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Cotilting and tilting modules over Prüfer domains

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Abstract. We give a characterization of cotilting modules over Prüfer domains, up to equivalence; moreover we show that tilting modules over Prüfer domains are of projective dimension at most one.

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Introduction

Tilting and cotilting modules have been introduced by S. Brenner, M. Butler, C. Ringel, D. Happel and others in the early eighties. Their definitions have been extended to the case of infinitely generated modules over arbitrary rings by many authors: K. Fuller, R. Colby, R. Colpi, J. Trlifaj, L. Angeleri Hügel, G. D'Este, A. Tonolo. A generalization to modules of projective or injective dimension greater than one was introduced by L. Angeleri Hügel and F. Coelho in [1]. So one can consider n-tilting and n-cotilting modules where n denotes the projective, respectively, injective dimension. Structure theorems for this type of modules are far from being known. The first complete description of cotilting modules was given by R. Göbel and J. Trlifaj [17] in the case of abelian groups. The author in [5], proved that cotilting modules over arbitrary rings are pure injective and in [10]. R. Göbel, L. Strüngmann and the author proved that over Prüfer domains n-cotilting modules have injective dimension at most one and thus over such rings all n-cotilting modules are pure injective. Very recently, after the submission of this paper, J. Šťovíček in [22] proved that all n-cotilting modules over arbitrary rings are pure injective.

In this paper we characterize cotilting modules over Prüfer domains. In Section 2 we prove that they are equivalent to direct products of cotilting modules over the localizations at the maximal ideals and in the remaining sections we characterize cotilting modules over valuation domains. Some results on decomposition of pure injective modules over a valuation domain are given in [16, XIII, 5.3], but about the structure of super decomposable pure injective modules very little is known. It is

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known that they exist if and only if the valuation domain is not strongly discrete, that is if it has a non zero idempotent prime ideal. We will prove that over valuation domains every cotilting module is equivalent to a direct product of suitable indecomposable pure injective modules, showing that even though there might exist super decomposable pure injective modules, cotilting modules can be assumed to have no super decomposable summands.

In Section 4 we characterize cotilting modules of cofinite type over valuation domains and we prove that every cotilting module is of cofinite type if and only if the valuation domain is strongly discrete. Thus we answer negatively the question whether all n-cotilting modules are of cofinite type (see definitions below). Not surprisingly, the description of cotilting modules over non strongly discrete valuation domains is more complicated. The characterization will be achieved by means of change of rings. Up to equivalence, we will reduce the problem to characterizing cotilting modules over suitable factors of localizations of the domain and also to the case of maximal valuation domains.

In Section 7 we show that for the class of Prüfer domains it is easy to prove that n-tilting modules are of projective dimension at most one and thus of finite type.

Added in proof. Recently, after this paper was submitted, it has been proved that all n-tilting modules over arbitrary rings are of finite type. The result for 1-tilting modules was obtained by D. Herbera and the author in [7]. The generalization to all n-tilting modules is proved in the paper by J. Šťovíček and the author [8] by using the results obtained by J. Šťovíček and J. Trlifaj [23].

Section 7 shows that in the case of Prüfer domains, the proof of the finite type of n-tilting modules can be obtained in a pretty easier way.

1 Preliminaries and definitions

For any left R-module M we define the following classes:

$$^{\perp}M = \{X \in R\text{-Mod} \mid \mathrm{Ext}^i_R(X, M) = 0, \forall i \geq 1\},$$

$$^{\intercal}M = \{X \in \text{Mod-}R \mid \mathrm{Tor}^R_i(X, M) = 0, \forall i \geq 1\}.$$

If M is a right R-module we define;

$$M^{\intercal} = \{X \in R\text{-Mod} \mid \mathrm{Tor}^R_i(M, X) = 0, \forall i \geq 1\}.$$

If M is an R-module, i.d. M will denote the injective dimension of M.

Over an arbitrary ring with 1 a R-module C is a n-cotilting module if the following conditions hold ([1]):

(C1) i.d. $C \leq n$;

(C2) $\mathrm{Ext}^i_R(C^{\lambda}, C) = 0$ for each $i > 0$ and for every cardinal λ;

(C3) there exists a long exact sequence:

$$0 \to C_r \to \cdots \to C_1 \to C_0 \cdots \to W \to 0,$$

where $C_i \in \operatorname{Prod} C$, for every $0 \leq i \leq r$ and W is an injective cogenerator of R-Mod.

If C is a n-cotilting module the class ${}^{\perp}C$ is called a n-cotilting class. n-cotilting classes have been characterized in [1]. In particular ${}^{\perp}C$ is closed under direct products.

In case $n = 1$ there is an alternative definition of 1-cotilting modules. A module C is 1-cotilting if and only if $\operatorname{Cogen} C = {}^{\perp}C$, where $\operatorname{Cogen} C$ denotes the class of modules cogenerated by C (cf [3, Prop. 2.3], [12, Prop. 1.7]). Moreover, if C is a 1-cotilting module, then $\operatorname{Cogen} C$ is a torsion free class. For results on torsion and torsion free classes we refer to [21].

Since every 1-cotilting module C is pure injective ([5]) we have that 1-cotilting classes are closed under direct products, direct limits and pure submodules. In other words they are definable classes, that is they are closed under elementary equivalence. Thus, if C is 1-cotilting, a module belongs to ${}^{\perp}C$ if and only if its pure injective envelope belongs to ${}^{\perp}C$ (see [18] or [13]).

The notion of n-cotilting modules of cofinite type was introduced in [2]. Since we will use this notion only for cotilting modules over Prüfer domains, we recall the definition in this particular case.

A 1-cotilting module C over a Prüfer domain R is of cofinite type provided there exists a set $\mathscr{S}$ of finitely presented R-modules such that ${}^{\perp}C = \bigcap_{S \in \mathscr{S}} S^{\mathsf{T}}$.

2 1-cotilting modules over commutative rings

In this section we prove that, up to equivalence, the study of 1-cotilting modules over commutative rings can be restricted to the local case. Recall that two cotilting modules C and C' are said to be equivalent if the corresponding cotilting classes, that is if ${}^{\perp}C$ and ${}^{\perp}C'$, coincide.

The following easy lemma will show to be very useful in the sequel.

Lemma 2.1. *Let $0 \neq I$ be an ideal of a commutative ring R and M an R-module. Let E be an injective module containing M and let $M[I]$ denote the submodule of M consisting of the elements annihilated by I, then:*

(1) $\operatorname{Ext}^1_R(R/I, M) \cong \dfrac{(E/M)[I]}{(E[I] + M)/M}$.

(2) $\operatorname{Ext}^1_R(R/I, M) = 0$, *if I is idempotent and $M[I] = M$.*

Proof. (1) Consider the exact sequence:

$$0 \to M \to E \to E/M \to 0,$$

and the induced sequence

$$\operatorname{Hom}(R/I, E) \xrightarrow{f} \operatorname{Hom}(R/I, E/M) \to \operatorname{Ext}^1_R(R/I, M) \to 0.$$

Identifying $\mathrm{Hom}(R/I, X)$ with $X[I]$ for every R-module X, then the image of f is $(E[I] + M)/M$ and the conclusion follows.

(2) Note that if $x + M \in (E/M)[I]$, then $Ix \leq M$. Hence $Ix = I^2x = 0$, so that $x + M \in (E[I] + M)/M$. □

Now we state a result which deals with properties of 1-cotilting modules with respect to change of rings. In what follows, if R and S are two rings and f is a ring homomorphism $f : R \to S$ we will view S-modules as R-modules via f.

Proposition 2.2. *Let R, S be commutative rings and let $f : R \to S$ be a ring homomorphism. If C is a 1-cotilting R-module such that $\mathrm{Ext}^1_R(S, C) = 0$, then $\mathrm{Hom}_R(S, C)$ is a pure injective R-module and a 1-cotilting S-module.*

Proof. $\mathrm{Hom}_R(S, C)$ is a pure injective R-module, since so is C. (See, for instance [16, XIII, 2.1]). By a result of Fuller [15], $\mathrm{Hom}_R(S, C)$ is a 1-cotilting S-module if $\mathrm{Ext}^1_R(S, C) = 0$ and $\mathrm{Hom}_R(S, C)$ is cogenerated by C. Clearly $\mathrm{Hom}_R(S, C)$ is cogenerated by C, since it is an R-submodule of C^S, so C is a 1-cotilting S-module. □

Useful tools in dealing with change of rings are the following homological formulas; they can be found in the book by Cartan Eilenberg ([11, VI, 4.1.3, 4.1.4]).

Assume $f : R \to S$ is a ring homomorphism.

(a) for every left R-module ${}_RA$ and left S-module ${}_SB$:

$$\mathrm{Ext}^1_S(S \otimes_R A, B) \cong \mathrm{Ext}^1_R(A, B),$$

if $\mathrm{Tor}^R_n(S, A) = 0$ for every n.

(b) for every left R-module ${}_RB$ and left S-module ${}_SA$:

$$\mathrm{Ext}^1_S(A, \mathrm{Hom}_R(S, B)) \cong \mathrm{Ext}^1_R(A, B),$$

if $\mathrm{Ext}^n_R(S, B) = 0$ for every n.

The next result is an easy consequence of formula **(b)**.

Lemma 2.3. *Let I be an idempotent ideal of a commutative ring R. Let A, B be R/I-modules with* i.d. $B \leq 1$ *as an R-module, then*

$$\mathrm{Ext}^1_R(A, B) \cong \mathrm{Ext}^1_{R/I}(A, B).$$

Proof. By Lemma 2.1 (2), $\mathrm{Ext}^1_R(R/I, B) = 0$, hence formula **(b)** yields $\mathrm{Ext}^1_{R/I}(A, \mathrm{Hom}_R(R/I, B)) \cong \mathrm{Ext}^1_R(A, B)$, and $\mathrm{Hom}_R(R/I, B) \cong B$ as R/I-modules. □

Recalling that 1-cotilting modules are pure injective (by [5]), we will make use of the following well known result by Auslander.

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Proposition 2.4 (Auslander [4]). *If R is an arbitrary ring and C is a pure injective R-module, then the functor $\mathrm{Ext}^1_R(-, C)$ sends direct limits into inverse limits. In particular, ${}^{\perp}C$ is closed under direct limits.*

For every R-module M over a commutative ring R and every maximal ideal $\mathfrak{m}$ of R denote by $M^{\mathfrak{m}}$ the $R_{\mathfrak{m}}$-module $\mathrm{Hom}_R(R_{\mathfrak{m}}, M)$.

Theorem 2.5. *Let R be a commutative domain and let C be a 1-cotilting R-module. Then for every maximal ideal $\mathfrak{m}$ of R, $C^{\mathfrak{m}}$ is a 1-cotilting $R_{\mathfrak{m}}$-module and $\prod_{\mathfrak{m} \in \mathrm{Max}\, R} C^{\mathfrak{m}}$ is a 1-cotilting R-module equivalent to C.*

Proof. For every maximal ideal $\mathfrak{m}$ of R, $C^{\mathfrak{m}}$ is a 1-cotilting $R_{\mathfrak{m}}$-module. In fact, Proposition 2.2 applies, since $R_{\mathfrak{m}}$ is a flat R-module and C is pure injective. Moreover $C^{\mathfrak{m}}$ is cogenerated by C as an R-module. Let $E = \prod_{\mathfrak{m} \in \mathrm{Max}\, R} C^{\mathfrak{m}}$, then $\mathrm{Cogen}\, E \subseteq \mathrm{Cogen}\, C = {}^{\perp}C$. We prove now that ${}^{\perp}E \subseteq \mathrm{Cogen}\, E$. In fact, a R-module M belongs to ${}^{\perp}E$ if and only if $\mathrm{Ext}^1_R(M, C^{\mathfrak{m}}) = 0$ for every maximal ideal $\mathfrak{m}$ and also, by formula **(a)**, if and only if $\mathrm{Ext}^1_{R_{\mathfrak{m}}}(R_{\mathfrak{m}} \otimes M, C^{\mathfrak{m}}) = 0$, for every maximal ideal $\mathfrak{m}$. Since $C^{\mathfrak{m}}$ is a cotilting $R_{\mathfrak{m}}$-module, we conclude that $M \in {}^{\perp}E$ if and only if $R_{\mathfrak{m}} \otimes M$ is cogenerated by $C^{\mathfrak{m}}$ as a $R_{\mathfrak{m}}$-module. From the embedding $0 \to M \to \prod_{\mathfrak{m} \in \mathrm{Max}\, R} R_{\mathfrak{m}} \otimes M$, we infer that M is cogenerated by E as a R-module. Therefore we have proved ${}^{\perp}E \subseteq \mathrm{Cogen}\, E \subseteq \mathrm{Cogen}\, C = {}^{\perp}C$. If we show that ${}^{\perp}C \subseteq {}^{\perp}E$, then the proof is complete. As noted above, M belongs to ${}^{\perp}E$ if and only if $\mathrm{Ext}^1_{R_{\mathfrak{m}}}(R_{\mathfrak{m}} \otimes M, C^{\mathfrak{m}}) = 0$ for every maximal ideal $\mathfrak{m}$ and so, by formula **(b)**, $M \in {}^{\perp}E$ if and only if $\mathrm{Ext}^1_R(R_{\mathfrak{m}} \otimes M, C) = 0$, or every maximal ideal $\mathfrak{m}$. So we must prove that $M \in {}^{\perp}C$ implies $R_{\mathfrak{m}} \otimes M \in {}^{\perp}C$. $R_{\mathfrak{m}}$ is isomorphic to the direct limit of $s^{-1}R$ where $s \in R \backslash \mathfrak{m}$. So, by Proposition 2.4 it is enough to show that $M \in {}^{\perp}C$ implies $s^{-1}R \otimes M \in {}^{\perp}C$, for every $s \in R \backslash \mathfrak{m}$. But this is true, since $s^{-1}R \cong R$ implies $s^{-1}R \otimes M \cong M$. □

In [10] it is proved that n-cotilting modules over Prüfer domains have injective dimension at most one. Thus, over such rings, we will simply write "cotilting modules" without any mention to the injective dimension. As an immediate consequence of the preceding theorem we have:

Corollary 2.6. *Let C be a cotilting module over a Prüfer domain R. Then for every maximal ideal $\mathfrak{m}$ of R, $C^{\mathfrak{m}}$ is a cotilting module over the valuation domain $R_{\mathfrak{m}}$ and $\prod_{\mathfrak{m} \in \mathrm{Max}\, R} C^{\mathfrak{m}}$ is a cotilting R-module equivalent to C.*

3 The set $\mathcal{G}$ associated to a cotilting module over a valuation domain

By Corollary 2.6 the study of cotilting modules over Prüfer domains can be restricted to the case of valuation domains.

So we will consider a valuation domain R with maximal ideal P and quotient field Q. For terminology and definitions on valuation domains and their modules, we refer to [16].

We state a result valid for all pure injective modules over valuation domains R; recall that pure injective modules over Prüfer domains have i.d. ≤ 1.

Lemma 3.1. *Let C a pure injective module over the valuation domain R. Then ${}^{\perp}C$ is determined by the cyclic modules that it contains. Moreover if R is an almost maximal valuation domain a module $M \in {}^{\perp}C$ if and only if every cyclic submodule of M belongs to ${}^{\perp}C$.*

Proof. First of all notice that ${}^{\perp}C$ contains the class of torsion free modules and that it is closed under submodules, since i.d. $C \leq 1$. Thus a module $M \in {}^{\perp}C$ if and only if its torsion submodule belongs to ${}^{\perp}C$. Now writing a module as a direct limit of its finitely generated submodules and applying Proposition 2.4, we conclude that $M \in {}^{\perp}C$ if and only if all of its finitely generated torsion submodules are in ${}^{\perp}C$. By [16, I, 7.8], a finitely generated torsion module A over a valuation domain admits a finite chain of pure submodules with cyclic successive factors. By the pure injectivity of C it is immediate to conclude that $A \in {}^{\perp}C$ if and only if all the cyclic factors are in ${}^{\perp}C$. The second statement follows analogously recalling that over almost maximal valuation domains the finitely generated torsion modules are direct sums of cyclics (see [16, V, 10.4]. □

Recall that a module C is 1-cotilting if and only if ${}^{\perp}C = \operatorname{Cogen} C$. It is well known that a 1-cotilting module C cogenerates a torsion theory whose torsion free class is the class $\operatorname{Cogen} C$. Moreover, for every R-module M the torsion submodule of M is the intersection of the kernels of all the homomorphisms from M to C. Thus, we have the following.

Lemma 3.2. *Let R be a valuation domain and C a cotilting module. For every ideal J of R the torsion submodule of R/J with respect to the torsion theory cogenerated by C is J'/J where $J' = \bigcap\{I \mid J \leq I \leq R, R/I \in \operatorname{Cogen} C\}$.*

In the case of cotilting modules C over a valuation domain, an important role will be played by the set $\mathscr{G}$ defined as follows:

$$\mathscr{G} = \{I < R \mid R/I \in {}^{\perp}C\} = \{I < R \mid R/I \in \operatorname{Cogen} C\}.$$

So $\mathscr{G}$ consists of the non-zero ideals of R such that R/I is torsion free in the torsion theory cogenerated by C. $\mathscr{G}$ will be called the set associated to C.

Recall that a module C is 1-cotilting if and only if ${}^{\perp}C = \operatorname{Cogen} C$. In the case of cotilting modules C over a valuation domain, an important role will be played by the set $\mathscr{G}$ defined as follows:

$$\mathscr{G} = \{I < R \mid R/I \in {}^{\perp}C\} = \{I < R \mid R/I \in \operatorname{Cogen} C\}.$$

$\mathscr{G}$ will be called the set associated to C.

Recall that, if I is a non-zero ideal of a valuation domain R, $I^{\#}$ denotes the prime ideal associated to I, that is $I^{\#} = \{r \in R \mid rI < I\}$. $I^{\#}$ is the union of the proper ideals of R isomorphic to I (see [16, p. 70 (g)]). If $I, J \leq Q$ denote by $J : I = \{x \in Q \mid xI \leq J\}$.

Lemma 3.3. *Let C be a cotilting R-module. The set $\mathscr{G}$ defined above has the following properties:*

(1) *$\mathscr{G}$ is closed under arbitrary sums and arbitrary intersections.*

(2) *$\mathscr{G}$ contains $\mathrm{Ann}(M)$ for every module $M \in {}^{\perp}C$.*

(3) *If $0 \neq I \in \mathscr{G}$, then for every $r \in R \backslash I$, $r^{-1}I \in \mathscr{G}$. Moreover, $I^{\#} \in \mathscr{G}$ and $R_{I^{\#}}/I^{\#} \in {}^{\perp}C$.*

(4) *If $rR_L \in \mathscr{G}$ for some prime ideal $L \in \mathscr{G}$ and some element $r \in L$, then $rL \in \mathscr{G}$.*

(5) *If $0 \neq I \in \mathscr{G}$ and $I < I^{\#}$, then for every $r \in I^{\#} \backslash I$, $rR_{I^{\#}}$ and $rI^{\#}$ belong to $\mathscr{G}$.*

(6) *If $rI \in \mathscr{G}$, for some $I \leq R$, then $r^n I \in \mathscr{G}$ for every n.*

Proof. (1) Let $I_\alpha \in \mathscr{G}$; then $R/\sum I_\alpha \cong \varinjlim R/I_\alpha$, since the ideals I_α are totally odered. Hence $\sum I_\alpha \in \mathscr{G}$, by Proposition 2.4. If $I_\alpha \in \mathscr{G}$, then R/I_α embeds in a direct product of copies of C. Let $I = \bigcap_\alpha I_\alpha$; then also R/I embeds in a direct product of copies of C, hence $I \in \mathscr{G}$.

(2) If $M \in {}^{\perp}C = \mathrm{Cogen}\, C$, the annihilators of the elements of M belong to $\mathscr{G}$, since $\mathrm{Cogen}\, C$ is closed under submodules. $\mathrm{Ann}(M)$ is the intersection of the annihilators of the elements of M; thus $\mathrm{Ann}(M)$ belongs to $\mathscr{G}$ by part 1.

(3) Let $0 \neq I \in \mathscr{G}$; then there exists a cyclic module $Rx \in \mathrm{Cogen}\, C$ such that $I = \mathrm{Ann}(x)$. If $r \in R \backslash I$, then $0 \neq Rrx \in \mathrm{Cogen}\, C$ and $\mathrm{Ann}(rx) = r^{-1}I$. Hence, $r^{-1}I \in \mathscr{G}$. Since $I^{\#} = \sum_{r \notin I} r^{-1}I$, $I^{\#} \in \mathscr{G}$ by part 1; that is $R/I^{\#} \in {}^{\perp}C$. Recalling that $r^{-1}I^{\#} = I^{\#}$ for every $r \in R \backslash I^{\#}$ we have that $r^{-1}R/I^{\#} \in {}^{\perp}C$. Moreover, $R_{I^{\#}}/I^{\#} = \sum_{r \in R \backslash I^{\#}} r^{-1}R/I^{\#}$; thus $R_{I^{\#}}/I^{\#} \in {}^{\perp}C$ by Proposition 2.4.

(4) Note that if L is a prime ideal and $r \in L$, then $rR_L \leq L$. Consider the exact sequence

$$0 \to \frac{rR_L}{rL} \to \frac{R}{rL} \to \frac{R}{rR_L} \to 0.$$

The first non-zero term is isomorphic to R_L/L, hence it is in ${}^{\perp}C$ by part 3; R/rR_L belongs to ${}^{\perp}C$ by hypothesis, thus also the middle term is in ${}^{\perp}C$ and we conclude that $rL \in \mathscr{G}$.

(5) Let $0 \neq I \in \mathscr{G}$ and let $r \in I^{\#} \backslash I$; then $r^{-1}I < R$ hence $r^{-1}I/I \in \mathrm{Cogen}\, C$. By part 2, $\mathrm{Ann}(r^{-1}I/I) = I : r^{-1}I = rR_{I^{\#}} \in \mathscr{G}$. By part (3) and (4) $rI^{\#} \in \mathscr{G}$.

(6) Consider the exact sequence

$$0 \to rI/r^2I \to R/r^2I \to R/rI \to 0.$$

$rI/r^2I \in {}^{\perp}C$, since it is isomorphic to a submodule of R/rI which is in ${}^{\perp}C$ by assumption. Hence, $R/r^2I \in {}^{\perp}C$, too. An easy induction completes the proof. □

In investigating the set $\mathcal{G}$, it is convenient to define the following.

Definition 1. Let $\mathcal{G}$ be the set associated to a cotilting module. We denote by $\sup\mathcal{G}$ the sum of all the ideals in $\mathcal{G}$ and, if $\mathcal{G} \neq \{0\}$, we denote by $\inf\mathcal{G}$ the intersection of the non-zero ideals in $\mathcal{G}$. Moreover, we denote by $\mathcal{G}'$ the set of non-zero prime ideals of $\mathcal{G}$.

To every R-module M over a valuation domain R one can associate two prime ideals:

$$M^{\#} = \{r \in R \mid rM < M\}; \quad M_{\#} = \{r \in R \mid \exists 0 \neq x \in M, rx = 0\}.$$

While over valuation domains $M^{\#}$ is crucial in characterizing tilting modules M (see [19]), the role of $M_{\#}$ will be of relevance in studying cotilting modules. Note that $M_{\#}$ is the union of the annihilators of the non zero elements of M.

Lemma 3.4. *Let C be a cotilting R-module with associated set $\mathcal{G}$. Then* $\sup\mathcal{G} = C_{\#}$ *and* $\inf\mathcal{G}$ *is an idempotent prime ideal.*

Proof. By definition $C_{\#} = \bigcup_{0 \neq c \in C} \mathrm{Ann}(c)$ and $\mathrm{Ann}(c) \in \mathcal{G}$ for every $0 \neq c \in C$, since C cogenerates its submodules. Thus, $C_{\#} \leq \sup\mathcal{G}$. If $I \in \mathcal{G}$, then R/I is cogenerated by C, hence I is an intersection of annihilators of elements of C. Thus, $I \leq C_{\#}$ and so $\sup\mathcal{G} = C_{\#}$. Suppose $\mathcal{G} \neq \{0\}$ and let $I_0 = \inf\mathcal{G}$. Assume, by way of contradiction that I_0 is not a prime ideal. Let $r \in R \backslash I_0$ and consider the exact sequence:

$$0 \to \frac{I_0}{rI_0} \to \frac{R}{rI_0} \to \frac{R}{I_0} \to 0.$$

By Lemma 3.3 (1) $I_0 \in \mathcal{G}$, hence $R/I_0 \in {}^{\perp}C$. Moreover, $I_0/rI_0 \cong r^{-1}I_0/I_0$ and $r^{-1}I_0/I_0$ is a submodule of $R/I_0 \in {}^{\perp}C$. Hence the midddle term of the sequence is in ${}^{\perp}C$, so $rI_0 \in \mathcal{G}$. By the minimality of I_0 we conclude that $rI_0 = I_0$, that is $I_0 = (I_0)^{\#}$ and thus I_0 is a prime ideal. Assume now that I_0 is not idempotent. Then $I_0 \cong rR_{I_0}$ for some $r \in I_0$. By Lemma 3.3 (4), $rI_0 \in \mathcal{G}$, contradicting the minimality of I_0. $\square$

Lemma 3.5. *Let C be a cotilting module with associated set $\mathcal{G}$. For every $L \in \mathcal{G}'$, let $J = \sum\{a^{-1}R \mid a^{-1}R/L \in {}^{\perp}C\}$. Then there is an idempotent prime ideal $L' \leq L$ such that $J = R_{L'}$ and $L' \in \mathcal{G}$. Moreover, $R_{L'}/L \in {}^{\perp}C$ and $L' = \inf\{N \in \mathcal{G}' \mid R_N/L \in {}^{\perp}C\}$.*

Proof. Let $X = \{a \in R \mid aL \in \mathcal{G}\}$; then $a^{-1}R/L \in {}^{\perp}C$ if and only if $a \in X$. If $a, b \in X$ the exact sequence

$$0 \to bL/abL \to R/abL \to R/bL \to 0$$

implies that $a^{-1}b^{-1}R \leq J$. Hence J is an overring of R, so $J = R_{L'}$ for some prime ideal L'. By Lemma 3.3 (3), $J \geq R_L$, hence $L' \leq L$. By Proposition 2.4, $R_{L'}/L \in {}^{\perp}C$;

moreover, $L' = \mathrm{Ann}(R_{L'}/L)$, hence $L' \in \mathscr{G}$, by Lemma 3.3 (2). Assume by way of contradiction that L' is not idempotent, that is $L' = rR_{L'}$ for some element $r \in L'$. From the exact sequence $0 \to R_{L'}/L \to r^{-1}R_{L'}/L \to R_{L'}/L' \to 0$, it follows that $r^{-1}R_{L'}/L \in {}^{\perp}C$, contradicting the maximality of J. The last statement follows by the fact that $N \le N'$ implies $R_N \ge R_{N'}$. □

4 Cotilting modules of cofinite type over valuation domains

In this section R will always denote a valuation domain.

Recall that a cotilting module C over a valuation domain is of cofinite type if and only if there exists a set $\mathscr{S}$ of finitely presented modules such that ${}^{\perp}C = \mathscr{S}^{\mathsf{T}} = \{X \in R\text{-Mod} \mid \mathrm{Tor}_1^R(S, X) = 0, \forall S \in \mathscr{S}\}$. If this is the case, then $\mathscr{S}$ can be chosen to be the intersection of ${}^{\mathsf{T}}({}^{\perp}C)$ with the class of finitely presented R-modules. In this section we prove that a cotilting module over a valuation domain R is of cofinite type if and only if R is strongly discrete.

For every R-module M, $E(M)$ and $\hat{M}$ denote the injective and the pure injective envelope of M, respectively.

Recall that a maximal immediate extension of a valuation domain R is a pure injective envelope of R and for fractional ideals I, J of R, $\widehat{J/I} \cong J\widehat{R_L}/I\widehat{R_L}$ where L denotes the prime ideal $J^{\#} \cup I^{\#}$ (see [16, XIII, 5.5]). Moreover, for every prime ideal L of R, $E(Q/L) \cong Q\widehat{R_L}/L\widehat{R_L} \cong \widehat{Q/L}$ (see [16, XIII, 4.3]).

The next results show that a cotilting module C of cofinite type is determined by the prime ideal $C_{\#}$.

We first consider the case in which C is a torsion free module. This is equivalent to $C_{\#} = 0$ and also to $\mathscr{G} = 0$.

Proposition 4.1. *Let C be a torsion free cotilting module. Then ${}^{\perp}C$ is the class of all torsion free modules and C is equivalent to $Q \oplus \hat{P}$. In particular C is of cofinite type.*

Proof. Denote by $\mathscr{TF}$ the class of all torsion free modules. Since $C \in \mathscr{TF}$, $\mathrm{Cogen}\, C \subseteq \mathscr{TF}$. The inclusion $\mathscr{TF} \subseteq {}^{\perp}C$ follows by the fact that C is pure injective. Consider the module $C_1 = Q \oplus \hat{P} \cong Q \oplus P\hat{R}$. C_1 is pure injective and clearly $\mathrm{Cogen}\, C_1 = \mathscr{TF}$; moreover ${}^{\perp}C_1 = {}^{\perp}P\hat{R}$. If I is a non-zero ideal of R, then, by Lemma 2.1, $\mathrm{Ext}_R^1(R/I, P\hat{R}) = 0$ if and only if $P\hat{R} : I = P\hat{R}$. But $P\hat{R} : I \ge \hat{R}$, thus there are no cyclic torsion modules in ${}^{\perp}C_1$. Hence we conclude that ${}^{\perp}C_1 = \mathscr{TF}$; so C and C_1 are equivalent cotilting modules. The last statement trivially follows by noting that $\mathscr{TF} = \mathscr{S}^{\mathsf{T}}$, where $\mathscr{S}$ is the class of all finitely presented R-modules. □

Before passing to consider the case $0 \neq C_{\#}$ we prove a lemma.

Lemma 4.2. *Let L be a non-zero prime ideal of R and $0 \neq I < R$. The following hold.*

(1) *$R/I \in {}^{\perp}\widehat{P/L}$ if and only if $I^{\#} \le L$.*

(2) *If L is idempotent, $R/I \in {}^{\perp}\widehat{R_L}$ if and only if $I \ge L$.*

Proof. (1) By [16, XIII, 5.5] $\widehat{P/L} \cong P\hat{R}/L\hat{R}$. Note that $Q\hat{R}/L\hat{R}$ is an injective R-module containing $P\hat{R}/L\hat{R}$. Thus, by Lemma 2.1, $\mathrm{Ext}^1_R(R/I, P\hat{R}/L\hat{R}) = 0$ if and only if $P\hat{R} : I = L\hat{R} : I + P\hat{R}$ as submodules of $Q\hat{R}$. This happens exactly if $I^\# \leq L$. In fact, if $I^\# > L$ the equality $I^\# = \bigcup_{r \in R \setminus I} r^{-1}I$ implies the existence of an element $r \notin I$ such that $r^{-1}I > L$. Hence $r^{-1} \in P\hat{R} : I$, but $r^{-1} \notin L\hat{R} : I + P\hat{R}$. Conversely, if $I^\# \leq L$ and $r^{-1}I \in P\hat{R}$, then $r^{-1}I \leq I^\#$, hence $r^{-1} \in L\hat{R} : I$.

(2) By Lemma 2.1, $R/I \in {}^\perp\widehat{R_L}$ if and only if $\widehat{R_L} : I = \widehat{R_L}$. This is equivalent to $I \geq L$. In fact, if there exists $r \in L \setminus I$, then $r^{-1}I \leq \widehat{R_L}$ and $r^{-1} \notin \widehat{R_L}$. Conversely, if $I \geq L$, then $I\widehat{R_L} \geq L\widehat{R_L}$, so $\widehat{R_L} : I = \widehat{R_L}$, since L is an idempotent prime ideal. □

Proposition 4.3. *Let C a cotilting module and let $0 \leq L = C_\#$. The following are equivalent:*

(1) *C is of cofinite type.*

(2) ${}^\perp C = \{M \in R\text{-Mod} \mid \forall 0 \neq x \in M, \mathrm{Ann}(x) \leq L\}$.

(3) *$E(Q/L) \oplus \widehat{P/L}$ is a cotilting module equivalent to C.*

Proof. (1) $\Leftrightarrow$ (2). Assume C is a cotilting module and let ${}^\perp C = \mathscr{F}$. C is of cofinite type if and only if $\mathscr{F} = \mathscr{S}^\intercal$ where $\mathscr{S}$ consists of the finitely presented modules which belong to ${}^\intercal\mathscr{F}$. Recalling that every finitely presented module over a valuation domain is a direct sum of modules of the form R/rR for some $r \in R$, we have that C is of cofinite type if and only if $\mathscr{F} = \mathscr{S}'^\intercal$ where $\mathscr{S}' = \{R/tR \mid R/tR \in {}^\intercal\mathscr{F}\}$. We have that $R/tR \in {}^\intercal\mathscr{F}$ if and only if $\mathrm{Tor}^R_1(R/tR, C) = 0$, since $\mathrm{Tor}^R_1(R/tR, -)$ commutes with direct products and $\mathrm{Tor}^R_2(R/tR, -) = 0$. Note that, for every $0 \neq t \in R$ and every R-module M, $\mathrm{Tor}^R_1(R/tR, M) \cong M[t]$. It follows that $R/tR \in {}^\intercal\mathscr{F}$ if and only if $t \in R \setminus L$. Moreover, it follows that $M \in \mathscr{S}'^\intercal$ if and only if for every $0 \neq x \in M$, $\mathrm{Ann}(x) \leq L$. Thus the equivalence of conditions (1) and (2) is proved.

(2) $\Leftrightarrow$ (3). Let $C_1 = E(Q/L) \oplus \widehat{P/L}$. First we prove that $\mathrm{Cogen}\, C_1 = \{M \in R\text{-Mod} \mid \forall 0 \neq x \in M, \mathrm{Ann}(x) \leq L\}$. By the description of the modules $E(Q/L)$ and $\widehat{P/L}$ given at the beginning of this section, it follows that the annihilators of non-zero elements in $E(Q/L)$ or in $\widehat{P/L}$ are, either of the form rL for some $r \in R$, or of the form $r^{-1}L$ for some $r \notin L$. Hence the annihilators are always contained in L, since L is a prime ideal. Consequently, if $0 \neq x \in C_1^\gamma$ for some cardinal γ, then $\mathrm{Ann}(x) \leq L$. Furthermore, if M is a module all of whose elements have annihilator contained in L, then M is an R-submodule of $M \otimes R_L$. Since $E(Q/L)$ is isomorphic to the R_L-injective envelope of the simple R_L-module R_L/L, C_1 cogenerates every R_L-module, thus in particular it cogenerates M as an R-module. We conclude that $\mathrm{Cogen}\, C_1 = \{M \in R\text{-Mod} \mid \forall 0 \neq x \in M, \mathrm{Ann}(x) \leq L\}$. Thus, assuming (3), condition (2) follows immediately.

Assume now that condition (2) holds. Then, $\mathrm{Cogen}\, C_1 = \mathrm{Cogen}\, C = {}^\perp C$. So condition (3) holds if and only ${}^\perp C_1 = {}^\perp C$. Since $E(Q/L)$ is an injective R-module, ${}^\perp C_1 = {}^\perp\widehat{P/L}$ and by Lemma 3.1 it is enough to show that ${}^\perp\widehat{P/L}$ and ${}^\perp C$ contain the same torsion cyclic modules. By Lemma 4.2, for every $0 \neq I \leq R$, $R/I \in {}^\perp\widehat{P/L}$ if and only if $I^\# \leq L$. By hypothesis, a cyclic torsion module R/I is in ${}^\perp C$ if and only if the annihilator of every non-zero element $r + I \in R/I$ is contained in L. If $r \notin I$, then $\mathrm{Ann}(r + I) = r^{-1}I$; so $R/I \in {}^{\perp}C$ if and only if $r^{-1}I \leq L$, for every $r \notin I$, that is if and only if $I^{\#} \leq L$. Hence we conclude that ${}^{\perp}C_1 = {}^{\perp}C$. □

As an application of the preceding results we obtain a characterization of cotilting modules over strongly discrete valuation domains. Recall that a valuation domain is called strongly discrete if every non-zero prime ideal is not idempotent; equivalently if every non-zero prime ideal L is a principal ideal of the localization R_L.

Proposition 4.4. *Let R be a strongly discrete valuation domain. Then every cotilting module C is of cofinite type. In particular C is equivalent to $E(Q/L) \oplus \widehat{P/L}$, where $L = C_{\#}$.*

Proof. Let $\mathscr{G}$ be the set associated to C. If $\mathscr{G} = \{0\}$, then the conclusion follows by Proposition 4.1. If $\mathscr{G} \neq \{0\}$, let $L = \sup \mathscr{G}$. Then, $L = C_{\#}$ by Lemma 3.4. Since there are no non-zero idempotent prime ideals, Lemma 3.5 yields that $Q/L \in {}^{\perp}C$. As noted in Section 1, ${}^{\perp}C$ is closed under pure injective envelopes; thus $\widehat{Q/L}$ belongs to ${}^{\perp}C$. $\widehat{Q/L}$ coincides with the injective envelope of Q/L and also with the R_L-injective envelope of the simple R_L-module R_L/L. We conclude that $\mathrm{Cogen}\, C$ contains all the R_L-modules. Let $\mathscr{A} = \{M \in R\text{-Mod} \mid \forall 0 \neq x \in M, \mathrm{Ann}(x) \leq L\}$. If $M \in \mathscr{A}$, then M is a R-submodule of $R_L \otimes M$, hence M is cogenerated by C. Conversely, $\mathrm{Cogen}\, C \subseteq \mathscr{A}$, since $L = C_{\#}$. Thus, condition 2 of Proposition 4.3 is satisfied and the conclusion follows. □

We are now in a position to show that over non strongly discrete valuation domains there exist cotilting modules which are not of cofinite type.

Proposition 4.5. *Let L be a non-zero idempotent prime ideal of R. The module*

$$C = Q \oplus \widehat{R_L} \oplus \widehat{R_L/L} \oplus \widehat{P/L}$$

is a cotilting module and

$${}^{\perp}C = \{M \in R\text{-Mod} \mid \forall 0 \neq x \in M, \mathrm{Ann}(x) = 0 \text{ or } \mathrm{Ann}(x) = L\}.$$

In particular C is not of cofinite type.

Proof. Let $\mathscr{A} = \{M \in R\text{-Mod} \mid \forall 0 \neq x \in M, \mathrm{Ann}(x) = 0 \text{ or } \mathrm{Ann}(x) = L\}$. We show first that ${}^{\perp}C \subseteq \mathscr{A}$. Let $M \in {}^{\perp}C$ and let $0 \neq I = \mathrm{Ann}(x)$ for some $0 \neq x \in M$. Then $R/I \in {}^{\perp}C$; in particular $R/I \in {}^{\perp}\widehat{R_L}$ and $R/I \in {}^{\perp}\widehat{P/L}$. By Lemma 4.2, $I = L$, so ${}^{\perp}C \subseteq \mathscr{A}$. To prove the inclusion $\mathscr{A} \subseteq {}^{\perp}C$, it is enough to show that every torsion module $M \in \mathscr{A}$ belongs to ${}^{\perp}C$, since C is pure injective and $\mathscr{A}$ is closed under submodules. If M is a non zero torsion module in $\mathscr{A}$, then M is an R-submodule of the localization $M \otimes R_L$. We show that $M \otimes R_L \in {}^{\perp}C$, so $M \in {}^{\perp}C$, too. $M \otimes R_L$ is a

R_L/L-module; hence isomorphic to a direct sum of copies of the field R_L/L. The R-module R_L/L is a direct limit of modules of the form $s^{-1}R/L$, $s \notin L$, thus isomorphic to $R/sL = R/L$. Therefore, by Proposition 2.4 we are led to show that $R/L \in {}^\perp C$. By Lemma 4.2, $R/L \in {}^\perp \widehat{R_L} \cap {}^\perp \widehat{P/L}$. It remains to show that $R/L \in {}^\perp(\widehat{R_L/L})$. Since $\widehat{R_L/L} \cong \widehat{R_L}/L\widehat{R_L}$, we consider the exact sequence

$$0 \to L\widehat{R_L} \to \widehat{R_L} \to \widehat{R_L}/L\widehat{R_L} \to 0.$$

Since $L\widehat{R_L}$ is the pure injective envelope of L, i.d. $L\widehat{R_L} \le 1$; thus from the above exact sequence we infer that ${}^\perp\widehat{R_L} \subseteq {}^\perp(\widehat{R_L}/L\widehat{R_L})$, hence $R/L \in {}^\perp(\widehat{R_L}/L\widehat{R_L})$ and we conclude that ${}^\perp C = \mathscr{A}$. We proceed now to check that $\operatorname{Cogen} C = \mathscr{A}$. It is easy to see that $C \in \mathscr{A}$, since L is a prime ideal; thus also $\operatorname{Cogen} C \subseteq \mathscr{A} = {}^\perp C$. In particular we obtain that $\operatorname{Cogen} C$ is closed under extensions and clearly $\operatorname{Cogen} C$ contains all torsion free modules. So, to prove that $\mathscr{A} \subseteq \operatorname{Cogen} C$ it is enough to verify that C cogenerates every torsion module $M \in \mathscr{A}$. By the above argument M is an R-submodule of a direct sum of copies of R_L/L and R_L/L is cogenerated by C, since it is a submodule of $\widehat{R_L/L}$. Thus we conclude that ${}^\perp C = \mathscr{A} = \operatorname{Cogen} C$. Proposition 4.3 implies that C is not of cofinite type. □

Propositions 4.4 and 4.5 imply the following result.

Corollary 4.6. *Let R be a valuation domain. Then every cotilting module is of cofinite type if and only if R is strongly discrete.*

5 Cotilting modules under change of rings

In this section we investigate the properties of cotilting modules over a valuation domain R with respect to localizations or factors of R.

Definition 2. Let $L_0 \le L$ be two prime ideals of a valuation domain R. We let

$$\langle L_0, L\rangle = \{I \le R \mid L_0 \le I \le I^\# \le L\}.$$

In particular $I \in \langle L_0, L\rangle$ if and only if $I \ge L_0$ and I is an ideal of R_L.

Lemma 5.1. *Let $L_0 \le L$ be two prime ideals of the valuation domain R with L_0 idempotent. Assume D is a R_L/L_0-module such that* i.d. $D \le 1$ *as a R-module. Then, for every ideal $I \in \langle L_0, L\rangle$*

$$\operatorname{Ext}^1_R(R/I, D) \cong \operatorname{Ext}^1_{R_L/L_0}(R_L/I, D).$$

Proof. By formula **(a)** in Section 2 we have $\operatorname{Ext}^1_R(R/I, D) \cong \operatorname{Ext}^1_{R_L}(R_L/I, D)$. Since L_0 is idempotent, Lemma 2.3 yields $\operatorname{Ext}^1_{R_L}(R_L/I, D) \cong \operatorname{Ext}^1_{R_L/L_0}(R_L/I, D)$. □

Proposition 5.2. *Let C be a cotilting R-module with associated set $\mathcal{G}$. Let $0 \neq L = \sup \mathcal{G}$ and $\inf \mathcal{G} = L_0$. Then C is equivalent to the cotilting module*

$$Q \oplus \widehat{R_{L_0}} \oplus \mathrm{Hom}_R\left(\frac{R_L}{L_0}, C\right) \oplus \widehat{P/L}.$$

Moreover, $\mathrm{Hom}_R(R_L/L_0, C)$ *is a* R_L/L_0*-cotilting module.*

Proof. Let $R' = R_L/L_0$. First of all notice that $\mathrm{Ext}^1_R(R', C) = 0$. In fact, by Lemma 3.3, $L_0 \in \mathcal{G}$ and $R_{L_0}/L_0 \in \mathrm{Cogen}\, C$, thus also $R_L/L_0 \in \mathrm{Cogen}\, C$, since $R_L \leq R_{L_0}$. Let $D = \mathrm{Hom}_R(R', C)$. By Proposition 2.2, D is a pure injective R-module and a cotilting R'-module. Let $C_1 = Q \oplus \widehat{R_{L_0}} \oplus D \oplus \widehat{P/L}$; C_1 is a pure injective R-module and we have to prove that C_1 is a cotilting module equivalent to C. We first show that ${}^\perp C = {}^\perp C_1$. By Lemma 3.1 it is enough to check that the two classes contain the same cyclic modules. By Lemma 4.2 and Lemma 5.1, $R/I \in {}^\perp C_1$ if and only if $I \in \langle L_0, L\rangle$ and $\mathrm{Ext}^1_{R'}(R_L/I, D) = 0$. Let now $R/I \in {}^\perp C$ for a nonzero ideal $I \leq R$; since $L = \sup \mathcal{G}$ and $L_0 = \inf \mathcal{G}$, $I \in \langle L_0, L\rangle$. Moreover, $R/I \in {}^\perp C$ if and only if $R_L/I \in {}^\perp C$. In fact, if $R/I \in {}^\perp C$ then Lemma 3.3 (3) implies $R_L/I^\# \in {}^\perp C$, since $R_L \leq R_{I^\#}$. From the exact sequence $0 \to I^\#/I \to R_L/I \to R_L/I^\# \to 0$ we conclude that $R_L/I \in {}^\perp C$. By formula **(b)**, $\mathrm{Ext}^1_R(R_L/I, C) \cong \mathrm{Ext}^1_{R'}(R_L/I, D)$. Thus, we conclude that $R/I \in {}^\perp C$ if and only if $R/I \in {}^\perp C_1$.

The summands Q, $\widehat{R_{L_0}}$ and $\mathrm{Hom}_R\left(\frac{R_L}{L_0}, C\right)$ of C_1 are clearly cogenerated by C as R-modules. By Lemma 3.3 $L \in \mathcal{G}$, so $P/L \in \mathrm{Cogen}\, C$ and thus $\widehat{P/L}$ is cogenerated by C, since the class $\mathrm{Cogen}\, C$ is closed under pure injective envelopes. So $\mathrm{Cogen}\, C_1 \subseteq \mathrm{Cogen}\, C$. It remains to show that C is cogenerated by C_1. $\mathrm{Cogen}\, C_1$ contains all the torsion free modules and the inclusions $\mathrm{Cogen}\, C_1 \leq \mathrm{Cogen}\, C = {}^\perp C = {}^\perp C_1$ imply that $\mathrm{Cogen}\, C_1$ is closed under extensions. Thus it is enough to show that the torsion submodule T of C is cogenerated by C_1. By assumption the annihilator of every non-zero torsion element of C contains L_0, thus T is an R/L_0-module. By Lemma 3.4 L_0 is idempotent, so **(a)** and Lemma 2.3 imply $\mathrm{Ext}^1_R(T, D) \cong \mathrm{Ext}^1_{R/L_0}(T, D) \cong \mathrm{Ext}^1_{R'}(T \otimes_{R/L_0} R', D)$. Now, $\mathrm{Ext}^1_R(T, D) = 0$ since $T \in {}^\perp C = {}^\perp C_1 \subseteq {}^\perp D$. Thus $T \otimes_{R/L_0} R'$, is cogenerated by D as an R'-module, since D is R'-cotilting. Moreover, $T_\# \leq L$ yields an exact sequence $0 \to T \to T \otimes_{R/L_0} R'$ which shows that T is an R-submodule of a product of copies of D. Thus T is cogenerated by D, hence by C_1 and we conclude that $\mathrm{Cogen}\, C = \mathrm{Cogen}\, C_1$. □

Remark 1. By the preceding result the investigation of cotilting modules over valuation domains can be reduced to the case in which $\sup \mathcal{G} = P$ and $\inf \mathcal{G} = 0$. In fact, the proof above shows that the set $\mathcal{G}_1$ associated to the R_L/L_0-cotilting module $\mathrm{Hom}_R(R_L/L_0, C)$ has inf 0 and sup the maximal ideal of R_L/L_0.

The next lemma shows that to characterize cotilting modules over a valuation domain R it is possible to assume that R is a maximal valuation domain.

Lemma 5.3. *Let C be a cotilting R-module and let S be a maximal immediate extension of R. Consider $D = \mathrm{Hom}_R(S, C)$. Then D is both a S- and a R-cotilting module; moreover C and D are equivalent cotilting R-modules.*

Proof. S is torsion free, hence $S \in {}^{\perp}C = \mathrm{Cogen}\, C$. Thus, by Proposition 2.2, D is a S-cotilting module and a pure injective R-module. We show that D is also a cotilting R-module. By formula **(a)** we have $\mathrm{Ext}^1_R(R/I, D) = 0$ if and only if $\mathrm{Ext}^1_S(S/IS, D) = 0$ and, by **(b)** if and only $\mathrm{Ext}^1_R(S/IS, C) = 0$. Now $\mathrm{Ext}^1_R(S/IS, C) = 0$ if and only if $\mathrm{Ext}^1_R(R/I, C) = 0$, since S/IS is isomorphic to the pure injective envelope of R/I and ${}^{\perp}C$ is closed under submodules and pure injective envelopes. By Lemma 3.1 we conclude that ${}^{\perp}C = {}^{\perp}D$ (as R-modules). D is clearly cogenerated by C and we show now that C is cogenerated by D. In fact, by **(a)** $\mathrm{Ext}^1_S(C \otimes S, D) \cong \mathrm{Ext}^1_R(C, D)$ and $\mathrm{Ext}^1_R(C, D) = 0$ since $C \in {}^{\perp}C = {}^{\perp}D$. Thus $C \otimes S$ is cogenerated by the D as an S-module, since D is S-cotilting. By the purity of the exact sequence $0 \to R \to S \to S/R \to 0$ we obtain the monomorphism $0 \to C \to C \otimes S$, which shows that C is an R-submodule of a product of copies of D. Thus we conclude that, as R-modules, ${}^{\perp}D = {}^{\perp}C = \mathrm{Cogen}\, C = \mathrm{Cogen}\, D$. □

6 A classification of cotilting modules over valuation domains

In order to complete the characterization of cotilting modules over valuation domains we need a more detailed investigation of the set $\mathcal{G}$ associated to a cotilting module (see Section 3). We will see that the complexity of the set $\mathcal{G}$ depends on the abundance of non-zero idempotent prime ideals that it contains. Recall that $\mathcal{G}'$ denotes the set of non-zero prime ideals of $\mathcal{G}$.

Definition 3. Let C be a cotilting module with associated set $\mathcal{G}$. Define

$$\phi : \mathcal{G}' \to \mathcal{G}, \quad \phi(L) = \inf\{N \in \mathcal{G}' \mid R_N/L \in {}^{\perp}C\},$$

$$\psi : \mathcal{G}' \to \mathcal{G}', \quad \psi(L) = \sup\{N \in \mathcal{G}' \mid R_{\phi(L)}/N \in {}^{\perp}C\}.$$

By Lemma 3.3 and 3.5 the two maps are well defined; by Lemma 3.5 is $\phi(L)$ is an idempotent prime ideal and it might be 0.

The two maps ϕ and ψ satisfy the following properties.

Lemma 6.1. *Let ϕ, ψ be defined as above. Then the following hold:*

(1) *For every $L \in \mathcal{G}'$, $R_{\phi(L)}/L$ and $R_{\phi(L)}/\psi(L)$ belong to ${}^{\perp}C$;*

(2) *ϕ, ψ are increasing maps; $\phi(L) \leq L$ and $L \leq \psi(L)$;*

(3) *$\phi(\psi(L)) = \phi(L)$ and $\phi(\phi(L)) = \phi(L)$;*

(4) *$\psi(\phi(L)) = \psi(L)$ and $\psi(\psi(L)) = \psi(L)$.*

Proof. (1) By Lemma 3.5, $R_{\phi(L)}/L \in {}^{\perp}C$. Let $\mathscr{A} = \{N \in \mathscr{G}' \mid R_{\phi(L)}/N \in {}^{\perp}C\}$. Then $R_{\phi(L)}/\psi(L)$ is isomorphic to $\varinjlim\{R_{\phi(L)}/N \mid N \in \mathscr{A}\}$, hence $R_{\phi(L)}/\psi(L) \in {}^{\perp}C$ by Proposition 2.4.

(2) $\phi(L) \leq L$ and $L \leq \psi(L)$ by Lemma 3.3 (3) and by (1) above.

Let $L_1 \leq L_2 \in \mathscr{G}'$. Consider the exact sequence

$$0 \to \frac{L_2}{L_1} \to \frac{R_{\phi(L_2)}}{L_1} \to \frac{R_{\phi(L_2)}}{L_2} \to 0.$$

The two outer terms belong to ${}^{\perp}C$, hence $R_{\phi(L_2)}/L_1 \in {}^{\perp}C$; thus $\phi(L_1) \leq \phi(L_2)$ by definition of ϕ. Moreover, $R_{\phi(L_2)} \leq R_{\phi(L_1)}$. By (1), $R_{\phi(L_1)}/\psi(L_1) \in {}^{\perp}C$; thus also $R_{\phi(L_2)}/\psi(L_1) \in {}^{\perp}C$. By definition of the map ψ we conclude that $\psi(L_1) \leq \psi(L_2)$.

(3) By (2) $L \leq \psi(L)$, so $\phi(L) \leq \phi(\psi(L))$. By (1), $R_{\phi(L)}/\psi(L) \in {}^{\perp}C$; thus, by the definition of ϕ we have $\phi(L) \geq \phi(\psi(L))$. So $\phi(\psi(L)) = \phi(L)$.

Clearly $\phi(\phi(L)) \leq \phi(L)$. By Lemma 3.5, $r \in \phi(L)$ if and only $r^{-1}R/L \notin {}^{\perp}C$, hence if and only if $rL \notin \mathscr{G}$. Assume $\phi(\phi(L)) < \phi(L)$ and choose $r \in \phi(L) \setminus \phi(\phi(L))$. Then $r\phi(L) \in \mathscr{G}$ and $rL \notin \mathscr{G}$. Since $r \notin r\phi(L)$, Lemma 3.3 (5) yields $rR_{\phi(L)} \in \mathscr{G}$. Moreover, $rL \leq rR_{\phi(L)} \leq R$. So, by Lemma 3.2 the torsion submodule of R/rL with respect to the torsion theory induced by C is contained in $rR_{\phi(L)}/rL$. But by (1), $R_{\phi(L)}/L$ is torsion free. So R/rL is torsion free, contradicting the hypothesis $rL \notin \mathscr{G}$.

(4) By (2) $\phi(L) \leq L$, so $\psi(\phi(L)) \leq \psi(L)$. By (1), $R_{\phi(L)}/\psi(L) \in {}^{\perp}C$; thus, by the definition of ψ and by the fact that $\phi^2 = \phi$, we have $\psi(L) \geq \psi(\phi(L))$. So $\psi(\phi(L)) = \psi(L)$.

Let $\psi(L) = N$; by (2) $\psi(L) \leq \psi(N)$. Using (3) we have $\phi(L) = \phi(N)$; therefore $R_{\phi(L)}/\psi(N) \in {}^{\perp}C$, by (1). By the definition of ψ we conclude that $\psi(N) \leq \psi(L)$, hence $\psi(\psi(L)) = \psi(N) = \psi(L)$. □

The following easy result on totally ordered sets will be useful.

Lemma 6.2. *Let X be a totally ordered set. Assume $\phi, \psi : X \to X$ are two increasing functions such that:*

(1) *for every $x \in X$, $\phi(x) \leq x$ and $x \leq \psi(x)$;*

(2) *$\phi \circ \phi = \phi$; $\psi \circ \psi = \psi$; $\phi \circ \psi = \phi$; $\psi \circ \phi = \psi$.*

Then, for every $a \in X$, the pre-image $\phi^{\leftarrow}(\phi(a))$ of $\phi(a)$ is the interval $[\phi(a), \psi(a)] = \{x \in X \mid \phi(a) \leq x \leq \psi(a)\}$. In particular, X is a disjoint union of intervals of the form $[\phi(a), \psi(a)]$, $a \in X$.

Proof. Let $\phi(a) \leq x \leq \psi(a)$. Then, $\phi(a) = \phi(\phi(a)) \leq \phi(x) \leq \phi(\psi(a)) = \phi(a)$. Thus, $\phi(x) = \phi(a)$ so x belongs to the pre-image of $\phi(a)$. Conversely, if $x < \phi(a)$, then $\phi(x) \leq x < \phi(a)$; so $\phi(x) \neq \phi(a)$. If $\psi(a) < x$, then $\psi(a) < x \leq \psi(x)$. So $\phi(x) \neq \phi(a)$, since otherwise $\psi(x) = \psi(\phi(x)) = \psi(\phi(a)) = \psi(a)$, a contradiction.

Clearly $\phi(a) \neq \phi(b)$ implies that $\phi^{\leftarrow}(\phi(a))$ and $\phi^{\leftarrow}(\phi(b))$ are disjoint. Moreover, every element $x \in X$ belongs to the pre-image of $\phi(x)$; hence X is a disjoint union of intervals of the form $[\phi(a), \psi(a)]$. □

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An immediate application of the two preceding results furnish the following corollary.

Corollary 6.3. *Let C be a cotilting module with associated set $\mathcal{G}$. If L, L' are two prime ideals in $\mathcal{G}'$ with $\phi(L) \neq \phi(L')$, then the intervals $[\phi(L), \psi(L)]$ and $[\phi(L'), \psi(L')]$ are disjoint and $\mathcal{G}'$ is the union of intervals of this form.*

Lemma 6.4. *If $L_0 \leq L$ are two prime ideals of a maximal valuation domain R, then R_{L_0}/L is an injective R_L/L_0-cogenerator.*

Proof. L/L_0 is the maximal ideal of R_L/L_0 and R_{L_0}/L_0 is the quotient field of R_L/L_0. Thus the conclusion follows since it is well known that the injective envelope of the simple module of a maximal valuation domain is the quotient field of the domain modulo its maximal ideal. □

Recalling the definition of $\langle L_0, L\rangle$ from Section 5, we are now in a position to describe a cotilting torsion free class.

Proposition 6.5. *Let C be a cotilting module over a maximal valuation domain R with associated set $\mathcal{G}$. A module M belongs to ${}^{\perp}C$ if and only if for every non-zero torsion element $x \in M$ there exists $L \in \mathcal{G}'$ such that* $\mathrm{Ann}(x) \in \langle\phi(L), \psi(L)\rangle$.

Proof. Let $M \in {}^{\perp}C$ and let $0 \neq x \in M$ be a torsion element. Then $0 \neq \mathrm{Ann}(x) = I \in \mathcal{G}$, so $I^{\#} \in \mathcal{G}'$. Let $L = I^{\#}$; we claim that $I \in \langle\phi(L), \psi(L)\rangle$. It is enough to show that $\phi(L) \leq I$. Assume $I < \phi(L)$ and let $r \in \phi(L)\backslash I$. By Lemma 3.3 (5), $rL \in \mathcal{G}$, hence $r^{-1}R/L \in {}^{\perp}C$. But $r^{-1}R > R_{\phi(L)}$, thus by Lemma 3.5 $r^{-1}R/L \notin {}^{\perp}C$, a contradiction.

To prove the converse, note that by Lemma 3.1, $M \in {}^{\perp}C$ if and only if every cyclic submodule of M belongs to ${}^{\perp}C$; so it is enough to show that for every $L \in \mathcal{G}'$, $I \in \langle\phi(L), \psi(L)\rangle$ implies $I \in \mathcal{G}$. Since $\phi(L) \leq I \leq I^{\#} \leq \psi(L)$, R/I is a $R/\phi(L)$-module and $IR_{\psi(L)} = I$. Hence R/I is a $R/\phi(L)$-submodule of the localization $R/I \otimes R_{\psi(L)}$. By Lemma 6.4, $R_{\phi(L)}/\psi(L)$ cogenerates the $R_{\psi(L)}/\phi(L)$-module $R/I \otimes R_{\psi(L)}$. Thus, R/I is cogenerated also as a R-module by $R_{\phi(L)}/\psi(L)$ and by Lemma 6.1, $R_{\phi(L)}/\psi(L)$ belongs to ${}^{\perp}C = \mathrm{Cogen}\, C$. □

We need now a technical lemma. For more details on the proof see [16].

Lemma 6.6. *Let R be a maximal valuation domain and $0 \neq I \leq R$. Assume L is a prime ideal of R and N is an idempotent prime ideal. Then:*

(1) *$R/I \in {}^{\perp}(R_N)$ if and only $I \geq N$.*

(2) *$R/I \in {}^{\perp}(R_N/L)$ if and only either:*
   2.a *$I \geq N$, or*
   2.b *$I < N$, $I \ncong R_N$ and $I^{\#} \leq L \cap N$.*

Proof. Recall that if R is a maximal valuation domain, then Q/I is an injective module for every $I \leq R$.

(1) See Lemma 4.2 (2).

(2) Consider the condition $R_N : I = L : I + R_N$ and label it by (*). By Lemma 2.1, $R/I \in {}^{\perp}(R_N/L)$ if and only if (*) holds. If $I \geq N$, then $IR_N \geq NR_N$, hence (*) is verified, since N is idempotent. Assume now $I < N$, then $R_N : I > R_N$, so (*) is satisfied if and only if $R_N : I = L : I$. If $I^{\#} > N$, then $I_N = rR_N$, for some element $r \in N$; hence $R_N : I = r^{-1}R_N$. But then $R_N : I > L : I$. So we must have $I^{\#} \leq N$. To show that $I \not\cong R_N$, assume by way of contradiction that $I = rR_N$ for some $r \in N$. Then, as before, (*) doesn't hold. If $L < I^{\#}$, there exists $r \notin I$ such that $r^{-1}I > L$. But $r^{-1}I \leq I^{\#} \leq N$; so $r^{-1} \in R_N : I \backslash L : I$, a contradiction. Thus, we conclude that $I^{\#} \leq L \cap N$, so the only if part of condition (2.b) is satisfied.

Conversely, assume $I \not\cong R_N$ and $I^{\#} \leq L \cap N$. Then $R_{I^{\#}} \geq R_L, R_N$ and $I = IR^{\#} = IR_N = IR_L$. Hence, $R_N : I = N : I = I^{\#} : I = L : I$. □

Lemma 6.7. *Let C be a cotilting R-module with associated set $\mathcal{G}$. For every $N \in \mathcal{G}'$ such that $\phi(N) \neq 0$ one and only one of the following conditions is satisfied:*

(1) $\phi(N) = \inf \mathcal{G}'$;

(2) $\phi(N) = \sup\{\psi(L) \mid L \in \mathcal{G}', L < \phi(N)\}$;

(3) *there exists $L \in \mathcal{G}'$ such that $\psi(L) < \phi(N)$ and there are no other primes of $\mathcal{G}'$ properly between $\psi(L)$ and $\phi(N)$.*

Proof. Assume (1) doesn't hold. Then the set $\mathcal{A} = \{\psi(L) \mid L \in \mathcal{G}', L < \phi(N)\}$ is non empty. Note that $L < \phi(N)$ implies $\psi(L) < \phi(N)$, since by Corollary 6.3 the intervals corresponding to L and N are disjoint. Let $L_0 = \sup \mathcal{A}$. If $L_0 = \phi(N)$, then condition (2) is satisfied. If $L_0 < \phi(N)$, then $\psi(L_0) < \phi(N)$ and clearly condition (3) is satisfied. □

Definition 4. If $N \in \mathcal{G}'$, $0 \neq \phi(N)$ and $\phi(N)$ satisfies condition (3) of Lemma 6.7 we say that $\phi(N)$ *covers* $\psi(L)$ and we write $\phi(N) \succ \psi(L)$. If $\phi(N)$ satisfies condition (1) we say that $\phi(N)$ *covers* 0 and we write $\phi(N) \succ 0$.

We are now not far from obtaining a characterization of cotilting modules over valuation domains. In view of Proposition 5.2 we may assume that $\sup \mathcal{G} = P$ and $\inf \mathcal{G} = 0$; moreover, by Lemma 5.3, we may assume that R is a maximal valuation domain. We split our final characterization result into two parts.

Proposition 6.8. *Let C be a cotilting module over a maximal valuation domain R with associated set $\mathcal{G}$. Assume $\sup \mathcal{G} = P$ and $\inf \mathcal{G} = 0$. Let*

$$E = Q \oplus \prod_{\phi(L) \in \mathcal{G}} \frac{R_{\phi(L)}}{\psi(L)} \oplus \prod_{\phi(N) \succ \psi(L)} \frac{R_{\phi(N)}}{\psi(L)} \bigoplus_{\phi(N) \succ 0} R_{\phi(N)}.$$

Then ${}^{\perp}C = {}^{\perp}E$.

Proof. C and E are pure injective modules, thus in view of Lemma 3.1 it is enough to show that ${}^{\perp}C$ and ${}^{\perp}E$ contain the same torsion cyclic modules.

CLAIM A: for every $0 \neq I \leq R$, $R/I \in {}^{\perp}C$ implies $R/I \in {}^{\perp}E$. By Proposition 6.5, there exists $L_0 \in \mathscr{G}'$ such that $I \in \langle \phi(L_0), \psi(L_0) \rangle$. By Lemma 6.6 (2.a), $R/I \in {}^{\perp}(R_{\phi(L_0)}/\psi(L_0))$. We show now that R/I belongs to ${}^{\perp}(R_{\phi(L)}/\psi(L))$ for every other $L \in \mathscr{G}'$ with $\phi(L_0) \neq \phi(L)$. If $I \geq \phi(L)$, again we are done by Lemma 6.6 (2.a). So assume $I < \phi(L)$; since $\phi(L_0) \leq I$ we have $\phi(L_0) < \phi(L)$ and by Corollary 6.3, the intervals $[\phi(L_0), \psi(L_0)]$ and $[\phi(L), \psi(L)]$ are disjoint, so $\psi(L_0) < \phi(L)$. Hence $I^{\#} \leq \psi(L_0) < \phi(L)$ and by Lemma 6.6 (2.b) $R/I \in {}^{\perp}(R_{\phi(L)}/\psi(L))$. Consider now the summands of E of the form $R_{\phi(N)}/\psi(L)$ where $\phi(N) > \psi(L)$, for some $L \in \mathscr{G}'$. If $\phi(L) = \phi(L_0)$, then the intervals $[\phi(L), \psi(L)]$ and $[\phi(L_0), \psi(L_0)]$ coincide, and clearly $\phi(N) > \psi(L_0)$. Since $I \in \langle \phi(L_0), \psi(L_0) \rangle$ and $\psi(L_0) < \phi(N)$, we have $I^{\#} \leq \psi(L_0) < \phi(N)$ and we conclude by Lemma 6.6 (2.b). If $\phi(L) \neq \phi(L_0)$ the intervals $[\phi(L), \psi(L)]$ and $[\phi(L_0), \psi(L_0)]$ are disjoint; so either $\psi(L) < \phi(L_0)$ or $\psi(L_0) < \phi(L)$. In the first case we must have $\phi(N) \leq \phi(L_0)$, since $\phi(N) > \psi(L)$; so $I \geq \phi(N)$ and Lemma 6.6 (2.a) applies. In the second case $I^{\#} \leq \psi(L_0) < \psi(L) < \phi(N)$ and Lemma 6.6 (2.b) applies. Thus, $R/I \in {}^{\perp}R_{\phi(N)}/\psi(L)$. It remains to consider the summand $R_{\phi(N)}$ for $\phi(N) > 0$, that is $0 \neq \phi(N) = \inf \mathscr{G}'$. In this case $\phi(N) \leq \phi(L_0)$ and $\phi(L_0) \leq I$. Hence $R/I \in {}^{\perp}R_{\phi(N)}$, by Lemma 6.6 (1).

CLAIM B: for every $0 \neq I \leq R$, $R/I \in {}^{\perp}E$ implies $R/I \in {}^{\perp}C$. By Proposition 6.5 it is enough to show that there exists a prime ideal $L_0 \in \mathscr{G}'$ such that $I \in \langle \phi(L_0), \psi(L_0) \rangle$. If $I \geq \phi(P)$, then $I \in \langle \phi(P), \psi(P) = P \rangle$ and we are done. Assuming $I < \phi(P)$ the set $\mathscr{A} = \{L \in \mathscr{G}' \mid I < \phi(L)\}$ is non-empty. Note that $L \in \mathscr{A}$ implies $I^{\#} \leq \phi(L)$, since by hypothesis $R/I \in {}^{\perp}R_{\phi(L)}/\psi(L)$ and so Lemma 6.6 (2.b) applies. Moreover, $L \in \mathscr{A}$ implies $\phi(L) \in \mathscr{A}$, since $\phi^2 = \phi$. Let $N = \inf \mathscr{A}$, then $I^{\#} \leq N$. Assuming $\phi(N) < N$ we have $I \geq \phi(N)$ and $I^{\#} \leq N \leq \psi(N)$; thus $I \in \langle \phi(N), \psi(N) \rangle$. It remains to consider the case $\phi(N) = N$. If $I = \phi(N)$, then $I \in \mathscr{G}$ and we are done. So we assume $I < \phi(N)$ and invoking Lemma 6.7, we consider the three distinct possibilities for $\phi(N)$. Condition (1) of Lemma 6.7 cannot be satisfied since otherwise $\phi(N) > 0$ and $R/I \in {}^{\perp}R_{\phi(N)}$ would imply $I \geq \phi(N)$, by Lemma 6.6 (1). Also condition (2) of Lemma 6.7 cannot be satisfied by $\phi(N)$, since otherwise there would exist $L \in \mathscr{G}'$, $L < \phi(N)$ such that $I < \psi(L)$. So $\psi(L) < \phi(N)$ since the intervals defined by L and N are disjoint. Thus there would also exist $L' \in \mathscr{G}'$, $L' < \phi(N)$ such that $\psi(L) < \psi(L') < \phi(N)$. Then the intervals $[\phi(L), \psi(L)]$ and $[\phi(L'), \psi(L')]$ are disjoint, so $I < \psi(L) < \phi(L')$. This would show that $L' \in \mathscr{A}$, contradicting $\phi(N) = \inf \mathscr{A}$. So condition (3) of Lemma 6.7 holds, that is $\phi(N) > \psi(L_0)$ for some $L_0 \in \mathscr{G}'$. Thus $R/I \in {}^{\perp}R_{\phi(N)}/\psi(L_0)$ and, by Lemma 6.6, $I^{\#} \leq \psi(L_0)$. In this case $I \in \langle \phi(L_0), \psi(L_0) \rangle$. □

Theorem 6.9. *Let C be a cotilting module over a maximal valuation domain R with associated set $\mathscr{G}$. Assume* $\sup \mathscr{G} = P$ *and* $\inf \mathscr{G} = 0$. *Then C is equivalent to the cotilting module*

$$E = Q \oplus \prod_{\phi(L) \in \mathscr{G}} \frac{R_{\phi(L)}}{\psi(L)} \oplus \prod_{\phi(N) > \psi(L)} \frac{R_{\phi(N)}}{\psi(L)} \bigoplus_{\phi(N) > 0} R_{\phi(N)}.$$

Proof. We first show that $E \in \operatorname{Cogen} C = {}^{\perp}C$. In fact, Q and $R_{\phi(N)}$ belong to ${}^{\perp}C$, since they are torsion free. For every $L \in \mathscr{G}'$, $R_{\phi(L)}/\psi(L) \in {}^{\perp}C$ by Lemma 6.1 and for every $\phi(N) > \psi(L)$ the exact sequence:

$$0 \to \frac{\psi(N)}{\psi(L)} \to \frac{R_{\phi(N)}}{\psi(L)} \to \frac{R_{\phi(N)}}{\psi(N)} \to 0$$

yields $R_{\phi(N)}/\psi(L) \in {}^{\perp}C$. Thus $\operatorname{Cogen} E \subseteq \operatorname{Cogen} C = {}^{\perp}C$ and by Proposition 6.8 ${}^{\perp}C = {}^{\perp}E$. Therefore to complete the proof we need to prove that ${}^{\perp}E \subseteq \operatorname{Cogen} E$. The inclusion $\operatorname{Cogen} E \subseteq {}^{\perp}E$ implies that $\operatorname{Cogen} E$ is closed under extensions. Therefore $\operatorname{Cogen} E$ is a torsion free class and a module A is torsion with respect to this torsion theory if and only if $\operatorname{Hom}_R(A, E) = 0$. Note that ${}^{\perp}E$ is closed under submodules, so to prove the inclusion ${}^{\perp}E \subseteq \operatorname{Cogen} E$ it is enough to show that if $M \in {}^{\perp}E$ and $\operatorname{Hom}_R(M, E) = 0$, then $M = 0$. By way of contradiction assume $0 \neq M$. Since Q is a direct summand of E, the hypothesis $\operatorname{Hom}_R(M, E) = 0$ implies that M has a non-zero torsion submodule (in the classical sense). So there exists $0 \neq x_0 \in M$ such that $0 \neq I = \operatorname{Ann}(x_0)$. Since $M \in {}^{\perp}E$, $R/I \in {}^{\perp}E = {}^{\perp}C$; hence by Proposition 6.5, there exists $L \in \mathscr{G}'$ such that $I \in \langle \phi(L), \psi(L) \rangle$. We claim that $\frac{M}{M[\phi(L)]} \in {}^{\perp}E$. By Lemma 3.1, it is enough to show that $R/\operatorname{Ann}(\bar{x}) \in {}^{\perp}E$ for every $0 \neq \bar{x} \in \frac{M}{M[\phi(L)]}$. Write $\bar{x} = x + M[\phi(L)]$; then $J = \operatorname{Ann}(x) < \phi(L)$ and $\operatorname{Ann}(\bar{x}) = J : \phi(L) = \operatorname{Ann}\left(\frac{\phi(L)}{J}\right)$. Since $x \in M \in {}^{\perp}E$ we know that $R/J \in {}^{\perp}E = {}^{\perp}C$; hence also $\frac{\phi(L)}{J} \in {}^{\perp}C$ and by Lemma 3.3 (2), $\operatorname{Ann}\left(\frac{\phi(L)}{J}\right) \in \mathscr{G}$. Thus $R/\operatorname{Ann}(\bar{x}) \in {}^{\perp}C = {}^{\perp}E$. Consider now the sequence

$$0 \to M[\phi(L)] \to M \to \frac{M}{M[\phi(L)]} \to 0.$$

Since $\frac{M}{M[\phi(L)]} \in {}^{\perp}E$ and $\operatorname{Hom}_R(M, E) = 0$, we conclude that also $\operatorname{Hom}_R(M[\phi(L)], E) = 0$. Consider the localization $M[\phi(L)] \otimes R_{\psi(L)}$ and let

$$f : M[\phi(L)] \to M[\phi(L)] \otimes R_{\psi(L)}$$

be the canonical map. Then $\operatorname{Ker} f = \{x \in M[\phi(L)] \mid sx = 0 \text{ for some } s \notin \psi(L)\}$. The condition $\operatorname{Hom}_R(M[\phi(L)], E) = 0$ yields $\operatorname{Hom}_R\left(\frac{M[\phi(L)]}{\operatorname{Ker} f}, E\right) = 0$. Now $M[\phi(L)] \otimes R_{\psi(L)}$ is an $R_{\psi(L)}/\phi(L)$-module, so, by Lemma 6.4, it is cogenerated by $R_{\phi(L)}/R_{\psi(L)}$ which is a direct summand of E. Thus the module $\frac{M[\phi(L)]}{\operatorname{Ker} f}$ is also cogenerated by E and the vanishing of $\operatorname{Hom}_R\left(\frac{M[\phi(L)]}{\operatorname{Ker} f}, E\right)$ gives $\frac{M[\phi(L)]}{\operatorname{Ker} f} = 0$. But the element $x_0 \in M$ we started with belongs $M[\phi(L)]$ and its annihilator is contained in $\psi(L)$, thus it doesn't belong to $\operatorname{Ker} f$, a contradiction. □

Collecting the results proved in the previous sections we can state the following.

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Theorem 6.10. *Let R be a Prüfer domain and C a cotilting R-module. Then C is equivalent to a cotilting module C' where C' is a direct product of indecomposable pure injective modules.*

Proof. By Corollary 2.6 C is equivalent to the direct product of the cotilting $R_{\mathfrak{m}}$-modules $C^{\mathfrak{m}} = \mathrm{Hom}_R(R_{\mathfrak{m}}, C)$ and in view of Proposition 5.2, Lemma 5.3 and Theorem 6.9, a cotilting module over a valuation domain is equivalent to a direct product of indecomposable pure injective modules and of modules of the form JS/IS for some $0 \le I \le J \le Q$, where S is a maximal immediate extension of the valuation domain. Recall that over a valuation domain a pure injective module is indecomposable if and only it is the pure injective envelope of a module of the form J/I (cf [16, XIII, 5.9]). For each summand JS/IS of $C^{\mathfrak{m}}$ we can argue like in the proof of [24, Theorem 3.4]. In fact, JS/IS is the pure injective envelope of a direct sum of κ copies of J/I for some cardinal κ (see [16, XIII, 5.4]. Let $M = \prod_{\alpha \in \kappa} \widehat{J/I}$. Then JS/IS is a direct summand of M and M is a summand of a direct product of copies of JS/IS. Thus in the decomposition of $C^{\mathfrak{m}}$ we can substitute every summand of the form JS/IS by suitable direct products of the indecomposable pure injective $R_{\mathfrak{m}}$-modules $\widehat{J/I}$. To conclude the proof we note that if A is a pure injective module over the localization $R_{\mathfrak{m}}$ of a domain R, then A is pure injective also as R-module, since pure monomorphisms are preserved under localization. Moreover, if A is indecomposable as $R_{\mathfrak{m}}$-module and $A_1 \oplus A_2$ is a R-direct sum decomposition of A, we must have $A_i \otimes R_{\mathfrak{m}} = 0$, for $i = 1$ or 2; so $A_i = 0$ and A is indecomposable also as a R-module. □

We conclude with the following observation.

Remark 2. Recall that a non zero module is called super decomposable if it doesn't have non zero indecomposable direct summands. It is known that if R is a valuation domain, then there exist super decomposable pure injective modules if and only if R is not strongly discrete (cf [16, XIII, 5.11]). We don't know whether a cotilting module C can have a super decomposable summand, say A; if so this summand should satisfy $\mathrm{Ext}^1_R(A^{\gamma}, A) = 0$, for every cardinal γ and we don't know whether this can happen. But, by the previous theorem, even if C has a super decomposable summand we can consider a cotilting module C' equivalent to C, such that C' has no super decomposable summands.

In [24] a set $\{M_\alpha \mid \alpha \in \kappa\}$ of pure injective modules with i.d. ≤ 1 satisfying the conditions $\mathrm{Ext}^1_R(M_\alpha, M_\beta) = 0$ for all $\alpha, \beta \in \kappa$ is said to be a rigid system if every M_α is indecomposable; and it is said to be an almost rigid system if M_0 is super decomposable and M_α is indecomposable for every $0 < \alpha \in \kappa$. In [24, Theorem 3.4] it is proved that for every 1-cotilting module C over an arbitrary ring there exists an almost rigid system $\{M_\alpha \mid \alpha \in \kappa\}$ such that the module $C' = \prod_{\alpha \in \kappa} M_\alpha$ is a cotilting module equivalent to C.

Our Theorem 6.10 shows that for cotilting modules over Prüfer domains we can improve the result of [24, Theorem 3.4] by stating that we can choose the almost rigid system to be rigid.

7 *n*-tilting modules over commutative and Prüfer domains

In this section we consider n-tilting modules and we prove that over Prüfer domains or over domains of i.d. ≤ 1, n-tilting modules have projective dimension at most one.

Symmetrically to the definitions in Section 1 we define, for any left R-module M:

$$M^{\perp} = \{X \in R\text{-Mod} \mid \operatorname{Ext}_R^i(M, X) = 0, \forall i \geq 1\}.$$

Let p.d. M denote the projective dimension of a R-module M. A R-module T is a n-tilting module if the following conditions hold ([1]):

(T1) p.d. $T \leq n$;

(T2) $\operatorname{Ext}_R^i(T, T^{(\lambda)}) = 0$ for each $i > 0$ and for every cardinal λ;

(T3) there exists a long exact sequence:

$$0 \to R \to T_0 \to T_1 \cdots \to T_r \to 0,$$

where $T_i \in \operatorname{Add} T$, for every $0 \leq i \leq r$.

If T is a n-tilting module the class $T^{\perp}$ is closed under direct sums ([1]). The notion of n-tilting modules of finite type was introduced in [2]. We recall that a 1-tilting module T is of finite type (countable) provided that there exists a set $\mathcal{S}$ of finitely presented (countably presented) R-modules of p.d. ≤ 1 such that $T^{\perp} = \bigcap_{s \in \mathcal{S}} S^{\perp}$. In [9] it has been proved that every 1-tilting module is of countable type and, moreover, that over Prüfer domains every 1-tilting module is of finite type.

In this section R will be a commutative domain with quotient field Q and K will denote the R-module Q/R.

Any direct sum of copies of K will be called a K-free module. We recall that every torsionfree divisible R-module is injective; thus, in particular, $Q^{(\alpha)}$ is injective for every cardinal α.

Proposition 7.1. *Let M be an R-module with* p.d. $M = n \geq 1$. *Then, there is a free R-module F such that* $\operatorname{Ext}_R^n(M, F) \neq 0$. *In particular, if $n > 1$,* $\operatorname{Ext}_R^{n-1}(M, D) \neq 0$, *for some K-free module D.*

Proof. The proof is by induction on n. Assume $n = 1$. By Eilenberg's trick there is an exact sequence $0 \to F_1 \to F_0 \to M \to 0$ where F_0 and F_1 are free R-modules. Then clearly $\operatorname{Ext}_R^1(M, F_1) \neq 0$. Assume $n > 1$ and consider a partial projective resolution of M

$$0 \to H_{n-1} \to P_{n-2} \to \cdots \to P_1 \to P_0 \to M \to 0$$

with projective modules P_i. By dimension shift we have that $\operatorname{Ext}_R^n(M, F) \cong \operatorname{Ext}_R^1(H_{n-1}, F)$ and also p.d. $M = n$ if and only if p.d. $H_{n-1} = 1$. Hence the result follows by the case $n = 1$. For the second statement note that, for every cardinal α, $\operatorname{Ext}_R^{n-1}(M, K^{(\alpha)}) \cong \operatorname{Ext}_R^n(M, R^{(\alpha)})$. □

As an application of the preceding results we obtain:

Proposition 7.2. *Assume T is a n-tilting R-module and* i.d. $R = 1$. *Then $n \leq 1$.*

Proof. The class $T^{\perp}$ is closed under direct sums; hence it contains all the K-free modules, since, by hypothesis, K is injective. By Proposition 7.1 we infer that p.d. T is at most 1. □

Assuming R is a Prüfer domain we obtain a result analogous to Proposition 7.2.

Proposition 7.3. *Let R be a Prüfer domain and T a n-tilting module. Then $n \leq 1$.*

Proof. Assume p.d. $T = n > 1$ and let H_{n-1} be the $(n-1)^{\text{st}}$-syzygy module of T. Then p.d. $H_{n-1} = 1$ and, by dimension shift, $H_{n-1}^{\perp} = \{X \in R\text{-Mod} \mid \mathrm{Ext}_R^n(T, X) = 0\}$. By [6, Lemma 3.4], $H_{n-1}^{\perp}$ is closed under direct sums and it is closed under epimorphic images, since p.d. $H_{n-1} = 1$. Thus, $H_{n-1}^{\perp}$ is a torsion class. By [3, Theorem 10] it is a special preenveloping class, hence by the characterization of tilting classes (see [3, Theorem 2.1]), $H_{n-1}^{\perp}$ is a 1-tilting class, that is $H_{n-1}^{\perp} = T_1^{\perp}$, for a 1-tilting module T_1. Note now that H_{n-1} is a torsionfree module and moreover flat since R is a Prüfer domain. Thus, $H_{n-1}^{\perp}$ contains all the cotorsion modules and ${}^{\perp}(H_{n-1}^{\perp})$ is contained in the class of flat modules. In particular T_1 is a flat module. By [9, Theorem 3.2] T_1 is of finite type. As noted before, the modules in ${}^{\perp}(T_1^{\perp})$ are flat, hence the finitely presented modules in ${}^{\perp}(T_1^{\perp})$ are projective. This implies $T_1^{\perp} = R\text{-Mod}$, hence $H_{n-1}^{\perp} = R\text{-Mod}$; that is H_{n-1} is projective, contradicting the assumption p.d. $T > 1$. □

Corollary 7.4. *If R is Prüfer domain, then all n-tilting modules are of finite type.*

Proof. It follows immediately by the preceding proposition and by [9, Theorem 3.4]. □

For a description of 1-tilting modules of finite type over Prüfer domains see [20].

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