is modified by the operators (65) to

$$[Y_L(F_2\tilde{l})h_d + \text{h.c.}]\delta(y - L), \quad (66)$$

where

$$\begin{pmatrix} \tilde{l}_1 \\ \tilde{l}_2 \\ \tilde{l}_3 \end{pmatrix} = \begin{pmatrix} 1 + z_1 + z_2 & 0 & 0 \\ 0 & 1 + \omega z_1 + \omega^2 z_2 & z_4 \\ 0 & z_3 & 1 + \omega^2 z_1 + \omega z_2 \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}, \quad (67)$$

$$z_{1,2} \equiv \frac{Z_{1,2}}{Y_L} \frac{v'^2}{\Lambda^2}, \quad z_{3,4} \equiv \frac{Z_{3,4}}{Y_L} \frac{uv'}{\Lambda^2}. \quad (68)$$

After integrating out the heavy modes (F_1, \bar{F}_2) in the limit of vanishing external momenta for the light modes, we obtain the effective four-dimensional Lagrangian

$$\frac{y_e}{\Lambda} e^c (\varphi\tilde{l})h_d + \frac{y_\mu}{\Lambda} \mu^c (\varphi\tilde{l})''h_d + \frac{y_\tau}{\Lambda} \tau^c (\varphi\tilde{l})'h_d, \quad (69)$$

$$\frac{y_f}{\Lambda} = 4Y_L Y_f e^{-ML} \quad (f = e, \mu, \tau). \quad (70)$$

The mass matrix for the charged leptons becomes

$$m_l = v_d \frac{v}{\Lambda} \begin{pmatrix} y_e(1 + z_1 + z_2) & y_e(1 + \omega z_1 + \omega^2 z_2 + z_3) & y_e(1 + \omega^2 z_1 + \omega z_2 + z_4) \\ y_\mu(1 + z_1 + z_2) & y_\mu \omega(1 + \omega z_1 + \omega^2 z_2 + \omega z_3) & y_\mu \omega^2(1 + \omega^2 z_1 + \omega z_2 + \omega^2 z_4) \\ y_\tau(1 + z_1 + z_2) & y_\tau \omega^2(1 + \omega z_1 + \omega^2 z_2 + \omega^2 z_3) & y_\tau \omega(1 + \omega^2 z_1 + \omega z_2 + \omega z_4) \end{pmatrix}. \quad (71)$$

6.3. Effects on masses and mixing angles

To first order in the small parameters b, c and z_i , the neutrino masses are modified into:

$$\begin{aligned} m_1 &= \left(a + d - \frac{1}{2}(b + c) \right) \frac{v_u^2}{\Lambda}, \\ m_2 &= (a + b + c) \frac{v_u^2}{\Lambda}, \\ m_3 &= \left(-a + d + \frac{1}{2}(b + c) \right) \frac{v_u^2}{\Lambda}, \end{aligned} \quad (72)$$

and the charged lepton masses are changed into

$$\begin{aligned} m_e &= \sqrt{3}y_e \left(1 + \frac{z_3}{3} + \frac{z_4}{3} \right) v_d \frac{v}{\Lambda}, \\ m_\mu &= \sqrt{3}y_\mu \left(1 + \omega \frac{z_3}{3} + \omega^2 \frac{z_4}{3} \right) v_d \frac{v}{\Lambda}, \\ m_\tau &= \sqrt{3}y_\tau \left(1 + \omega^2 \frac{z_3}{3} + \omega \frac{z_4}{3} \right) v_d \frac{v}{\Lambda}. \end{aligned} \quad (73)$$

To the same order, but neglecting terms like $z_i y_{e,\mu}/y_\tau$, we get:

$$\begin{aligned}
 |U_{e3}| &= \left| \frac{(\bar{b} - \bar{c})(d - a) + (b - c)(\bar{d} + \bar{a})}{2\sqrt{2}(a\bar{d} + \bar{a}d)} + \frac{1}{\sqrt{2}} \left(-\bar{z}_1 + \bar{z}_2 + \frac{i}{\sqrt{3}}z_3 - \frac{i}{\sqrt{3}}z_4 \right) \right|, \\
 |\tan^2 \theta_{23}| &= 1 + \frac{(\bar{b} - \bar{c})d + (b - c)\bar{d}}{(a\bar{d} + \bar{a}d)} + 2 \left[z_2 + \bar{z}_2 + \frac{1}{3}(z_3 + \bar{z}_3 + z_4 + \bar{z}_4) \right], \\
 |\tan^2 \theta_{12}| &= \frac{1}{2} \left[1 + \frac{3}{2} \left(-z_1 - \bar{z}_1 - z_2 - \bar{z}_2 + \frac{z_3 + \bar{z}_3}{3} + \frac{z_4 + \bar{z}_4}{3} \right) \right]. \quad (74)
 \end{aligned}$$

These relations explicitly show that the corrections induced by the higher-dimensional operators are of order uv'/Λ^2 or v'^2/Λ^2 . From our estimate in Eq. (35) we see that these parameters can be as small as 2×10^{-5} . If the cut-off Λ is one order of magnitude larger than the VEVs of the model, the resulting corrections are at the level of one percent, already beyond any planned experimental test. If on the contrary, the VEVs are anomalously close to the cut-off Λ , then Eq. (74) show that deviations roughly of the same size are expected in U_{e3} , $\tan^2 \theta_{23}$ and $\tan^2 \theta_{12}$. How much close to Λ can the VEVs be? We expect that the subleading corrections do not spoil the leading order form of the neutrino mass spectrum, Eq. (28). This implies that $v'^2/\Lambda^2 \ll r$, so that r sets the natural upper bound to the expected deviations from the leading order results.

6.4. Corrections from one-loop F -exchange

Further corrections to lepton mass matrices and to the scalar potential can arise from one-loop exchange of (F_1, \bar{F}_2) in the static limit. Consider, for instance, the following operators localized at $y = 0$ and at $y = L$:

$$\frac{1}{\Lambda} \varphi F_1 F_2 \delta(y), \quad \frac{1}{\Lambda^6} \xi l l F_1 F_2 h_u h_u \delta(y - L). \quad (75)$$

By integrating out, at one-loop order, the heavy modes contained in (F_1, \bar{F}_2) we get:

$$\frac{1}{\Lambda^7} \xi \varphi l l h_u h_u \int d^4 k \Delta_F(k, 0, L) \Delta_F(k, L, 0), \quad (76)$$

where k is the four-momentum running in the loop and $\Delta_F(k, y, y')$ is the adimensional propagator of (F_1, \bar{F}_2) in a mixed momentum-space representation. Since the loop integral is convergent, we get

$$\frac{1}{\Lambda^3} \frac{f(ML)}{\Lambda^4 L^4} \xi \varphi l l h_u h_u, \quad (77)$$

where $f(ML)$ is a function of the adimensional combination ML . Thus the resulting local operator is suppressed by four additional powers of the cut-off scale. This behavior is quite generic and similar suppressions are found for other operators originating from one-loop exchange of (F_1, \bar{F}_2) .

The corrections that modify the scalar potential discussed in the previous section are of this type. As an example, consider the localized interactions:

$$\frac{1}{\Lambda^2} \varphi \varphi F_1 F_2 \delta(y), \quad \frac{1}{\Lambda^2} \varphi' \varphi' F_1 F_2 \delta(y - L). \quad (78)$$

Also in this case, after integrating over (F_1, \bar{F}_2) in the limit of vanishing external momenta, we get:

$$\frac{f(ML)}{\Lambda^4 L^4} \varphi \varphi \varphi' \varphi'. \quad (79)$$

Due to their large suppression, these corrections are negligible compared to those discussed above.

7. Conclusion

There are by now several theoretical mechanisms that can qualitatively explain the observed large lepton mixing angles [19]. They are sufficiently flexible to quantitatively accommodate the measured parameters. They are also compatible with our ideas on quarks masses and mixing angles so that they can be nicely embedded into a unified picture of fermion properties, such as, for instance, a grand unified theory. Many of these mechanisms predict a generically large atmospheric mixing angle and a generically small θ_{13} angle, without favouring any specific value for these parameters. The best values of global fits are currently very close to $\theta_{23} = \pi/4$ and $\theta_{13} = 0$, but the experimental errors still allow for large deviations from these remarkable values. Indeed, according to many of the above mentioned mechanisms, deviations from $\theta_{23} = \pi/4$ and $\theta_{13} = 0$ are expected at the observable level. It may take a long time before such deviations can be actually observed. A sensitivity on θ_{13} around 0.05 is foreseen in about ten years from now, with the full exploitation of high-intensity neutrino beams. A reduction by a factor of two of the present error on θ_{23} will also require special neutrino beams and a similar time scale.

It might happen that after all this experimental effort, $(\theta_{23} - \pi/4)$ and θ_{13} still remain close to zero, within errors. At this point it would be legitimate to suspect that such special values are produced by a highly symmetric flavour dynamics. Given the already good experimental precision on θ_{12} , the so-called Harrison–Perkins–Scott mixing scheme, where $\theta_{23} = \pi/4$, $\theta_{13} = 0$ and $\sin^2 \theta_{12} = 1/3$, would fit very well the data. In this paper we have proposed a model that reproduces accurately the HPS mixing pattern. We started by discussing whether such a pattern can be obtained from an exact flavour symmetry. We showed that, under general conditions, an exactly maximal atmospheric mixing angle cannot arise from an exact flavour symmetry. The flavour symmetry should be necessarily broken and a maximal θ_{23} is the result of a special alignment between the breaking effects in the neutrino sector and those occurring in the charged lepton sector. If the flavour symmetry is spontaneously broken, this corresponds to a non-trivial vacuum alignment. Our model gives rise to the HPS mixing scheme in the context of a spontaneously broken A_4 flavour symmetry, A_4 being the discrete subgroup of $SO(3)$ leaving a tetrahedron invariant.

At leading order, that is by neglecting symmetric operators of higher dimension, neutrino masses only depend on two complex Yukawa coupling constants. Due to the unknown phase difference between these two constants, we cannot determine the absolute scale of neutrino masses. We expect that the neutrino spectrum is of the normal hierarchy type but not too far from degenerate. At leading order the model predicts $|m_3|^2 = |m_{ee}|^2 + 10/9 \Delta m_{\text{atm}}^2 (1 - r/2)$. A remarkable feature of our model is that at the leading

order the lepton mixing angles are completely independent from these two parameters, so that the HPS mixing pattern is always obtained. The lepton mixing depends entirely on the relative alignment between the VEVs giving masses to the neutrino sector and those giving masses to the lepton sector. We discuss in detail the problem of vacuum alignment. To avoid the proliferation of Higgs doublets, the scalar fields breaking A_4 are gauge singlets in our model. We propose an unconventional solution to the vacuum alignment problem, where an extra dimension described by a spatial interval plays an important role. Two scalar sectors live at the opposite ends of the interval and their respective scalar potentials are minimized by the desired field configurations, for natural values of the implied parameters. Such a mechanism only works in the case of discrete symmetries, since in the continuous case the large symmetry of the total potential energy would make the relative orientations of the two scalar sectors undetermined. We have also extensively discussed how this lowest order picture is modified by the introduction of higher-dimensional operators. The induced corrections are parametrically small, of second order in the expansion parameter VEV/Λ , Λ being the cut-off of the theory, and they can be made numerically negligible. Last but not least, the hierarchy of the charged lepton masses can be reproduced by the usual Froggatt–Nielsen mechanism within the context of an Abelian flavour symmetry, which turns out to be fully compatible with the present scheme.

We believe that, from a purely technical point of view, we have fulfilled our goal to realize a completely natural construction of the HPS mixing scheme. But to construct our model we had to introduce a number of special dynamical tricks (like a peculiar set of discrete symmetries in extra dimensions). Apparently this is the price to pay for a “special” model where all mixing angles are fixed to particular values. Perhaps this exercise can be taken as a hint that it is more plausible to expect that, in the end, experiment will select a “normal” model with θ_{13} not too small and θ_{23} not too close to maximal.

Acknowledgement

We thank Zurab Berezhiani, Isabella Masina and Luigi Pilo for useful discussions. F.F. thanks the CERN Theory Division for hospitality in summer 2004, when this project started. This project is partially supported by the European Program MRTN-CT-2004-503369.

Appendix A

Here we discuss a SUSY solution to the vacuum alignment problem. In a supersymmetric context, the right-hand side of Eq. (16) should be interpreted as the superpotential w_l of the theory, in the lepton sector. A key observation is that this superpotential is invariant not only with respect to the gauge symmetry $SU(2) \times U(1)$ and the flavour symmetry $U(1)_F \times A_4$, but also under a discrete Z_3 symmetry and a continuous $U(1)_R$ symmetry under which the fields transform as shown in Table 1.

We see that the Z_3 symmetry explains the absence of the term (ll) in w_l : such a term transforms as ω^2 under Z_3 and need to be compensated by the field ξ in our construction.

Table 1

Field	1	e^c	μ^c	τ^c	$h_{u,d}$	φ	φ'	ξ	φ_0	φ'_0	ξ_0
A_4	3	1	$1'$	$1''$	1	3	3	1	3	3	1
Z_3	ω	ω^2	ω^2	ω^2	1	1	ω	ω	1	ω	ω
$U(1)_R$	1	1	1	1	0	0	0	0	2	2	2

At the same time Z_3 does not allow the interchange between φ' and φ , which transform differently under Z_3 . Charged leptons and neutrinos acquire masses from two independent sets of fields. If the two sets of fields develop VEVs according to the alignment described in Eq. (17), then the desired mass matrices follow.

Finally, there is a continuous $U(1)_R$ symmetry that contains the usual R -parity as a subgroup. Suitably extended to the quark sector, this symmetry forbids the unwanted dimension two and three terms in the superpotential that violate baryon and lepton number at the renormalizable level. The $U(1)_R$ symmetry allows us to classify fields into three sectors. There are “matter fields” such as the leptons l , e^c , μ^c and τ^c , which occur in the superpotential through bilinear combinations. There is a “symmetry breaking sector” including the Higgs doublets $h_{u,d}$ and the flavons φ' , φ and ξ . As we will see these fields acquire non-vanishing vacuum expectation values (VEVs) and break the symmetries of the model. Finally, there are “driving fields” such as φ'_0 , φ_0 and ξ_0 that allows to build a non-trivial scalar potential in the symmetry breaking sector. Since driving fields have R -charge equal to two, the superpotential is linear in these fields.

The full superpotential of the model is

$$w = w_l + w_d, \tag{A.1}$$

where, at leading order in a $1/\Lambda$ expansion, w_l is given by the right-hand side of Eq. (16) and the “driving” term w_d reads:

$$w_d = M(\varphi_0\varphi) + g(\varphi_0\varphi\varphi) + g_1(\varphi'_0\varphi'\varphi') + g_2\xi(\varphi'_0\varphi') + g_3\xi_0(\varphi'\varphi') + g_4\xi_0\xi^2. \tag{A.2}$$

We notice that at the leading order there are no terms involving the Higgs fields $h_{u,d}$. We assume that the electroweak symmetry is broken by some mechanism, such as radiative effects when supersymmetry (SUSY) is broken. It is interesting that at the leading order the electroweak scale does not mix with the potentially large scales u , v and v' . The scalar potential is given by:

$$V = \sum_i \left| \frac{\partial w}{\partial \phi_i} \right|^2 + m_i^2 |\phi_i|^2 + \dots, \tag{A.3}$$

where ϕ_i denote collectively all the scalar fields of the theory, m_i^2 are soft masses and dots stand for D -terms for the fields charged under the gauge group and possible additional soft breaking terms. Since m_i are expected to be much smaller than the mass scales involved in w_d , it makes sense to minimize V in the supersymmetric limit and to account for soft breaking effects subsequently. From the driving sector we have:

$$\frac{\partial w}{\partial \varphi_0} = M\varphi_1 + g\varphi_2\varphi_3 = 0,$$

$$\begin{aligned}
\frac{\partial w}{\partial \varphi_{02}} &= M\varphi_2 + g\varphi_3\varphi_1 = 0, \\
\frac{\partial w}{\partial \varphi_{03}} &= M\varphi_3 + g\varphi_1\varphi_2 = 0, \\
\frac{\partial w}{\partial \varphi'_{01}} &= g_1\varphi'_2\varphi'_3 + g_2\xi\varphi'_1 = 0, \\
\frac{\partial w}{\partial \varphi'_{02}} &= g_1\varphi'_3\varphi'_1 + g_2\xi\varphi'_2 = 0, \\
\frac{\partial w}{\partial \varphi'_{03}} &= g_1\varphi'_1\varphi'_2 + g_2\xi\varphi'_3 = 0, \\
\frac{\partial w}{\partial \xi_0} &= g_3(\varphi'\varphi') + g_4\xi^2 = 0.
\end{aligned} \tag{A.4}$$

The first three equations are solved by (up to irrelevant sign ambiguities):

$$\varphi = (v, v, v), \quad v = -\frac{M}{g}. \tag{A.5}$$

The remaining equations are solved, in general, by:

$$\varphi' = (0, 0, 0), \quad \xi = 0, \tag{A.6}$$

unless some further relation is imposed on the coefficients g_1, \dots, g_4 . If $g_2 = 0$, then, up to an irrelevant reordering, we have

$$\varphi' = (v', 0, 0), \quad \xi = u = -\frac{g_3}{g_4}(\varphi'\varphi') \tag{A.7}$$

with v' and u undetermined. In this case we find that, for $m_{\varphi_0}^2, m_{\varphi'_0}^2, m_{\xi_0}^2 > 0$, the driving fields φ_0, φ'_0 and ξ_0 vanish at the minimum. Moreover, if $m_{\varphi'_0}^2, m_{\xi_0}^2 < 0$, then u and v' slide to large scales, eventually stabilized by one-loop radiative corrections. The supersymmetric case is better than the non-supersymmetric case in two respects. First of all, at least from a technical viewpoint, the absence of a term in the superpotential is radiatively stable. Moreover, as we have seen, once g_2 has been set to zero, the equations selecting (17) as the correct minimum are consistent.

References

- [1] A. Strumia, F. Vissani, hep-ph/0503246;
G.L. Fogli, E. Lisi, A. Marrone, A. Melchiorri, A. Palazzo, P. Serra, J. Silk, Phys. Rev. D 70 (2004) 113003, hep-ph/0408045;
J.N. Bahcall, M.C. Gonzalez-Garcia, C. Pena-Garay, JHEP 0408 (2004) 016, hep-ph/0406294;
M. Maltoni, T. Schwetz, M.A. Tortola, J.W.F. Valle, New J. Phys. 6 (2004) 122, hep-ph/0405172.
- [2] B. Aharmim, et al., SNO Collaboration, nucl-ex/0502021.
- [3] LSND Collaboration, hep-ex/0104049.
- [4] KARMEN Collaboration, Nucl. Phys. B (Proc. Suppl.) 91 (2000) 191.
- [5] P. Spentzouris, Nucl. Phys. B (Proc. Suppl.) 100 (2001) 163.

- [6] C. Kraus, et al., hep-ex/0412056;
V.M. Lobashev, Phys. At. Nucl. 63 (2000) 962, Yad. Fiz. 63 (2000) 1037 (in Russian).
- [7] C.L. Bennett, et al., Astrophys. J. Suppl. 148 (2003) 1;
D.N. Spergel, et al., Astrophys. J. Suppl. 148 (2003) 175.
- [8] W. Grimus, L. Lavoura, JHEP 0107 (2001) 045;
W. Grimus, L. Lavoura, Acta Phys. Pol. B 32 (2001) 3719;
W. Grimus, L. Lavoura, Eur. Phys. J. C 28 (2003) 123;
W. Grimus, L. Lavoura, Phys. Lett. B 572 (2003) 189;
W. Grimus, L. Lavoura, hep-ph/0305309;
W. Grimus, L. Lavoura, Acta Phys. Pol. B 34 (2003) 5393;
W. Grimus, A.S. Joshipura, S. Kaneko, L. Lavoura, M. Tanimoto, JHEP 0407 (2004) 078;
W. Grimus, A.S. Joshipura, S. Kaneko, L. Lavoura, H. Sawanaka, M. Tanimoto, Nucl. Phys. B 713 (2005) 151;
S. Morisi, M. Picariello, hep-ph/0505113;
F. Caravaglios, M. Morisi, hep-ph/0503234.
- [9] C. Wetterich, Phys. Lett. B 451 (1999) 397;
R. Barbieri, L.J. Hall, G.L. Kane, G.G. Ross, hep-ph/9901228;
O. Vives, hep-ph/0504079.
- [10] E. Ma, G. Rajasekaran, Phys. Rev. D 64 (2001) 113012, hep-ph/0106291;
K.S. Babu, E. Ma, J.W.F. Valle, Phys. Lett. B 552 (2003) 207;
M. Hirsch, J.C. Romao, S. Skadhauge, J.W.F. Valle, A. Villanova del Moral, hep-ph/0312244;
M. Hirsch, J.C. Romao, S. Skadhauge, J.W.F. Valle, A. Villanova del Moral, hep-ph/0312265;
E. Ma, hep-ph/0404199.
- [11] E. Ma, Phys. Rev. D 70 (2004) 031901;
E. Ma, hep-ph/0409075;
E. Ma, New J. Phys. 6 (2004) 104.
- [12] S.L. Chen, M. Frigerio, E. Ma, hep-ph/0404084.
- [13] P.F. Harrison, D.H. Perkins, W.G. Scott, Phys. Lett. B 530 (2002) 167, hep-ph/0202074;
P.F. Harrison, W.G. Scott, Phys. Lett. B 535 (2002) 163, hep-ph/0203209;
P.F. Harrison, W.G. Scott, Phys. Lett. B 547 (2002) 219;
P.F. Harrison, W.G. Scott, Phys. Lett. B 557 (2003) 76;
P.F. Harrison, W.G. Scott, hep-ph/0402006;
P.F. Harrison, W.G. Scott, hep-ph/0403278.
- [14] J. Kubo, A. Mondragon, M. Mondragon, E. Rodriguez-Jauregui, Prog. Theor. Phys. 109 (2003) 795;
T. Ohlsson, G. Seidl, Phys. Lett. B 537 (2002) 95;
T. Ohlsson, G. Seidl, Nucl. Phys. B 643 (2002) 247;
S.F. King, G.G. Ross, Phys. Lett. B 520 (2001) 243.
- [15] L.J. Hall, H. Murayama, N. Weiner, Phys. Rev. Lett. 84 (2000) 2572;
N. Haba, H. Murayama, Phys. Rev. D 63 (2001) 053010;
M.S. Berger, K. Siyeon, Phys. Rev. D 63 (2001) 057302;
M. Hirsch, hep-ph/0102102;
M. Hirsch, S.F. King, Phys. Lett. B 516 (2001) 103, hep-ph/0102103;
F. Vissani, Phys. Lett. B 508 (2001) 79;
G. Altarelli, F. Feruglio, I. Masina, JHEP 0301 (2003) 035;
A. de Gouvea, H. Murayama, hep-ph/0301050;
J.R. Espinosa, hep-ph/0306019.
- [16] H. Fritzsch, Z.Z. Xing, Phys. Lett. B 372 (1996) 265;
H. Fritzsch, Z.Z. Xing, Phys. Lett. B 440 (1998) 313;
H. Fritzsch, Z.Z. Xing, Prog. Part. Nucl. Phys. 45 (2000) 1;
M. Fukugita, M. Tanimoto, T. Yanagida, Phys. Rev. D 57 (1998) 4429;
M. Fukugita, M. Tanimoto, T. Yanagida, Phys. Rev. D 59 (1999) 113016;
M. Tanimoto, T. Watari, T. Yanagida, Phys. Lett. B 461 (1999) 345;
S.K. Kang, C.S. Kim, Phys. Rev. D 59 (1999) 091302;
M. Tanimoto, Phys. Lett. B 483 (2000) 417;

- N. Haba, Y. Matsui, N. Okamura, T. Suzuki, *Phys. Lett. B* 489 (2000) 184;
Y. Koide, A. Ghosal, *Phys. Lett. B* 488 (2000) 344;
E.K. Akhmedov, G.C. Branco, F.R. Joaquim, J.I. Silva-Marcos, *Phys. Lett. B* 498 (2001) 237.
- [17] R. Gatto, G. Morchio, G. Sartori, F. Strocchi, *Nucl. Phys. B* 163 (1980) 221;
J.I. Silva-Marcos, *JHEP* 0307 (2003) 012;
C.I. Low, R.R. Volkas, *Phys. Rev. D* 68 (2003) 033007;
C.I. Low, hep-ph/0404017;
Y. Koide, *Phys. Rev. D* 71 (2005) 016010;
F. Feruglio, *Nucl. Phys. B (Proc. Suppl.)* 143 (2005) 184.
- [18] See, for instance S.F. King, *JHEP* 0209 (2002) 011.
- [19] For a review, see G. Altarelli, F. Feruglio, *New J. Phys.* 6 (2004) 106.