Multi-Instanton Measure from Recursion Relations in $N = 2$ Supersymmetric Yang-Mills Theory

JHEP04(2001)041

(http://iopscience.iop.org/1126-6708/2001/04/041)

HYPER VERSION

RECEIVED: *March 30, 2001,* ACCEPTED: *April 30, 2001*

Multi-instanton measure from recursion relations in $N = 2$ supersymmetric Yang-Mills theory

Marco Matone

Department of Physics "G. Galilei" - Istituto Nazionale di Fisica Nucleare
University of Padova, Via Marzolo, 8 - 35131 Padova, Italy
E-mail: matone@pd.infn.it

ABSTRACT: By using the recursion relations found in the framework of $N = 2$ Super Yang-Mills theory with gauge group SU(2), we reconstruct the structure of the instanton moduli space and its volume form for all winding numbers.

KEYWORDS: Solitons Monopoles and Instantons, Models of Quantum Gravity, Supersymmetry and Duality, Differential and Algebraic Geometry.

JHEP04(2001)041

In [1] the entire nonperturbative contribution to the holomorphic part of the Wilsonian effective action was computed for $N = 2$ globally supersymmetric (SUSY) theories with gauge group SU(2), using ansätze dictated by physical intuitions. There are several aspects of the Seiberg-Witten (SW) model [1] which are related to the theory of moduli spaces of Riemann surfaces. In particular, here, we will consider the recursion relations for nonperturbative (instanton) contributions to the $N = 2$ Super Yang-Mills (SYM) effective prepotential [2] and will compare them with the recursion relations for the Weil-Petersson volumes of punctured Riemann spheres. In the Seiberg-Witten model there exists a relation the modulus $u = \langle \mathrm{Tr}\, \phi^2 \rangle$ and the effective prepotential [2] (see also [3]). This allowed to prove the SW conjecture by using the reflection symmetry of vacua [4]. On the other hand, it is rather surprising that, while on one side all the instanton coefficients have been computed in [2], explicit calculations have been performed only in the one and two-instanton background [5, 6, 7], while the above mentioned relation has been shown to hold to all instanton orders [8, 9]. The problem for instanton number $k \geq 3$ seems extremely difficult to solve. Indeed, the ADHM constraint equations become nonlinear and have not been explicitly solved up to now. Moreover, neither the structure of the moduli space, nor the volume form are known. The instanton measure for all winding numbers has been written in [10], but only in an implicit form (i.e. by implementing the bosonic and fermionic ADHM constraints through the use of Dirac delta functions), which in some special cases allows to extract information on the instanton moduli space [11]. However, the mathematical challenging problem of finding the explicit structure of the instanton moduli space for generic winding numbers still remains unsolved. On the other hand, the simple way in which the recursion relations have been derived, strongly suggests that there may be some mechanism which should make the explicit calculations possible. The investigation of such mechanism would provide important information on the structure of the instanton moduli space (of which only the boundary à la Donaldson-Uhlenbeck is known for generic winding number [12, 13, 14]) and of the associated volume form. In particular, even if the integrals seem impossible to compute, (actually, as we stated before we know neither the structure of the space nor the volume form), the existence of recursion relations and the simple way in which they arise, seem to suggest that these integrals could be easy to compute because of some underlying geometrical recursive structure. It has been claimed for some time, but only recently proven [15], that the nonperturbative contributions to u actually can be written as total derivatives, i.e. as pure boundary terms, on the moduli space. If the boundary is composed by moduli spaces of instantons of lower winding number times zero-size instantons moduli spaces, as it happens in the Donaldson-Uhlenbeck compactification, this would immediately provide, in the case of a suitable volume form, a recursion relation.

We will now see how the similar problem one finds in computing the Weil-Petersson (WP) volumes of punctured spheres has been solved thanks to the recursive structure of the Deligne-Knudsen-Mumford boundary and to the peculiar nature of the WP 2-form. The main analogy we will display, concerns the volume of moduli space of n-punctured Riemann spheres $\Sigma_{0,n} = \widehat{\mathbb{C}}\backslash\{z_1,\ldots,z_n\}$, $n \geq 3$, where $\widehat{\mathbb{C}} \equiv \mathbb{C}\cup\{\infty\}$. Their moduli space is the space of classes of isomorphic $\Sigma_{0,n}$'s, that is

$$\mathcal{M}_{0,n} = \{(z_1,\ldots,z_n) \in \widehat{\mathbb{C}}^n | z_j \neq z_k \text{ for } j \neq k\}/\mathrm{Symm}(n) \times PSL(2,\mathbb{C})\,, \tag{1}$$

where $\mathrm{Symm}(n)$ acts by permuting $\{z_1,\ldots,z_n\}$ whereas $PSL(2,\mathbb{C})$ acts as a linear fractional transformation. Using $PSL(2,\mathbb{C})$ symmetry we can recover the "standard normalization": $z_{n-2} = 0$, $z_{n-1} = 1$ and $z_n = \infty$. The classical Liouville tensor or Fuchsian projective connection is

$$T^F(z) = \left\{J_H^{-1}, z\right\} = \varphi_{cl\,zz} - \frac{1}{2}\varphi_{cl\,z}^2\,. \tag{2}$$

In the case of the punctured Riemann sphere we have

$$T^F(z) = \sum_{k=1}^{n-1}\left(\frac{1}{2(z-z_k)^2} + \frac{c_k}{z-z_k}\right), \tag{3}$$

where the coefficients $c_1, \ldots c_{n-1}$, called *accessory parameters*, satisfy the constraints

$$\sum_{j=1}^{n-1} c_j = 0\,, \qquad \sum_{j=1}^{n-1} z_j c_j = 1 - \frac{n}{2}\,. \tag{4}$$

These parameters are defined on the space

$$V^{(n)} = \{(z_1,\ldots,z_{n-3}) \in \mathbb{C}^{n-3} | z_j \neq 0,1 \;\; ; z_j \neq z_k\,,\ \text{for } j \neq k\}\,. \tag{5}$$

Note that

$$\mathcal{M}_{0,n} \cong V^{(n)}/\mathrm{Symm}(n)\,, \tag{6}$$

where the action of $\mathrm{Symm}(n)$ on $V^{(n)}$ is defined by comparing (1) with (6).

Let us now consider the compactification $\overline{V}^{(n)}$ à la Deligne-Knudsen-Mumford [16, 17]. The divisor at the boundary

$$D = \overline{V}^{(n)}\backslash V^{(n)}\,, \tag{7}$$

decomposes in the sum of divisors D_1,...,$D_{[n/2]-1}$, which are subvarieties of real dimension $2n-8$. The locus D_k consists of surfaces that split, upon removal of the node, into two Riemann spheres with $k+2$ and $n-k$ punctures. In particular, D_k consists of $C(k)$ copies of the space $\overline{V}^{(k+2)} \times \overline{V}^{(n-k)}$ where $C(k) = \binom{n}{k+1}$, for $k = 1,\ldots,(n-3)/2$, n odd. In the case of even n the unique difference is for $k = n/2-1$, for which we have $C(n/2-1) = \frac{1}{2}\binom{n}{n/2}$. An important property of the divisors D_k's is that their image provides a basis in $H_{2n-8}(\overline{\mathcal{M}}_{0,n}, \mathbb{R})$. The Weil-Petersson volume is

$$\mathrm{Vol}_{WP}\left(\mathcal{M}_{0,n}\right) = \frac{1}{(n-3)!}\int_{\overline{\mathcal{M}}_{0,n}} \omega_{WP}^{(n)}{}^{\,n-3} = \frac{1}{(n-3)!}\left[\omega_{WP}^{(n)}\right]^{n-3} \cap \left[\overline{\mathcal{M}}_{0,n}\right]. \quad (8)$$

It has been shown that [17]

$$\mathrm{Vol}_{WP}\left(\mathcal{M}_{0,n}\right) = \frac{1}{n!}\mathrm{Vol}_{WP}\left(V^{(n)}\right) = \frac{\pi^{2(n-3)}V_n}{n!(n-3)!}, \qquad n \geq 4, \quad (9)$$

where $V_n = \pi^{2(3-n)}\left[\omega_{WP}^{(n)}\right]^{n-3} \cap \left[\overline{V}^{(n)}\right]$ satisfies the recursion relations

$$V_3 = 1, \qquad V_n = \frac{1}{2}\sum_{k=1}^{n-3}\frac{k(n-k-2)}{n-1}\binom{n}{k+1}\binom{n-4}{k-1}V_{k+2}V_{n-k}, \qquad n \geq 4. \quad (10)$$

These recursive relations are a consequence of two basic properties. The first one is the fact that the boundary of the moduli space in the Deligne-Knudsen-Mumford compactification is the union of product of moduli spaces of lower order. The second one is the restriction phenomenon satisfied by the Weil-Petersson 2-form. A property discovered by Wolpert in [18] (see also [19, appendix]). The basic idea is to start with the natural embedding

$$i : \overline{V}^{(m)} \to \overline{V}^{(m)} \times * \to \overline{V}^{(m)} \times \overline{V}^{(n-m+2)} \to \partial\overline{V}^{(n)} \to \overline{V}^{(n)}, \qquad n > m, \quad (11)$$

where $*$ is an arbitrary point in $\overline{V}^{(n-m+2)}$, it follows that [18]

$$\left[\omega_{WP}^{(m)}\right] = i^*\left[\omega_{WP}^{(n)}\right], \qquad n > m. \quad (12)$$

There is a similarity between the above recursion relations for the WP volumes and the recursion relations satisfied by the instanton coefficients. To see this let us recall that in the case of the WP volumes, it has been derived in [19] a nonlinear differential equation satisfied by the generating function for the Weil-Petersson volumes

$$g(x) = \sum_{k=3}^{\infty} a_k x^{k-1}, \quad (13)$$

where

$$a_k = \frac{V_k}{(k-1)((k-3)!)^2}, \qquad k \geq 3, \quad (14)$$

so that (10) assumes the simple form

$$a_3 = 1/2, \qquad a_n = \frac{1}{2}\frac{n(n-2)}{(n-1)(n-3)}\sum_{k=1}^{n-3}a_{k+2}a_{n-k}, \qquad n \geq 4. \quad (15)$$

One can check that (15) implies that the function g satisfies the differential equation [19]

$$x(x-g)g'' = xg'^2 + (x-g)g' \,. \tag{16}$$

Remarkably, it has been shown in [20], that this nonlinear differential equation is essentially the inverse of a linear differential (Bessel) equation. More precisely, defining $g = x^2\partial_x x^{-1}h$, one has that (16) implies

$$xh'' - h' = (xh' - h)h'' \,. \tag{17}$$

Differentiating (17) we get

$$yy'' = xy^3 \,, \tag{18}$$

where $y = h'$. Then, interchanging the rôles of x and y, (18) transforms into the Bessel equation

$$y\frac{d^2x}{dy^2} + x = 0 \,. \tag{19}$$

It has been suggested in [20] that the appearance of such a linear differential equation may be related to the "mirror phenomenon".

The above structure is reminiscent of the above derived in Seiberg-Witten theory. In particular, in the case of WP volumes one starts evaluating the recursion relations by means of the Deligne-Knudsen-Mumford compactification and the Wolpert restriction phenomenon [17], then derives the associated nonlinear ODE [19] and end to a linear ODE [20] which is obtained by essentially inverting it. In the Seiberg-Witten model, one starts by observing that the $a^D(u)$ and $a(u)$ moduli satisfy a linear ODE [21], inverts the equation to obtain a nonlinear one satisfied by $u(a)$ then finds recursion relations for the coefficients of the expansion of $u(a)$ [2]. The final point stems from the observation that u and $\mathcal{F}$ are related in a simple way which allows one to consider the derived recursion relation as a relation for the instanton contributions to the preportential $\mathcal{F}$. The above similarity suggests to reconstruct the instanton moduli space and its measure starting from the recursion relations [2]

$$\mathcal{G}_{n+1} = \frac{1}{8\mathcal{G}_0^2(n+1)^2}\Bigg[(2n-1)(4n-1)\mathcal{G}_n + 2\mathcal{G}_0\sum_{k=0}^{n-1} c_{k,n}\mathcal{G}_{n-k}\mathcal{G}_{k+1} - \\ - 2\sum_{j=0}^{n-1}\sum_{k=0}^{j+1} d_{j,k,n}\mathcal{G}_{n-j}\mathcal{G}_{j+1-k}\mathcal{G}_k\Bigg] \,, \tag{20}$$

where $n \geq 0$, $\mathcal{G}_0 = 1/2$ and

$$c_{k,n} = 2k(n-k-1)+n-1 \,, \qquad d_{j,k,n} = [2(n-j)-1][2n-3j-1+2k(j-k+1)] \,. \tag{21}$$

It is still possible to rewrite some apparently cubic terms in the third term on the r.h.s. as quadratic ones and absorb them in the second term on the r.h.s. of (20), obtaining thus

$$\mathcal{G}_{n+1} = \frac{1}{2(n+1)^2}\Bigg[(2n-1)(4n-1)\mathcal{G}_n + \sum_{k=0}^{n-1} b_{k,n}\mathcal{G}_{n-k}\mathcal{G}_{k+1} - \\ - 2\sum_{j=1}^{n-1}\sum_{k=1}^{j} d_{j,k,n}\mathcal{G}_{n-j}\mathcal{G}_{j+1-k}\mathcal{G}_k\Bigg], \tag{22}$$

where $b_{k,n} = c_{k,n} - 2d_{k,0,n}$ and we have exploited the fact that $d_{k,0,n} = d_{k,k+1,n}$. Let us now consider the volume $\mathcal{G}_n$ of the moduli space of an instanton configuration of winding number n. In order to reproduce the recursion relation, we assume that $\mathcal{G}_n$ can be written as

$$\mathcal{G}_n = \int_{\overline{V}_I^{(n)}} \bigwedge_{k=1}^{X(n)} \omega_I^{(n)} = \left[\omega_I^{(n)}\right]^{X(n)} \cap \left[\overline{V}_I^{(n)}\right], \tag{23}$$

where $\cap$ is the topological cup product, $\omega_I^{(n)}$ is the natural 2-form defined on the n-instanton moduli space and $\overline{V}_I^{(n)}$ is a suitable compactification of $V_I^{(n)}$, which we will make explicit later. The function $X(n)$, representing the complex dimension of $\overline{V}_I^{(n)}$, will be fixed later. It is possible to recast (23) in the form

$$\begin{aligned}\mathcal{G}_{n+1} &= \left[\omega_I^{(n+1)}\right]^{X(n+1)-1} \cap \left(\left[\omega_I^{(n+1)}\right] \cap \left[\overline{V}_I^{(n+1)}\right]\right) \\ &= \left[\omega_I^{(n+1)}\right]^{X(n+1)-1} \cap \left[\mathcal{D}_\omega^{(n+1)} \cdot \overline{V}_I^{(n+1)}\right] \\ &= \left[\omega_I^{(n+1)}\right]^{X(n+1)-1} \cap \left[\mathcal{D}_\omega^{(n+1)}\right],\end{aligned} \tag{24}$$

where $\cdot$ denotes the topological intersection and $\mathcal{D}_\omega^{(n+1)}$ is the $[2X(n+1)-2]$-cycle Poincaré dual to the "instanton" class $[\omega_I^{(n+1)}]$. The divisor at the boundary

$$\mathcal{D}^{(n+1)} = \overline{V}_I^{(n+1)}/V_I^{(n+1)}, \tag{25}$$

decomposes in the sum of divisors $\mathcal{D}_{1,j}$, $\mathcal{D}_{2,j,k}$ and $\mathcal{D}_{3,n}$. In order to make contact with the recursion relation for the $\mathcal{G}_n$'s, we set

$$\begin{aligned}\mathcal{D}_{1,j} &= c_{n,j}^{(1)}\overline{V}_I^{(n-j)} \times \overline{V}_I^{(j+1)}, \\ \mathcal{D}_{2,j,k} &= c_{n,j,k}^{(2)}\overline{V}_I^{(n-j)} \times \overline{V}_I^{(j+1-k)} \times \overline{V}_I^{(k)} \times \overline{V}_I^{(1)}, \\ \mathcal{D}_{3,n} &= c_n^{(3)}\overline{V}_I^{(n)} \times \overline{V}_I^{(1)}.\end{aligned} \tag{26}$$

Let us now expand $\mathcal{D}_\omega^{(n+1)}$ in terms of the divisors at the boundary of the moduli space, namely

$$\mathcal{D}_\omega^{(n+1)} = \sum_{j=0}^{n-1} d_{n,j}^{(1)}\mathcal{D}_{1,j} + \sum_{j=1}^{n-1}\sum_{k=1}^{j} d_{n,j,k}^{(2)}\mathcal{D}_{2,j,k} + d_n^{(3)}\mathcal{D}_{3,n}. \tag{27}$$

One can see that consistency requirements on the outlined procedure uniquely determine $X(n)$ to be

$$X(n) = 2n - 1\,. \tag{28}$$

Let us consider the following natural embedding

$$i : \overline{V}_I^{(m)} \to \overline{V}_I^{(m)} \times * \to \overline{V}_I^{(m)} \times \overline{V}_I^{(n-m)} \to \partial \overline{V}_I^{(n)} \to \overline{V}_I^{(n)}\,, \qquad n > m\,, \tag{29}$$

where $*$ is an arbitrary point in $\overline{V}_I^{(n-m)}$. We now impose the following constraint

$$\left[\omega_I^{(m)}\right] = i^* \left[\omega_I^{(n)}\right], \qquad n > m\,. \tag{30}$$

Let us elaborate the three terms on the r.h.s. of (24): the first term is

$$\begin{aligned}
&\left[\omega_I^{(n+1)}\right]^{2(n+1)-2} \cap \left[\overline{V}_I^{(n-j)} \times \overline{V}_I^{(j+1)}\right] = \\
&= \left[\omega_I^{(n-j)} + \omega_I^{(j+1)}\right]^{2(n+1)-2} \cap \left[\overline{V}_I^{(n-j)} \times \overline{V}_I^{(j+1)}\right] \\
&= \binom{2(n+1)-2}{2(n-j)-1} \left(\left[\omega_I^{(n-j)}\right]^{2(n-j)-1} \cap \left[\overline{V}_I^{(n-j)}\right]\right)\left(\left[\omega_I^{(j+1)}\right]^{2(j+1)-1} \cap \left[\overline{V}_I^{(j+1)}\right]\right) \\
&= \binom{2(n+1)-2}{2(n-j)-1} \mathcal{G}_{n-j}\mathcal{G}_{j+1}\,.
\end{aligned} \tag{31}$$

The second term has the form

$$\begin{aligned}
&\left[\omega_I^{(n+1)}\right]^{2(n+1)-2} \cap \left[\overline{V}_I^{(n-j)} \times \overline{V}_I^{(j+1-k)} \times \overline{V}_I^{(k)} \times \overline{V}_I^{(1)}\right] = \\
&= \left[\omega_I^{(n-j)} + \omega_I^{(j+1-k)} + \omega_I^{(k)} + \omega_I^{k}(1)\right]^{2(n+1)-2} \cap \left[\overline{V}_I^{(n-j)} \times \overline{V}_I^{(j+1-k)} \times \overline{V}_I^{(k)} \times \overline{V}_I^{(1)}\right] \\
&= 2k \binom{2(n+1)-2}{2k} \binom{2(n+1)-2-2k}{2(n-j)-1} \left(\left[\omega_I^{(n-j)}\right]^{2(n-j)-1} \cap \left[\overline{V}_I^{(n-j)}\right] \times \right. \\
&\quad \left. \times \left(\left[\omega_I^{(j+1-k)}\right]^{2(j+1-k)-1} \cap \left[\overline{V}_I^{(j+1-k)}\right]\right)\left(\left[\omega_I^{(k)}\right]^{2k} \cap \left[\overline{V}_I^{(k)}\right]\right)\left(\left[\omega_I^{(1)}\right] \cap \left[\overline{V}_I^{(1)}\right]\right)\right. \\
&= \frac{k}{2} \binom{2(n+1)-2}{2k} \binom{2(n+1)-2-2k}{2(n-j)-1} \mathcal{G}_{n-j}\mathcal{G}_{j+1-k}\mathcal{G}_k\,,
\end{aligned} \tag{32}$$

where we used the fact that $\mathcal{G}_1 = 1/4$. Finally, the last term is

$$\left[\omega_I^{(n+1)}\right]^{2(n+1)-2} \cap \left[\overline{V}_I^{(n)} \times \overline{V}_I^{(1)}\right] = \frac{n}{2}\mathcal{G}_n\,. \tag{33}$$

In this way we can recast the recursion relations as

$$\begin{aligned}
\mathcal{G}_{n+1} = &\sum_{k=0}^{n-1} \binom{2n}{2(n-k)-1} d_{n,k}^{(1)} c_{n,k}^{(1)} \mathcal{G}_{n-k}\mathcal{G}_{k+1} + \sum_{j=1}^{n-1}\sum_{k=1}^{j} \frac{k}{2}\binom{2n}{2k} \times \\
&\times \binom{2(n-k)}{2(n-j)-1} d_{n,j,k}^{(2)} c_{n,j,k}^{(2)} \mathcal{G}_{n-j}\mathcal{G}_{j+1-k}\mathcal{G}_k + \frac{n}{2} d_n^{(3)} c_n^{(3)} \mathcal{G}_n\,,
\end{aligned} \tag{34}$$

which can be straightforwardly compared to (22) and gives

$$d^{(1)}_{n,k}c^{(1)}_{n,k}\begin{pmatrix} 2n \\ 2(n-k)-1\end{pmatrix} = \frac{b_{k,n}}{2(n+1)^2}\,,$$
$$d^{(2)}_{n,j,k}c^{(2)}_{n,j,k}\begin{pmatrix} 2n \\ 2k\end{pmatrix}\begin{pmatrix} 2(n-k) \\ 2(n-j)-1\end{pmatrix} = -\frac{2d_{j,k,n}}{k(n+1)^2}\,,$$
$$d^{(3)}_{n}c^{(3)}_{n} = \frac{(2n-1)(4n-1)}{n(n+1)^2}\,. \qquad (35)$$

Acknowledgments

We would like to thank D. Bellisai for many fruitful discussions. Work partly supported by the European Commission TMR programme ERBFMRX–CT96–0045.

References

[1] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation and confinement in $N=2$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 426** (1994) 19 [hep-th/9407087]; *ibid.* **430** (1994) 485

[2] M. Matone, *Instantons and recursion relations in $N=2$ SUSY gauge theory*, *Phys. Lett.* **B 357** (1995) 342 [hep-th/9506102]; *Koebe 1/4 theorem and inequalities in $N=2$ superQCD*, *Phys. Rev.* **D 53** (1996) 7354 [hep-th/9506181].

[3] J. Sonnenschein, S. Theisen and S. Yankielowicz, *On the relation between the holomorphic prepotential and the quantum moduli in SUSY gauge theories*, *Phys. Lett.* **B 367** (1996) 145 [hep-th/9510129];
T. Eguchi and S.-K. Yang, *Prepotentials of $n=2$ supersymmetric gauge theories and soliton equations*, *Mod. Phys. Lett.* **A 11** (1996) 131 [hep-th/9510183];
G. Bonelli and M. Matone, *Nonperturbative renormalization group equation and beta function in $N=2$ SUSY Yang-Mills*, *Phys. Rev. Lett.* **76** (1996) 4107 [hep-th/9602174]; *Nonperturbative relations in $N=2$ SUSY Yang-Mills and WDVV equation*, *Phys. Rev. Lett.* **77** (1996) 4712 [hep-th/9605090];
E. D'Hoker, I.M. Krichever and D.H. Phong, *The renormalization group equation in $N=2$ supersymmetric gauge theories*, *Nucl. Phys.* **B 494** (1997) 89 [hep-th/9610156];
J.D. Edelstein, M. Mariño and J. Mas, *Whitham hierarchies, instanton corrections and soft supersymmetry breaking in $N=2\,\mathrm{SU}(N)$ super Yang-Mills theory*, *Nucl. Phys.* **B 541** (1999) 671 [hep-th/9805172];
J.D. Edelstein, M. Gomez-Reino and J. Mas, *Instanton corrections in $N=2$ supersymmetric theories with classical gauge groups and fundamental matter hypermultiplets*, *Nucl. Phys.* **B 561** (1999) 273 [hep-th/9904087];
G. Chan and E. D'Hoker, *Instanton recursion relations for the effective prepotential in $N=2$ superYang-Mills*, *Nucl. Phys.* **B 564** (2000) 503 [hep-th/9906193];

J.D. Edelstein, M. Gomez-Reino, M. Marino and J. Mas, $N = 2$ *supersymmetric gauge theories with massive hypermultiplets and the Whitham hierarchy*, *Nucl. Phys.* **B 574** (2000) 587 [hep-th/9911115].

[4] G. Bonelli, M. Matone and M. Tonin, *Solving* $N = 2$ *SYM by reflection symmetry of quantum vacua*, *Phys. Rev.* **D 55** (1997) 6466 [hep-th/9610026].

[5] D. Finnell and P. Pouliot, *Instanton calculations versus exact results in four-dimensional SUSY gauge theories*, *Nucl. Phys.* **B 453** (1995) 225 [hep-th/9503115].

[6] N. Dorey, V.V. Khoze and M.P. Mattis, *Multi-instanton calculus in* $N = 2$ *supersymmetric gauge theory*, *Phys. Rev.* **D 54** (1996) 2921 [hep-th/9603136];
N. Dorey, V.V. Khoze and M.P. Mattis, *Multi-instanton calculus in* $N = 2$ *supersymmetric gauge theory. II: coupling to matter*, *Phys. Rev.* **D 54** (1996) 7832 [hep-th/9607202].

[7] F. Fucito and G. Travaglini, *Instanton calculus and nonperturbative relations in* $N = 2$ *supersymmetric gauge theories*, *Phys. Rev.* **D 55** (1997) 1099 [hep-th/9605215].

[8] N. Dorey, V.V. Khoze and M.P. Mattis, *Multi-instanton check of the relation between the prepotential* f *and the modulus* u *in* $N = 2$ *SUSY Yang-Mills theory*, *Phys. Lett.* **B 390** (1997) 205 [hep-th/9606199].

[9] P.S. Howe and P.C. West, *Superconformal ward identities and* $N = 2$ *Yang-Mills theory*, *Nucl. Phys.* **B 486** (1997) 425 [hep-th/9607239].

[10] N. Dorey, V.V. Khoze and M.P. Mattis, *Supersymmetry and the multi-instanton measure*, *Nucl. Phys.* **B 513** (1998) 681 [hep-th/9708036];
N. Dorey, T.J. Hollowood, V.V. Khoze and M.P. Mattis, *Supersymmetry and the multi-instanton measure, 2. From* $N = 4$ *to* $N = 0$, *Nucl. Phys.* **B 519** (1998) 470 [hep-th/9709072].

[11] N. Dorey, V.V. Khoze, M.P. Mattis and S. Vandoren, *Yang-Mills instantons in the large-N limit and the AdS/CFT correspondence*, *Phys. Lett.* **B 442** (1998) 145 [hep-th/9808157];
N. Dorey, T.J. Hollowood, V.V. Khoze, M.P. Mattis and S. Vandoren, *Multi-instantons and Maldacena's conjecture*, *J. High Energy Phys.* **06** (1999) 023 [hep-th/9810243].

[12] D. S. Freed and K. K. Uhlenbeck, *Instantons and Four-Manifolds*, Mathematical Sciences Research Institute Publications, 1, Springer, New York 1984.

[13] S. K. Donaldson and P. B. Kronheimer, *The Geometry of Four-Manifolds*, Oxford University Press 1990.

[14] A. Maciocia, *Metrics on the moduli spaces of instantons over euclidean four space*, *Comm. Math. Phys.* **135** (1991) 467.

JHEP04(2001)041

[15] D. Bellisai, F. Fucito, A. Tanzini and G. Travaglini, *Instanton calculus, topological field theories and $N = 2$ super Yang-Mills theories*, *J. High Energy Phys.* **07** (2000) 017 [hep-th/0003272]; *Multi-instantons, supersymmetry and topological field theories*, *Phys. Lett.* **B 480** (2000) 365 [hep-th/0002110].

[16] P. Deligne and D. Mumford, *The irreducibility of the space of curves of given genus*, *Institut des Hautes Études Scientifiques, Publication Mathématiques* **36** (1969) 75
F. Knudsen and D. Mumford, *The projectivity of the moduli space of stable curves, 1. Preliminaries on "det" and "Div"*, *Math. Scand.* **39** (1976) 19
F. Knudsen, *The projectivity of the moduli space of stable curves, 2. The stacks $M_{g,n}$*, *Math. Scand.* **52** (1983) 200; *The projectivity of the moduli space of stable curves, 3. The line bundles on $M_{g,n}$, and a proof of the projectivity of $\overline{M}_{0,n}$ in characteristic 0*, *Math. Scand.* **52** (1983) 200.

[17] P.G. Zograf, *The Weil-Petersson volume of the moduli space of punctured spheres*, *Cont. Math. Amer. AMS* **150** (1993) 267.

[18] S.A. Wolpert, *On the homology of the moduli space of stable curves*, *Ann. of Math.* **118** (1983) 491; *On the Weil-Petersson geometry of the moduli space of curves*, *Amer. J. Math.* **107** (1985) 969

[19] M. Matone, *Nonperturbative model of liouville gravity*, *J. Geom. Phys.* **21** (1997) 381 [hep-th/9402081].

[20] M. Kontsevich and Yu. Manin (with appendix by R. Kaufmann), *Quantum cohomology of a product*, *Inv. Math.* **124** (1996) 313 [q-alg/9502009];
R. Kaufmann, Yu. Manin and D. Zagier, *Higher Weil-Petersson volumes of moduli spaces of stable n-pointed curves*, *Comm. Math. Phys.* **181** (1996) 763.

[21] A. Klemm, W. Lerche and S. Theisen, *Nonperturbative effective actions of $N = 2$ supersymmetric gauge theories*, *Int. J. Mod. Phys.* **A 11** (1996) 1929 [hep-th/9505150].