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LIE ALGEBRAIC OBSTRUCTIONS TO Γ-CONVERGENCE OF OPTIMAL CONTROL PROBLEMS*

ARIELA BRIANI† AND FRANCO RAMPAZZO‡

Abstract. We investigate the possibility of describing the "limit problem" of a sequence of optimal control problems $(\mathcal{P})_{(b_n)}$, each of which is characterized by the presence of a time dependent vector valued coefficient $b_n = (b_{n_1}, \dots, b_{n_M})$. The notion of "limit problem" is intended in the sense of Γ-convergence, which, roughly speaking, prescribes the convergence of both the minimizers and the infimum values. Due to the type of growth involved in each problem $(\mathcal{P})_{(b_n)}$ the (weak) limit of the functions $(b^2_{n_1}, \dots, b^2_{n_M})$—beside the limit $(b_1, \dots, b_M)$ of the $(b_{n_1}, \dots, b_{n_M})$—is crucial for the description of the limit problem. Of course, since the b_n are L^2 maps, the limit of the $(b^2_{n_1}, \dots, b^2_{n_M})$ may well be a (vector valued) measure $\mu = (\mu_1, \dots, \mu_M)$. It happens that when the problems $(\mathcal{P})_{(b_n)}$ enjoy a certain commutativity property, then the pair (b, μ) is sufficient to characterize the limit problem.

This is no longer true when the commutativity property is not in force. Indeed, we construct two sequences of problems $(\mathcal{P})_{(b_n)}$ and $(\mathcal{P})_{(\tilde{b}_n)}$ which are equal except for the coefficient $b_n(\cdot)$ and $\tilde{b}_n(\cdot)$, respectively. Moreover, both the sequences (b_n, b_n^2) and $(\tilde{b}_n, \tilde{b}_n^2)$ converge to the same pair (b, μ). However, the infimum values of the problems $(\mathcal{P})_{(b_n)}$ tend to a value which is different from the limit of the infimum values of the $(\mathcal{P})_{(\tilde{b}_n)}$. This means that the mere information contained in the pair (b, μ) is not sufficient to characterize the limit problem. We overcome this drawback by embedding the problems in a more general setting where limit problems can be characterized by triples of functions (B_0, B, γ) with $B_0 \geq 0$.

Key words. Γ-convergence, optimal control problems, Lie brackets

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1. Introduction.

The general goal in the various theories of variational convergence consists in singling out a notion of limit problem $(\mathcal{P})$ for a sequence of minimum problems $(\mathcal{P}_n)$. Loosely speaking, this means that both the minimizers (provided they exist) of the problems $(\mathcal{P}_n)$ and the corresponding minimum values should converge (in some sense) to the minimizers and the minimum value of $(\mathcal{P})$, respectively.

In this paper we shall deal with the case where the minimum problems $(\mathcal{P}_n)$ have the form of the optimal control problems $(\mathcal{P})_{(b_n)}$ considered below. More precisely, the dependence on n follows by the fact that the dynamics of these problems contain n-dependent time functions b_n. We are motivated to study this particular problem essentially for two reasons. The first one is related to the general problem of homogenization (see, e.g., [BLP78], [LPV85], [SP80]). More specifically, for a control system one could think to the case where the dynamic contains a quite irregular time dependent coefficient. This would motivate the interest in looking for suitable topologies such that the approximation of this coefficient with regular functions would provide a "good" approximation of the given optimal control problem.

The second reason why we are studying the particular class of problems specified below is twofold. On one hand, this class of problems is general enough to display

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†Dipartimento di Matematica, Università di Pisa, Via Buonarroti 2, 56127 Pisa, Italy (briani@dm.unipi.it).

‡Dipartimento di Matematica Pura ed Applicata, Università di Padova, Via Belzoni 7, Padova, Italy (rampazzo@math.unipd.it).

the pathology related to Lie brackets of the involved vector fields (see below). On the other hand, the relatively simple structure of these problems allows one to avoid unessential technicalities which would obscure the nature of the question at issue.

Referring to the appendix for some basic tools of the general issue of variational convergence, let us specify the class of optimal control problems we are going to deal with.

Let $g_0, g_1, \ldots, g_M$ be smooth vector fields, and let l, k_i, h_i be given real functions. We shall consider sequences of optimal control problems of the form

$(\mathcal{P})_{(b_n)}$

$$\begin{cases} \dot{x} = g_0(t,x) + \displaystyle\sum_{i=1}^{M} g_i(x) b_{n_i}(t) u_i(t), \;\; x(0) = x_0, \\ \displaystyle\min_u \left\{ J(x,u) = \int_0^T \left(l(t,x) + \sum_{i=1}^{M} k_i(t,x) b_{n_i}(t) u_i(t) + \sum_{i=1}^{M} h_i^2(x) u_i^2(t) \right) dt \right\}, \end{cases}$$

where $(b_n)_{n\in\mathbb{N}}$ is a sequence of $\mathbb{R}^M$-valued, time dependent coefficients.

We will investigate the Γ-limit (see the appendix) of problems $(\mathcal{P})_{(b_n)}$ when

$$\begin{aligned} &\lim_{n\to\infty} b_{n_i}(\cdot) = b_i(\cdot) \qquad \text{weakly in } L^2(0,T), \\ &\lim_{n\to\infty} b_{n_i}^2(\cdot) = \mu_i(\cdot) \qquad \text{weakly}^* \text{ in } \mathcal{M}([0,T]) \end{aligned} \tag{1.1}$$

for $i = 1, \ldots, M$ (where $L^2(0,T)$ and $\mathcal{M}([0,T])$ denote the space of 2-integrable functions and the space of Borel measures, respectively).

We shall assume the following set of hypotheses on the data.

(Hg0) The function $g_0 : (0,T) \times \mathbb{R}^N \to \mathbb{R}^N$ is continuous. Moreover, for every compact subset $Q \subset \mathbb{R}^N$ there exists a continuous function $\gamma_0(t)$ such that, for every $t \in [0,T]$ and for every $x, y \in Q$, one has

$$|g_0(t,x) - g_0(t,y)| \leq \gamma_0(t)|x-y|.$$

(Hg1) For each $i = 1, \ldots, M$ the vector fields g_i from $\mathbb{R}^N$ into $\mathbb{R}^N$ are of class C^2, and the trajectories of the equations $\dot{x} = g_i(x)$ exist globally.

(Hb) For each $n \in N$, $b_n(t) = (b_{n_1}(t), \ldots, b_{n_M}(t)) \in L^2(0,T;\mathbb{R}^M)$.

(Hu) The controls $u(t) = (u_1(t), \ldots, u_M(t))$ belong to $L^2(0,T;\mathbb{R}^M)$.

(Hl) The function $l : [0,T] \times \mathbb{R}^N \to [0,\infty]$ is a Borel function, and for every compact subset $Q \subset \mathbb{R}^N$ there exists an L^1 function $\eta(t)$ such that, for every $t \in [0,T]$ and for every $x, y \in Q$,

$$|l(t,x) - l(t,y)| \leq \eta(t)\,|x-y|\,.$$

Moreover, the function $l(t,0)$ belongs to $L^1(0,T)$.

(Hk) For each $i = 1, \ldots, M$, $k_i : [0,T] \times \mathbb{R}^N \to [0,\infty]$ is a continuous function. There exists a constant $C > 0$ such that, for each $t \in [0,T]$ and for each $y \in \mathbb{R}^N$

$$|k_i(t,y)| \leq C, \qquad i = 1, \ldots, M. \tag{1.2}$$

Moreover, there exists a constant $L_k > 0$ such that, for each $t \in [0,T]$ and for each $y, z \in \mathbb{R}^N$,

$$|k_i(t,y) - k_i(t,z)| \leq L_k|y-z|, \qquad i = 1, \ldots, M. \tag{1.3}$$

(Hh) For each $i = 1, \dots, M$, $h_i : \mathbb{R}^N \to [0, \infty]$ is a Borel function, and for every compact subset $Q \subset \mathbb{R}^N$ there exists a constant L_h such that

$$|h_i(x) - h_i(y)| \le L_h |x - y|, \quad i = 1, \dots, M$$

for every $x, y \in Q$. Moreover, we assume the following *coercivity* hypothesis. There exists a constant $K > 0$ such that, for every $x \in \mathbb{R}^N$,

$$\sum_{i=1}^{M} h_i^2(x) u_i^2 \ge K |u|^2$$

for every $u \in L^2(0, T)$.

Remark 1.1. Some of these hypotheses can be weakened further. For example, in view of section 5, the constants C and L_k in (1.2), (1.3) may be replaced by two functions in $L^1(0, T)$. Moreover, at the cost of some technical complications in the computation of the Γ-limit in Definition 2.3 below, the maps l, h_i, and k_i may be allowed to depend on n as well.

Let us begin by remarking that some authors (see, e.g., [BC89], [BF93], [Fr98]) studied this problem when the maps $g_1, \dots, g_M$, $h_1, \dots, h_M$ are *constant* and $k_i = 0$, $i = 1, \dots, M$. In particular, in [BF93], [Fr98] one studies the limit of these problems when the L^2 structural parameters $b_n(\cdot) = (b_{n_1}, \dots, b_{n_M})(\cdot)$ converge, say, weakly, to an L^2 map $b(\cdot) = (b_1, \dots, b_M)(\cdot)$. It turns out that in order to single out the limit problem one needs to know the (weak) limit $\mu = (\mu_1, \dots, \mu_M)(\cdot)$ of the maps $b_n^2 = (b_{n_1}^2, \dots, b_{n_M}^2)(\cdot)$ as well. Let us recall that this limit, when it exists, can well be different from $b^2(\cdot)$. (Actually, one has $\mu \ge b^2$.) Moreover, in general, it is not an L^1 function. Actually, it is a measure on $[0, T]$. The main point established in the quoted papers consists in the fact that the pair (b, μ) does single out the limit problem. This result relies upon a crucial assumption, namely, the fact that the g_i and the h_i are independent of x, which, in turn, allows one to regard the limit equation and the limit payoff as relations *in measure*. On the contrary, as soon as the g_i actually depend on x—and a certain commutativity assumption (see below) is not verified—the measure-theoretical approach does not work, as shown by the simple example in section 3.

In this paper we shall study the limit of problem $(\mathcal{P})_{(b_n)}$ when both *the g_i and the h_i can depend on x* and the k_i do not vanish.

Our aim is threefold. To begin with, in section 2 we assume a *commutativity* hypothesis, which generalizes the case where the g_i are constant. Namely, we assume that $[g_i, g_j] = 0$ for all $i, j = 1, \dots, M$ (plus the fact that the k_i and h_i are constant), where $[g_i, g_j]$ denotes the Lie bracket of the fields g_i and g_j. It is remarkable that, under this assumption, one can prove the same result as in the case where the g_i are x-independent. In other words, the limit problem of the $(\mathcal{P})_{(b_n)}$ is still singled out by the limit (b, μ) of the pairs (b_n, b_n^2). This limit is denoted by $\Phi^{-1}(Q_{(b,\mu)})$, for it is the preimage of a simpler problem $(\mathcal{Q})_{(b,\mu)}$ via a diffeomorphism Φ, which, in turn, is determined by the (commutative) fields g_i. The result in this section allows one to get a geometric insight into the results in [BC89], [BF93], [Fr98] as well, for, while the property of "being independent of x" is not chart-invariant, commutativity has an intrinsic meaning.

Second, in section 3 we present an example that reveals the crucial difference between the case with vanishing Lie brackets and the general case. Actually, in this example (the Lie brackets do not vanish and) two sequences $((b_n, b_n^2))_{n \in N}$ and

$((\tilde{b}_n, \tilde{b}_n^2))_{n \in N}$ converge to the *same* pair (b, μ), while the corresponding problems $(\mathcal{P})_{(b_n)}$ and $(\mathcal{P})_{(\tilde{b}_n)}$ converge to *different* limit problems. Hence, provided a limit problem exists (in some possibly extended sense), in order to characterize it one needs some "extra information" beside that contained in the assignment of the pair (b, μ).

The construction of an extended setting for problems with no commutativity assumptions is, in fact, the third aim of the paper. We pursue this objective in section 4 by redefining the minimum problems in the space of the graphs. Within this extended setting every minimum problem is identified by a triple of functions (B_0, B, γ) defined on $[0, 1]$, this triple replacing the role of the pair (b, μ). The map B_0, whose square root is the derivative of time t with respect to a *pseudotime parameter* s in the interval $[0, 1]$, assumes values greater than or equal to zero. A particular case is represented by the original problems $(\mathcal{P})_{(b_n)}$, which are identified with problems corresponding to triples of the form (B_{0_n}, B_n, B_n^2) with B_{0_n} *strictly greater than zero almost everywhere (a.e.) in* $[0, 1]$ and $B_n \doteq b_n B_{0_n}$. On the other hand, the extra information needed in order to single out the limit problem is provided by the restriction of γ to the subintervals of $[0, 1]$, where B_0 is equal to zero.

Last, in section 5 we prove some statements aiming to compose the (apparent) discrepancy between the case with vanishing Lie brackets—which is treated in section 2 in terms of the original time t—and the general case—which is addressed in section 4 in an extended framework. The key points consist in a projection of the set of triples (B_0, B, γ) onto the set of the pairs (b, μ) and in the consequent partition of the set of triples. Roughly speaking, when the commutativity hypothesis holds, all extended problems in a class of this partition correspond to a unique problem, namely, the one singled out by the (unique) projection (b, μ) of the triples in the class.

For the sake of self-consistency we conclude the paper with an appendix, where some basic facts from the general theory of Γ-convergence are briefly recalled.

Let us point out that a reader interested only in the case with vanishing Lie brackets may read just section 2. On the other hand, the construction of the extended setting for the general case, which is performed in section 4, is self-contained and independent of the antecedent material of the paper.

Notation. We will write $L^p(0, T; \mathbb{R}^M)$ to denote the space of p-integrable functions from $[0, T]$ into $\mathbb{R}^M$ endowed with the usual norm $\|\cdot\|_p$. Moreover, $\mathcal{M}([0, T]; \mathbb{R}^M)$ and $BV([0, T]; \mathbb{R}^M)$ will denote the space of $\mathbb{R}^M$-valued Borel measure on $[0, T]$ and the space of $\mathbb{R}^M$-valued functions with bounded variation on $[0, T]$, respectively. If $M = 1$, we write $L^p(0, T)$, $\mathcal{M}([0, T])$, $BV([0, T])$ instead of $L^p(0, T; \mathbb{R})$, $\mathcal{M}([0, T]; \mathbb{R})$, $BV([0, T]; \mathbb{R})$, respectively.

If $\mu \in \mathcal{M}([0, T])$, μ^a and μ^s stand for the absolutely continuous and the singular part of μ with respect to the Lebesgue measure dt, respectively. If μ_1 and μ_2 are a vector measure and a scalar measure on $[0, T]$, respectively, we write $\mu_1 << \mu_2$ to mean that μ_1 is absolutely continuous with respect to μ_2. Moreover, we denote the derivative of μ_1 with respect to μ_2 (in the sense of the Radon–Nikodym theorem) by $\frac{d\mu_1}{d\mu_2}$. Finally, by $\operatorname{supp} \mu$ we mean the support of the measure μ.

2. Null Lie brackets. We assume here the *commutativity condition* (HC) below, which, in particular, states that all Lie brackets $[g_i, g_j]$, $i, j = 1, \dots, M$, are identically equal to zero. This hypothesis is crucial in order to prove a result of Γ-convergence (see the appendix for the definition of Γ-limit) analogous to the one proved in [Fr98], where the vectors multiplying the control were assumed x-independent. This fact allows one to get a geometric insight into the question, since the case with constant g_i

is nothing but a particular occurrence of the commutativity condition. We will see in the next sections that such a result does not hold when the commutativity assumption is not assumed.

Commutativity condition (HC). For every $i, j = 1, \dots, M$ the Lie bracket

$$[g_i, g_j](x) = Dg_j(x)g_i(x) - Dg_i(x)g_j(x)$$

(where $Dg(x)$ denotes the derivative of g at x) is identically equal to zero. Moreover, the maps h_i and k_i are constant. (See Remark 2.12 below for a comment on this latter condition.)

In order to define the Γ-limit, we introduce a suitable coordinate transformation which is induced by the fields $g_1, \dots, g_M$. This transformation is made possible by the crucial commutativity assumption (HC). Let us begin by adding the auxiliary equations $z_i(t) = \int_0^t b_{n_i}(s)u_i(s)ds$, $i = 1, \dots, M$. Then the state equation of $(\mathcal{P})_{(b_n)}$ reads as

$$\begin{pmatrix} \dot{z} \\ \dot{x} \end{pmatrix} = \tilde{g}_0(t, x) + \sum_{i=1}^{M} \tilde{g}_i(x) b_{n_i}(t) u_i(t),$$

where

$$\begin{array}{cccc} \tilde{g}_0 : & [0, T] \times \mathbb{R}^N & \to & \mathbb{R}^M \times \mathbb{R}^N, \\ & (t, x) & \mapsto & \begin{pmatrix} 0_M \\ g_0^1(t, x) \\ \cdot \\ \cdot \\ g_0^N(t, x) \end{pmatrix}, \end{array}$$

and, for every $i = 1, \dots, M$,

$$\begin{array}{cccc} \tilde{g}_i : & \mathbb{R}^N & \to & \mathbb{R}^M \times \mathbb{R}^N, \\ & x & \mapsto & \begin{pmatrix} e_i \\ g_i^1(x) \\ \cdot \\ \cdot \\ g_i^N(x) \end{pmatrix}, \end{array}$$

0_M and e_i being the zero vector and the ith (column) vector of the canonical basis in $\mathbb{R}^M$, respectively.

In the extended state space $\mathbb{R}^M \times \mathbb{R}^N$, problem $(\mathcal{P})_{(b_n)}$ is now formulated as $(\mathcal{P})_{(b_n)}$

$$\begin{cases} \begin{pmatrix} \dot{z} \\ \dot{x} \end{pmatrix} = \tilde{g}_0(t, x) + \sum_{i=1}^{M} \tilde{g}_i(x) b_{n_i}(t) u_i(t), \ (z(0), x(0)) = (0, x_0), \\ \min_u \left\{ J_n((z, x), u) = \int_0^T \left(l(t, x) + \sum_{i=1}^{M} k_i b_{n_i}(t) u_i(t) + \sum_{i=1}^{M} h_i^2 u_i^2(t) \right) dt \right\}. \end{cases}$$

(Notice that we use the same notation, namely, $(\mathcal{P})_{(b_n)}$, to mean both the problem in $\mathbb{R}^N$ and the corresponding one in $\mathbb{R}^M \times \mathbb{R}^N$.)

Let us set

$$\Phi_1(z, x) = z,$$
$$\Phi_2(z, x) = \exp(-z_M g_M) \circ \cdots \circ \exp(-z_1 g_1) x$$

(where $\exp(sg)x$ stands for the value at time s of the solution of the Cauchy problem $\dot{y}(s) = g(y(s))$, $y(0) = x$), and let us consider the map Φ defined by

$$\begin{pmatrix} z \\ y \end{pmatrix} = \Phi(z,x) \doteq \begin{pmatrix} \Phi_1(z,x) \\ \Phi_2(z,x) \end{pmatrix}.$$

We shall also use the notations $(z, x(z,y))$ and $(z, y(z,x))$ instead of $\Phi^{-1}(z,y)$ and $\Phi(z,x)$, respectively. Notice that, since the maps g_i are of class C^2, Φ is a local diffeomorphism. Actually, Φ is a global diffeomorphism.

Let us define the vector fields $\check{g}_0 : (0,T) \times \mathbb{R}^M \times \mathbb{R}^N \to \mathbb{R}^M \times \mathbb{R}^N$ and $\check{g}_i : \mathbb{R}^M \times \mathbb{R}^N \to \mathbb{R}^M \times \mathbb{R}^N$, $i = 1, \dots, M$, by setting

$$\begin{aligned} \check{g}_0(t,z,y) &\doteq D\Phi(z,x)\, \tilde{g}_0(t,x), \\ \check{g}_i(z,y) &\doteq D\Phi(z,x)\, \tilde{g}_i(x), \end{aligned}$$

where $(z,x) = \Phi^{-1}(z,y)$. Notice that $\check{g}_0$ and $\check{g}_i$ are the expressions of $\tilde{g}_0$ and $\tilde{g}_i$, respectively, in the new coordinate (z,y).

Proposition 2.1. *The first components of the vector field $\check{g}_0 : (0,T) \times \mathbb{R}^M \times \mathbb{R}^N \to \mathbb{R}^M \times \mathbb{R}^N$ are equal to zero, that is,*

$$\check{g}_0(t,z,y) = \begin{pmatrix} 0_M \\ g_0^\sharp(t,z,y) \end{pmatrix},$$

where the (column) vector field $g_0^\sharp(t,z,y)$ is given by $g_0^\sharp(t,z,y) = D_x\Phi_2(z,x)\tilde{g}_0(t,x)$ with $(z,x) = \Phi^{-1}(z,y)$. In particular, $\check{g}_0$ verifies (Hg0) *(with N replaced by $M+N$). Moreover, one has, for $i = 1, \dots, M$,*

$$\check{g}_i(z,y) = \begin{pmatrix} e_i \\ 0_N \end{pmatrix},$$

where 0_N stands for the (column) zero vector of $\mathbb{R}^N$.

A proof of this trivial proposition can be found in [BR91].

By means of this coordinate change, problem $(\mathcal{P})_{(b_n)}$ is transformed into the problem

$$(\mathcal{Q})_{(b_n)} \qquad \begin{cases} \begin{pmatrix} \dot{z} \\ \dot{y} \end{pmatrix} = \check{g}_0(t,z,y) + \displaystyle\sum_{i=1}^{M} \check{g}_i b_{n_i}(t) u_i(t), \qquad (z(0), y(0)) = (0, x_0), \\ \min\limits_u \left\{ \check{J}_n((z,y),u) \right\}, \end{cases}$$

$$\check{J}_n((z,y),u) = \int_0^T \left(l(t, x(z,y)) + \sum_{i=1}^{M} k_i b_{n_i}(t) u_i(t) + \sum_{i=1}^{M} h_i^2 u_i^2(t) \right) dt,$$

which, thanks to Proposition 2.1, displays the following, particularly simple, form:

$$(\mathcal{Q})_{(b_n)} \qquad \begin{cases} \dot{z}_1(t) = b_{n_1}(t) u_1(t), \\ \cdot \\ \cdot \\ \dot{z}_M(t) = b_{n_M}(t) u_M(t), \\ \dot{y}(t) = g_0^\sharp(t,z,y), \\ \min\limits_u \left\{ \check{J}_n((z,y),u) \right\}. \end{cases}$$

Remark 2.2. By saying that "$(\mathcal{P})_{(b_n)}$ is transformed into $(\mathcal{Q})_{(b_n)}$" we mean the following.

(i) A trajectory-control pair $((z,y),u)$ is admissible for the problem $(\mathcal{Q})_{(b_n)}$ if and only if the trajectory-control pair $((z,x),u) \doteq (\Phi^{-1}(z,y),u)$ is admissible for $(\mathcal{P})_{(b_n)}$.

(ii) For each trajectory-control pair $((z,y),u)$, if $((z,x),u) \doteq (\Phi^{-1}(z,y),u)$, then

$$J_n((z,x),u) = \check{J}_n(\Phi(z,x),u)$$

for every $u \in L^2(0,T;\mathbb{R}^M)$.

In particular, a trajectory-control pair $((z^\sharp,x^\sharp),u^\sharp)$ is optimal for $(\mathcal{P})_{(b_n)}$ if and only if $((z^\sharp,y^\sharp),u^\sharp)$ is optimal for $(\mathcal{Q})_{(b_n)}$, where $y^\sharp = \Phi_2(z^\sharp,x^\sharp)$.

In order to provide a representation of the Γ-limit of problems $(\mathcal{P})_{(b_n)}$ we shall be concerned with the set of *data pairs*

$$A = \left\{(b,\mu) \in L^2(0,T;\mathbb{R}^M) \times \mathcal{M}([0,T];\mathbb{R}^M) \;:\; \mu \geq b^2\right\},$$

where $\mu = (\mu_1,\ldots,\mu_M)$, $b = (b_1,\ldots,b_M)$, and the inequality has to be interpreted as $\mu_i \geq b_i^2$ for all $i = 1,\ldots,M$ (in the measure-theoretical sense). In particular, we shall consider the subset $A_s \subset A$ defined by

$$A_s = \left\{(b,\mu) \in A \;:\; \mu = b^2\right\},$$

which we call the subset of *simple data pairs* of A. (We recall that b^2 denotes the vector $(b_1^2,\ldots,b_M^2)$.)

Definition 2.3. *Let $(b,\mu) \in A$, and let us set $\sigma = \sum_{i=1}^M \mu_i^s$. We consider the variational problem*

$$(\mathcal{Q})_{(b,\mu)} \qquad \min_{((z,y),u)} \left\{\check{J}\left((z,y),u\right) \;:\; \dot{z} << dt + \sigma, \quad \dot{y} = g_0^\sharp(t,z,y)\right\},$$

where the minimum is searched over the trajectory-control pairs $((z,y),u)$ in $BV([0,T];\mathbb{R}^M \times \mathbb{R}^N) \times L^2(0,T;\mathbb{R}^M)$ and the cost functional $\check{J}$ is defined by

$$\begin{aligned}
&\check{J}\left((z,y),u\right)\\
&\quad \doteq \int_0^T \left[l(t,w) + \sum_{i=1}^M \left(k_i \dot{z}_i^a(t) + h_i^2 u_i^2(t) + h_i^2 \frac{\left(b_i(t)u_i(t) - \dot{z}_i^a(t)\right)^2}{\left(\mu_i^a(t) - b_i^2(t)\right)}\right)\right] dt\\
&\quad + \int_{\Omega_s \setminus \{0,T\}} \sum_{i=1}^M \left(h_i^2 \left|\frac{d\dot{z}_i^s}{d\sigma}\right|^2 + k_i \left|\frac{d\dot{z}_i^s}{d\sigma}\right|\right) d\sigma\\
&\quad + \sum_{i=1}^M \left(h_i^2 \frac{\left|z_i(0^+) - z_i(0^-)\right|^2}{\sigma(\{0\})} + k_i \left|z_i(0^+) - z_i(0^-)\right|\right)\\
&\quad + \sum_{i=1}^M \left(h_i^2 \frac{\left|z_i(T) - z_i(T^-)\right|^2}{\sigma(\{T\})} + k_i \left|z_i(T^+) - z_i(T^-)\right|\right),
\end{aligned}$$

where we have set $w = x(z,y)$ and $\Omega_s = \operatorname{supp}\sigma$. (See section 1 for the notations in the above formula.)

Remark 2.4. We adopt here the convention (already used in [BC89], [BF93], [Fr98]) according to which the fractions appearing in the definition of $\check{J}$ are zero as soon as their denominators are zero.

Remark 2.5. If one has $b_i^2 = \mu_i$ for $i = 1, \ldots, M$, then the limit problem $(\mathcal{Q})_{(b,\mu)}$ reduces to the standard form

$$\begin{cases} \begin{pmatrix} \dot{z} \\ \dot{y} \end{pmatrix} = \check{g}_0(t, z, y) + \sum_{i=1}^{M} \check{g}_i(z, y) b_i(t) u_i(t), \quad (z(0), y(0)) = (0, x_0), \\ \min_u \left\{ \check{J}((z, y), u) \right\}, \end{cases}$$

$$\check{J}((z, y), u) = \int_0^T \left(l(t, x(z, y)) + \sum_{i=1}^{M} k_i b_i(t) u_i(t) + \sum_{i=1}^{M} h_i^2 u_i^2(t) \right) dt.$$

Definition 2.6. *Let us rewrite problem $(\mathcal{Q})_{(b,\mu)}$ in the form*

$$(\mathcal{Q})_{(b,\mu)} \qquad \min \left\{ \check{F}((z, y), u) : (z, y) \in BV([0, T]; \mathbb{R}^M \times \mathbb{R}^N),\ u \in L^2(0, T; \mathbb{R}^M) \right\},$$

where $\check{F}((z, y), u) \doteq \check{J}\,((z, y), u) + \chi_{\{\dot{z} << dt + \sigma\ ,\ \dot{y} = g_0^\sharp(t,w)\}}$. *We define problem* $\Phi^{-1}((\mathcal{Q})_{(b,\mu)})$ *as follows:*

$$\Phi^{-1}((\mathcal{Q})_{(b,\mu)}) \qquad \min\{\check{F}(\Phi((z, x)), u) : (z, x) \in BV([0, T]; \mathbb{R}^M \times \mathbb{R}^N), u \in L^2(0, T; \mathbb{R}^M)\}.$$

The next result states that problems $(\mathcal{P})_{(b_n)}$ converge to the variational problem $\Phi^{-1}((\mathcal{Q})_{(b,\mu)})$. For the basic facts concerning the Γ-convergence, see the appendix and the references therein.

Theorem 2.7. *If the* (b_n, b_n^2) *converge to* (b, μ) *as in* (1.1), *then the problems* $(\mathcal{P})_{(b_n)}$ Γ*-converge to* $\Phi^{-1}((\mathcal{Q})_{(b,\mu)})$.

Proof. In view of Lemma 2.8 below we have to prove only that the $(\mathcal{Q})_{(b_n)}$ Γ-converge to $(\mathcal{Q})_{(b,\mu)}$. Now the optimal control problems $(\mathcal{Q})_{(b_n)}$ verify hypotheses (7.1)–(7.5) in [Fr98]. Moreover, assumption (1.1) here implies (7.17) and (7.18) therein. Hence, in view of the results in [Fr98], problems $((\mathcal{Q})_{(b_n)})_{n\in N}$ Γ-converge to the problem $(\mathcal{Q})_{(b,\mu)}$ introduced in Definition 2.3. ∎

Lemma 2.8. *If the sequence of problems* $((\mathcal{Q})_{(b_n)})_{n\in N}$ Γ*-converges to* $(\mathcal{Q})_{(b,\mu)}$, *then the sequence* $((\mathcal{P})_{(b_n)})_{n\in N}$ Γ*-converges to* $\Phi^{-1}((\mathcal{Q})_{(b,\mu)})$.

Proof. To begin with, for each $n \in N$ we set

$$\check{F}_n((z, y), u) \doteq \check{J}_n((z, y), u) + \chi_{\check{C}_n}((z, y), u),$$

where $\check{C}_n$ is the set of admissible trajectory-control pairs for $(\mathcal{Q})_{(b_n)}$ (see the appendix). By assumption we have (see Definition A.3 in the appendix)

$$\Gamma(N, U^-, Y^-) \lim_{n\to\infty} \check{F}_n((z, y), u) = \check{F}((z, y), u). \tag{2.1}$$

Now (see Remark 2.2)

$$\begin{aligned} F_n((z, x), u) &\doteq J_n((z, x), u) + \chi_{C_n}((z, x), u) \\ &= \check{J}_n(\Phi(z, x), u) + \chi_{\check{C}_n}(\Phi(z, x), u)) = \check{F}_n(\Phi(z, x), u), \end{aligned}$$

where C_n is the set of admissible trajectory-control pairs for $(\mathcal{P})_{(b_n)}$. Hence, by (2.1),

$$\begin{aligned} &\Gamma(N, U^-, Y^-) \lim_{n\to\infty} F_n((z, x), u) \\ &\quad = \Gamma(N, U^-, Y^-) \lim_{n\to\infty} \check{F}_n(\Phi(z, x), u) = \check{F}(\Phi(z, x), u), \end{aligned}$$

which proves the lemma. ◻

Theorem 2.7 says that the Γ-limit of a sequence of problems $(\mathcal{P})_{(b_n)}$ has the form $\Phi^{-1}((\mathcal{Q})_{(b,\mu)})$. Conversely, we have the following.

THEOREM 2.9. *For each problem $\Phi^{-1}((\mathcal{Q})_{(b,\mu)})$ with $(b,\mu) \in A$, there exists a sequence of problems $((\mathcal{P})_{(b_n)})_{n\in N}$ which Γ-converges to $\Phi^{-1}((\mathcal{Q})_{(b,\mu)})$.*

In order to prove this theorem, we need the following result.

LEMMA 2.10. *For each $(b,\mu) \in A$ (with $M = 1$) there exists a sequence $(b_n)_{n\in N} \in L^2(0,T)$ such that $b_n \to b$ weakly in $L^2(0,T)$ and $b_n^2 \to \mu$ weakly* in $\mathcal{M}([0,T])$.*

In the case where μ is an L^∞-function we can sharpen the above result as follows.

LEMMA 2.11. *If $(b,\mu) \in A$ (with $M = 1$) and $\mu \in L^\infty(0,T)$, then there exists a sequence $(b_n)_{n\in N} \in L^2(0,T)$ such that $b_n \to b$ weakly in $L^2(0,T)$ and $b_n^2 \to \mu$ weakly* in $L^\infty(0,T)$.*

We omit the proofs of both Lemmas 2.10 and 2.11, for they are mostly based on the same arguments as in the proof of Theorem 3.2 in [BR93].

Proof of Theorem 2.9. In view of Lemma 2.10, for each $(b,\mu) \in L^2(0,T;\mathbb{R}^M) \times \mathcal{M}([0,T];\mathbb{R}^M)$ such that $(b,\mu) \in A$ there exist sequences $(b_{n_i})_{n\in N}$ in $L^2(0,T)$ such that $b_{n_i} \to b_i$ weakly in $L^2(0,T)$ and $b_{n_i}^2 \to \mu_i$ weakly* in $\mathcal{M}([0,T])$ for $i = 1,\ldots,M$. Hence, in view of Theorem 2.7, the sequence of problems $((\mathcal{P})_{(b_n)})_{n\in N}$ Γ-converges to $\Phi^{-1}((\mathcal{P})_{(b,\mu)})$. ◻

Remark 2.12. By the above arguments it is clear that we could replace hypothesis (HC) with the following more general assumption (GHC), which, on one hand, does not assume that the functions k_i and h_i are constant and, on the other hand, involves these functions in the zero-Lie bracket condition.

Generalized commutativity condition (GHC). For every $\alpha,\beta = 1,\ldots,2M$

$$[\gamma_\alpha,\gamma_\beta] = 0,$$

where the vector fields γ_δ are defined on $\mathbb{R}^{N+2}$ by

$$\gamma_\delta = \begin{pmatrix} g_i^1 \\ \cdot \\ \cdot \\ g_i^N \\ k_\delta \\ 0 \end{pmatrix}$$

when $\delta = 1,\ldots,M$, and

$$\gamma_\delta = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ 0 \\ h_\delta \end{pmatrix}$$

when $\delta = M+1,\ldots,2M$.

3. Nonvanishing Lie brackets: An example. In the previous section it has been shown that whenever the vector fields commute the Γ-limit of problems $(\mathcal{P})_{(b_n)}$ for (b_n, b_n^2) converging to (b,μ) *does exist*. However, this is no longer true whenever some Lie bracket is not vanishing, as shown in the example below. In the next sections we will provide a theoretical framework from which it will be clear that, in general, there exist infinitely many limit problems corresponding to the pair (b, μ).

In order to get rid of the suspicion that having a state's dimension larger than the control's dimension might matter with the convergence question, the state in this example is one-dimensional.

Let $N = 1$, $M = 2$, and consider the state equation

$$\begin{cases} \dot{x}(t) = b_{n_1}(t)u_1(t) + a(x(t))b_{n_2}(t)u_2(t), \\ x(0) = 0, \end{cases}$$

where $a(x)$ is a bounded C^2 function coinciding with the identity map in the interval $[-4, 4]$. Hence $g_1(x)$ coincides with the constant 1, and $g_2(x) = a(x)$. Let us assume (Hb), (Hu), $T = 1$, and let us consider the cost functional

$$J_{b_n}(x, u) = \int_0^1 \left(|u(t)|^2 + b_{n_1}(t)u_1(t) + a(x(t))b_{n_2}(t)u_2(t)\right) dt \left(= \int_0^1 |u(t)|^2 \, dt + x(1) \right).$$

If we set $h_1(x) = 1$, $h_2(x) = 1$, $k_1(t, x) = 1$, $k_2(t, x) = a(x)$, and $l(t, x) = 0$, the hypotheses in section 2 turns out to be satisfied.

Since $[g_1, g_2](0) = -1$, neither the commutativity condition (HC) nor its generalization (GHC) are fulfilled.

Let us consider the two sequences of coefficients

$$(b_{n_1}(t), b_{n_2}(t)) \doteq \left(\sqrt{2n}, 0\right) \mathrm{I}_{\left[1-\frac{1}{n}, 1-\frac{1}{2n}\right]}(t) + \left(0, \sqrt{2n}\right) \mathrm{I}_{\left[1-\frac{1}{2n}, 1\right]}(t),$$

$$\left(\tilde{b}_{n_1}(t), \tilde{b}_{n_2}(t)\right) \doteq \left(0, \sqrt{2n}\right) \mathrm{I}_{\left[1-\frac{1}{n}, 1-\frac{1}{2n}\right]}(t) + \left(\sqrt{2n}, 0\right) \mathrm{I}_{\left[1-\frac{1}{2n}, 1\right]}(t),$$

where $\mathrm{I}_{[a,b]} = 1$ if $t \in [a, b]$ and $\mathrm{I}_{[a,b]} = 0$ if $t \notin [a, b]$. Let us observe that

$$(b_{n_1}(t), b_{n_2}(t)) \to (0, 0) \qquad \text{weakly in } L^2(0, T),$$

$$\left(\tilde{b}_{n_1}(t), \tilde{b}_{n_2}(t)\right) \to (0, 0) \qquad \text{weakly in } L^2(0, T),$$

$$\left(b^2_{n_1}(t), b^2_{n_2}(t)\right) \to (\delta_1, \delta_1) \qquad \text{weakly}^* \text{ in } \mathcal{M}([0, T]),$$

$$\left(\tilde{b}^2_{n_1}(t), \tilde{b}^2_{n_2}(t)\right) \to (\delta_1, \delta_1) \qquad \text{weakly}^* \text{ in } \mathcal{M}([0, T]),$$

where δ_1 denotes the Dirac measure at $T = 1$. Hence the two sequences fulfill the convergence assumption (1.1) with the *same* limit $(b_1, b_2) = (0, 0)$ and $(\mu_1, \mu_2) = (\delta_1, \delta_1)$. Yet the corresponding sequences $((\mathcal{P})_{(b_n)})_{n \in N}$ and $((\mathcal{P})_{(\tilde{b}_n)})_{n \in N}$ *cannot* converge to the same Γ-limit. Indeed, if we implement the control

$$(u_1^n(t), u_2^n(t)) = \left(-\sqrt{\frac{n}{2}}\, \mathrm{I}_{\left[1-\frac{1}{n}, 1-\frac{1}{2n}\right]}(t), 0\right) + \left(0, \sqrt{n}\, \mathrm{I}_{\left[1-\frac{1}{2n}, 1\right]}(t)\right)$$

in the system driven by the (b_n), we obtain a trajectory x_n verifying

$$x_n(1) = -\frac{1}{2} \exp(2^{-1/2}).$$

Thus

$$J_{b_n}(x_n, u_n) = K \doteq -\frac{1}{2}\exp(2^{-1/2}) + \frac{3}{4}\left(< -\frac{1}{4}\right).$$

On the contrary, if we consider $(\mathcal{P})_{(\tilde{b}_n)}$, a simple application of the Pontryagin maximum principle shows that

$$(\hat{u}_1^n(t), \hat{u}_2^n(t)) = \left(-\frac{1}{2}\,\tilde{b}_{n_1}, 0\right)$$

is an optimal control. The corresponding optimal trajectory $\hat{x}_n$ solves

$$\dot{\hat{x}}_n(t) = -\frac{1}{2}\,\tilde{b}^2_{n_1}, \qquad \hat{x}_n(0) = 0.$$

Hence

$$-\frac{1}{4} = J_{\tilde{b}_n}(\hat{x}_n, \hat{u}_n) = \min_u\{J_{\tilde{b}_n}(x, u)\}.$$

In particular, one has

$$\liminf_{n\to\infty}\left(\inf_u\{J_{b_n}(x, u)\}\right) \le K < -\frac{1}{4} = \liminf_{n\to\infty}\left(\min_u\{J_{\tilde{b}_n}(x, u)\}\right).$$

Hence, although the (b_{n_1}, b_{n_2}) and $(\tilde{b}_{n_1}, \tilde{b}_{n_2})$ converge to the same (b, μ) in the sense of (1.1), in view of Theorem A.2 in the appendix the Γ-limit of the $(\mathcal{P})_{(b_n)}$ and $(\mathcal{P})_{(\tilde{b}_n)}$ are necessarily different.

4. Nonvanishing Lie brackets: An extended setting. In this section we still assume hypotheses (Hg0), (Hg1), (Hl), (Hk), and (Hh), but we *do not* assume the commutativity hypothesis (HC) made in section 2. The previous example shows that in order to determine the limit problem *it is not enough* to assume that $b_n \to b$ weakly in $L^2(0, T; \mathbb{R}^M)$ and $b_n^2 \to \mu$ weakly* in $\mathcal{M}([0, T]; \mathbb{R}^M)$. In fact, due to the noncommutativity of the vector fields g_i $(i = 1, \ldots, M)$, some extra information—related to the choice of the particular sequence (b_n, b_n^2) approximating (b, μ)—is needed. It turns out that this extra information can be represented neatly by first embedding the problem in the (t, x)-space and then reparameterizing time with a nondecreasing map whose derivative is zero for those values of t where μ is concentrated. In particular, this embedding allows one to keep track of the particular sequence (b_n, b_n^2) approximating (b, μ).

Let us begin with some definitions.

Definition 4.1. *The set of* data triples *is defined as*

$$\begin{aligned}\mathcal{A} \;\doteq\; \Big\{ &(B_0, B, \gamma) :\ B_0 : [0, 1] \to \mathbb{R}^+ \cup \{0\}\ ,\ B : [0, 1] \to \mathbb{R}^M, \\ &\gamma : [0, 1] \to (\mathbb{R}^+ \cup \{0\})^M \textit{ are Borel functions in } L^\infty(0, T; \mathbb{R}^M)\ : \\ &\gamma_i \ge B_i^2 \ \textit{ for all } i = 1, \ldots, M \ \textit{ and } \int_0^1 B_0^2(s)ds = T \Big\}.\end{aligned}$$

The subset A_{NI} of nonimpulsive data triples *is defined as*

$$\mathcal{A}_{NI} \doteq \{(B_0, B, \gamma) \in \mathcal{A}\ :\ B_0 > 0 \ \textit{a.e. on } [0, 1]\}.$$

The subset of simple data triples *is defined as*

$$\mathcal{A}_s \doteq \left\{(B_0, B, \gamma) \in \mathcal{A} \,:\, \gamma_i = B_i^2 \text{ a.e. on } [0,1]\,,\; i = 1, \dots, M \right\}.$$

We will denote the vector $(B_1^2, \dots, B_M^2)$ by B^2.

For each triple $(B_0, B, \gamma) \in \mathcal{A}$ let us consider the *space-time* optimal control problem

$$(\mathcal{P})_{(B_0,B,\gamma)} \qquad \begin{cases} y'(s) = g_0(s,y)B_0^2(s) + \displaystyle\sum_{i=1}^{M} g_i(y)V_i(s), \quad y(0) = y_0, \\ \displaystyle\min_{U,V} \left\{ \hat{J}(y,U,V) \right\}, \end{cases}$$

$$\begin{aligned} \hat{J}(y,U,V) \;&=\; \int_0^1 \Bigg(l(s,y)B_0^2(s) + \sum_{i=1}^{M} k_i(s,y)V_i(s) + \sum_{i=1}^{M} h_i^2(y)U_i^2(s) \\ &\quad + \sum_{i=1}^{M} h_i^2(y) \frac{(B_i(s)U_i(s) - V_i(s))^2}{(\gamma_i(s) - B_i^2(s))} \Bigg) ds, \end{aligned}$$

where $U \in L^2(0,1;\mathbb{R}^M)$ and $V \in L^2(0,1;\mathbb{R}^M)$.

Remark 4.2. When $(B_0, B, \gamma) \in \mathcal{A}_s$, that is, $\gamma_i = B_i^2$, $i = 1, \dots, M$, the optimal control problem $(\mathcal{P})_{(B_0,B,\gamma)}$ reduces to the following standard form:

$$(\mathcal{P})_{(B_0,B,B^2)} \qquad \begin{cases} y'(s) = g_0(s,y)B_0^2(s) + \displaystyle\sum_{i=1}^{M} g_i(y)B_i(s)U_i(s), \quad y(0) = y_0, \\ \displaystyle\min_{U} \left\{ \hat{J}(y,U) \right\}, \end{cases}$$

$$\hat{J}(y,U) = \int_0^1 \left(l(s,y)B_0^2(s) + \sum_{i=1}^{M} k_i(s,y)B_i(s)U_i(s) + \sum_{i=1}^{M} h_i^2(y)U_i^2(s) \right) ds.$$

We shall show that the class of problems $(\mathcal{P})_{(b)}$—where $(\mathcal{P})_{(b)}$ stands for a problem like $(\mathcal{P})_{(b_n)}$ when b_n is replaced by b—can be put into one-to-one correspondence with the class of space-time problems $\{(\mathcal{P})_{(B_0,B,B^2)} \;:\; (B_0,B,B^2) \in \mathcal{A}_s \cap \mathcal{A}_{NI}\}$. Then we shall give sufficient conditions for the Γ-convergence of a sequence of problems $(\mathcal{P})_{(B_{0_n},B_n,B_n^2)}$ to a problem $(\mathcal{P})_{(B_0,B,\gamma)}$. Last, we shall see that *every* such problem is the Γ-limit of a suitable sequence of problems $(\mathcal{P})_{(B_{0_n},B_n,B_n^2)}$.

Definition 4.3. *Given $(B_0, B, \gamma) \in \mathcal{A}$, let us define $\alpha(B_0, B, \gamma) \doteq (b, \mu)$ by setting the following.*

(i) $$t(s) \doteq \int_0^s B_0^2(u)du,$$

and, whenever there exists $\delta > 0$ such that $B_0 > 0$ a.e. on $[s-\delta, s+\delta] \cap [0,1]$,

$$b_i(t(s)) \doteq \frac{B_i(s)}{B_0(s)} \qquad (i = 1, \dots, M).$$

(ii) *For each Borel subset* $E \subseteq [0,T]$

$$\mu_i(E) \doteq \int_I \gamma_i(s)ds \qquad (i = 1,\dots,M)$$

when $E = t(I)$.

Remark 4.4. (a) The function $b(\cdot)$ is well defined. Indeed, the set of values of t such that t^{-1} is not a singleton is at most countable.

(b) For each $(B_0, B, \gamma) \in \mathcal{A}$ the pair $(b,\mu) = \alpha(B_0, B, \gamma)$ is in A; in particular, if $(B_0, B, B^2) \in \mathcal{A}_{NI} \cap \mathcal{A}_s$, then $\alpha(B_0, B, B^2) = (b, b^2) \in A_s$.

(c) The definition of μ is equivalent to

$$\int_{[0,T]} \phi(t) d\mu = \int_0^1 \langle \gamma_i(s), \phi(t(s)) \rangle ds \quad \text{for all } \phi \in C([0,T];\mathbb{R}^M), \tag{4.1}$$

where $\langle \cdot,\cdot \rangle$ denotes the scalar product in $\mathbb{R}^M$.

(d) The map α is not injective, unless it is restricted to $\mathcal{A}_{NI} \cap \mathcal{A}_s$. Let us show that it is surjective. Indeed, for every (b,μ) let us set

$$s(t) = \begin{cases} 0 & t = 0, \\ \dfrac{t + \int_{]0,t]} d\mu}{T + \int_{]0,T]} d\mu} & 0 < t < T, \\ 1 & t = T, \end{cases}$$

and let us define $t(s)$ as the unique nondecreasing continuous map such that $t \circ s(\tau) = \tau$ for all $\tau \in [0,T]$. Correspondingly, let us set $B_0^2(s) = t'(s)$, $B_i(s) = b_i(t(s))B_0(s)$, $s \in [0,1]$, $i = 1,\dots,M$. Finally, let us choose $\gamma_i(s)$ such that

$$\int_{s_1}^{s_2} \gamma_i(s) ds = \int_{t(s_1,s_2)} d\mu_i, \quad i = 1,\dots,M,$$

for each subinterval (s_1, s_2) of $[0,1]$. Then $\alpha(B_0, B, \gamma) = (b,\mu)$. We will call this data triple *the canonical preimage* of (b,μ). Let us notice that for every (b,μ), $\alpha^{-1}(b,\mu)$ turns out to be the class of data triples in $\mathcal{A}$ such that

$$(B_0, B, \gamma), (\tilde{B}_0, \tilde{B}, \tilde{\gamma}) \in \alpha^{-1}(b,\mu) \quad \Leftrightarrow \quad \begin{cases} B_0 = \tilde{B}_0 \text{ a.e. }, \\ B = \tilde{B} \text{ a.e. }, \\ \gamma_i(s) = \tilde{\gamma}_i(s) \text{ for a.e. } s \in [0,1] \setminus \cup I_j \\ \text{and} \displaystyle\int_{I_j} \gamma_i(s) ds = \int_{I_j} \tilde{\gamma}_i(s) ds \quad \text{for all } j, \end{cases}$$

where $\{I_j\}$ is the (countable) family of (disjoint) subintervals of $[0,1]$ such that $B_0 = \tilde{B}_0 = 0$ on each I_j.

In the following two theorems we establish a one-to-one correspondence between the class of problems $(\mathcal{P})_{(b)}$, $b \in L^2(0,T;\mathbb{R}^M)$, and the class of problems $(\mathcal{P})_{(B_0,B,\gamma)}$, $(B_0, B, \gamma) \in \mathcal{A}_{NI} \cap \mathcal{A}_s$ (i.e., $B^2 = \gamma$ and $B_0 > 0$). Before stating these results let us notice that α is one-to-one from $\mathcal{A}_{NI} \cap \mathcal{A}_s$ onto A_s.

THEOREM 4.5. *Let b and u satisfy* (Hb) *and* (Hu), *and let $x(\cdot)$ be the corresponding solution of the state equation of* $(\mathcal{P})_{(b)}$

$$\begin{cases} \dot{x} = g_0(t,x) + \displaystyle\sum_{i=1}^M g_i(x) b_i(t) u_i(t), \\ x(0) = x_0. \end{cases} \tag{4.2}$$

Let $(B_0, B, B^2) = \alpha^{-1}(b, b^2)$*, and set* $U(s) \doteq [u \circ t(s)]B_0(s)$*. Let* y *be the solution of the state equation of* $(\mathcal{P})_{(B_0,B,B^2)}$

$$\begin{cases} y'(s) = g_0(s, y)B_0^2(s) + \displaystyle\sum_{i=1}^{M} g_i(y)B_i(s)U_i(s), \\ y(0) = y_0. \end{cases} \tag{4.3}$$

Then

$$y(s) = x(t(s)) \quad \text{for all } s \in [0, 1].$$

Conversely, let $(B_0, B, B^2) \in \mathcal{A}_s \cap \mathcal{A}_{NI}$*,* $U \in L^2(0, 1; \mathbb{R}^M)$*, and let* $y(\cdot)$ *be the corresponding solution of* (4.3)*. Setting* $(b, b^2) = \alpha(B_0, B, B^2)$*, let us define*

$$u_i(t) \doteq \frac{U_i(s(t))}{B_0(s(t))}, \quad i = 1, \dots, M.$$

If $x(\cdot)$ *is the solution of* (4.2) *corresponding to these* b_i *and* u_i*, then*

$$x(t) = y(s(t)) \quad \text{for all } t \in [0, T].$$

Proof. The proof of this theorem relies essentially on the uniqueness properties of (4.2) and (4.3). For this reason we omit it. □

An analogous result holds for the payoffs J and $\hat{J}$.

THEOREM 4.6. *Consider* b, u, x, B_0, B, U*, and* y *as in the first part of Theorem* 4.5*, and set*

$$J(x, u) = \int_0^T \left(l(t, x) + \sum_{i=1}^{M} k_i(t, x)b_i(t)u_i(t) + \sum_{i=1}^{M} h_i^2(x)u_i^2(t) \right) dt,$$

$$\hat{J}(y, U) = \int_0^1 \left(l(s, y)B_0(s) + \sum_{i=1}^{M} k_i(s, y)B_i(s)U_i(s) + \sum_{i=1}^{M} h_i^2(y)U_i^2(s) \right) ds.$$

Then $\hat{J}(y, U) = J(x, u)$*. Conversely, if* B_0, B, U, y *and* b, u, x *are as in the second part of Theorem* 4.5*, then* $J(x, u) = \hat{J}(y, U)$*.*

Proof. In view of Theorem 4.5 the proof of this theorem is straightforward. □

When the problems $(\mathcal{P})_{(b)}$ and $(\mathcal{P})_{(B_0,B,B^2)}$ are related as in the previous result, we say that they are *isomorphic*. In view of Theorems 4.5 and 4.6 the map $(\mathcal{P})_{(b)} \mapsto (\mathcal{P})_{(B_0,B,B^2)}$ with $\alpha(B_0, B, B^2) = (b, b^2)$ establishes a *one-to-one correspondence between the class of problems* $\{(\mathcal{P})_{(b)}\ ,\ b \in L^2(0, T; \mathbb{R}^M)\}$ *and the subset* $\{(\mathcal{P})_{(B_0,B,\gamma)}\ ,\ (B_0, B, \gamma) \in \mathcal{A}_s \cap \mathcal{A}_{NI}\} \subset \{(\mathcal{P})_{(B_0,B,\gamma)}\ ,\ (B_0, B, \gamma) \in \mathcal{A}\}$. This one-to-one correspondence can be regarded as an embedding of the original class of problems $\{(\mathcal{P})_{(b)}\ :\ b \in L^2(0, T; \mathbb{R}^M)\}$ in the larger class $\{(\mathcal{P})_{(B_0,B,\gamma)}\ ,\ (B_0, B, \gamma) \in \mathcal{A}\}$. In this extended setting we are now able to provide a convergence result, so giving an answer to the question raised with the example in section 3. In other words, we are going to replace the assumptions on the sequence $((\mathcal{P})_{(b_n)})_{n\in N}$ with hypotheses on the sequence of isomorphic problems $(\mathcal{P}_{(B_{0_n},B_n,B_n^2)})_{n\in N}$. And assigning the limit of the triples (B_{0_n}, B_n, B_n^2), we actually provide the extra information whose lack was revealed by the example of section 3.

Here is the main result.

THEOREM 4.7. *Let* $(B_{0_n}, B_n, B_n^2) \in \mathcal{A}_s \cap \mathcal{A}_{NI}$ *and* $(B_0, B, \gamma) \in \mathcal{A}$ *verify*

$$\lim_{n\to\infty} B_{0_n}(\cdot) = B_0(\cdot) \text{ a.e. on } [0,1], \tag{4.4}$$

$$\lim_{n\to\infty} B_n(\cdot) = B(\cdot) \text{ weakly in } L^1(0,1;\mathbb{R}^M), \tag{4.5}$$

$$\lim_{n\to\infty} B_n^2(\cdot) = \gamma(\cdot) \text{ weakly in } L^1(0,1;\mathbb{R}^M). \tag{4.6}$$

Then problems $\mathcal{P}_{(B_{0_n}, B_n, B_n^2)}$ Γ*-converge to problem* $(\mathcal{P})_{(B_0,B,\gamma)}$.

Remark 4.8. In the previous statement one possibly has $\alpha(B_{0_n}, B_n, B_n^2) = (b_n, b_n^2)$ with $b_n \in L^2(0,T;\mathbb{R}^M)$. So, in particular, Theorem 4.7 can be regarded as a convergence result concerning the original problems $(\mathcal{P})_{(b_n)}$.

Proof of Theorem 4.7. Thanks to the performed rescaling of the problem, we can exploit the general results proved by Buttazzo and Cavazzuti in [BC89]. Actually, hypotheses (Hg0), (Hg1), (Hl), (Hk), (Hh), and (4.4)–(4.6) imply (3.6)–(3.10) and (3.12)–(3.15) in [BC89], respectively. Hence Propositions 3.2 and 3.3 of [BC89] state that $(\mathcal{P})_{(B_0,B,\gamma)}$ is the Γ-limit of the $\mathcal{P}_{(B_{0_n}, B_n, B_n^2)}$. ∎

Similarly to what has been done in the case where the Lie brackets vanish, we now prove that *each* problem $(\mathcal{P})_{(B_0,B,\gamma)}$ with $(B_0, B, \gamma) \in \mathcal{A}$ is indeed the Γ-limit of a sequence of problems of the form $\mathcal{P}_{(B_{0_n}, B_n, B_n^2)}$ with $B_{0_n} > 0$ (which, up to the introduced one-to-one correspondence, means that $(\mathcal{P})_{(B_0,B,\gamma)}$ is the Γ-limit of problems $(\mathcal{P})_{(b_n)}$ with $(b_n, b_n^2) = \alpha(B_{0_n}, B_n, B_n^2)$).

THEOREM 4.9. *For every* $(B_0, B, \gamma) \in \mathcal{A}$*, the problem* $(\mathcal{P})_{(B_0,B,\gamma)}$ *is the* Γ*-limit of a suitable sequence* $(\mathcal{P}_{(B_{0_n}, B_n, B_n^2)})_{n\in N}$ *with* $(B_{0_n}, B_n, B_n^2) \in \mathcal{A}_s \cap \mathcal{A}_{NI}$.

Proof. By Lemma 2.11 for each $(B_0, B, \gamma) \in A$ there is a sequence $((B_0, B_n, B_n^2))_{n\in N}$ $(\in \mathcal{A}_s)$ such that $B_{n_i} \to B_i$ weakly in $L^2(0,1)$ and $B_{n_i}^2 \to \gamma_i$ weakly in $L^1(0,1)$ for $i = 1, \dots, M$.

Moreover, by setting

$$B_{0_n}(s) = \sqrt{\frac{T}{T + \frac{1}{n}}\left(B_0^2(s) + \frac{1}{n}\right)},$$

we find that the triples (B_{0_n}, B_n, B_n^2) (belong to $\mathcal{A}_s \cap \mathcal{A}_{NI}$ and) verify (4.4)–(4.6). Hence one concludes by Theorem 4.7. ∎

5. Revisiting sections 2 and 3 in the light of the extended setting. On one hand, sections 2 and 3 reveal a crucial discrepancy between the case when all the brackets $[g_i, g_j]$ vanish identically and the general case. On the other hand, in section 4 we have introduced an extended setting in order to state a convergence result in the general case. In this section we are going to revisit both the positive result of section 2 and the counterexample of section 3 in light of the theory developed in section 4. Let us recall that the map $\alpha : \mathcal{A} \to A$ induces a one-to-one correspondence between the subset $\mathcal{A}_{NI} \cap \mathcal{A}_s \subset \mathcal{A}$ and $A_s \subset A$.

Null Lie brackets. In Theorem 5.1 below we show—under the commutativity hypothesis (HC) in section 2—that when problems $(\mathcal{P})_{(B_{0_n}, B_n, B_n^2)}$ Γ-converge to a problem $(\mathcal{P})_{(B_0,B,\gamma)}$, then the space-projected problems $(\mathcal{P})_{(b_n)}$ with $(b_n, b_n^2) = \alpha(B_{0_n}, B_n, B_n^2)$ Γ-converge to the projected limit $\Phi^{-1}((\mathcal{Q})_{(b,\mu)})$, where $(b, \mu) = \alpha(B_0, B, \gamma)$.

The most relevant point of this theorem consists in the fact that two sequences $(B_{0_n}, B_n, B_n^2), (\tilde{B}_{0_n}, \tilde{B}_n, \tilde{B}_n^2)$ converging to two *different* triples $(B_0, B, \gamma), (\tilde{B}_0, \tilde{B}, \tilde{\gamma})$ such that $(b, \mu) = \alpha(B_0, B, \gamma) = \alpha(\tilde{B}_0, \tilde{B}, \tilde{\gamma})$ give rise to problems $(\mathcal{P})_{(b_n)}$ and $(\mathcal{P})_{(\tilde{b}_n)}$ Γ-converging to the *same* limit problem $\Phi^{-1}((\mathcal{Q})_{(b,\mu)})$. In particular, this explains why as soon as all the brackets vanish there is in fact no need of the extended setting.

THEOREM 5.1. *Let us assume the hypotheses of section* 2 *(in particular, the commutativity hypothesis* (HC)*). Given* $(B_0, B, \gamma) \in \mathcal{A}$*, let us consider* any *sequence* $(B_{0_n}, B_n, B_n^2) \in \mathcal{A}_{NI} \cap \mathcal{A}_s$ *such that the* B_{0_n} *are equibounded and*

$$\lim_{n\to\infty} B_{0_n}(\cdot) = B_0(\cdot) \text{ a.e. on } [0,1], \tag{5.1}$$

$$\lim_{n\to\infty} B_n(\cdot) = B(\cdot) \text{ weakly in } L^1(0,1;\mathbb{R}^M), \tag{5.2}$$

$$\lim_{n\to\infty} B_n^2(\cdot) = \gamma(\cdot) \text{ weakly in } L^1(0,1;\mathbb{R}^M) \tag{5.3}$$

(so that, by Theorem 4.7*,* $(\mathcal{P})_{(B_{0_n}, B_n, B_n^2)}$ Γ*-converges to* $(\mathcal{P})_{(B_0,B,\gamma)}$*). Then, setting* $(b_n, b_n^2) = \alpha(B_{0_n}, B_n, B_n^2)$ *and* $(b, \mu) = \alpha(B_0, B, \gamma)$*, one has that the problems* $(\mathcal{P})_{(b_n)}$ Γ*-converge to* $\Phi^{-1}((\mathcal{Q})_{(b,\mu)})$.

Proof. In view of Theorem 2.7, we have to prove only that

$$b_n \to b \text{ weakly in } L^2(0,T;\mathbb{R}^M), \tag{5.4}$$

$$b_n^2 \to \mu \text{ weakly}^* \text{ in } \mathcal{M}([0,T];\mathbb{R}^M). \tag{5.5}$$

We begin by observing that hypotheses (5.1)–(5.3) imply

$$\lim_{n\to\infty} B_{0_n}^2(\cdot) = B_0^2(\cdot) \text{ a.e. on } (0,1),$$

$$\lim_{n\to\infty} B_n(\cdot) B_{0_n}(\cdot) = B(\cdot) B_0(\cdot) \text{ weakly in } L^1(0,1;\mathbb{R}^M),$$

and

$$\lim_{n\to\infty} B_n^2(\cdot) = \gamma(\cdot) \text{ weakly in } L^1(0,T;\mathbb{R}^M).$$

Set

$$t_n(s) = \int_0^s B_{0_n}^2(u)du \quad \text{and} \quad t(s) = \int_0^s B_0^2(u)du.$$

Since the t_n tend to t pointwise and each t_n is increasing, the t_n tend to t uniformly on [0,1]. Hence, for each $\varphi \in C([0,T];\mathbb{R}^M)$,

$$\begin{aligned}\lim_{n\to\infty} &\left| \int_0^T \langle b_n^2(t), \varphi(t)\rangle dt - \int_{[0,T]} \varphi(t) d\mu \right| \\ &= \lim_{n\to\infty} \left| \int_0^1 \langle B_n^2(s), \varphi(t_n(s))\rangle ds - \int_0^1 \langle \gamma(s), \varphi(t(s))\rangle ds \right| = 0,\end{aligned}$$

which proves (5.5).

In order to prove (5.4), let us observe that for any function $\varphi \in C_c^\infty(0,T;\mathbb{R}^M)$

$$\lim_{n\to\infty}\left|\int_0^T \langle b_n(t),\varphi(t)\rangle dt - \int_0^T \langle b(t),\varphi(t)\rangle dt\right|$$
$$= \lim_{n\to\infty}\left|\int_0^1 \langle B_n(s)B_{0_n}(s),\varphi(s)\rangle ds - \int_0^1 \langle B(s)B_0(s),\varphi(s)\rangle ds\right| = 0.$$

Since the B_n are uniformly bounded in $L^2(0,T;\mathbb{R}^M)$, (5.4) follows by the density of $C_c^\infty(0,T;\mathbb{R}^M)$ in $L^2(0,T;\mathbb{R}^M)$. ◻

A compactness result. We now examine the converse situation, where a sequence $(b_n)_{n\in N}$ is given such that $((b_n,b_n^2))_{n\in N}$ converges—with some regularity—to $(b,\mu)\in A$. We do not assume here that the Lie brackets vanish. It turns out that a subsequence of the corresponding triples (B_{0_n},B_n,B_n^2) converges to an element $(B_0,B,\gamma)\in\alpha^{-1}(b,\mu)$.

THEOREM 5.2. *Assume the hypotheses of section* 2, *with the exclusion of the commutativity hypothesis* (HC). *Let $\mathcal{T}=\{t_i, i\in N\}$, a (countable) subset of $[0,T]$, and let $(b,\mu)\in A$ and $(b_n)_{n\in N}$ be given such that*

(i) *$\mu = b^2+\mu^\tau$ with μ^τ a (positive) measure concentrated in $\mathcal{T}$;*

(ii) *for each $n\in N$, $b_n\in C([0,T]\setminus\mathcal{T};\mathbb{R}^M)$, and*

$$b_n(\cdot)\to b(\cdot) \text{ uniformly on the compact subsets of } [0,T]\setminus\mathcal{T}, \tag{5.6}$$

$$b_n \to b \text{ weakly in } L^2(0,T;\mathbb{R}^M),$$

$$b_n^2\to\mu \text{ weakly}^* \text{ in } \mathcal{M}([0,T];\mathbb{R}^M). \tag{5.7}$$

(So, if the commutative hypothesis (HC) *is in force, $(\mathcal{P})_{(b_n)}\Gamma$-converges to $(\mathcal{P})_{(b,\mu)}$.) Then, setting $(B_{0_n},B_n,B_n^2)\doteq\alpha^{-1}(b_n,b_n^2)$, there exists a subsequence $(\check B_{0_n},\check B_n,\check B_n^2)$ and a data triple $(B_0,B,\gamma)\in\alpha^{-1}(b,\mu)$, such that the problems $(\mathcal{P})_{(\check B_{0_n},\check B_n,\check B_n^2)}$ Γ-converge to $(\mathcal{P})_{(B_0,B,\gamma)}$.*

The proof of this theorem relies essentially on the following lemma.

LEMMA 5.3. *Assume the hypotheses of Theorem* 5.2. *Let us set*

$$s(t)\doteq\begin{cases} 0, & t=0,\\ \dfrac{t+\int_{]0,t]}d\mu}{T+\int_{]0,T]}d\mu}, & 0<t<T,\\ 1, & t=T,\end{cases}$$

$$s_n(t)\doteq\frac{\int_0^t(1+|b_n|^2(s))ds}{\int_0^T(1+|b_n|^2(s))ds},$$

and let us define $t_n(s)$ and $t(s)$ as the inverse of $s_n(t)$ and the unique nondecreasing continuous map such that $t\circ s(\tau)=id_{[0,T]}$, respectively. Then $t_n(\cdot)$ and $t(\cdot)$ are equi-Lipschitz continuous and

$$\lim_{n\to\infty}t_n(s)=t(s) \tag{5.8}$$

uniformly on $[0,1]$. *Moreover, setting* $B_0(s) \doteq \sqrt{t'(s)}$, *one has*

$$B_0(s) = 0 \quad \textit{for all } s \in \operatorname{int}(t^{-1}(\mathcal{T})) \tag{5.9}$$

and

$$B_{0_n}(s) \to B_0 \ \textit{a.e. on } [0,1]. \tag{5.10}$$

Proof. Since $t^{-1}(\mathcal{T}) = \cup_{t_i \in \mathcal{T}} t^{-1}(t_i)$, we immediately obtain (5.9). Let us observe that hypothesis (5.7) implies that

$$\lim_{n\to\infty} s_n(t) = s(t) \quad \text{a.e. on } [0,T]$$

(see, e.g., Proposition 7.19 in [Fo84]).

Actually, by the continuity of $s(t)$ on $[0,T] \setminus \mathcal{T}$, one has

$$\lim_{n\to\infty} s_n(t) = s(t) \quad \text{ for all } t \in [0,T] \setminus \mathcal{T}.$$

Moreover, by the monotonicity of the $s_n(\cdot)$ and of $s(\cdot)$, it follows that

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} (s_n(t+\varepsilon) - s_n(t-\varepsilon)) = s(t^+) - s(t^-),$$

which yields

$$\lim_{n\to\infty} t_n(s) = t(s) \quad \text{ uniformly on } [0,1]. \tag{5.11}$$

Since $B_{0_n}^2 = t_n' \geq 0$ for every n, there exists a subsequence, still denoted with $(B_{0_n})_{n\in N}$, such that

$$B_{0_n}^2 = t_n' \to 0 \quad \text{a.e. on } \operatorname{int}(t^{-1}(\mathcal{T})).$$

The convergence of B_{0_n} to B_0 on the set $\operatorname{int}(t^{-1}(\mathcal{T}))$ is proved. In order to conclude, let us prove this convergence for every $s \in [0,1] \setminus t^{-1}(\mathcal{T})$. Indeed, by (5.6) and (5.11) one has

$$\lim_{n\to\infty} B_{0_n}^2(s) = \lim_{n\to\infty} t_n'(s) = \lim_{n\to\infty} \frac{1}{\dot{s}(t_n(s))} = \lim_{n\to\infty} \frac{\int_0^T (1 + b_n(u)^2) du}{1 + b_n^2(t_n(s))}$$

$$= \frac{T + \int_{[0,T]} d\mu}{1 + b^2(t(s))} = B_0^2(s).$$

The lemma is proved. □

Proof of Theorem 5.2. By Theorem 4.7 in section 4 we have to prove only that the there exists a subsequence $(\check{B}_{0_n}, \check{B}_n, \check{B}_n^2)$ of (B_{0_n}, B_n, B_n^2) and a data triple $(B_0, B, \gamma) \in \alpha^{-1}(b, \mu)$ such that

$$\lim_{n\to\infty} \check{B}_{0_n}(\cdot) = B_0(\cdot) \text{ a.e. on } [0,1], \tag{5.12}$$

$$\lim_{n\to\infty} \check{B}_n(\cdot) = B(\cdot) \text{ weakly in } L^1(0,1;\mathbb{R}^M), \tag{5.13}$$

$$\lim_{n\to\infty} \check{B}_n^2(\cdot) = \gamma(\cdot) \text{ weakly in } L^1(0,1;\mathbb{R}^M). \tag{5.14}$$

Let us define B_0 as in Lemma 5.3, which yields (5.12).

Moreover, since the B_n are equibounded, there is a subsequence $(\tilde{B}_n)_{n\in N}$ of $(B_n)_{n\in N}$ converging to a map B weakly in $L^1(0,1;\mathbb{R}^M)$.

By Ascoli–Arzela's theorem there exists a subsequence $(\phi_n)_{n\in N}$ of

$$\tilde{\phi}_n(s) \doteq \int_0^s \tilde{B}_n(\sigma)^2 d\sigma$$

converging to a Lipschitz continuous map ϕ. Then the subsequence $\check{B}_n^2(s) \doteq \frac{d\phi_n(s)}{ds}$ converges weakly* in $L^\infty(0,T;\mathbb{R}^M)$ to $\gamma(s) \doteq \frac{d\phi(s)}{ds}$, which implies (5.14).

We claim that $(B_0, B, \gamma) \in \alpha^{-1}(b,\mu)$. Indeed, $\check{B}_{0_n}(s)$ tends to $B_0(s)$ for every s such that $B_0 > 0$ a.e. on $[s-\delta, s+\delta]$ for a sufficiently small δ. Moreover, thanks to (5.11) and hypothesis (5.6), $b_n(t_n(s))$ converges to $b(t(s))$ for every such point s. Since $\check{B}_n(s)$ tends a.e. to $B(s)$, (i) in the definition of the mapping α (Definition 4.3) turns out to be satisfied. Finally, in view of (5.14) and (4.1), (ii) in the definition of α holds true as well. ∎

Revisiting the example of section 3. Let us conclude by framing the example of section 3 in the extended setting. This will clarify that the distinct limiting behavior of problems $(\mathcal{P})_{(b_n)}$ and problems $(\mathcal{P})_{(\tilde{b}_n)}$ arises from the fact that the corresponding triples $(B_{0_n}, B_n, B_n^2), (\tilde{B}_{0_n}, \tilde{B}_n, \tilde{B}_n^2)$ converge to different limits.

Let us recall that the state equation and the cost functional were given by

$$\begin{cases} \dot{x}(t) = b_{n_1}(t)u_1(t) + a(x(t))b_{n_2}(t)u_2(t), \\ x(0) = 0 \end{cases}$$

and

$$J_n(x,u) = \int_0^1 \left(|u(t)|^2 + b_{n_1}(t)u_1(t) + a(x(t))b_{n_2}(t)u_2(t)\right) dt \left(= \int_0^1 |u(t)|^2 dt + x(1)\right),$$

respectively. The problem of minimizing $J_n(x,u)$ over the controls $u \in L^2(0,T)$ was denoted by $(\mathcal{P})_{(b_n)}$ and $(\mathcal{P})_{(\tilde{b}_n)}$ when the parameters were identified with

$$(b_{n_1}(t), b_{n_2}(t)) = \left(\sqrt{2n}, 0\right)\mathrm{I}_{\left[1-\frac{1}{n}, 1-\frac{1}{2n}\right]}(t) + \left(0, \sqrt{2n}\right)\mathrm{I}_{\left[1-\frac{1}{2n}, 1\right]}(t)$$

and

$$\left(\tilde{b}_{n_1}(t), \tilde{b}_{n_2}(t)\right) = \left(0, \sqrt{2n}\right)\mathrm{I}_{\left[1-\frac{1}{n}, 1-\frac{1}{2n}\right]}(t) + \left(\sqrt{2n}, 0\right)\mathrm{I}_{\left[1-\frac{1}{2n}, 1\right]}(t),$$

respectively. Following the construction performed in section 4, let us compute the isomorphic problems $\mathcal{P}_{(B_0, B_n, B_n^2)}$ and $\mathcal{P}_{(\tilde{B}_0, \tilde{B}_n, \tilde{B}_n^2)}$ with $(B_0, B_n, B_n^2) = \alpha^{-1}(b_n, b_n^2)$ and $(\tilde{B}_0, \tilde{B}_n, \tilde{B}_n^2) = \alpha^{-1}(\tilde{b}_n, \tilde{b}_n^2)$. In both cases the optimal control problem turns out to have the form

$$\begin{cases} y'(s) = B_{n_1}(s)U_1(s) + a(y(s))B_{n_2}(s)U_2(s)\ ,\ y(0) = 0, \\ \min\limits_U\{\hat{J}_n(y,U)\}, \end{cases}$$

$$\hat{J}_n(y,U) = \int_0^1 \left(|U(s)|^2 + B_{n_1}(s)U_1(s) + a(y(s))B_{n_2}(s)U_2(s)\right) ds$$

with the parameters B identified with

$$
\begin{aligned}
B_{0_n}(s) &= \sqrt{3}\,\mathrm{I}_{\left[0,\frac{1}{3}\left(1-\frac{1}{n}\right)\right]} + \sqrt{\frac{3}{1+2n}}\,\mathrm{I}_{\left[\frac{1}{3}\left(1-\frac{1}{n}\right),1\right]},\\
B_n(s) &= (B_{n_1}(s), B_{n_2}(s))\\
&= \left(\sqrt{\frac{6n}{1+2n}},0\right)\,\mathrm{I}_{\left[\frac{1}{3}\left(1-\frac{1}{n}\right),\frac{1}{3}\left(2-\frac{1}{2n}\right)\right]} + \left(0,\sqrt{\frac{6n}{1+2n}}\right)\,\mathrm{I}_{\left[\frac{1}{3}\left(2-\frac{1}{2n}\right),1\right]},
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{B}_{0_n}(s) &= \sqrt{3}\,\mathrm{I}_{\left[0,\frac{1}{3}\left(1-\frac{1}{n}\right)\right]} + \sqrt{\frac{3}{1+2n}}\,\mathrm{I}_{\left[\frac{1}{3}\left(1-\frac{1}{n}\right),1\right]},\\
\tilde{B}_n(s) &= (\tilde{B}_{n_1}(s), \tilde{B}_{n_2}(s))\\
&= \left(0,\sqrt{\frac{6n}{1+2n}}\right)\,\mathrm{I}_{\left[\frac{1}{3}\left(1-\frac{1}{n}\right),\frac{1}{3}\left(2-\frac{1}{2n}\right)\right]} + \left(\sqrt{\frac{6n}{1+2n}},0\right)\,\mathrm{I}_{\left[\frac{1}{3}\left(2-\frac{1}{2n}\right),1\right]},
\end{aligned}
$$

respectively. In order to find the Γ-limit of problems $\mathcal{P}_{(B_0,B_n,B_n^2)}$ and $\mathcal{P}_{(\tilde{B}_0,\tilde{B}_n,\tilde{B}_n^2)}$, we need to compute the limits appearing in hypotheses (4.4), (4.5), and (4.6).

For the data triples (B_0, B_n, B_n^2), we obtain

$$
\begin{array}{llll}
\lim_{n\to\infty} B_{0_n}(s) = & B_0(s) \doteq & \sqrt{3}\,\mathrm{I}_{[0,1/3]} & \text{a.e. } [0,1],\\
\lim_{n\to\infty} B_{n_1}(s) = & B_1(s) \doteq & \sqrt{3}\,\mathrm{I}_{[1/3,2/3]} & \text{in } L^1(0,1),\\
\lim_{n\to\infty} B_{n_2}(s) = & B_2(s) \doteq & \sqrt{3}\,\mathrm{I}_{[2/3,1]} & \text{in } L^1(0,1),\\
\lim_{n\to\infty} B_{n_1}^2(s) = & B_1^2(s) = & 3\ \mathrm{I}_{[1/3,2/3]} & \text{in } L^1(0,1),\\
\lim_{n\to\infty} B_{n_2}^2(s) = & B_2^2(s) = & 3\,\mathrm{I}_{[2/3,1]} & \text{in } L^1(0,1),
\end{array}
$$

while, for the data triples $(\tilde{B}_0, \tilde{B}_n, \tilde{B}_n^2)$, we have

$$
\begin{array}{llll}
\lim_{n\to\infty} \tilde{B}_{0_n}(s) = & \tilde{B}_0(s) \doteq & \sqrt{3}\,\mathrm{I}_{[0,1/3]} & \text{a.e. } [0,1],\\
\lim_{n\to\infty} \tilde{B}_{n_1}(s) = & \tilde{B}_1(s) \doteq & \sqrt{3}\,\mathrm{I}_{[2/3,1]} & \text{in } L^1(0,1),\\
\lim_{n\to\infty} \tilde{B}_{n_2}(s) = & \tilde{B}_2(s) \doteq & \sqrt{3}\,\mathrm{I}_{[1/3,2/3]} & \text{in } L^1(0,1),\\
\lim_{n\to\infty} \tilde{B}_{n_1}^2(s) = & \tilde{B}_1^2(s) = & 3\,\mathrm{I}_{[2/3,1]} & \text{in } L^1(0,1),\\
\lim_{n\to\infty} \tilde{B}_{n_2}^2(s) = & \tilde{B}_2^2(s) = & 3\,\mathrm{I}_{[1/3,2/3]} & \text{in } L^1(0,1).
\end{array}
$$

Let us remark that the limits of (B_0, B_n, B_n^2) and $(\tilde{B}_0, \tilde{B}_n, \tilde{B}_n^2)$ are *different*. This explains why problems $(\mathcal{P})_{(b_n)}$ and $(\mathcal{P})_{(\tilde{b}_n)}$ in the example cannot converge to the same limit problem. More precisely, in view of Theorem 4.7, problems $\mathcal{P}_{(B_0,B_n,B_n^2)}$ and $\mathcal{P}_{(\tilde{B}_0,\tilde{B}_n,\tilde{B}_n^2)}$ Γ-converge to the optimal control problems

$$
\begin{cases}
y'(s) = B_1(s)U_1(s) + a(y(s))B_2(s)U_2(s), \qquad y(0) = 0,\\
\min_U \{\hat{J}(y,U)\},
\end{cases}
$$

$$
\hat{J}(y,U) = \int_0^1 \left(|U(s)|^2 + B_1(s)U_1(s) + a(y(s))B_2(s)U_2(s)\right) ds
$$

and

$$\begin{cases} y'(s) = \tilde{B}_1(s)U_1(s) + a(y(s))\tilde{B}_2(s)U_2(s), \qquad y(0) = 0, \\ \min_U \{\hat{J}(y, U)\}, \end{cases}$$

$$\hat{J}(y, U) = \int_0^1 \big(|U(s)|^2 + \tilde{B}_1(s)U_1(s) + a(y(s))\tilde{B}_2(s)U_2(s)\big)ds,$$

respectively.

Appendix. Basic tools from Γ-convergence applied to control theory. Since the work of Wijsman [Wi64], [Wi66], many different concepts of convergence for sequences of functionals and operators have been appearing in the literature. These concepts were especially designed to approach the limit of sequences of variational problems. Each type of variational problem (minimization, maximization, min-max, etc.) has been associated to a particular concept of convergence.

In the case of the *minimization problem,* the first concept of convergence was the so-called *epiconvergence.* The epiconvergence of a sequence of functionals is equivalent to set-convergence of the corresponding epigraphs.

In turn, this concept was placed in the general framework of Γ-convergence theory by De Giorgi. The theory of Γ-convergence aims to deduce the asymptotic behavior of the solutions of a sequence of variational problems from the asymptotic behavior of the corresponding functionals. Typical examples of applications of Γ-convergence are the theories of homogenization, of singular perturbations, and of the limit behavior of elliptic problems with various obstacles. (We refer, e.g., to the books of Attouch [At84], Bensoussan, Lions, and Papanicolau [BLP78], Sanchez-Palencia [SP80], Dal Maso [DM93], and Buttazzo [Bu89].)

In this paper we have studied the Γ-convergence of sequences of optimal control problems. Let us sketch the general framework of this branch of the theory of Γ-convergence. For each $n \in N$ let $C_n \subseteq Y \times \mathcal{U}$ denote the set of *admissible trajectory-control pairs* defined by

$$C_n \doteq \{(y, u) \in Y \times \mathcal{U} \;:\; A_n(y) = B_n(u)\},$$

where A_n and B_n map Y and $\mathcal{U}$, respectively, in a third topological space V. Correspondingly, let us consider the optimal control problems

$$(\mathcal{P}_n) \qquad \min\{J_n(y, u) \;:\; (y, u) \in C_n\},$$

where J_n is a real operator defined on $Y \times \mathcal{U}$.

Setting $F_n(y, u) = J_n(y, u) + \chi_{C_n}(y, u)$ (where χ_E is 1 on E and $+\infty$ on $(Y \times \mathcal{U}) \setminus E$), let us rephrase problems $(\mathcal{P}_n)$ as follows:

$$(\mathcal{P}_n) \qquad \min\{F_n(y, u) \;:\; (y, u) \in Y \times \mathcal{U}\}.$$

We will say that (y_n, u_n) is an *optimal pair* for the problem $(\mathcal{P}_n)$ if

$$F_n(y_n, u_n) = \min_{Y \times \mathcal{U}} F_n.$$

Via Theorem A.2 below, the theory of Γ-convergence provides a notion of the limit problem guaranteeing the following property. *If* (y_n, u_n) *is an optimal pair of*

$(\mathcal{P}_n)$, or simply a minimizing sequence, and if (y_n, u_n) tends to (y, u) in $Y \times \mathcal{U}$, then (y, u) is an optimal pair for the limit problem $(\mathcal{P})$.

Theorem A.2 below provides a notion of Γ-*limit* problem $(\mathcal{P})$ such that this property holds. In order to state this theorem, we recall the definition of the multiple Γ-limit operator (see [BDM82]). We shall denote the "sup" and the "inf" operators by $Z(+)$ and $Z(-)$, respectively.

Definition A.1. *Let X and W be two topological spaces, and let $F_n : X\times W \to \overline{\mathbb{R}}$ be a sequence of functions. For every $x \in X$, $w \in W$, and $\alpha, \beta, \gamma \in \{+, -\}$, let us define the* Γ-limit of the F_n *by setting*

$$\Gamma(N^\alpha, X^\beta, W^\gamma) \lim_{n\to\infty} F_n(w, x) = \underset{(x_n)\in S(x)}{\mathrm{Z}(\beta)} \; \underset{(w_n)\in S(w)}{\mathrm{Z}(\gamma)} \; \underset{k\in N}{\mathrm{Z}(-\alpha)} \, \underset{n\geq k}{\mathrm{Z}(\alpha)} \, F_n(w_n, x_n),$$

where $S(x)$ and $S(w)$ denote the sets of all sequences $x_n \to x$ in X and $w_n \to w$ in W, respectively. When the Γ-limit does not depend on the sign $+$ or $-$, this sign is omitted. For example, if

$$\Gamma(N^+, X^-, W^+) \lim_{n\to\infty} F_n(w, x) = \Gamma(N^+, X^+, W^+) \lim_{n\to\infty} F_n(w, x),$$

their common value will be indicated by $\Gamma(N^+, X, W^+) \lim_{n\to\infty} F_n(w, x)$.

In particular,

$$\Gamma(N, \mathcal{U}^-, Y^-) \lim_{n\to\infty} F_n(y, u) = \inf_{(u_n)\in S(u)} \; \inf_{(y_n)\in S(y)} \lim_{n\to\infty} F_n(y_n, u_n).$$

Theorem A.2. *Let Y and $\mathcal{U}$ be two topological spaces, and let $F_n \colon Y \times \mathcal{U} \to \overline{\mathbb{R}}$ be a sequence of functions. For each $n \in N$, let (y_n, u_n) be a minimum point for F_n or simply a pair such that*

$$\lim_{n\to\infty} F_n(y_n, u_n) = \lim_{n\to\infty} [\inf_{Y\times\mathcal{U}} F_n].$$

Assume that the (y_n, u_n) converge to (y, u) in $Y \times \mathcal{U}$ and there exists

$$F(y, u) \doteq \Gamma(N, \mathcal{U}^-, Y^-) \lim_{n\to\infty} F_n(y_n, u_n). \tag{A.1}$$

Then

(i) *(y, u) is a minimum point for F on $Y \times \mathcal{U}$;*

(ii) *$\lim_{n\to\infty}[\inf_{Y\times\mathcal{U}} F_n] = \min_{Y\times\mathcal{U}} F(y, u)$.*

(For the proof see [BDM82, Proposition 2.1, p. 388].)

Note that if $F_n(y, u) \doteq F(u)$, then the Γ-limit $F(y, u)$ in (A.1) coincides with the so-called relaxed functional $\overline{F}$ (see, e.g., [Bu89]).

The above theorem motivates the following definition of the Γ-limit problem.

Definition A.3. *When* (A.1) *is verified we say that the problem*

$$\min\{F(y, u) \;:\; (y, u) \in Y \times \mathcal{U}\} \tag{$\mathcal{P}$}$$

is the Γ-limit of problems $(\mathcal{P}_n)$.

See, e.g., [BDM82], [BC89], [BF93], [BF95], and [Fr98] for the explicit calculation of the Γ-limits in various interesting situations.

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