Publ. RIMS, Kyoto Univ.
40 (2004), 1093–1111

On Twisted Microdifferential Modules I. Non-existence of Twisted Wave Equations†

By

Andrea D'Agnolo* and Pierre Schapira**

Abstract

Using the notion of subprincipal symbol, we give a necessary condition for the existence of twisted $\mathcal{D}$-modules simple along a smooth involutive submanifold of the cotangent bundle to a complex manifold. As an application, we prove that there are no generalized massless field equations with non-trivial twist on grassmannians, and in particular that the Penrose transform does not extend to the twisted case.

Introduction

Let $\mathbb{T}$ be a 4-dimensional complex vector space, $\mathbb{P}$ the 3-dimensional projective space of lines in $\mathbb{T}$, and $\mathbb{G}$ the 4-dimensional grassmannian of 2-planes in $\mathbb{T}$. According to Penrose, $\mathbb{G}$ is a conformal compactification of the complexified Minkowski space. Denote by $\mathcal{M}_{(h)}$ the $\mathcal{D}_{\mathbb{G}}$-modules associated with the massless field equations of helicity $h \in \frac{1}{2}\mathbb{Z}$. The Penrose correspondence realizes $\mathcal{M}_{(1+m/2)}$ as the transform of the $\mathcal{D}_{\mathbb{P}}$-module associated with the line bundle $\mathcal{O}_{\mathbb{P}}(m)$, for $m \in \mathbb{Z}$. For $\lambda \in \mathbb{C}$, $\mathcal{O}_{\mathbb{P}}(\lambda)$ makes sense in the theory of twisted sheaves. It is then a natural question to ask whether the Penrose correspondence extends to the twisted case. In particular, are there "massless field equations" of complex helicity $h \notin \frac{1}{2}\mathbb{Z}$?

The $\mathcal{D}_{\mathbb{G}}$-modules $\mathcal{M}_{(h)}$ are simple along a smooth involutive submanifold V of the cotangent bundle to $\mathbb{G}$ which is given by the geometry of the integral

Communicated by M. Kashiwara. Received January 19, 2004.
2000 Mathematics Subject Classification(s): 35A27; 32C38; 14A20

†This article is an invited contribution to a special issue of Publications of RIMS commemorating the fortieth anniversary of the founding of the Research Institute for Mathematical Sciences.

*Dipartimento di Matematica, Università di Padova, via G. Belzoni, 7, 35131 Padova, Italy
e-mail: dagnolo@math.unipd.it; http://www.math.unipd.it/˜dagnolo/
**Institut de Mathématiques, Analyse Algébrique, Université Pierre et Marie Curie, 175, rue du Chevaleret, 75013 Paris, France
e-mail: schapira@math.jussieu.fr; http://www.math.jussieu.fr/˜schapira/

transform. In this paper we give a negative answer to the question raised above: for topological reasons, there are no simple $\mathcal{D}_{\mathbb{G}}$-modules along V with non-trivial twist. Indeed, this is a corollary of the following more general result.

Let X be a complex manifold, and V a conic involutive submanifold of its cotangent bundle. Denote by $\mathcal{D}_{\Omega^{1/2}_{V/X}}$ the ring of differential operators on V acting on relative half-forms and by $\mathcal{D}^{bic}_{\Omega^{1/2}_{V/X}}(0)$ its subring of operators homogeneous of degree 0 and commuting with the functions which are locally constant on the bicharacteristic leaves. The ring of microdifferential operators $\mathcal{E}_X$ is endowed with the so-called V-filtration $\{\mathsf{F}^V_k\mathcal{E}_X\}_{k\in\mathbb{Z}}$ and by a result of Kashiwara-Oshima, there is a natural isomorphism of rings $\mathsf{F}^V_0\mathcal{E}_X/\mathsf{F}^V_{-1}\mathcal{E}_X \xrightarrow{\sim} \mathcal{D}^{bic}_{\Omega^{1/2}_{V/X}}(0)$.

Let $\mathfrak{S}$ be a stack of twisted sheaves on X, and consider the category of twisted microdifferential modules $\mathsf{Mod}(\mathcal{E}_X;\mathfrak{S})$. One says that a twisted microdifferential module is simple along V if it can be endowed with a good V-filtration whose associated graded module is locally isomorphic to $\mathcal{O}_V(0)$.

Let Σ be a smooth bicharacteristic leaf of V. Recall that stacks of twisted sheaves on X are classified by $H^2(X;\mathbb{C}^\times_X)$, and denote by $[\mathfrak{S}]$ the class of $\mathfrak{S}$. Our main result (see Theorem 7.1) consists in associating to $[\mathfrak{S}]$ a class in $H^2(\Sigma;\mathbb{C}^\times_\Sigma)$ whose triviality is a necessary condition for the existence of a globally simple module along V in $\mathsf{Mod}(\mathcal{E}_X;\mathfrak{S})$.

Let us briefly describe our construction. Let $\mathcal{M}$ be a globally simple module along V in $\mathsf{Mod}(\mathcal{E}_X;\mathfrak{S})$. By definition, $\mathcal{M}$ has a good V-filtration, and we denote by $\overline{\mathcal{M}}$ its associated graded module.

(i) By Kashiwara-Oshima's result mentioned above, we consider $\overline{\mathcal{M}}$ as an object of $\mathsf{Mod}(\mathcal{D}^{bic}_V(0);\mathfrak{T})$. Here, $\mathfrak{T}$ is a stack of twisted sheaves on V whose class $[\mathfrak{T}]\in H^2(V;\mathbb{C}^\times_V)$ is the product of the pull back of $[\mathfrak{S}]$ by the class of the stack containing the inverse relative half-forms $\Omega^{-1/2}_{V/X}$.

(ii) The restriction $\overline{\mathcal{M}}_\Sigma$ of $\overline{\mathcal{M}}$ to Σ is a line bundle with flat connection in the category of twisted differential modules $\mathsf{Mod}(\mathcal{D}_\Sigma;\mathfrak{U})$, where $\mathfrak{U}$ is a stack of twisted sheaves on Σ whose class $[\mathfrak{U}]\in H^2(\Sigma;\mathbb{C}^\times_\Sigma)$ is the restriction of $[\mathfrak{T}]$.

(iii) By the Riemann-Hilbert correspondence, $\overline{\mathcal{M}}_\Sigma$ is associated with a local system of rank one in $\mathfrak{U}(\Sigma)$. Since there are no local systems of rank one with non-trivial twist, the triviality of $[\mathfrak{U}]$ is a necessary condition for the existence of a globally simple module along V in $\mathsf{Mod}(\mathcal{E}_X;\mathfrak{S})$.

We would like to thank Masaki Kashiwara for extremely useful conversations and helpful insights.

§1. Review of Twisted Sheaves

In this section we briefly review the notion of twisted sheaves. References are made to [7, 8], see also [2].

Let X be a complex analytic manifold, $\mathcal{O}_X$ its structure sheaf, and denote by $\mathbb{C}_X$ the constant sheaf with stalk $\mathbb{C}$. If $\mathcal{A}$ is a sheaf of $\mathbb{C}$-algebras on X, we denote by $\mathsf{Mod}(\mathcal{A})$ the category of sheaves of $\mathcal{A}$-modules on X and by $\mathfrak{Mod}(\mathcal{A})$ the corresponding $\mathbb{C}$-stack, $U \mapsto \mathsf{Mod}(\mathcal{A}|_U)$. We denote by $\mathcal{A}^\times$ the sheaf of invertible sections of $\mathcal{A}$.

The short exact sequence of abelian groups

$$1 \to \mathbb{C}_X^\times \to \mathcal{O}_X^\times \to \mathcal{O}_X^\times/\mathbb{C}_X^\times \to 1$$

induces the exact sequence

$$H^1(X;\mathbb{C}_X^\times) \underset{\alpha}{\to} H^1(X;\mathcal{O}_X^\times) \underset{\beta}{\to} H^1(X;\mathcal{O}_X^\times/\mathbb{C}_X^\times) \underset{\delta}{\to} H^2(X;\mathbb{C}_X^\times). \tag{1.1}$$

Note that the isomorphism $d\log\colon \mathcal{O}_X^\times/\mathbb{C}_X^\times \xrightarrow{\sim} d\mathcal{O}_X$ induces an isomorphism

$$\iota\colon H^1(X;\mathcal{O}_X^\times/\mathbb{C}_X^\times) \xrightarrow{\sim} H^1(X;d\mathcal{O}_X). \tag{1.2}$$

The $\mathbb{C}$-vector space structure of $H^1(X;d\mathcal{O}_X)$ thus gives a meaning to $\lambda \cdot c$ for $c \in H^1(X;\mathcal{O}_X^\times/\mathbb{C}_X^\times)$ and $\lambda \in \mathbb{C}$.

We will consider several characteristic classes with values in these cohomology groups.

- A local system is a $\mathbb{C}_X$-module locally free of finite rank. To a local system L of rank one corresponds a class $[L] \in H^1(X;\mathbb{C}_X^\times)$ which characterizes L up to isomorphisms of $\mathbb{C}_X$-modules.
- A line bundle is an $\mathcal{O}_X$-module locally free of rank one. To a line bundle $\mathcal{L}$ on X corresponds a class $[\mathcal{L}] \in H^1(X;\mathcal{O}_X^\times)$ which characterizes $\mathcal{L}$ up to isomorphisms of $\mathcal{O}_X$-modules.
- A stack of twisted sheaves is a $\mathbb{C}$-stack locally $\mathbb{C}$-equivalent to $\mathfrak{Mod}(\mathbb{C}_X)$. To a stack of twisted sheaves $\mathfrak{S}$ corresponds a class $[\mathfrak{S}] \in H^2(X;\mathbb{C}_X^\times)$ which characterizes $\mathfrak{S}$ up to $\mathbb{C}$-equivalences. Objects of $\mathfrak{S}(X)$ are called twisted sheaves.

Recall that $[\mathfrak{S}]$ has the following description using Cech cohomology. Let $X = \bigcup_i U_i$ be an open covering such that there are $\mathbb{C}$-equivalences $\varphi_i\colon \mathfrak{S}|_{U_i} \to \mathfrak{Mod}(\mathbb{C}_{U_i})$. By Morita theory, the auto-equivalence $\varphi_i \circ \varphi_j^{-1}$ of $\mathfrak{Mod}(\mathbb{C}_{U_{ij}})$ are given by $G \mapsto G \otimes L_{ij}$ for a local system L_{ij} of rank one. By refining the covering we may assume that $L_{ij} \simeq \mathbb{C}_{U_{ij}}$. The isomorphisms $L_{ij} \otimes L_{jk} \simeq L_{ik}$ on U_{ijk} are then multiplication by locally constant functions $c_{ijk} \in \Gamma(U_{ijk}; \mathbb{C}_X^\times)$. The class $[\mathfrak{S}]$ is described by the Cech cocycle $\{c_{ijk}\}$. A twisted sheaf $F \in \mathfrak{S}(X)$ is then described by a family of sheaves $F_i \in \mathsf{Mod}(\mathbb{C}_{U_i})$ and isomorphisms $\theta_{ij} \colon F_j|_{U_{ij}} \to F_i|_{U_{ij}}$ satisfying $\theta_{ij} \circ \theta_{jk} = c_{ijk}\theta_{ik}$.

Let $\mathfrak{S}$ be a stack of twisted sheaves on X and let $\mathcal{A}$ be a sheaf of $\mathbb{C}$-algebras on X. We denote by $\mathfrak{Mod}(\mathcal{A}; \mathfrak{S})$ the stack of left $\mathcal{A}$-modules in $\mathfrak{S}$.

- A twisted line bundle is a pair $(\mathfrak{S}, \mathcal{F})$ of a stack of twisted sheaves $\mathfrak{S}$ and an object $\mathcal{F} \in \mathsf{Mod}(\mathcal{O}_X; \mathfrak{S})$ locally free of rank one over $\mathcal{O}_X$. To a twisted line bundle corresponds a class $[\mathfrak{S}, \mathcal{F}] \in H^1(X; \mathcal{O}_X^\times/\mathbb{C}_X^\times)$ which characterizes it up to the following equivalence relation: two twisted line bundles $(\mathfrak{S}, \mathcal{F})$ and $(\mathfrak{T}, \mathcal{G})$ are equivalent if there exist a $\mathbb{C}$-equivalence $\varphi \colon \mathfrak{S} \to \mathfrak{T}$ and an isomorphism $\varphi(\mathcal{F}) \simeq \mathcal{G}$ in $\mathsf{Mod}(\mathcal{O}_X; \mathfrak{T})$.

Let $(\mathfrak{S}, \mathcal{F})$ be a twisted line bundle and let $X = \bigcup_i U_i$ be an open covering such that there are $\mathbb{C}$-equivalences $\varphi_i \colon \mathfrak{S}|_{U_i} \to \mathfrak{Mod}(\mathbb{C}_{U_i})$, and denote by $\{c_{ijk}\}$ the Cech cocycle of $[\mathfrak{S}]$. These induce equivalences φ_i: $\mathfrak{Mod}(\mathcal{O}_{U_i}; \mathfrak{S}|_{U_i}) \to \mathfrak{Mod}(\mathcal{O}_{U_i})$ and $\mathcal{F}$ is described by a family of line bundles $\mathcal{F}_i \in \mathsf{Mod}(\mathcal{O}_{U_i})$ and isomorphisms θ_{ij}: $\mathcal{F}_j|_{U_{ij}} \to \mathcal{F}_i|_{U_{ij}}$. By refining the covering, we may assume that there are nowhere vanishing sections $s_i \in \Gamma(U_i; \mathcal{F}_i)$, so that $\mathcal{F}_i \simeq \mathcal{O}_{U_j}$. Hence θ_{ij} are multiplications by the sections $f_{ij} = s_i/\theta_{ij}(s_j) \in \Gamma(U_{ij}; \mathcal{O}_X^\times)$, so that $f_{ij}f_{jk} = c_{ijk}f_{ik}$. The class $[\mathfrak{S}, \mathcal{F}]$ in $H^1(X; \mathbb{C}_X^\times \to \mathcal{O}_X^\times)$ is thus described by the Cech hyper-cocycle $\{f_{ij}, c_{ijk}\}$.

The characteristic classes constructed above are related (up to sign) as follows, using the exact sequence (1.1):

1. if L is a local system of rank one, then $\alpha([L]) = [L \otimes \mathcal{O}_X]$,
2. if $\mathcal{L}$ is a line bundle, then $\beta([\mathcal{L}]) = [\mathfrak{Mod}(\mathbb{C}_X), \mathcal{L}]$,
3. if $(\mathfrak{S}, \mathcal{F})$ is a twisted line bundle, then $\delta([\mathfrak{S}, \mathcal{F}]) = [\mathfrak{S}]$.

The next result will play an essential role in the proof of Theorem 7.1. It immediately follows from the Morita theory for stacks.

Proposition 1.1. *A stack of twisted sheaves $\mathfrak{S}$ is globally $\mathbb{C}$-equivalent to $\mathfrak{Mod}(\mathbb{C}_X)$ if and only if there exists an object $F \in \mathfrak{S}(X)$ locally free of rank one over $\mathbb{C}$.*

Example 1. For $\mathcal{L}$ an untwisted line bundle, and $\lambda \in \mathbb{C}$, there is a twisted line bundle $(\mathfrak{S}_{\mathcal{L}^\lambda}, \mathcal{L}^\lambda)$ whose class $[\mathfrak{S}_{\mathcal{L}^\lambda}, \mathcal{L}^\lambda]$ is described as follows. Let

$X = \bigcup_i U_i$ be an open covering such that there are nowhere vanishing sections $s_i \in \Gamma(U_i; \mathcal{L})$, and set $g_{ij} = s_i/s_j$. Choose a determination f_{ij} for the ramified function g_{ij}^λ on U_{ij}. Then $f_{ij}f_{jk}$ and f_{ik} are different determinations of g_{ik}^λ, so that $f_{ij}f_{jk} = c_{ijk}f_{ik}$ for some $c_{ijk} \in \Gamma(U_{ijk}; \mathbb{C}_X^\times)$. Then $[\mathfrak{S}_{\mathcal{L}^\lambda}, \mathcal{L}^\lambda]$ is described by the Cech hyper-cocycle $\{f_{ij}, c_{ijk}\}$. Since $d\log f_{ij} = \lambda d\log g_{ij}$, we have

$$[\mathfrak{S}_{\mathcal{L}^\lambda}, \mathcal{L}^\lambda] = \lambda \cdot \beta([\mathcal{L}]) \quad \text{in } H^1(X; \mathcal{O}_X^\times/\mathbb{C}_X^\times),$$

where the action of λ on $\beta([\mathcal{L}])$ is induced by the isomorphism (1.2). Note that $\mathcal{L}^\lambda$ is unique up to tensoring by a local system of rank one.

Consider two stacks $\mathfrak{S}$ and $\mathfrak{S}'$ of twisted sheaves on X (here, X is simply a topological space, or even a site). There are stacks of twisted sheaves $\mathfrak{S} \circledast \mathfrak{S}'$ and $\mathfrak{S}^{\circledast -1}$ on X such that if $F \in \mathfrak{S}(X)$ and $F' \in \mathfrak{S}'(X)$ are twisted sheaves, then $F \otimes F' \in (\mathfrak{S} \circledast \mathfrak{S}')(X)$ and if F is a local system of rank one, then $F^{-1} = \mathcal{H}om(F, \mathbb{C}_X) \in \mathfrak{S}^{\circledast -1}$. Moreover,

$$[\mathfrak{S} \circledast \mathfrak{S}'] = [\mathfrak{S}] \cdot [\mathfrak{S}']$$
$$[\mathfrak{S}^{\circledast -1}] = ([\mathfrak{S}])^{-1}.$$

If $f\colon Y \to X$ is a morphism of topological spaces (or of sites), there exists a stack of twisted sheaves $f^\circledast \mathfrak{S}$ on Y such that if $F \in \mathfrak{S}(X)$, then $f^{-1}F \in (f^\circledast\mathfrak{S})(Y)$. Moreover,

$$[f^\circledast \mathfrak{S}] = f^\sharp([\mathfrak{S}]).$$

Here, for $\mathsf{t}, \mathsf{t}' \in H^2(X; \mathbb{C}_X^\times)$, we denote by $\mathsf{t} \cdot \mathsf{t}'$ and t^{-1} the product and the inverse in $H^2(X; \mathbb{C}_X^\times)$, respectively, and by $f^\sharp \mathsf{t} \in H^2(Y; \mathbb{C}_Y^\times)$ the pull-back.

Let $(\mathfrak{S}_\mathcal{F}, \mathcal{F})$ and $(\mathfrak{S}_\mathcal{G}, \mathcal{G})$ be twisted line bundles on X, and consider the associated twisted line bundles $(\mathfrak{S}_{\mathcal{F}^{-1}}, \mathcal{F}^{-1})$ and $(\mathfrak{S}_{\mathcal{F}\otimes_\mathcal{O}\mathcal{G}}, \mathcal{F} \otimes_\mathcal{O} \mathcal{G})$ on X, and $(\mathfrak{S}_{f^*\mathcal{F}}, f^*\mathcal{F})$ on Y. Then there are $\mathbb{C}$-equivalences

$$\mathfrak{S}_{\mathcal{F}^{-1}} \simeq \mathfrak{S}_\mathcal{F}^{\circledast -1},$$
$$\mathfrak{S}_{\mathcal{F}\otimes_\mathcal{O}\mathcal{G}} \simeq \mathfrak{S}_\mathcal{F} \circledast \mathfrak{S}_\mathcal{G},$$
$$\mathfrak{S}_{f^*\mathcal{F}} \simeq f^\circledast \mathfrak{S}_\mathcal{F}.$$

§2. Review of Twisted Differential Operators

In this section we briefly review the notions of twisted differential operators. References are made to [7, 1] (see also [2] for an exposition).

Let X be a complex analytic manifold, and $\mathcal{D}_X$ the sheaf of finite order differential operators on X. Recall that automorphisms of $\mathcal{D}_X$ as an $\mathcal{O}_X$-ring are described by closed one-forms.

- A ring of twisted differential operators (a t.d.o. ring for short) is a sheaf of $\mathcal{O}_X$-rings locally isomorphic to $\mathcal{D}_X$. To a t.d.o. ring $\mathcal{A}$ corresponds a class $[\mathcal{A}] \in H^1(X; d\mathcal{O}_X)$ which characterizes $\mathcal{A}$ up to isomorphism of $\mathcal{O}_X$-rings.

Let $(\mathfrak{S}, \mathcal{F})$ be a twisted line bundle. An example of t.d.o. ring is given by

$$\mathcal{D}_{\mathcal{F}} = \mathcal{F} \otimes_{\mathcal{O}} \mathcal{D}_X \otimes_{\mathcal{O}} \mathcal{F}^{-1},$$

where $\mathcal{F}^{-1} = \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{O}_X)$. Notice that $\mathcal{F}^{-1} \in \mathsf{Mod}(\mathcal{O}_X; \mathfrak{S}^{\circledast -1})$, so that $\mathcal{D}_{\mathcal{F}}$ is untwisted as a sheaf.

As we recalled, a twisted line bundle $(\mathfrak{S}, \mathcal{F})$ can be described by an open covering $X = \bigcup_i U_i$, $\mathbb{C}$-equivalences $\varphi_i \colon \mathfrak{S}|_{U_i} \to \mathfrak{Mod}(\mathbb{C}_{U_i})$, line bundles $\mathcal{F}_i \in \mathsf{Mod}(\mathcal{O}_{U_i})$, and isomorphisms $\theta_{ij} \colon \mathcal{F}_j|_{U_{ij}} \to \mathcal{F}_i|_{U_{ij}}$. For nowhere vanishing sections $s_i \in \Gamma(U_i; \mathcal{F}_i)$, and $f_{ij} = s_i/\theta_{ij}(s_j) \in \Gamma(U_{ij}; \mathcal{O}_X^\times)$, sections of $\mathcal{D}_{\mathcal{F}}$ are described by families $s_i \otimes P_i \otimes s_i^{-1}$, where $P_i \in \Gamma(U_i; \mathcal{D}_X)$ and

$$P_i = f_{ji} \cdot P_j \cdot f_{ij} \quad \text{in } \Gamma(U_{ij}; \mathcal{D}_X). \tag{2.1}$$

The isomorphism ι in (1.2) is then described by $\iota([\mathfrak{S}, \mathcal{F}]) = [\mathcal{D}_{\mathcal{F}}]$. In particular, to any t.d.o. ring $\mathcal{A}$ is associated a twisted line bundle $\mathcal{F}$, unique up to tensoring by a local system of rank one, such that $\mathcal{A} \simeq \mathcal{D}_{\mathcal{F}}$ as an $\mathcal{O}_X$-ring.

Let $(\mathfrak{S}, \mathcal{F})$ be a twisted line bundle and $\mathfrak{T}$ a stack of twisted sheaves on X. There is a $\mathbb{C}$-equivalence

$$\begin{aligned} \mathfrak{Mod}(\mathcal{D}_{\mathcal{F}}; \mathfrak{T}) &\to \mathfrak{Mod}(\mathcal{D}_X; \mathfrak{S}^{\circledast -1} \circledast \mathfrak{T}) \\ \mathcal{M} &\mapsto \mathcal{F}^{-1} \otimes_{\mathcal{O}} \mathcal{M}. \end{aligned} \tag{2.2}$$

Denote by Θ_X the sheaf of vector fields and by Ω_X the sheaf of forms of maximal degree. We end this section by giving an explicit description, which will be of use later on, of the t.d.o. ring $\mathcal{D}_{\Omega_X^\lambda}$ for $\lambda \in \mathbb{C}$. Let $v \in \Theta_X$. Recall that the Lie derivative $\mathsf{L}(v)$ acts on differential forms of any degree, in particular on $\mathcal{O}_X$, where $\mathsf{L}(v)(a) = v(a)$, and on Ω_X. Let ω be a nowhere vanishing local section of Ω_X. One checks that the morphism

$$\begin{aligned} \mathsf{L}^{(\lambda)} \colon \Theta_X &\to \mathcal{D}_{\Omega_X^\lambda} = \Omega_X^\lambda \otimes_{\mathcal{O}} \mathcal{D}_X \otimes_{\mathcal{O}} \Omega_X^{-\lambda} \\ v &\mapsto \omega^\lambda \otimes \left(v + \lambda \frac{\mathsf{L}(v)(\omega)}{\omega} \right) \otimes \omega^{-\lambda} \end{aligned} \tag{2.3}$$

is well defined and independent from the choice of ω. (Here $\mathsf{L}(v)(\omega)/\omega = a$, where $a \in \mathcal{O}_X$ is such that $\mathsf{L}(v)(\omega) = a\omega$.) Then $\mathcal{D}_{\Omega_X^\lambda}$ is generated by $\mathcal{O}_X$ and $\mathsf{L}^{(\lambda)}(\Theta_X)$ with the relations

$$\mathsf{L}^{(\lambda)}(av) = a \cdot \mathsf{L}^{(\lambda)}(v) + \lambda v(a), \tag{2.4}$$

$$[\mathsf{L}^{(\lambda)}(v), a] = v(a), \tag{2.5}$$

$$[\mathsf{L}^{(\lambda)}(v), \mathsf{L}^{(\lambda)}(w)] = \mathsf{L}^{(\lambda)}([v, w]), \tag{2.6}$$

for $a \in \mathcal{O}_X$, and $v, w \in \Theta_X$. Of course, $\mathsf{L}^{(0)}(v) = v$ and $\mathsf{L}^{(1)}(v) = \mathsf{L}(v)$.

§3. Microdifferential Operators on Involutive Submanifolds

In this section we recall the notion of V-filtration on microdifferential operators. References are made to [11, 12] (see also [6, 9, 13] for expositions).

Let W be a complex manifold. In this paper, by a submanifold of W, we mean a smooth locally closed complex submanifold.

Let X be a complex manifold, and denote by $\pi\colon T^*X \to X$ its cotangent bundle. Identifying X with the zero-section of T^*X, one sets $\dot{T}^*X = T^*X \setminus X$.

The canonical 1-form α_X induces a homogeneous symplectic structure on T^*X. Denote by $\{f, g\} \in \mathcal{O}_{T^*X}$ the Poisson bracket of two functions $f, g \in \mathcal{O}_{T^*X}$ and by

$$H\colon T^*T^*X \xrightarrow{\sim} TT^*X$$

the Hamiltonian isomorphism. For $k \in \mathbb{Z}$, denote by $\mathcal{O}_{T^*X}(k) \subset \mathcal{O}_{T^*X}$ the subsheaf of functions φ homogeneous of order k, that is, satisfying $eu(\varphi) = k\cdot\varphi$. Here, $eu = -H(\alpha_X)$ denotes the Euler vector field on T^*X, the infinitesimal generator of the action of $\mathbb{C}^\times$.

Denote by $\mathcal{E}_X$ the ring of microdifferential operators on T^*X. It is endowed with the order filtration $\{\mathsf{F}_m\mathcal{E}_X\}_{m\in\mathbb{Z}}$, where $\mathsf{F}_m\mathcal{E}_X$ is the subsheaf of microdifferential operators of order at most m. There is a canonical morphism

$$\sigma_m\colon \mathsf{F}_m\mathcal{E}_X \to \mathcal{O}_{T^*X}(m)$$

called the principal symbol of order m. This morphism induces an isomorphism of graded rings $\mathcal{G}r\,\mathcal{E}_X \simeq \bigoplus_k \mathcal{O}_{T^*X}(k)$. If $P \in \mathsf{F}_m\mathcal{E}_X$, $Q \in \mathsf{F}_l\mathcal{E}_X$, one has

$$\sigma_{m+l}(PQ) = \sigma_m(P)\sigma_l(Q), \tag{3.1}$$

$$\sigma_{m+l-1}([P, Q]) = \{\sigma_m(P), \sigma_l(Q)\}. \tag{3.2}$$

Let $V \subset T^*X$ be a submanifold and denote by $\mathcal{J}_V \subset \mathcal{O}_{T^*X}$ its annihilating ideal. Recall that V is called homogeneous, or conic, if $eu\,\mathcal{J}_V \subset \mathcal{J}_V$. In this case, $eu_V := eu|_V$ is tangent to V, and one defines $\mathcal{O}_V(k) \subset \mathcal{O}_V$ similarly to $\mathcal{O}_{T^*X}(k) \subset \mathcal{O}_{T^*X}$. A conic submanifold $V \subset T^*X$ is called involutive if for any pair $f, g \in \mathcal{J}_V$ of holomorphic functions vanishing on V, the Poisson bracket $\{f, g\}$ vanishes on V. A conic involutive submanifold V is called regular if $\alpha_X|_V$ never vanishes.

Let $V \subset T^*X$ be a conic involutive submanifold, and set

$$\mathcal{I}_V = \{P \in \mathsf{F}_1\mathcal{E}_X|_V; \sigma_1(P)|_V = 0\} \subset \mathcal{E}_X|_V.$$

Note that $[\mathcal{I}_V, \mathcal{I}_V] \subset \mathcal{I}_V$.

Definition 3.1. Let $V \subset \dot{T}^*X$ be a conic involutive submanifold. One denotes by $\mathcal{E}_V$ the subring of $\mathcal{E}_X|_V$ generated by $\mathcal{I}_V$, and one sets $\mathsf{F}^V_m\mathcal{E}_X := \mathsf{F}_m\mathcal{E}_X|_V \cdot \mathcal{E}_V$.

One easily checks that $\mathsf{F}^V_m\mathcal{E}_X = \mathcal{E}_V \cdot \mathsf{F}_m\mathcal{E}_X|_V$, and $\mathsf{F}^V_m\mathcal{E}_X \cdot \mathsf{F}^V_l\mathcal{E}_X \subset \mathsf{F}^V_{m+l}\mathcal{E}_X$. In particular, $\{\mathsf{F}^V_k\mathcal{E}_X\}_{k\in\mathbb{Z}}$ is an exhaustive filtration of $\mathcal{E}_X|_V$, called the V-filtration, and $\mathsf{F}^V_{-1}\mathcal{E}_X$ is a two-sided ideal of $\mathcal{E}_V = \mathsf{F}^V_0\mathcal{E}_X$.

Example 2. Let $(x) = (x_1, \dots, x_n)$ be a local coordinate system on X and denote by $(x;\xi) = (x_1, \dots, x_n; \xi_1, \dots, \xi_n)$ the associated homogeneous symplectic local coordinate system on T^*X. Recall that locally, any conic regular involutive submanifold V of codimension d may be written after a homogeneous symplectic transformation as:

$$V = \{(x;\xi); \xi_1 = \cdots = \xi_d = 0\}.$$

In such a case,

$$\mathsf{F}^V_m\mathcal{E}_X \simeq (\mathsf{F}_m\mathcal{E}_X|_V)[\partial_{x_1}, \dots, \partial_{x_d}].$$

§4. Systems with Simple Characteristics

In this section we recall the notion of systems with simple characteristics. References are made to [11, 12]. See also [9, 13] for an exposition.

Definition 4.1. Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module. A lattice in $\mathcal{M}$ is a coherent $\mathsf{F}_0\mathcal{E}_X$-submodule $\mathcal{M}_0$ which generates $\mathcal{M}$ over $\mathcal{E}_X$.

Recall that if an $\mathsf{F}_0\mathcal{E}_X$-submodule $\mathcal{M}_0$ of $\mathcal{M}$ defined on an open subset of $\dot{T}^*X$ is locally of finite type, then it is coherent. A lattice $\mathcal{M}_0$ endows $\mathcal{M}$ with the filtration

$$\mathsf{F}_k\mathcal{M} = \mathsf{F}_k\mathcal{E}_X \cdot \mathcal{M}_0.$$

If $\mathcal{M}$ is endowed with a filtration $\{\mathsf{F}_k\mathcal{M}\}_k$, its associated symbol module is given by

$$\widetilde{\mathcal{G}r}(\mathcal{M}) := \mathcal{O}_{T^*X} \otimes_{\mathcal{G}r(\mathcal{E}_X)} \mathcal{G}r(\mathcal{M}),$$

where $\mathcal{G}r(\mathcal{M}) = \oplus_{k\in\mathbb{Z}}(\mathsf{F}_k\mathcal{M}/\mathsf{F}_{k-1}\mathcal{M})$.

Definition 4.2. Let $V \subset \dot{T}^*X$ be a conic involutive submanifold.

(a) A coherent $\mathcal{E}_X$-module $\mathcal{M}$ is simple along V if it is locally generated by a section $u \in \mathcal{M}$, called a simple generator, such that denoting by $\mathcal{I}_u$ the annihilator ideal of u in $\mathcal{E}_X$, the symbol ideal $\widetilde{\mathcal{G}r}(\mathcal{I}_u)$ is reduced and coincides with the annihilator ideal $\mathcal{J}_V$ of V in $\mathcal{O}_{T^*X}$.

(b) A coherent $\mathcal{E}_X$-module $\mathcal{M}$ is globally simple along V if it admits a lattice $\mathcal{M}_0$ such that $\mathcal{E}_V\mathcal{M}_0 \subset \mathcal{M}_0$ and $\mathcal{M}_0/\mathsf{F}_{-1}\mathcal{M}$ is locally isomorphic to $\mathcal{O}_V(0)$. Such an $\mathcal{M}_0$ is called a V-lattice in $\mathcal{M}$.

Lemma 4.1. *If $\mathcal{M}$ is globally simple, then it is simple.*

Proof. Let $\mathcal{M}_0$ be a V-lattice. Choose a local section $u \in \mathcal{M}_0$ whose image in $\mathcal{M}_0/\mathsf{F}_{-1}\mathcal{M}$ is a generator of $\mathcal{O}_V(0)$. Then $\mathcal{M}_0 = \mathsf{F}_0\mathcal{E}_X u + \mathsf{F}_{-1}\mathcal{M}$ and it follows that for all $k \le 0$

$$\mathcal{M}_0 = \mathsf{F}_0\mathcal{E}_X u + \mathsf{F}_k\mathcal{M}.$$

Since the filtration on $\mathcal{M}$ is separated (see [12]), u generates $\mathcal{M}_0$ over $\mathsf{F}_0\mathcal{E}_X$. ☐

Let $(t) \in \mathbb{C}$ be a coordinate, and denote by $(t;\tau) \in T^*\mathbb{C}$ the associated homogeneous symplectic coordinate system. Let $V \subset T^*X$ be a conic involutive submanifold, non necessarily regular. The trick of the dummy variable consists in replacing V with the conic involutive submanifold $\widetilde{V} = V \times \dot{T}^*\mathbb{C}$, which is regular. Let $p \in V$ and $q \in \dot{T}^*\mathbb{C}$. If Σ is the bicharacteristic leaf of V through p, then $\Sigma \times \{q\}$ is the bicharacteristic leaf of $\widetilde{V}$ through (p,q).

Proposition 4.1. *If $\mathcal{M}$ is a globally simple $\mathcal{E}_X$-module along V, then $\widetilde{\mathcal{M}} = \mathcal{E}_{X\times\mathbb{C}} \otimes_{\mathcal{E}_X\boxtimes\mathcal{E}_\mathbb{C}} (\mathcal{M}\boxtimes\mathcal{E}_\mathbb{C})$ is globally simple along $\widetilde{V}$.*

Proof. Let $\mathcal{M}_0$ be a V-lattice in $\mathcal{M}$, and set

$$\widetilde{\mathcal{M}}_0 = \mathsf{F}_0\mathcal{E}_{X\times\mathbb{C}} \otimes_{\mathsf{F}_0\mathcal{E}_X\boxtimes\mathsf{F}_0\mathcal{E}_\mathbb{C}} (\mathcal{M}_0\boxtimes\mathsf{F}_0\mathcal{E}_\mathbb{C}).$$

Clearly, $\widetilde{\mathcal{M}}_0$ is a lattice in $\widetilde{\mathcal{M}}$ and moreover, $\mathcal{E}_{\widetilde{V}}\widetilde{\mathcal{M}}_0 \subset \widetilde{\mathcal{M}}_0$. Note that

$$\mathsf{F}_{-1}\widetilde{\mathcal{M}} = \mathsf{F}_0\mathcal{E}_{X\times\mathbb{C}} \otimes_{\mathsf{F}_0\mathcal{E}_X\boxtimes\mathsf{F}_0\mathcal{E}_\mathbb{C}} (\mathsf{F}_{-1}\mathcal{M}\boxtimes\mathsf{F}_0\mathcal{E}_\mathbb{C} + \mathcal{M}_0\boxtimes\mathsf{F}_{-1}\mathcal{E}_\mathbb{C}).$$

Set $\mathcal{M}_{-1} = \mathsf{F}_{-1}\mathcal{M}$, $\overline{\mathcal{M}}_0 = \mathcal{M}_0/\mathcal{M}_{-1}$, and consider the commutative exact diagram of $\mathsf{F}_0\mathcal{E}_X \boxtimes \mathsf{F}_0\mathcal{E}_{\mathbb{C}}$-modules:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{M}_{-1} \boxtimes \mathsf{F}_{-1}\mathcal{E}_{\mathbb{C}} & \to & \mathcal{M}_0 \boxtimes \mathsf{F}_{-1}\mathcal{E}_{\mathbb{C}} & \to & \overline{\mathcal{M}}_0 \boxtimes \mathsf{F}_{-1}\mathcal{E}_{\mathbb{C}} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{M}_{-1} \boxtimes \mathsf{F}_0\mathcal{E}_{\mathbb{C}} & \to & \mathcal{M}_0 \boxtimes \mathsf{F}_0\mathcal{E}_{\mathbb{C}} & \to & \overline{\mathcal{M}}_0 \boxtimes \mathsf{F}_0\mathcal{E}_{\mathbb{C}} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{M}_{-1} \boxtimes (\mathsf{F}_0\mathcal{E}_{\mathbb{C}}/\mathsf{F}_{-1}\mathcal{E}_{\mathbb{C}}) & \to & \mathcal{M}_0 \boxtimes (\mathsf{F}_0\mathcal{E}_{\mathbb{C}}/\mathsf{F}_{-1}\mathcal{E}_{\mathbb{C}}) & \to & \overline{\mathcal{M}}_0 \boxtimes (\mathsf{F}_0\mathcal{E}_{\mathbb{C}}/\mathsf{F}_{-1}\mathcal{E}_{\mathbb{C}}) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

It follows that the sequence

$$0 \to \mathcal{M}_{-1} \boxtimes \mathsf{F}_0\mathcal{E}_{\mathbb{C}} + \mathcal{M}_0 \boxtimes \mathsf{F}_{-1}\mathcal{E}_{\mathbb{C}} \to \mathcal{M}_0 \boxtimes \mathsf{F}_0\mathcal{E}_{\mathbb{C}} \to \overline{\mathcal{M}}_0 \boxtimes \mathsf{F}_0\mathcal{E}_{\mathbb{C}}/\mathsf{F}_{-1}\mathcal{E}_{\mathbb{C}} \to 0$$

is exact. Since $\mathsf{F}_0\mathcal{E}_{X\times\mathbb{C}}$ is flat over $\mathsf{F}_0\mathcal{E}_X \boxtimes \mathsf{F}_0\mathcal{E}_{\mathbb{C}}$, we locally have

$$\begin{aligned}
\mathsf{F}_0\widetilde{\mathcal{M}}/\mathsf{F}_{-1}\widetilde{\mathcal{M}} &\simeq \mathsf{F}_0\mathcal{E}_{X\times\mathbb{C}} \otimes_{\mathsf{F}_0\mathcal{E}_X\boxtimes\mathsf{F}_0\mathcal{E}_{\mathbb{C}}} (\overline{\mathcal{M}}_0 \boxtimes \mathsf{F}_0\mathcal{E}_{\mathbb{C}}/\mathsf{F}_{-1}\mathcal{E}_{\mathbb{C}}) \\
&\simeq \mathsf{F}_0\mathcal{E}_{X\times\mathbb{C}} \otimes_{\mathsf{F}_0\mathcal{E}_X\boxtimes\mathsf{F}_0\mathcal{E}_{\mathbb{C}}} (\mathcal{O}_V(0) \boxtimes \mathcal{O}_{\dot{T}^*\mathbb{C}}(0)) \\
&\simeq \mathcal{O}_{\widetilde{V}}(0).
\end{aligned}$$

□

Remark. Let $\mathfrak{S}$ be a $\mathbb{C}$-stack of twisted sheaves on X. Then Definition 4.2, Lemma 4.1 and Proposition 4.1 extend to objects of $\mathsf{Mod}(\mathcal{E}_X; \pi^{\circledast}\mathfrak{S})$.

§5. Differential Operators on Involutive Submanifolds

We recall here the construction of the ring of homogeneous twisted differential operators invariant by the bicharacteristic flow.

Let $V \subset T^*X$ be a conic regular involutive submanifold and denote by $TV^\perp \subset TV$ the symplectic orthogonal to TV. Denote by $\Theta_V^\perp \subset \Theta_V$ the sheaf of sections of the bundle $TV^\perp \to V$, and let

$$\begin{aligned}
\mathcal{O}_V^{bic} &:= \{a \in \mathcal{O}_V; v(a) = 0 \text{ for any } v \in \Theta_V^\perp\}, \\
\mathcal{O}_V^{bic}(k) &:= \mathcal{O}_V^{bic} \cap \mathcal{O}_V(k).
\end{aligned}$$

Then $\mathcal{O}_V^{bic}$ is the sheaf of holomorphic functions locally constant along the bicharacteristic leaves of V. Consider the ring

$$\mathcal{D}_V^{bic} := \{P \in \mathcal{D}_V; [a, P] = 0 \text{ for any } a \in \mathcal{O}_V^{bic}\},$$

and the subring of operators homogeneous of degree zero

$$\mathcal{D}_V^{bic}(0) := \{P \in \mathcal{D}_V^{bic}; [eu_V, P] = 0\}.$$

Example 3. Let $(x;\xi) = (x_1,\dots,x_n;\xi_1,\dots,\xi_n)$ be a local homogeneous symplectic coordinate system on T^*X and assume that

$$V = \{(x;\xi); \xi_1 = \cdots = \xi_d = 0\}.$$

Set $x' = (x_1,\dots,x_d)$, $x'' = (x_{d+1},\dots x_n)$, and similarly set $\xi = (\xi', \xi'')$. One has $(x', x'', \xi'') \in V$, and the bicharacteristic leaves of V are the submanifolds defined by

$$\Sigma = \{(x', x''; \xi''); (x''; \xi'') = (x''_0; \xi''_0)\}.$$

The Euler field eu_V is given by

$$eu_V = \sum_{d+1}^{n} \xi_i \partial_{\xi_i} = \xi'' \partial_{\xi''}.$$

Hence a function locally constant along the bicharacteristic leaves depends only on (x'', ξ''). A section of $\mathcal{O}_V(0)$ is a holomorphic functions in the variable (x', x'', ξ''), homogeneous of degree 0 with respect to ξ''. Moreover a section of $\mathcal{D}_V^{bic}(0)$ is uniquely written as a finite sum

$$\sum_{\alpha\in\mathbb{N}^d} a_\alpha \partial_{x'}^\alpha, \text{ with } a_\alpha \in \mathcal{O}_V(0). \tag{5.1}$$

Let $j_\Sigma\colon \Sigma \to V$ be the embedding of a bicharacteristic leaf. Assume that V is regular, and that Σ is a locally closed submanifold of V. Denote by $\mathcal{J}_\Sigma^{bic}(0)$ the annihilator ideal of Σ in $\mathcal{O}_V^{bic}(0)$, and note that $\mathcal{O}_\Sigma \simeq \mathcal{O}_V^{bic}(0)/\mathcal{J}_\Sigma^{bic}(0)|_\Sigma$. Since $\mathcal{O}_V^{bic}(0)$ is in the center of $\mathcal{D}_V^{bic}(0)$, there is a restriction map

$$\begin{aligned} j_\Sigma^*\colon \mathsf{Mod}(\mathcal{D}_V^{bic}(0)) &\to \mathsf{Mod}(\mathcal{D}_\Sigma) \\ &\mathcal{M} \mapsto \mathbb{C}_\Sigma \otimes_{\mathcal{O}_V^{bic}(0)|_\Sigma} \mathcal{M}|_\Sigma. \end{aligned}$$

We will be interested in the twisted analogue of the above construction. Namely, set

$$\begin{aligned} \mathcal{D}^{bic}_{\Omega_V^{1/2}} &:= \{P \in \mathcal{D}_{\Omega_V^{1/2}}; [a, P] = 0 \text{ for any } a \in \mathcal{O}_V^{bic}\}, \\ \mathcal{D}^{bic}_{\Omega_V^{1/2}}(0) &:= \{P \in \mathcal{D}^{bic}_{\Omega_V^{1/2}}; [\mathsf{L}^{(1/2)}(eu_V), P] = 0\}. \end{aligned}$$

For $p \in \Sigma$, the quotient $T_pV/T_p\Sigma \simeq T_pV/T_pV^\perp$ is a symplectic space. Hence $j_\Sigma^*\Omega_V \simeq \Omega_\Sigma$. Thus, there is a restriction morphism

$$j_\Sigma^*\colon \mathsf{Mod}(\mathcal{D}^{bic}_{\Omega_V^{1/2}}(0)) \to \mathsf{Mod}(\mathcal{D}_{\Omega_\Sigma^{1/2}}). \tag{5.2}$$

§6. Subprincipal Symbol

In this section we recall the notion of subprincipal symbol, and prove the regular involutive analogue of an isomorphism obtained in [10, Lemma 1.5.1] for the Lagrangian case. References are made to [6, 9, 10, 11] (see [4] for the corresponding constructions in the C^∞ case).

As we will recall, the subprincipal symbol is intrinsically defined for microdifferential operators twisted by half-forms. We will thus consider here the ring

$$\mathcal{E}_{\Omega_X^{1/2}} = \pi^{-1}\Omega_X^{1/2} \otimes_{\pi^{-1}\mathcal{O}} \mathcal{E}_X \otimes_{\pi^{-1}\mathcal{O}} \pi^{-1}\Omega_X^{-1/2},$$

instead of $\mathcal{E}_X$. All the notions recalled in Section 3 extend to this ring. In particular, its V-filtration is defined by

$$\begin{aligned}
\mathcal{I}_V^{\Omega_X^{1/2}} &= \{P \in \mathsf{F}_1\mathcal{E}_{\Omega_X^{1/2}}|_V ; \sigma_1(P)|_V = 0\} \\
&\simeq \pi_V^{-1}\Omega_X^{1/2} \otimes_{\pi_V^{-1}\mathcal{O}} \mathcal{I}_V \otimes_{\pi_V^{-1}\mathcal{O}} \pi_V^{-1}\Omega_X^{-1/2}, \\
\mathsf{F}_m^V\mathcal{E}_{\Omega_X^{1/2}} &= \pi_V^{-1}\Omega_X^{1/2} \otimes_{\pi_V^{-1}\mathcal{O}} \mathsf{F}_m^V\mathcal{E}_X \otimes_{\pi_V^{-1}\mathcal{O}} \pi_V^{-1}\Omega_X^{-1/2}, \\
\mathcal{E}_{V,\Omega_X^{1/2}} &= \mathsf{F}_0^V\mathcal{E}_{\Omega_X^{1/2}},
\end{aligned}$$

where $\pi_V = \pi|_V$.

Let (x) be a local coordinate system on X, and denote by $(x;\xi)$ the associated homogeneous symplectic coordinate system on T^*X. A microdifferential operator $P \in \mathsf{F}_m\mathcal{E}_{\Omega_X^{1/2}}$ is then described by its total symbol $\{p_k(x;\xi)\}_{k\le m}$, where $p_k \in \mathcal{O}_{T^*X}(k)$. The functions p_k depend on the local coordinate system (x) on X, except the top degree term $p_m = \sigma_m(P)$ which does not. Recall that the subprincipal symbol

$$\sigma'_{m-1} \colon \mathsf{F}_m\mathcal{E}_{\Omega_X^{1/2}} \longrightarrow \mathcal{O}_{T^*X}(m-1)$$

given by

$$\sigma'_{m-1}((dx)^{1/2} \otimes P \otimes (dx)^{-1/2}) = p_{m-1}(x,\xi) - \frac{1}{2}\sum_i \partial_{x_i}\partial_{\xi_i}p_m(x,\xi),$$

does not depend on the local coordinate system (x) on X. For $P \in \mathsf{F}_m\mathcal{E}_{\Omega_X^{1/2}}$, $Q \in \mathsf{F}_l\mathcal{E}_{\Omega_X^{1/2}}$, one has

$$\begin{aligned}
\text{(6.1)}\qquad \sigma'_{m+l-1}(PQ) &= \sigma_m(P)\sigma'_{l-1}(Q) + \sigma'_{m-1}(P)\sigma_l(Q) \\
&\quad + \frac{1}{2}\{\sigma_m(P), \sigma_l(Q)\},
\end{aligned}$$

$$\text{(6.2)}\qquad \sigma'_{m+l-2}([P,Q]) = \{\sigma_m(P), \sigma'_{l-1}(Q)\} + \{\sigma'_{m-1}(P), \sigma_l(Q)\}.$$

Let $V \subset T^*X$ be a conic involutive submanifold. For $f \in \mathcal{O}_{T^*X}$, denote by $H_f = H(df) \in TT^*X$ its Hamiltonian vector field. Recall that H induces an isomorphism

$$H\colon T^*_V T^*X \xrightarrow{\sim} TV^{\perp}. \tag{6.3}$$

In particular, $H_f|_V$ is tangent to V for $f \in \mathcal{J}_V$. With notations (2.3), consider the transport operator

$$\begin{aligned}\mathcal{L}^0_V\colon \mathcal{I}_V^{\Omega_X^{1/2}} &\to \mathsf{F}_1\mathcal{D}_{\Omega_V^{1/2}},\\ P &\mapsto \mathsf{L}^{(1/2)}(H_{\sigma_1(P)}|_V) + \sigma'_0(P)|_V.\end{aligned} \tag{6.4}$$

Using the above relations, one checks that the morphism $\mathcal{L}^0_V$ does not depend on the choice of coordinates, and satisfies the relations

$$\begin{aligned}\mathcal{L}^0_V(AP) &= \sigma_0(A)\mathcal{L}^0_V(P),\\ \mathcal{L}^0_V(PA) &= \mathcal{L}^0_V(P)\sigma_0(A),\\ \mathcal{L}^0_V([P,Q]) &= [\mathcal{L}^0_V(P), \mathcal{L}^0_V(Q)],\end{aligned}$$

for $P, Q \in \mathcal{I}_V^{\Omega_X^{1/2}}$ and $A \in \mathsf{F}_0\mathcal{E}_{\Omega_X^{1/2}}$ (see [11, §2] or [9, §8.3]). It follows that $\mathcal{L}^0_V$ extends as a ring morphism

$$\mathcal{L}_V\colon \mathcal{E}_{V,\Omega_X^{1/2}} \to \mathcal{D}_{\Omega_V^{1/2}} \tag{6.5}$$

by setting $\mathcal{L}_V(P_1 \cdots P_r) = \mathcal{L}^0_V(P_1) \cdots \mathcal{L}^0_V(P_r)$, for $P_i \in \mathcal{I}_V^{\Omega_X^{1/2}}$.

Theorem 6.1. *Let $V \subset \dot{T}^*X$ be a conic regular involutive submanifold. The morphism* (6.5) *induces a ring isomorphism*

$$\mathcal{L}_V\colon \mathcal{E}_{V,\Omega_X^{1/2}}/\mathsf{F}^V_{-1}\mathcal{E}_{\Omega_X^{1/2}} \xrightarrow{\sim} \mathcal{D}^{bic}_{\Omega_V^{1/2}}(0). \tag{6.6}$$

It is possible to show that the above statement holds even without the assumption of regularity for V (for example, the Lagrangian case is obtained in [10, Lemma 1.5.1]).

Proof. The statement is local. We may thus assume that $\Omega_X \simeq \mathcal{O}_X$ and $\Omega_V \simeq \mathcal{O}_V$, so that we are reduced to prove the isomorphism

$$\mathcal{L}_V\colon \mathcal{E}_V/\mathsf{F}^V_{-1}\mathcal{E}_X \xrightarrow{\sim} \mathcal{D}^{bic}_V(0).$$

Moreover, since V is regular we may assume that we are in the situation of Example 3. By Example 2, sections of $\mathcal{E}_V$ are uniquely written as finite sums

$$\sum_{\alpha\in\mathbb{N}^d} A_\alpha \partial^\alpha_{x'}, \text{ with } A_\alpha \in \mathsf{F}_0\mathcal{E}_X|_V. \tag{6.7}$$

One concludes using (5.1) since, by definition of $\mathcal{L}_V$,

$$\mathcal{L}_V\left(\sum_\alpha A_\alpha \partial_{x'}^\alpha\right) = \sum_\alpha \sigma_0(A_\alpha)\partial_{x'}^\alpha.$$

□

Let $\Omega_{V/X} = \Omega_V \otimes_{\mathcal{O}} \pi_V^* \Omega_X^{-1}$ be the sheaf of relative forms. Recall from Example 1 that $\mathfrak{S}_{\Omega_{V/X}^{-1/2}}$ denotes a stack of twisted sheaves such that $\Omega_{V/X}^{-1/2} \in \mathsf{Mod}(\mathcal{O}_V; \mathfrak{S}_{\Omega_{V/X}^{-1/2}})$.

Corollary 6.1. *Let $V \subset \dot{T}^*X$ be a conic regular involutive submanifold, and $\mathfrak{T}$ be a stack of twisted sheaves on V. Then there is an equivalence of categories*

$$\mathsf{Mod}(\mathcal{E}_V/\mathsf{F}_{-1}^V\mathcal{E}_X; \mathfrak{T}) \simeq \mathsf{Mod}(\mathcal{D}_V^{bic}(0); \mathfrak{T} \circledast \mathfrak{S}_{\Omega_{V/X}^{-1/2}}).$$

§7. Statement of the Result

We can now state our main result. For a submanifold $\Sigma \subset \dot{T}^*X$, set $\pi_\Sigma = \pi|_\Sigma$, and denote by $\pi_\Sigma^\sharp \colon H^2(X; \mathbb{C}_X^\times) \to H^2(\Sigma; \mathbb{C}_\Sigma^\times)$ the pull-back.

Theorem 7.1. *Let $V \subset \dot{T}^*X$ be a conic involutive submanifold, $\Sigma \subset V$ a bicharacteristic leaf, and $\mathfrak{T}$ a stack of twisted sheaves on X. Assume that Σ is a locally closed submanifold of V, and that there exists a globally simple module along V in $\mathsf{Mod}(\mathcal{E}_X; \pi^\circledast\mathfrak{T})$. Then*

$$\pi_\Sigma^\sharp([\mathfrak{T}]) = [\mathfrak{S}_{\Omega_{\Sigma/X}^{1/2}}] \quad \text{in } H^2(\Sigma; \mathbb{C}_\Sigma^\times).$$

Proof. The proof follows the same lines as in [10, §I.5.2]. Let us first reduce to the regular involutive case by the trick of the dummy variable. Let $p\colon \widetilde{X} = X \times \mathbb{C} \to X$ be the projection. With the notations of Proposition 4.1, replace X with $\widetilde{X}$, $\mathfrak{T}$ with $\widetilde{\mathfrak{T}} = p^\circledast\mathfrak{T}$, V with $\widetilde{V} = V \times \dot{T}^*\mathbb{C}$, $\mathcal{M}$ with $\widetilde{\mathcal{M}}$, and Σ with $\widetilde{\Sigma} = \Sigma \times \{(0;1)\}$. Under the isomorphism $H^2(\Sigma; \mathbb{C}_\Sigma^\times) \simeq H^2(\widetilde{\Sigma}; \mathbb{C}_{\widetilde{\Sigma}}^\times)$ one has $\pi_\Sigma^\sharp([\mathfrak{T}]) = \pi_{\widetilde{\Sigma}}^\sharp([\widetilde{\mathfrak{T}}])$. Hence we may assume that V is regular involutive.

Let $\mathcal{M}$ be a globally simple module along V in $\mathsf{Mod}(\mathcal{E}_X; \pi^\circledast\mathfrak{T})$, and let $\mathcal{M}_0$ be a V-lattice in $\mathcal{M}$. Then $\overline{\mathcal{M}}_0 = \mathcal{M}_0/\mathsf{F}_{-1}\mathcal{M} \in \mathsf{Mod}(\mathcal{E}_V/\mathsf{F}_{-1}^V\mathcal{E}_X; \pi_V^\circledast\mathfrak{T})$ is locally isomorphic to $\mathcal{O}_V(0)$. By Corollary 6.1 we may further consider $\overline{\mathcal{M}}_0$ as an object of $\mathsf{Mod}(\mathcal{D}_V^{bic}(0); \pi_V^\circledast\mathfrak{T} \circledast \mathfrak{S}_{\Omega_{V/X}^{-1/2}})$.

By the restriction functor (5.2) and the equivalence (2.2), $j_\Sigma^*(\overline{\mathcal{M}}_0)$ is an object of $\mathsf{Mod}(\mathcal{D}_\Sigma; \pi_\Sigma^\circledast\mathfrak{T} \circledast \mathfrak{S}_{\Omega_{\Sigma/X}^{-1/2}})$ locally isomorphic to $\mathcal{O}_\Sigma$. Hence its solution sheaf $\mathcal{H}om_{\mathcal{D}_\Sigma}(j_\Sigma^*(\overline{\mathcal{M}}_0), \mathcal{O}_\Sigma) \in (\pi_\Sigma^\circledast\mathfrak{T} \circledast \mathfrak{S}_{\Omega_{\Sigma/X}^{-1/2}})^{\circledast-1}(\Sigma)$ is a local system of rank 1. It follows by Proposition 1.1 that the class $[(\pi_\Sigma^{\circledast}\mathfrak{T} \circledast \mathfrak{S}_{\Omega_{\Sigma/X}^{-1/2}})^{\circledast -1}] = [\mathfrak{S}_{\Omega_{\Sigma/X}^{1/2}}] \cdot \pi_\Sigma^\sharp([\mathfrak{T}])^{-1}$ is trivial in $H^2(\Sigma; \mathbb{C}_\Sigma^\times)$. □

Remark. Let us say that a coherent $\mathcal{E}_X$-module $\mathcal{M}$ is globally r-simple along V if it admits a lattice $\mathcal{M}_0$ such that $\mathcal{E}_V\mathcal{M}_0 \subset \mathcal{M}_0$ and $\mathcal{M}_0/\mathsf{F}_{-1}\mathcal{M}$ is locally isomorphic to $\mathcal{O}_V(0)^r$. Theorem 7.1 extends to globally r-simple modules as follows. If there exists a globally r-simple module along V in $\mathsf{Mod}(\mathcal{E}_X; \pi^{\circledast}\mathfrak{T})$, then

$$\pi_\Sigma^\sharp([\mathfrak{T}])^r = \left([\mathfrak{S}_{\Omega_{\Sigma/X}^{1/2}}]\right)^r \quad \text{in } H^2(\Sigma; \mathbb{C}_\Sigma^\times).$$

The proof goes along the same lines as the one above, recalling the following fact. Let $\mathfrak{S}$ be a stack of twisted sheaves on X, and let $F \in \mathfrak{S}(X)$ be a local system of rank r. Then $\det F$ is a local system of rank 1 in $\mathfrak{S}^{\circledast r}(X)$, so that $\mathfrak{S}^{\circledast r}$ is globally $\mathbb{C}$-equivalent to $\mathfrak{Mod}(\mathbb{C}_X)$.

Corollary 7.1. *Let $V \subset \dot{T}^*X$ be a conic involutive submanifold, $\Sigma \subset V$ a bicharacteristic leaf, and $\mathfrak{T}$ a stack of twisted sheaves on X. Assume that Σ is a locally closed submanifold of V, that $\pi_\Sigma^\sharp \colon H^2(X; \mathbb{C}_X^\times) \to H^2(\Sigma; \mathbb{C}_\Sigma^\times)$ is injective, that $[\mathfrak{S}_{\Omega_{\Sigma/X}^{1/2}}] = 1$ in $H^2(\Sigma; \mathbb{C}_\Sigma^\times)$, and that there exists a globally simple module along V in $\mathsf{Mod}(\mathcal{E}_X; \pi^{\circledast}\mathfrak{T})$. Then $\mathfrak{T}$ is globally $\mathbb{C}$-equivalent to $\mathfrak{Mod}(\mathbb{C}_X)$.*

Proof. By Theorem 7.1, $\pi_\Sigma^\sharp([\mathfrak{T}]) = 1$ in $H^2(\Sigma; \mathbb{C}_\Sigma^\times)$. Since $\pi_\Sigma^\sharp$ is injective, $[\mathfrak{T}] = 1$ in $H^2(X; \mathbb{C}_X^\times)$, and this implies that the stack $\mathfrak{T}$ is globally $\mathbb{C}$-equivalent to $\mathfrak{Mod}(\mathbb{C}_X)$. □

§8. Application: Non Existence of Twisted Wave Equations

Let $\mathbb{T}$ be an $(n+1)$-dimensional complex vector space, $\mathbb{P}$ the projective space of lines in $\mathbb{T}$, and $\mathbb{G}$ the Grassmannian of $(p+1)$-dimensional subspaces in $\mathbb{T}$. Assume $n \geq 3$ and $1 \leq p \leq n-2$. The Penrose correspondence (see [5]) is associated with the double fibration

(8.1) $$\mathbb{P} \underset{f}{\leftarrow} \mathbb{F} \underset{g}{\rightarrow} \mathbb{G}$$

where $\mathbb{F} = \{(y, x) \in \mathbb{P} \times \mathbb{G}; y \subset x\}$ is the incidence relation, and f, g are the natural projections. The double fibration (8.1) induces the maps

$$\dot{T}^*\mathbb{P} \underset{p}{\leftarrow} \dot{T}^*_{\mathbb{F}}(\mathbb{P} \times \mathbb{G}) \underset{q}{\rightarrow} \dot{T}^*\mathbb{G},$$

where $T^*_{\mathbb{F}}(\mathbb{P} \times \mathbb{G}) \subset T^*(\mathbb{P} \times \mathbb{G})$ denotes the conormal bundle to $\mathbb{F}$, and p and q are the natural projections. Note that p is smooth surjective, and q is a closed embedding. Set

$$V = q(\dot{T}^*_{\mathbb{F}}(\mathbb{P} \times \mathbb{G})).$$

Then V is a closed conic regular involutive submanifold of $\dot{T}^*\mathbb{G}$, and q identifies the fibers of p with the bicharacteristic leaves of V.

For $m \in \mathbb{Z}$, let $\mathcal{O}_{\mathbb{P}}(m)$ be the line bundle on $\mathbb{P}$ corresponding to the sheaf of homogeneous functions of degree m on $\mathbb{T}$, and denote by $\mathcal{N}_{(m)} := \mathcal{D}_{\mathbb{P}} \otimes_{\mathcal{O}} \mathcal{O}_{\mathbb{P}}(-m)$ the associated $\mathcal{D}_{\mathbb{P}}$-module. Denote by $\mathbb{D}g_*$ and $\mathbb{D}f^*$ the direct and inverse image in the derived categories of $\mathcal{D}$-modules and consider the family of $\mathcal{D}_{\mathbb{G}}$-modules

$$\mathcal{M}_{(1+m/2)} := H^0(\mathbb{D}g_* \mathbb{D}f^* \mathcal{N}_{(m)}).$$

For $n = 3$ and $p = 1$, Penrose identifies $\mathbb{G}$ with a conformal compactification of the complexified Minkowski space, and the $\mathcal{D}_{\mathbb{G}}$-module $\mathcal{M}_{(1+m/2)}$ corresponds to the massless field equation of helicity $1 + m/2$.

By [3], the microlocalization $\mathcal{E}_{\mathbb{G}} \otimes_{\pi^{-1}\mathcal{D}_{\mathbb{G}}} \pi^{-1}\mathcal{M}_{(1+m/2)}$ of $\mathcal{M}_{(1+m/2)}$ is globally simple along V.

Theorem 8.1. *Let $\mathfrak{S}$ be a stack of twisted sheaves on $\mathbb{G}$ and $\mathcal{M}$ an object of $\mathsf{Mod}(\mathcal{D}_{\mathbb{G}}; \mathfrak{S})$ whose microlocalization $\mathcal{E}_{\mathbb{G}} \otimes_{\pi^{-1}\mathcal{D}_{\mathbb{G}}} \pi^{-1}\mathcal{M}$ is globally simple along V. Then $\mathfrak{S}$ is globally $\mathbb{C}$-equivalent to $\mathfrak{Mod}(\mathbb{C}_{\mathbb{G}})$, so that $\mathsf{Mod}(\mathcal{D}_{\mathbb{G}}; \mathfrak{S})$ is $\mathbb{C}$-equivalent to $\mathsf{Mod}(\mathcal{D}_{\mathbb{G}})$.*

In other words, $\mathcal{M}$ is untwisted.

Proof. Let us start by recalling the microlocal geometry underlying the double fibration (8.1). There are identifications

$$\begin{aligned} T^*\mathbb{P} &= \{(y;\eta); \quad y \subset \mathbb{T}, \eta \in \mathrm{Hom}\,(\mathbb{T}/y, y)\}, \\ T^*\mathbb{G} &= \{(x;\xi); \quad x \subset \mathbb{T}, \xi \in \mathrm{Hom}\,(\mathbb{T}/x, x)\}, \\ T^*_{\mathbb{F}}(\mathbb{P} \times \mathbb{G}) &= \{(y,x;\tau); \quad y \subset x \subset \mathbb{T}, \tau \in \mathrm{Hom}\,(\mathbb{T}/x, y)\}. \end{aligned}$$

The maps p and q are described as follows:

$$\begin{array}{ccccc} \dot{T}^*\mathbb{P} & \xleftarrow[p]{} & \dot{T}^*_{\mathbb{F}}(\mathbb{P} \times \mathbb{G}) & \xrightarrow[q]{} & \dot{T}^*\mathbb{G} \\ (y; \tau \circ j) & \longleftarrow\!\!| & (y,x;\tau) & \longmapsto & (x; i \circ \tau), \end{array}$$

where $i \colon y \rightarrowtail x$ and $j \colon \mathbb{T}/y \twoheadrightarrow \mathbb{T}/x$ are the natural maps. We thus get

$$V = \{(x;\xi); \quad \mathrm{rk}(\xi) = 1\},$$

where $\mathrm{rk}(\xi)$ denotes the rank of the linear map ξ. In order to describe the bicharacteristic leaves of V, denote by $\mathbb{P}^*$ the dual projective space consisting of hyperplanes $z \subset \mathbb{T}$, and consider the incidence relation

$$\mathbb{A} = \{(y, z) \in \mathbb{P} \times \mathbb{P}^*; y \subset z \subset \mathbb{T}\}.$$

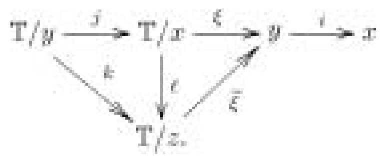
Then

$$\dot{T}^*_{\mathbb{A}}(\mathbb{P} \times \mathbb{P}^*) = \{(y, z; \theta); y \subset z \subset \mathbb{T},\ \theta\colon \mathbb{T}/z \xrightarrow{\sim} y\}.$$

There is an isomorphism

$$\begin{aligned} \dot{T}^*_{\mathbb{A}}(\mathbb{P} \times \mathbb{P}^*) &\xrightarrow{\sim} \dot{T}^*\mathbb{P} \\ (y, z; \theta) &\mapsto (y; \theta \circ k), \end{aligned}$$

where $k\colon \mathbb{T}/y \twoheadrightarrow \mathbb{T}/z$ is the natural map. Set $y = \operatorname{im}\xi$, $z = x + \ker\xi$, and consider the commutative diagram of linear maps



We thus get the following description of the composite map

$$\begin{array}{lccccccc} \widetilde{p}\colon & V & \xrightarrow{\sim} & \dot{T}^*_{\mathbb{F}}(\mathbb{P} \times \mathbb{G}) & \xrightarrow{p} & \dot{T}^*\mathbb{P} & \xrightarrow{\sim} & \dot{T}^*_{\mathbb{A}}(\mathbb{P} \times \mathbb{P}^*) \\ & (x;\xi) & \mapsto & (\operatorname{im}\xi, x; \xi) & \mapsto & (\operatorname{im}\xi; \xi \circ j) & \mapsto & (\operatorname{im}\xi, x + \ker\xi; \widetilde{\xi}). \end{array}$$

It follows that the bicharacteristic leaf $\Sigma_{(y,z,\theta)} := \widetilde{p}^{-1}(y, z, \theta)$ of V is given by

$$\begin{aligned} \Sigma_{(y,z,\theta)} &= \{(x;\xi); y = \operatorname{im}\xi,\ z = x + \ker\xi,\ \theta \circ \ell = \xi\} \\ &= \{(x;\xi); y \subset x \subset z,\ \xi = \theta \circ \ell\}, \end{aligned} \tag{8.2}$$

where $\ell\colon \mathbb{T}/x \twoheadrightarrow \mathbb{T}/z$ is the natural map. Thus, $\Sigma_{(y,z,\theta)}$ is the Grassmannian of p-dimensional linear subspaces in the $(n-1)$-dimensional vector space z/y.

Let us fix a point $(y, z, \theta) \in \dot{T}^*_{\mathbb{A}}(\mathbb{P} \times \mathbb{P}^*)$, and set $\Sigma = \Sigma_{(y,z,\theta)}$. In order to apply Corollary 7.1, we need to compute the map $\pi^\sharp_\Sigma$ and the class $[\mathfrak{S}_{\Omega^{1/2}_{\Sigma/\mathbb{G}}}]$.

The universal bundle $U_{\mathbb{G}} \to \mathbb{G}$ is the sub-bundle of the trivial bundle $\mathbb{G} \times \mathbb{T}$ whose fiber at $x \in \mathbb{G}$ is the $(p+1)$-dimensional linear subspace $x \subset \mathbb{T}$ itself.

Consider the line bundle $D_{\mathbb{G}} = \det U_{\mathbb{G}}$, and denote by $\mathcal{O}_{\mathbb{G}}(-1)$ the sheaf of its sections. Recall the isomorphisms

$$\begin{aligned}
&H^1(\mathbb{G};\mathbb{C}^\times_{\mathbb{G}}) \simeq H^2(\mathbb{G};\mathcal{O}^\times_{\mathbb{G}}) \simeq 0,\\
&H^1(\mathbb{G};\mathcal{O}^\times_{\mathbb{G}}) \simeq \mathbb{Z} \text{ with generator } [\mathcal{O}_{\mathbb{G}}(-1)],\\
&H^1(\mathbb{G};\mathcal{O}^\times_{\mathbb{G}}/\mathbb{C}^\times_{\mathbb{G}}) \simeq \mathbb{C} \text{ with generator } [\mathfrak{Mod}(\mathbb{C}_{\mathbb{G}}), \mathcal{O}_{\mathbb{G}}(-1)],
\end{aligned}$$

so that the sequence of abelian groups

$$H^1(\mathbb{G};\mathbb{C}^\times_{\mathbb{G}}) \underset{\alpha}{\longrightarrow} H^1(\mathbb{G};\mathcal{O}^\times_{\mathbb{G}}) \underset{\beta}{\longrightarrow} H^1(\mathbb{G};\mathcal{O}^\times_{\mathbb{G}}/\mathbb{C}^\times_{\mathbb{G}}) \underset{\delta}{\longrightarrow} H^2(\mathbb{G};\mathbb{C}^\times_{\mathbb{G}}) \longrightarrow H^2(\mathbb{G};\mathcal{O}^\times_{\mathbb{G}}),$$

is isomorphic to the sequence of additive abelian groups

$$0 \longrightarrow \mathbb{Z} \underset{\beta}{\longrightarrow} \mathbb{C} \underset{\delta}{\longrightarrow} \mathbb{C}/\mathbb{Z} \longrightarrow 0.$$

Similar results hold for Σ, which is also a grassmannian.

By Lemma 8.1 below one has $\pi_\Sigma^*\mathcal{O}_{\mathbb{G}}(-1) \simeq \mathcal{O}_\Sigma(-1)$. Hence $\pi_\Sigma^\sharp$ is the isomorphism

$$\pi_\Sigma^\sharp \colon H^2(\mathbb{G};\mathbb{C}^\times_{\mathbb{G}}) \simeq \mathbb{C}/\mathbb{Z} \simeq H^2(\Sigma;\mathbb{C}^\times_\Sigma).$$

There are isomorphisms

$$\Omega_{\mathbb{G}} \simeq \mathcal{O}_{\mathbb{G}}(-n-1), \qquad \Omega_\Sigma \simeq \mathcal{O}_\Sigma(-n+1).$$

Again by Lemma 8.1, we thus have

$$\pi_\Sigma^*\Omega_{\mathbb{G}} \simeq \pi_\Sigma^*\mathcal{O}_{\mathbb{G}}(-n-1) \simeq \mathcal{O}_\Sigma(-n-1).$$

It follows that $\Omega_{\Sigma/\mathbb{G}} \simeq \mathcal{O}_\Sigma(2)$, and thus

$$[\Omega_{\Sigma/\mathbb{G}}] = 2 \quad \text{in } \mathbb{Z} \simeq H^1(\Sigma;\mathcal{O}^\times_\Sigma).$$

Therefore

$$[\mathfrak{S}_{\Omega^{1/2}_{\Sigma/\mathbb{G}}}, \Omega^{1/2}_{\Sigma/\mathbb{G}}] = 1 \quad \text{in } \mathbb{C} \simeq H^1(\Sigma;\mathcal{O}^\times_\Sigma/\mathbb{C}^\times_\Sigma),$$

so that

$$[\mathfrak{S}_{\Omega^{1/2}_{\Sigma/\mathbb{G}}}] = \delta\left([\mathfrak{S}_{\Omega^{1/2}_{\Sigma/\mathbb{G}}}, \Omega^{1/2}_{\Sigma/\mathbb{G}}]\right) = 0 \quad \text{in } \mathbb{C}/\mathbb{Z} \simeq H^2(\Sigma;\mathbb{C}^\times_\Sigma).$$

The statement follows by Corollary 7.1. □

Lemma 8.1. *There is a natural isomorphism* $\pi_\Sigma^*\mathcal{O}_{\mathbb{G}}(-1) \simeq \mathcal{O}_\Sigma(-1)$*.*

Proof. Recall that $D_{\mathbb{G}}$ denotes the determinant of the universal bundle on $\mathbb{G}$. Geometrically, we have to prove that there is an isomorphism $\delta\colon D_\Sigma \xrightarrow{\sim} D_{\mathbb{G}}|_\Sigma$.

Recall the description (8.2), and let $(x;\xi) \in \Sigma$ for $p = (y, z, \theta) \in \dot{T}^*\mathbb{P}$. Then $(D_\Sigma)_{(x;\xi)} = \det(x/y)$, $(D_{\mathbb{G}})_{(x;\xi)} = \det x$, and δ is obtained by a trivialization of $\det y \simeq \mathbb{C}$. □

References

[1] Beilinson, A. and Bernstein, J., *A proof of Jantzen conjectures*, in: I. M. Gelfand Seminar, *Adv. Soviet Math.*, **16** part 1, *Amer. Math. Soc.*, (1993), 1-50.

[2] D'Agnolo, A. and Polesello, P., *Stacks of twisted modules and integral transforms*, to appear in: Geometric Aspects of Dwork's Theory, de Gruyter; see also `arXiv:math.AG/0307387`.

[3] D'Agnolo, A. and Schapira, P., Radon-Penrose transform for $\mathcal{D}$-modules, *J. Funct. Anal.*, **139** (1996), 349-382.

[4] Duistermaat, J. J. and Hörmander, L., Fourier integral operators. II, *Acta Math.*, **128** (1972), 183-269.

[5] Eastwood, M. G., Penrose, R., and Wells, R. O. Jr., Cohomology and massless fields, *Comm. Math. Phys.*, **78** (1980/81), 55-71.

[6] Kashiwara, M., Introduction to microlocal analysis, *Enseign. Math.*, **32** (1986), 227-259.

[7] ———, Representation theory and $\mathcal{D}$-modules on flag varieties, in: Orbites unipotentes et représentations, III, *Astérisque*, **173-174** (1989), 55-109.

[8] ———, Quantization of contact manifolds, *Publ. RIMS, Kyoto Univ.*, **32** (1996), 1-7.

[9] ———, *D-modules and microlocal calculus*, Transl. Math. Monogr., **217**, *Amer. Math. Soc.*, (2003).

[10] Kashiwara, M. and Kawai, T., On holonomic systems of microdifferential equations. III. Systems with regular singularities, *Publ. RIMS, Kyoto Univ.*, **17** (1981), 813-979.

[11] Kashiwara, M. and Oshima, T., Systems of differential equations with regular singularities and their boundary value problems, *Ann. of Math.*, **106** (1977), 145-200.

[12] Sato, M., Kawai, T. and Kashiwara, M., Hyperfunctions and pseudo-differential equations, in: Hyperfunctions and Pseudo-Differential Equations, H. Komatsu, editor. *Lecture Notes in Math.*, **287**, Springer (1973), 265-529.

[13] Schapira, P., *Microdifferential Systems in the Complex Domain*, Grundlehren Math. Wiss. **269**, Springer, 1985.