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Radon and Fourier transforms for $\mathscr{D}$-modules

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0. Introduction

The Fourier and Radon hyperplane transforms are closely related, and one such relation was established by Brylinski [4] in the framework of holonomic $\mathscr{D}$-modules. The integral kernel of the Radon hyperplane transform is associated with the hypersurface $\mathbb{S} \subset \mathbb{P} \times \mathbb{P}^*$ of pairs (x, y), where x is a point in the n-dimensional complex projective space $\mathbb{P}$ belonging to the hyperplane $y \in \mathbb{P}^*$. As it turns out, a useful variant is obtained by considering the integral transform associated with the open complement $\mathbb{U}$ of $\mathbb{S}$ in $\mathbb{P} \times \mathbb{P}^*$. In the first part of this paper, we generalize Brylinski's result in order to encompass this variant of the Radon transform, and also to treat arbitrary quasi-coherent $\mathscr{D}$-modules, as well as (twisted) abelian sheaves. Our proof is entirely geometrical, and consists in a reduction to the one-dimensional case by the use of homogeneous coordinates.

The second part of this paper applies the above result to the quantization of the Radon transform, in the sense of [7]. First we deal with line bundles. More precisely, let $\mathbb{P} = \mathbb{P}(\mathbb{V})$ be the projective space of lines in the vector space $\mathbb{V}$, denote by $(\bullet) \overset{\mathbb{D}}{\circ} \mathscr{R}$ the Radon transform associated with $\mathbb{U} \subset \mathbb{P} \times \mathbb{P}^*$, and for $m \in \mathbb{Z}$ set

$$m^* = -m - n - 1, \quad \mathscr{D}_{\mathbb{P}}(m) = \mathscr{D}_{\mathbb{P}} \otimes_{\mathscr{O}} \mathscr{O}_{\mathbb{P}}(m),$$

where $\mathscr{O}_{\mathbb{P}}(m)$ is the $-m$th tensor power of the tautological line bundle $\mathscr{O}_{\mathbb{P}}(-1)$. In [7], it was shown that the natural morphism

$$\mathscr{D}_{\mathbb{P}}(-m^*) \overset{\mathbb{D}}{\circ} \mathscr{R} \otimes \det \mathbb{V} \to \mathscr{D}_{\mathbb{P}^*}(-m)$$

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is an isomorphism for $m<0$. Using the Fourier transform we give a different proof of this result in Theorem 6, as well as a description of the Radon transform of $\mathscr{D}_{\mathbb{P}}(-m^*)$ for $m\geqslant 0$. Then we consider differential forms. More precisely, denote by $\mathscr{S}p^{\mathbb{P}}_{\bullet}$ the Spencer complex. Recall that the Spencer and de Rham complexes are interchanged by the solution functor, so that the shifted subcomplex $\mathscr{S}p^{\mathbb{P}}_{\geqslant q}[q]$ describes the sheaf of closed q-forms. We establish in Theorem 7 the isomorphism

$$\mathscr{S}p^{\mathbb{P}}_{\leqslant q}[q] \overset{\mathbb{D}}{\circ} \mathscr{R} \xleftarrow{\sim} \mathscr{S}p^{\mathbb{P}^*}_{\geqslant n-q}[n-q]. \tag{$*$}$$

Consider the maps $\mathbb{P} \xleftarrow{\pi} \mathbb{V}\backslash\{0\} \xrightarrow{j} \mathbb{V}$. Denoting by θ the Euler vector field, the sheaf $\pi^{-1}\Omega^q_{\mathbb{P}}$ is identified with the subsheaf of $j^{-1}\Omega^q_{\mathbb{V}}$ whose sections ω satisfy

$$L_\theta\omega = \theta \lrcorner \omega = 0,$$

where L_θ denotes the Lie derivative, and $\lrcorner$ the interior product. We obtain $(*)$ by first relating in Theorem 8 the Radon transform of the sheaf of q-forms with the subsheaf of $j^{-1}\Omega^{n+1-q}_{\mathbb{V}^*}$ whose sections σ satisfy the Fourier transform of the above relations, namely

$$L_\theta\sigma = d\sigma = 0.$$

1. Radon and Fourier transforms for $\mathscr{D}$-modules

Let V and W be mutually dual $(n+1)$-dimensional real vector spaces, P and P^* the associated projective spaces, and $x=(x_0,\dots,x_n)$ and $y=(y_0,\dots,y_n)$ dual systems of homogeneous coordinates. Consider the Leray form on P given by

$$\omega(x) = \sum_{j=0}^{n} (-1)^j x_j\, dx_0\wedge\cdots\widehat{dx_j}\cdots\wedge dx_n,$$

and note that, setting $\tilde{x}=tx$, one has $d\tilde{x} = t^n\omega(x)\,dt + t^{n+1}\,dx$. Let $u(t)$ be one of the distributions 1, $Y(t)$, $1/t$, or $\delta(t)$ on the real line, so that $\hat{u}(t)=\delta(t),\ 1/t,\ Y(t),\ 1$, respectively. Let $\varphi(x)$ be a homogeneous function with homogeneity degree such that $\varphi(x)\hat{u}(\langle x,y\rangle)\omega(x)$ descends to a relative density on $\mathsf{P}\times\mathsf{P}^*$ (e.g. if $u=1$, then $\hat{u}=\delta$, and φ must satisfy the homogeneity relation $\varphi(tx)=\operatorname{sgn}(t)^{-n}t^{-n}\varphi(x)$). One then has the following formal relation between the Radon and Fourier transforms, the usual Radon hyperplane transform corresponding to the case $u=1$,

$$\begin{aligned}\int \varphi(x)\hat{u}(\langle x,y\rangle)\omega(x) &= \int \varphi(x)\left(\int u(t)e^{-t\langle x,y\rangle}\,dt\right)\omega(x)\\ &= \int \psi(\tilde{x})e^{-\langle\tilde{x},y\rangle}\,d\tilde{x} \quad \text{for } \psi(\tilde{x})=\varphi(x)t^{-n}u(t).\end{aligned}$$

(It is quite delicate to make the above formula precise for functions, but [14] provides a convenient framework.) The aim of this section is to establish the corresponding relation for $\mathscr{D}$-modules, thus generalizing a result of Brylinski [4].

1.1. Review on algebraic $\mathscr{D}$-modules

For the reader's convenience, we recall here the notions and results from the theory of $\mathscr{D}$-modules that we need. Refer e.g. to [10,12,17] for the analytic case, and to [2,3] for the algebraic case.

Let X be a smooth algebraic variety over a field $\mathbf{k}$ of characteristic zero, and let $\mathscr{O}_X$ and $\mathscr{D}_X$ be its structure sheaf and the ring of differential operators, respectively. Let $\mathrm{Mod}(\mathscr{D}_X)$ be the abelian category of left $\mathscr{D}_X$-modules, $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$ its bounded derived category, and $\mathsf{D}^{\mathrm{b}}_{\text{q-coh}}(\mathscr{D}_X)$ (resp. $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$) the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$ whose objects have quasi-coherent (resp. coherent) cohomologies. To $\mathscr{M}\in\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ one associates its characteristic variety $\mathrm{char}(\mathscr{M})$, a closed involutive subvariety of the cotangent bundle T^*X.

We use the following notations for the operations of external tensor product, inverse image, and direct image for $\mathscr{D}$-modules:

$$\overset{\mathbb{D}}{\boxtimes} : \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)\times\mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y)\to\mathsf{D}^{\mathrm{b}}(\mathscr{D}_{X\times Y}),$$

$$\mathbb{D}f^* : \mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y)\to\mathsf{D}^{\mathrm{b}}(\mathscr{D}_X),$$

$$\mathbb{D}f_* : \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)\to\mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y),$$

where $f : X\to Y$ is a map of smooth algebraic varieties. More precisely, denoting by $\mathscr{D}_{X\to Y}$ and $\mathscr{D}_{Y\leftarrow X}$ the transfer bimodules, one has

$$\mathbb{D}f^*\mathscr{N} = \mathscr{D}_{X\to Y}\overset{L}{\underset{f^{-1}\mathscr{D}_Y}{\otimes}} f^{-1}\mathscr{N},$$
$$\mathbb{D}f_*\mathscr{M} = Rf_*(\mathscr{D}_{Y\leftarrow X}\overset{L}{\underset{\mathscr{D}_X}{\otimes}}\mathscr{M}).$$

Recall that these operations preserve quasi-coherency, and if $g : Y\to Z$ is another map of smooth algebraic varieties, then there are natural isomorphisms $\mathbb{D}g_*\mathbb{D}f_*\mathscr{M}\simeq\mathbb{D}(g\circ f)_*\mathscr{M}$ and $\mathbb{D}f^*\mathbb{D}g^*\mathscr{P}\simeq\mathbb{D}(g\circ f)^*\mathscr{P}$. Moreover, to any Cartesian square is attached a canonical isomorphism as follows:

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "h'"'] \arrow[dr, phantom, "\square"] & Y' \arrow[d, "h"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad \mathbb{D}h^*\mathbb{D}f_*\mathscr{M}[d_{Y'}-d_Y]\simeq\mathbb{D}f'_*\mathbb{D}h'^*\mathscr{M}[d_{X'}-d_X],\qquad \mathscr{M}\in\mathsf{D}^{\mathrm{b}}_{\text{q-coh}}(\mathscr{D}_X),$$

where d_X denotes the dimension of X.

The internal tensor product

$$\overset{\mathbb{D}}{\otimes} : \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X) \times \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X) \to \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$$

is defined by $\mathscr{M}_1 \overset{\mathbb{D}}{\otimes} \mathscr{M}_2 = \mathbb{D}\delta^*(\mathscr{M}_1 \overset{\mathbb{D}}{\boxtimes} \mathscr{M}_2)$, where $\delta : X \hookrightarrow X \times X$ is the diagonal embedding. Recall that $\mathscr{M}_1 \overset{\mathbb{D}}{\otimes} \mathscr{M}_2 \simeq \mathscr{M}_1 \otimes^L_{\mathscr{O}_X} \mathscr{M}_2$ as $\mathscr{O}_X$-modules, and $\mathbb{D}f^*(\mathscr{M}_1 \overset{\mathbb{D}}{\otimes} \mathscr{M}_2) \simeq \mathbb{D}f^*\mathscr{M}_1 \overset{\mathbb{D}}{\otimes} \mathbb{D}f^*\mathscr{M}_2$. Moreover, one has the projection formula

$$\mathbb{D}f_*(\mathscr{M} \overset{\mathbb{D}}{\otimes} \mathbb{D}f^*\mathscr{N}) \simeq \mathbb{D}f_*\mathscr{M} \overset{\mathbb{D}}{\otimes} \mathscr{N}, \quad \mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\text{q-coh}}(\mathscr{D}_X), \ \mathscr{N} \in \mathsf{D}^{\mathrm{b}}_{\text{q-coh}}(\mathscr{D}_Y).$$

The duality functor

$$\mathbb{D}_X : \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)^{\mathrm{op}} \to \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$$

is defined by $\mathbb{D}_X\mathscr{M} = R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X \otimes_{\mathscr{O}_X} \Omega_X^{\otimes -1})[d_X]$, where Ω_X denotes the sheaf of forms of maximal degree. Duality preserves coherency, but it does not preserve quasi-coherency, in general. The functor

$$\mathbb{D}f_! : \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X) \to \mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y)$$

is defined by $\mathbb{D}f_!\mathscr{M} = \mathbb{D}_Y\mathbb{D}f_*\mathbb{D}_X\mathscr{M}$.

Consider the microlocal correspondence associated with f:

$$T^*X \overset{f_d}{\leftarrow} X \times_Y T^*Y \overset{f_\pi}{\to} T^*Y.$$

One says that $\mathscr{N} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_Y)$ is non-characteristic for f if

$$f_d^{-1}(T^*_XX) \cap f_\pi^{-1}(\mathrm{char}(\mathscr{N})) \subset X \times_Y T^*_YY,$$

where T^*_XX denotes the zero section of T^*X. Recall the following results.

Theorem 1. (i) *The exterior tensor product* $\overset{\mathbb{D}}{\boxtimes}$ *preserves coherency and commutes with duality.*

(ii) *If f is proper, then $\mathbb{D}f_*$ preserves coherency and commutes with duality. In particular, $\mathbb{D}f_*\mathscr{M} \simeq \mathbb{D}f_!\mathscr{M}$ for $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$.*

(iii) *If $\mathscr{N} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_Y)$ is non-characteristic for f, then $\mathbb{D}f^*\mathscr{N}$ is coherent and $\mathbb{D}_X\mathbb{D}f^*\mathscr{N} \simeq \mathbb{D}f^*\mathbb{D}_Y\mathscr{N}$. In particular, if f is smooth then $\mathbb{D}f^*$ preserves coherency and commutes with duality.*

Let $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$ (resp. $\mathsf{D}^{\mathrm{b}}_{\text{r-hol}}(\mathscr{D}_X)$) be the full-triangulated subcategory of $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ consisting of holonomic (resp. regular holonomic) objects. Holonomy is stable for all of the above operations, and regular holonomy is stable under tensor product, inverse image, and proper direct image.

1.2. Review on the Fourier–Laplace transform

Let $\mathbb{V}$ be the affine space associated with an $(n+1)$-dimensional vector space over $\mathbf{k}$, and let $\mathbb{V}^*$ be the dual affine space. Denote by $D(\mathbb{V}) = \Gamma(\mathbb{V}; \mathscr{D}_{\mathbb{V}})$ the Weyl algebra, and recall that since $\mathbb{V}$ is affine the two functors

$$\mathsf{D}^{\mathrm{b}}_{\text{q-coh}}(\mathscr{D}_{\mathbb{V}}) \underset{\mathscr{D}_{\mathbb{V}} \otimes^{L}_{D(\mathbb{V})}(\bullet)}{\overset{\mathrm{R}\Gamma(\mathbb{V};\bullet)}{\rightleftarrows}} \mathsf{D}^{\mathrm{b}}_{\text{q-coh}}(D(\mathbb{V}))$$

are quasi-inverse to each other. The formal relation

$$P(x, \partial_x)e^{-\langle x,y \rangle} = Q(y, \partial_y)e^{-\langle x,y \rangle}$$

associates to each $Q \in D(\mathbb{V}^*)$ a unique $P \in D(\mathbb{V})$, called its Fourier transform. Since $P_1 P_2 e^{-\langle x,y \rangle} = P_1 Q_2 e^{-\langle x,y \rangle} = Q_2 P_1 e^{-\langle x,y \rangle} = Q_2 Q_1 e^{-\langle x,y \rangle}$, this gives a $\mathbf{k}$-algebra isomorphism

$$D(\mathbb{V}^*) \xrightarrow{\sim} D(\mathbb{V})^{\mathrm{op}}.$$

(Note that, choosing dual systems of coordinates $\mathbb{V} = \mathrm{Spec}(\mathbf{k}[x_0, \ldots, x_n])$ and $\mathbb{V}^* = \mathrm{Spec}(\mathbf{k}[y_0, \ldots, y_n])$, the above isomorphism is described by $y_i \mapsto -\partial_{x_i}$, $\partial_{y_i} \mapsto -x_i$.) Moreover, one has algebra isomorphisms

$$\begin{aligned} D(\mathbb{V})^{\mathrm{op}} &\simeq \Gamma(\mathbb{V}; \Omega_{\mathbb{V}} \otimes_{\mathcal{O}} \mathscr{D}_{\mathbb{V}} \otimes_{\mathcal{O}} \Omega_{\mathbb{V}}^{\otimes -1}) \\ &\simeq \det \mathbb{V}^* \otimes D(\mathbb{V}) \otimes \det \mathbb{V}, \end{aligned}$$

the identification $\Omega_{\mathbb{V}} \simeq \mathcal{O}_{\mathbb{V}} \otimes \det \mathbb{V}^*$ being induced by $T^*\mathbb{V} = \mathbb{V} \times \mathbb{V}^*$. It is then possible to consider the functor associating to a quasi-coherent $D(\mathbb{V})$-module M the quasi-coherent $D(\mathbb{V}^*)$-module $M^\wedge = \det \mathbb{V}^* \otimes M$. Since this functor is exact, it induces a functor

$$\wedge : \mathsf{D}^{\mathrm{b}}_{\text{q-coh}}(\mathscr{D}_{\mathbb{V}}) \to \mathsf{D}^{\mathrm{b}}_{\text{q-coh}}(\mathscr{D}_{\mathbb{V}^*}) \tag{1.1}$$

called the Fourier–Laplace transform. The Fourier–Laplace transform is an equivalence, it preserves coherency and holonomy, but it does not preserve regular holonomy, in general. (For references see e.g. [4,14,16].)

1.3. Review on the Radon transform(s)

Let $\mathbb{P} = \mathbb{P}(\mathbb{V})$ be the n-dimensional projective space associated with $\mathbb{V}$, and $\mathbb{P}^* = \mathbb{P}(\mathbb{V}^*)$ the dual projective space. Let us denote by $\mathbb{S}$ the smooth hypersurface of $\mathbb{P} \times \mathbb{P}^*$ defined by the homogeneous equation $\langle x, y \rangle = 0$, and set $\mathbb{U} = (\mathbb{P} \times \mathbb{P}^*) \setminus \mathbb{S}$. Identifying $\mathbb{P}^*$ with the family of hyperplanes in $\mathbb{P}$, the set $\mathbb{S}$ describes the incidence relation "the point $x \in \mathbb{P}$ belongs to the hyperplane $y \in \mathbb{P}^*$." Consider the smooth maps

$$\mathbb{P} \xleftarrow{p_{\mathbb{S}}} \mathbb{S} \xrightarrow{q_{\mathbb{S}}} \mathbb{P}^*, \quad \mathbb{P} \xleftarrow{p_{\mathbb{U}}} \mathbb{U} \xrightarrow{q_{\mathbb{U}}} \mathbb{P}^*,$$

defined by restriction of the natural projections p and q from $\mathbb{P} \times \mathbb{P}^*$. To these maps are attached the pull-back–push-forward functors

$$\mathbb{D}q_{\mathbb{S}*}\mathbb{D}p_{\mathbb{S}}^*, \ \mathbb{D}q_{\mathbb{U}*}\mathbb{D}p_{\mathbb{U}}^* : \mathsf{D}^{\mathrm{b}}_{\text{q-coh}}(\mathscr{D}_{\mathbb{P}}) \to \mathsf{D}^{\mathrm{b}}_{\text{q-coh}}(\mathscr{D}_{\mathbb{P}^*}). \tag{1.2}$$

The first functor is the $\mathscr{D}$-module analogue of the usual Radon transform, consisting in "integrating along hyperplanes." The second functor (cf. [1,15,17]) is a small variation[1] on the first one which has, amongst others, the advantage of giving an equivalence of categories.

Note that since $p_{\mathbb{S}}$ and $q_{\mathbb{S}}$ are smooth and proper, the first functor preserves coherency. Even though $q_{\mathbb{U}}$ is not proper, it follows e.g. from Lemma 1 below that also $\mathbb{D}q_{\mathbb{U}*}\mathbb{D}p_{\mathbb{U}}^*$ preserves coherency, as does the functor

$$\mathbb{D}q_{\mathbb{U}!}\mathbb{D}p_{\mathbb{U}}^* : \mathsf{D}^{\mathrm{b}}(\mathscr{D}_{\mathbb{P}}) \to \mathsf{D}^{\mathrm{b}}(\mathscr{D}_{\mathbb{P}^*}). \tag{1.3}$$

(For references see e.g. [7].)

1.4. Review on the blow-up transform(s)

Let $\dot{\mathbb{V}} = \mathbb{V} \setminus \{0\}$ and consider the natural projection and embedding

$$\mathbb{P} \xleftarrow{\pi} \dot{\mathbb{V}} \overset{j}{\hookrightarrow} \mathbb{V}.$$

They induce an embedding (π, j) of $\dot{\mathbb{V}}$ as a locally closed subvariety of $\mathbb{P} \times \mathbb{V}$. Let $\widetilde{\mathbb{V}_0}$ be the closure of $\dot{\mathbb{V}}$ in $\mathbb{P} \times \mathbb{V}$, a smooth subvariety, and consider the maps

$$\mathbb{P} \xleftarrow{\tilde{\pi}} \widetilde{\mathbb{V}_0} \xrightarrow{\tilde{j}} \mathbb{V}$$

obtained by restriction of the natural projections from $\mathbb{P} \times \mathbb{V}$. Note that $\tilde{j}$ is the blow-up of the origin 0 in $\mathbb{V}$, $\tilde{j}$ is proper, and $\tilde{\pi}$ is smooth. To these maps are

[1] As follows e.g. from (1.9) and Lemma 1 below, there is a distinguished triangle $\mathcal{O}_{\mathbb{P}^*} \otimes \mathrm{R}\Gamma(\mathbb{P}; \Omega_{\mathbb{P}} \otimes^L_{\mathscr{D}_{\mathbb{P}}} \mathscr{M}) \to \mathbb{D}q_{\mathbb{U}*}\mathbb{D}p_{\mathbb{U}}^*\mathscr{M} \to \mathbb{D}q_{\mathbb{S}*}\mathbb{D}p_{\mathbb{S}}^*\mathscr{M} \xrightarrow{+1}$.

attached the functors

$$\mathbb{D}j_*\mathbb{D}\pi^*, \quad \mathbb{D}\tilde{\jmath}_*\mathbb{D}\tilde{\pi}^* : \mathsf{D}^{\mathrm{b}}_{\text{q-coh}}(\mathscr{D}_{\mathbb{P}}) \to \mathsf{D}^{\mathrm{b}}_{\text{q-coh}}(\mathscr{D}_{\mathbb{V}}). \tag{1.4}$$

Using similar remarks as for the Radon transform one checks that these functors preserve coherency, as does the functor

$$\mathbb{D}j_!\mathbb{D}\pi^* : \mathsf{D}^{\mathrm{b}}(\mathscr{D}_{\mathbb{P}}) \to \mathsf{D}^{\mathrm{b}}(\mathscr{D}_{\mathbb{V}}). \tag{1.5}$$

1.5. A first statement of the result

As a last piece of notation, let $\dot{\mathbb{V}}^* = \mathbb{V}^* \setminus \{0\}$ and consider the natural projection and embedding

$$\mathbb{P}^* \overset{\pi}{\leftarrow} \dot{\mathbb{V}}^* \overset{j}{\hookrightarrow} \mathbb{V}^*.$$

The next theorem generalizes a result of Brylinski [4, Théorème 7.27], who obtained the isomorphism (1.7) assuming $\mathscr{M}$ regular holonomic. In order to help the reader in following the pull-back–push-forward procedures, let us summarize in the next diagram the maps that we will use. The starting point is $\mathbb{P}$, and the target is $\dot{\mathbb{V}}^*$.

$$\begin{tikzcd}
& \mathbb{V} \arrow[rr, dash, dotted, "\wedge"] & & \mathbb{V}^* \\
\widetilde{\mathbb{V}_0} \arrow[ur, "\tilde{\jmath}"] \arrow[dr, "\tilde{\pi}"'] & \dot{\mathbb{V}} \arrow[l, hook'] \arrow[u, hook, "j"] \arrow[d, "\pi"'] & \mathbb{S} \arrow[dl, "p_{\mathbb{S}}"'] \arrow[dr, "q_{\mathbb{S}}"] & \dot{\mathbb{V}}^* \arrow[u, hook, "j"'] \arrow[d, "\pi"] \\
& \mathbb{P} & \mathbb{U} \arrow[l, "p_{\mathbb{U}}"] \arrow[r, "q_{\mathbb{U}}"'] & \mathbb{P}^*
\end{tikzcd} \tag{1.6}$$

Theorem 2. *For $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_{\mathbb{P}})$ there are natural isomorphisms in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_{\dot{\mathbb{V}}^*})$*

$$\mathbb{D}\pi^*(\mathbb{D}q_{\mathbb{U}*}\mathbb{D}p^*_{\mathbb{U}}\mathscr{M}) \simeq \mathbb{D}j^*[(\mathbb{D}j_!\mathbb{D}\pi^*\mathscr{M})^\wedge],$$

$$\mathbb{D}\pi^*(\mathbb{D}q_{\mathbb{U}!}\mathbb{D}p^*_{\mathbb{U}}\mathscr{M}) \simeq \mathbb{D}j^*[(\mathbb{D}j_*\mathbb{D}\pi^*\mathscr{M})^\wedge].$$

For $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\text{q-coh}}(\mathscr{D}_{\mathbb{P}})$ there is a natural isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_{\dot{\mathbb{V}}^*})$

$$\mathbb{D}\pi^*(\mathbb{D}q_{\mathbb{S}*}\mathbb{D}p^*_{\mathbb{S}}\mathscr{M}) \simeq \mathbb{D}j^*[(\mathbb{D}\tilde{\jmath}_*\mathbb{D}\tilde{\pi}^*\mathscr{M})^\wedge]. \tag{1.7}$$

The statement may be visualized by the commutative diagram:

$$\begin{tikzcd}
\mathsf{D}^{\mathrm{b}}(\mathscr{D}_{\mathbb{V}}) \arrow[r, "\text{Fourier}"] & \mathsf{D}^{\mathrm{b}}(\mathscr{D}_{\mathbb{V}^*}) \arrow[dr, "\text{pull-back}"] & \\
& & \mathsf{D}^{\mathrm{b}}(\mathscr{D}_{\dot{\mathbb{V}}^*}) \\
\mathsf{D}^{\mathrm{b}}(\mathscr{D}_{\mathbb{P}}) \arrow[uu, "\text{Blow-up}"] \arrow[r, "\text{Radon}"'] & \mathsf{D}^{\mathrm{b}}(\mathscr{D}_{\mathbb{P}^*}) \arrow[ur, "\text{pull-back}"'] &
\end{tikzcd}$$

In order to prove this theorem we will first restate it, using the language of integral kernels, as Theorem 3. This has the advantage of applying to quasi-coherent modules, and gives a reason for the strange-looking pattern of $*$'s and !'s in the above formulae.

1.6. Review on integral kernels

Let X and Y be smooth algebraic varieties, and consider the projections

$$X \overset{p}{\leftarrow} X \times Y \overset{q}{\rightarrow} Y.$$

For $\mathscr{K} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_{X\times Y})$ the functor

$$(\bullet) \overset{\mathbb{D}}{\circ} \mathscr{K} : \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X) \to \mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y)$$

$$\mathscr{M} \mapsto \mathscr{M} \overset{\mathbb{D}}{\circ} \mathscr{K} = \mathbb{D}q_*(\mathbb{D}p^*\mathscr{M} \overset{\mathbb{D}}{\otimes} \mathscr{K})$$

is called integral transform with kernel $\mathscr{K}$. More generally, if Z is another smooth algebraic variety and $\mathscr{L} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_{Y\times Z})$, one sets

$$\mathscr{K} \overset{\mathbb{D}}{\circ} \mathscr{L} = \mathbb{D}q_{13*}(\mathbb{D}q_{12}^*\mathscr{K} \overset{\mathbb{D}}{\otimes} \mathbb{D}q_{23}^*\mathscr{L}) \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_{X\times Z}),$$

where q_{ij} denotes the projection from $X \times Y \times Z$ to the corresponding factors, so that for example $q_{13}(x,y,z) = (x,z)$. The bifunctor $\overset{\mathbb{D}}{\circ}$ preserves quasi-coherency, is associative in the sense that $(\mathscr{M} \overset{\mathbb{D}}{\circ} \mathscr{K}) \overset{\mathbb{D}}{\circ} \mathscr{L} \simeq \mathscr{M} \overset{\mathbb{D}}{\circ} (\mathscr{K} \overset{\mathbb{D}}{\circ} \mathscr{L})$, and the identity functor corresponds to the regular holonomic kernel $\mathscr{B}_{X|X\times X} = \mathbb{D}\delta_*\mathscr{O}_X$, where $\delta : X \hookrightarrow X \times X$ is the diagonal embedding.

One says that $\mathscr{K} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_{X\times Y})$ and $\mathscr{L} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_{Y\times Z})$ are transversal if

$$(\mathrm{char}(\mathscr{K}) \times T_Z^*Z) \cap (T_X^*X \times \mathrm{char}(\mathscr{L})) \subset T^*_{X\times Y\times Z}(X \times Y \times Z).$$

In particular, $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ is transversal to $\mathscr{K}$ if

$$(\mathrm{char}(\mathscr{M}) \times T^*_Y Y) \cap \mathrm{char}(\mathscr{K}) \subset T^*_{X\times Y}(X \times Y).$$

In this case, assuming moreover that $\mathrm{supp}(\mathscr{K})$ is proper over Y, it follows from Theorem 1 that $\mathscr{M} \overset{\mathbb{D}}{\circ} \mathscr{K}$ is coherent, and

$$\mathbb{D}_Y(\mathscr{M} \overset{\mathbb{D}}{\circ} \mathscr{K}) \simeq \mathbb{D}_X \mathscr{M} \overset{\mathbb{D}}{\circ} \mathbb{D}_{X\times Y}\mathscr{K}. \tag{1.8}$$

1.7. Basic regular holonomic kernels

Let S be a smooth variety, let Z be a closed smooth subvariety of S of codimension d, set $U = S \backslash Z$, and consider the embeddings

$$j_Z : Z \hookrightarrow S, \quad j_U : U \hookrightarrow S.$$

The simplest regular holonomic $\mathscr{D}_S$-modules attached to the stratification $S = Z \sqcup U$ are

$$\mathscr{O}_S, \quad \mathscr{B}_{Z|S} = \mathbb{D}j_{Z*}\mathscr{O}_Z, \quad \mathscr{B}_{U|S} = \mathbb{D}j_{U*}\mathscr{O}_U, \quad \mathbb{D}_S\mathscr{B}_{U|S} = \mathbb{D}j_{U!}\mathscr{O}_U.$$

As an alternative description, one has

$$\mathscr{B}_{Z|S} = \mathrm{R}\Gamma_{[Z]}\mathscr{O}_S[d], \quad \mathscr{B}_{U|S} = \mathrm{R}\Gamma_{[U]}\mathscr{O}_S,$$

where $\mathrm{R}\Gamma_{[Z]}\mathscr{M} \simeq \mathbb{D}j_{Z*}\mathbb{D}j_Z^*\mathscr{M}[-d]$, and $\mathrm{R}\Gamma_{[U]}\mathscr{M} \simeq \mathbb{D}j_{U*}\mathbb{D}j_U^*\mathscr{M}$. Recall that one has a distinguished triangle

$$\mathrm{R}\Gamma_{[Z]}\mathscr{M} \to \mathscr{M} \to \mathrm{R}\Gamma_{[U]}\mathscr{M} \xrightarrow{+1}. \tag{1.9}$$

The basic model is the stratification $\mathbb{A}^1_{\mathbf{k}} = \{0\} \sqcup \dot{\mathbb{A}}^1_{\mathbf{k}}$ of the affine line $\mathbb{A}^1_{\mathbf{k}} = \mathrm{Spec}(\mathbf{k}[t])$, where one has the regular holonomic modules

$$\begin{aligned} \mathscr{O}_{\mathbb{A}^1_{\mathbf{k}}} &= \mathscr{D}_{\mathbb{A}^1_{\mathbf{k}}}/\langle \partial_t \rangle = \mathscr{D}_{\mathbb{A}^1_{\mathbf{k}}} \cdot 1, \\ \mathscr{B}_{0|\mathbb{A}^1_{\mathbf{k}}} &= \mathscr{D}_{\mathbb{A}^1_{\mathbf{k}}}/\langle t \rangle = \mathscr{D}_{\mathbb{A}^1_{\mathbf{k}}} \cdot \delta, \\ \mathscr{B}_{\dot{\mathbb{A}}^1_{\mathbf{k}}|\mathbb{A}^1_{\mathbf{k}}} &= \mathscr{D}_{\mathbb{A}^1_{\mathbf{k}}}/\langle \partial_t\, t \rangle = \mathscr{D}_{\mathbb{A}^1_{\mathbf{k}}} \cdot 1/t, \\ \mathbb{D}_{\mathbb{A}^1_{\mathbf{k}}}\mathscr{B}_{\dot{\mathbb{A}}^1_{\mathbf{k}}|\mathbb{A}^1_{\mathbf{k}}} &= \mathscr{D}_{\mathbb{A}^1_{\mathbf{k}}}/\langle t\, \partial_t \rangle = \mathscr{D}_{\mathbb{A}^1_{\mathbf{k}}} \cdot Y. \end{aligned} \tag{1.10}$$

Here we used the pattern

$$\mathscr{M} = \mathscr{D}_{\mathbb{A}^1_{\mathbf{k}}}/\langle P \rangle = \mathscr{D}_{\mathbb{A}^1_{\mathbf{k}}} \cdot u$$

to indicate that $\mathscr{M}$ is a cyclic $\mathscr{D}_{\mathbb{A}^1_{\mathbf{k}}}$-module with generator u and relation $Pu = 0$.

Let now S be a closed smooth subvariety of $X \times Y$, and consider the embedding

$$i : S \hookrightarrow X \times Y,$$

and the maps

$$X \overset{p_S}{\leftarrow} S \overset{q_S}{\rightarrow} Y, \quad X \overset{p_U}{\leftarrow} U \overset{q_U}{\rightarrow} Y,$$

obtained by restriction of the natural projections p and q from $X \times Y$. Note that $\mathbb{D}i_* \mathcal{O}_S \simeq \mathcal{B}_{S|X \times Y}$, $\mathbb{D}i_* \mathcal{B}_{Z|S} \simeq \mathcal{B}_{Z|X \times Y}$.

Lemma 1. *For $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\text{q-coh}}(\mathcal{D}_X)$, there are natural isomorphisms in $\mathsf{D}^{\mathrm{b}}(\mathcal{D}_Y)$:*

$$\mathcal{M} \overset{\mathbb{D}}{\circ} \mathcal{B}_{S|X \times Y} \simeq \mathbb{D}q_{S*} \mathbb{D}p_S^* \mathcal{M},$$

$$\mathcal{M} \overset{\mathbb{D}}{\circ} \mathbb{D}i_* \mathcal{B}_{U|S} \simeq \mathbb{D}q_{U*} \mathbb{D}p_U^* \mathcal{M},$$

$$\mathcal{M} \overset{\mathbb{D}}{\circ} \mathcal{O}_{X \times Y} \simeq \mathcal{O}_Y \otimes \mathrm{DR}(\mathcal{M}),$$

where $\mathrm{DR}(\mathcal{M}) = \mathrm{R}\Gamma(X; \Omega_X \otimes^L_{\mathcal{D}_X} \mathcal{M})$. If moreover $\mathcal{M}$ is coherent and transversal to $\mathbb{D}i_* \mathcal{B}_{U|S}$, and S is proper over Y, then there is an isomorphism of functors from $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ to $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_Y)$:

$$\mathcal{M} \overset{\mathbb{D}}{\circ} \mathbb{D}i_* \mathbb{D}_S \mathcal{B}_{U|S} \simeq \mathbb{D}q_{U!} \mathbb{D}p_U^* \mathcal{M}.$$

In order to check the transversality condition, note that

$$\mathrm{char}(\mathbb{D}i_* \mathcal{B}_{U|S}) \subset T^*_Z(X \times Y) \cup T^*_S(X \times Y).$$

Proof. The first isomorphism is a particular case of the second one for $Z = \emptyset$, $S = U$. To prove the second isomorphism, note that for $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\text{q-coh}}(\mathcal{D}_X)$ there is the chain of isomorphisms

$$\begin{aligned}\mathcal{M} \overset{\mathbb{D}}{\circ} \mathbb{D}i_* \mathcal{B}_{U|S} &\simeq \mathbb{D}q_*(\mathbb{D}p^* \mathcal{M} \overset{\mathbb{D}}{\otimes} \mathbb{D}i_* \mathbb{D}j_{U*} \mathcal{O}_U) \\ &\simeq \mathbb{D}q_* \mathbb{D}i_* \mathbb{D}j_{U*}(\mathbb{D}j_U^* \mathbb{D}i^* \mathbb{D}p^* \mathcal{M} \overset{\mathbb{D}}{\otimes} \mathcal{O}_U) \\ &\simeq \mathbb{D}q_{U*} \mathbb{D}p_U^* \mathcal{M}.\end{aligned}$$

As for the third isomorphism, using the first one with $S = X \times Y$ we get

$$\begin{aligned}\mathcal{M} \overset{\mathbb{D}}{\circ} \mathcal{O}_{X \times Y} &\simeq \mathbb{D}q_* \mathbb{D}p^* \mathcal{M} \\ &\simeq \mathbb{D}a_Y^* \mathbb{D}a_{X*} \mathcal{M},\end{aligned}$$

where $a_X : X \to \{\mathrm{pt}\}$ denotes the map to the variety reduced to a point. Finally, for $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$, the last isomorphism follows from the second one by (1.8), as follows:

$$\begin{aligned}
\mathscr{M} \overset{\mathbb{D}}{\circ} \mathbb{D}i_* \mathbb{D}_S \mathscr{B}_{U|S} &\simeq \mathscr{M} \overset{\mathbb{D}}{\circ} \mathbb{D}_{X\times Y} \mathbb{D}i_* \mathscr{B}_{U|S} \\
&\simeq \mathbb{D}_Y((\mathbb{D}_X \mathscr{M}) \overset{\mathbb{D}}{\circ} \mathbb{D}i_* \mathscr{B}_{U|S}) \\
&\simeq \mathbb{D}_Y \mathbb{D}q_{U*} \mathbb{D}p_U^* \mathbb{D}_X \mathscr{M} \\
&\simeq \mathbb{D}_Y \mathbb{D}q_{U*} \mathbb{D}_U \mathbb{D}p_U^* \mathscr{M} \\
&= \mathbb{D}q_{U!} \mathbb{D}p_U^* \mathscr{M}. \qquad \square
\end{aligned}$$

1.8. Radon and Fourier transforms for $\mathscr{D}$-modules

Consider the holonomic kernel (irregular at infinity)

$$\mathscr{L} = \mathscr{D}_{\mathbb{V}\times\mathbb{V}^*}/\mathscr{I} = \mathscr{D}_{\mathbb{V}\times\mathbb{V}^*} e^{-\langle x,y\rangle}, \tag{1.11}$$

where $\mathscr{I}$ is the left ideal of differential operators $P \in \mathscr{D}_{\mathbb{V}\times\mathbb{V}^*}$ such that, formally, $Pe^{-\langle x,y\rangle} = 0$. Then $\mathscr{L}$ is the kernel attached to the Fourier–Laplace transform, since one has (see [16, Section 7.5])

$$\mathscr{M}^{\wedge} \simeq \mathscr{M} \overset{\mathbb{D}}{\circ} \mathscr{L}, \quad \mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\text{q-coh}}(\mathscr{D}_{\mathbb{V}}).$$

Concerning the Radon transform, it follows from Lemma 1 that the functors in (1.2) and (1.3) are given by composition with the regular holonomic kernels attached to the stratification $\mathbb{P} \times \mathbb{P}^* = \mathbb{S} \sqcup \mathbb{U}$. According to (1.10), let us give these kernels the following names:

$$\mathscr{R}_1 = \mathscr{O}_{\mathbb{P}\times\mathbb{P}^*}, \quad \mathscr{R}_Y = \mathbb{D}_{\mathbb{P}\times\mathbb{P}^*} \mathscr{B}_{\mathbb{U}|\mathbb{P}\times\mathbb{P}^*}, \quad \mathscr{R}_{1/t} = \mathscr{B}_{\mathbb{U}|\mathbb{P}\times\mathbb{P}^*}, \quad \mathscr{R}_\delta = \mathscr{B}_{\mathbb{S}|\mathbb{P}\times\mathbb{P}^*}. \tag{1.12}$$

As for the blow-up, let $\mathbb{E} = \widetilde{\mathbb{V}_0} \backslash \dot{\mathbb{V}}$ be its exceptional divisor, a smooth hypersurface of $\widetilde{\mathbb{V}_0}$. It follows from Lemma 1 that the functors in (1.4) and (1.5) are given by composition with the regular holonomic kernels attached to the stratification $\widetilde{\mathbb{V}_0} = \mathbb{E} \sqcup \dot{\mathbb{V}}$. According to (1.10), let us give these kernels the following names:

$$\mathscr{S}_1 = \mathscr{O}_{\widetilde{\mathbb{V}_0}}, \quad \mathscr{S}_Y = \mathbb{D}_{\widetilde{\mathbb{V}_0}} \mathscr{B}_{\dot{\mathbb{V}}|\widetilde{\mathbb{V}_0}}, \quad \mathscr{S}_{1/t} = \mathscr{B}_{\dot{\mathbb{V}}|\widetilde{\mathbb{V}_0}}, \quad \mathscr{S}_\delta = \mathscr{B}_{\mathbb{E}|\widetilde{\mathbb{V}_0}}. \tag{1.13}$$

Summarizing, one has

$u =$	1	Y	$1/t$	δ
$\mathcal{R}_u =$	$\mathcal{O}_{\mathbb{P}\times\mathbb{P}^*}$	$\mathbb{D}_{\mathbb{P}\times\mathbb{P}^*}\mathcal{B}_{\mathbb{U}\vert\mathbb{P}\times\mathbb{P}^*}$	$\mathcal{B}_{\mathbb{U}\vert\mathbb{P}\times\mathbb{P}^*}$	$\mathcal{B}_{\mathbb{S}\vert\mathbb{P}\times\mathbb{P}^*}$
$(\bullet)\overset{\mathbb{D}}{\circ}\mathcal{R}_u \simeq$	$\mathcal{O}_{\mathbb{P}^*}\otimes \mathrm{DR}(\bullet)$	$\mathbb{D}q_{\mathbb{U}!}\mathbb{D}p_{\mathbb{U}}{}^*$	$\mathbb{D}q_{\mathbb{U}*}\mathbb{D}p_{\mathbb{U}}{}^*$	$\mathbb{D}q_{\mathbb{S}*}\mathbb{D}p_{\mathbb{S}}{}^*$
$\mathcal{S}_u =$	$\mathcal{O}_{\widetilde{\mathbb{V}_0}}$	$\mathbb{D}_{\widetilde{\mathbb{V}_0}}\mathcal{B}_{\dot{\mathbb{V}}\vert\widetilde{\mathbb{V}_0}}$	$\mathcal{B}_{\dot{\mathbb{V}}\vert\widetilde{\mathbb{V}_0}}$	$\mathcal{B}_{\mathbb{E}\vert\widetilde{\mathbb{V}_0}}$
$(\bullet)\overset{\mathbb{D}}{\circ}\mathbb{D}\tilde{\imath}_*\mathcal{S}_u \simeq$	$\mathbb{D}\tilde{\jmath}_*\mathbb{D}\tilde{\pi}^*$	$\mathbb{D}j_!\mathbb{D}\pi^*$	$\mathbb{D}j_*\mathbb{D}\pi^*$	$\mathcal{B}_{0\vert\mathbb{V}}\otimes \mathrm{DR}(\bullet)$

Consider the maps

$$\widetilde{\mathbb{V}_0} \overset{\tilde{\imath}}{\hookrightarrow} \mathbb{P}\times\mathbb{V}, \quad \mathbb{P}^* \overset{\pi}{\leftarrow} \dot{\mathbb{V}}^* \overset{j}{\rightarrow} \mathbb{V}^*.$$

Theorem 3. *Let* $\mathcal{M}\in \mathsf{D}^{\mathrm{b}}_{\text{q-coh}}(\mathcal{D}_{\mathbb{P}})$, *and let u be one of the four generators in* (1.10), *so that*

$$u = 1, Y, 1/t, \delta, \qquad \hat{u} = \delta, 1/t, Y, 1,$$

respectively. Then there is a natural isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathcal{D}_{\dot{\mathbb{V}}^*})$:

$$\mathbb{D}\pi^*(\mathcal{M}\overset{\mathbb{D}}{\circ}\mathcal{R}_{\hat{u}})\simeq \mathbb{D}j^*(\mathcal{M}\overset{\mathbb{D}}{\circ}\mathbb{D}\tilde{\imath}_*\mathcal{S}_u\overset{\mathbb{D}}{\circ}\mathcal{L}).$$

As we already pointed out, this statement implies Theorem 2.

Proof. Consider the maps

$$\mathbb{P}\times\mathbb{P}^* \overset{\pi''}{\leftarrow} \mathbb{P}\times\dot{\mathbb{V}}^* \overset{j''}{\rightarrow} \mathbb{P}\times\mathbb{V}^*$$

induced by $\mathbb{P}^* \overset{\pi}{\leftarrow} \dot{\mathbb{V}}^* \overset{j}{\rightarrow} \mathbb{V}^*$. Denote by $\mathbb{S}''$ the hypersurface of $\mathbb{P}\times\mathbb{V}^*$ defined by the equation $\langle x, y\rangle = 0$, let $\mathbb{U}'' = (\mathbb{P}\times\mathbb{V}^*)\setminus\mathbb{S}''$, and set

$$\mathcal{R}''_1 = \mathcal{O}_{\mathbb{P}\times\mathbb{V}^*}, \quad \mathcal{R}''_Y = \mathbb{D}_{\mathbb{P}\times\mathbb{V}^*}\mathcal{B}_{\mathbb{U}''\vert\mathbb{P}\times\mathbb{V}^*}, \quad \mathcal{R}''_{1/t} = \mathcal{B}_{\mathbb{U}''\vert\mathbb{P}\times\mathbb{V}^*}, \quad \mathcal{R}''_\delta = \mathcal{B}_{\mathbb{S}''\vert\mathbb{P}\times\mathbb{V}^*}.$$

One has

$$\mathbb{D}\pi^*(\mathcal{M}\overset{\mathbb{D}}{\circ}\mathcal{R}_{\hat{u}})\simeq \mathcal{M}\overset{\mathbb{D}}{\circ}\mathbb{D}\pi''^*\mathcal{R}_{\hat{u}}\simeq \mathcal{M}\overset{\mathbb{D}}{\circ}\mathbb{D}j''^*\mathcal{R}''_{\hat{u}}\simeq \mathbb{D}j^*(\mathcal{M}\overset{\mathbb{D}}{\circ}\mathcal{R}''_{\hat{u}}).$$

Then the statement is a corollary of the following proposition. $\square$

Proposition 1. *There is an isomorphism in* $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_{\mathbb{P}\times\mathbb{V}^*})$:

$$\mathscr{R}''_{\hat{u}} \simeq \mathbb{D}\tilde{\imath}_*\mathscr{S}_u \overset{\mathbb{D}}{\circ} \mathscr{L}.$$

Proof. Let us start by observing that $\widetilde{\mathbb{V}_0}$ is the quotient of $\mathbb{A}^1_{\mathbf{k}} \times \dot{\mathbb{V}} \simeq \dot{\mathbb{V}} \times_{\mathbb{P}} \widetilde{\mathbb{V}_0}$ by the action of the multiplicative group $\mathbb{G}_m$ given by $c(t,x) = (c^{-1}t, cx)$. Let us denote by $[\![t,x]\!]$ the equivalence class of (t,x). Consider the commutative diagram

$$\begin{tikzcd}
\mathbb{A}^1_{\mathbf{k}} & \mathbb{A}^1_{\mathbf{k}} \times \dot{\mathbb{V}} \arrow[l, "t"'] \arrow[r, "\tau"] \arrow[dr, phantom, "\square"] & \widetilde{\mathbb{V}_0} \arrow[r, hook, "\tilde{\imath}"] \arrow[dr, phantom, "\square"] & \mathbb{P} \times \mathbb{V} \\
\mathbb{A}^1_{\mathbf{k}} \times \mathbb{A}^1_{\mathbf{k}} \arrow[u, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\square"] & \mathbb{A}^1_{\mathbf{k}} \times \dot{\mathbb{V}} \times \mathbb{V}^* \arrow[l, "\dot{\gamma}''"'] \arrow[r, "\tau'"] \arrow[u, "p_{12}"] \arrow[d, "p_{23}"] \arrow[dr, phantom, "\square"] & \widetilde{\mathbb{V}_0} \times \mathbb{V}^* \arrow[r, hook, "\tilde{\imath}'"] \arrow[u, "q_1"] \arrow[d, "\tilde{\pi}'"] \arrow[dr, "\tilde{\jmath}'"] & \mathbb{P} \times \mathbb{V} \times \mathbb{V}^* \arrow[u, "q_{12}"'] \arrow[d, "q_{23}"] \arrow[dl, "q_{13}"'] \\
\mathbb{A}^1_{\mathbf{k}} & \dot{\mathbb{V}} \times \mathbb{V}^* \arrow[l, "\dot{\gamma}"'] \arrow[r, "\pi'"] & \mathbb{P} \times \mathbb{V}^* & \mathbb{V} \times \mathbb{V}^*
\end{tikzcd}$$

where p_i, q_i, p_{ij}, and q_{ij} are the natural projections,

$$t(t,x) = t, \quad \tau(t,x) = [\![t,x]\!], \quad \dot{\gamma}(x,y) = \langle x,y \rangle,$$

$$\tilde{\imath}([\![t,x]\!]) = ([x], tx), \quad \tilde{\jmath}([\![t,x]\!]) = tx, \quad \tilde{\pi}([\![t,x]\!]) = [x],$$

$\dot{\gamma}'' = \mathrm{id}_{\mathbb{A}^1_{\mathbf{k}}} \times \dot{\gamma}$, and $f' = f \times \mathrm{id}_{\mathbb{V}^*}$ for $f = \tau, \tilde{\imath}, \tilde{\pi}, \tilde{\jmath}, \pi$. There are natural isomorphisms in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_{\mathbb{P}\times\mathbb{V}^*})$:

$$\begin{aligned}
\mathbb{D}\tilde{\imath}_*\mathscr{S}_u \overset{\mathbb{D}}{\circ} \mathscr{L} &\simeq \mathbb{D}q_{13*}(\mathbb{D}q_{12}^*\mathbb{D}\tilde{\imath}_*\mathscr{S}_u \overset{\mathbb{D}}{\otimes} \mathbb{D}q_{23}^*\mathscr{L}) \\
&\simeq \mathbb{D}q_{13*}(\mathbb{D}\tilde{\imath}'_*\mathbb{D}q_1^*\mathscr{S}_u \overset{\mathbb{D}}{\otimes} \mathbb{D}q_{23}^*\mathscr{L}) \\
&\simeq \mathbb{D}q_{13*}\mathbb{D}\tilde{\imath}'_*(\mathbb{D}q_1^*\mathscr{S}_u \overset{\mathbb{D}}{\otimes} \mathbb{D}\tilde{\imath}'^*\mathbb{D}q_{23}^*\mathscr{L}) \\
&\simeq \mathbb{D}\tilde{\pi}'_*(\mathbb{D}q_1^*\mathscr{S}_u \overset{\mathbb{D}}{\otimes} \mathbb{D}\tilde{\jmath}'^*\mathscr{L}).
\end{aligned}$$

There are natural isomorphisms in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_{\dot{\mathbb{V}}\times\mathbb{V}^*})$:

$$\begin{aligned}
\mathbb{D}\pi'^*\mathbb{D}\tilde{\pi}'_*(\mathbb{D}q_1^*\mathscr{S}_u \overset{\mathbb{D}}{\otimes} \mathbb{D}\tilde{\jmath}'^*\mathscr{L}) &\simeq \mathbb{D}p_{23*}\mathbb{D}\tau'^*(\mathbb{D}q_1^*\mathscr{S}_u \overset{\mathbb{D}}{\otimes} \mathbb{D}\tilde{\jmath}'^*\mathscr{L}) \\
&\simeq \mathbb{D}p_{23*}(\mathbb{D}\tau'^*\mathbb{D}q_1^*\mathscr{S}_u \overset{\mathbb{D}}{\otimes} \mathbb{D}\tau'^*\mathbb{D}\tilde{\jmath}'^*\mathscr{L}) \\
&\simeq \mathbb{D}p_{23*}(\mathbb{D}p_{12}^*\mathbb{D}\tau^*\mathscr{S}_u \overset{\mathbb{D}}{\otimes} \mathbb{D}\tau'^*\mathbb{D}\tilde{\jmath}'^*\mathscr{L}) \\
&\simeq \mathbb{D}p_{23*}(\mathbb{D}p_{12}^*\mathbb{D}t^*(\mathscr{D}_{\mathbb{A}^1_{\mathbf{k}}} \cdot u) \overset{\mathbb{D}}{\otimes} \mathbb{D}\dot{\gamma}''^*\mathscr{L}_1)
\end{aligned}$$

$$\simeq \mathbb{D}p_{23*}(\mathbb{D}\dot{\gamma}''^{*}\mathbb{D}p_1^*(\mathscr{D}_{\mathbb{A}^1_{\mathbf{k}}}\cdot u) \overset{\mathbb{D}}{\otimes} \mathbb{D}\dot{\gamma}''^{*}\mathscr{L}_1)$$

$$\simeq \mathbb{D}p_{23*}\mathbb{D}\dot{\gamma}''^{*}(\mathbb{D}p_1^*(\mathscr{D}_{\mathbb{A}^1_{\mathbf{k}}}\cdot u) \overset{\mathbb{D}}{\otimes} \mathscr{L}_1)$$

$$\simeq \mathbb{D}\dot{\gamma}^*((\mathscr{D}_{\mathbb{A}^1_{\mathbf{k}}}\cdot u) \overset{\mathbb{D}}{\circ} \mathscr{L}_1)$$

$$\simeq \mathbb{D}\dot{\gamma}^*(\mathscr{D}_{\mathbb{A}^1_{\mathbf{k}}}\cdot \hat{u})$$

$$\simeq \mathbb{D}\pi'^*\mathscr{R}''_{\hat{u}},$$

where $\mathscr{L}_1 = \mathscr{D}_{\mathbb{A}^1_{\mathbf{k}}\times\mathbb{A}^1_{\mathbf{k}}}e^{-tu}$ is the one-dimensional Fourier–Laplace kernel, and $\mathscr{D}_{\mathbb{A}^1_{\mathbf{k}}}\cdot u$ is the cyclic module defined in (1.10). Summarizing, we have an isomorphism

$$\mathbb{D}\pi'^*(\mathbb{D}\tilde{\imath}_*\mathscr{S}_u \overset{\mathbb{D}}{\circ} \mathscr{L}) \simeq \mathbb{D}\pi'^*\mathscr{R}''_{\hat{u}}.$$

One concludes by the following lemma. □

Lemma 2. *Let $f: X \to Y$ be a fibration with fiber $\dot{\mathbb{A}}^1_{\mathbf{k}} = \mathbb{A}^1_{\mathbf{k}}\setminus\{0\}$. Then the functor $\mathbb{D}f^*: \mathrm{Mod}_{\text{q-coh}}(\mathscr{D}_Y) \to \mathrm{Mod}_{\text{q-coh}}(\mathscr{D}_X)$ is exact and fully faithful.*

Proof. Since f is smooth, $\mathbb{D}f^*$ is exact. Moreover, one has an isomorphism:

$$\mathrm{RHom}_{\mathscr{D}_X}(\mathbb{D}f^*\mathscr{N}_1, \mathbb{D}f^*\mathscr{N}_2) \simeq \mathrm{RHom}_{\mathscr{D}_Y}(\mathscr{N}_1, \mathbb{D}f_*\mathbb{D}f^*\mathscr{N}_2)[-1]. \tag{1.14}$$

By the projection formula, $\mathbb{D}f_*\mathbb{D}f^*\mathscr{N}_2 \simeq \mathbb{D}f_*\mathscr{O}_X \overset{\mathbb{D}}{\otimes} \mathscr{N}_2$, and one has

$$\mathbb{D}f_*\mathscr{O}_X \simeq Rf_*\left(\mathscr{O}_X \xrightarrow{d_{X/Y}} \Omega^1_{X/Y}\right),$$

where $\Omega^1_{X/Y}$, the sheaf of relative one-forms, sits in degree zero. Hence, locally on Y one has $\mathbb{D}f_*\mathscr{O}_X \simeq \mathscr{O}_Y \oplus \mathscr{O}_Y[1]$. Taking zeroth cohomology, (1.14) gives

$$\mathrm{Hom}_{\mathscr{D}_X}(\mathbb{D}f^*\mathscr{N}_1, \mathbb{D}f^*\mathscr{N}_2) \simeq \mathrm{Hom}_{\mathscr{D}_Y}(\mathscr{N}_1, \mathscr{N}_2). \qquad \square$$

1.9. Twisted case

For $\mathbf{k} = \mathbb{C}$ and $\lambda \in \mathbb{C}$, one can replace the ring $\mathscr{D}_{\mathbb{P}}$ with the ring of twisted differential operators (TDO-ring):

$$\mathscr{D}_{\mathbb{P},\lambda} = \mathscr{O}_{\mathbb{P}}(\lambda) \otimes_{\mathscr{O}} \mathscr{D}_{\mathbb{P}} \otimes_{\mathscr{O}} \mathscr{O}_{\mathbb{P}}(-\lambda),$$

whose sections, by definition, are locally of the form $s^{-\lambda} \otimes P \otimes s^{\lambda}$, where s is a nowhere vanishing section of the tautological line bundle $\mathscr{O}_{\mathbb{P}}(-1)$, with the glueing condition $s_1^{-\lambda} \otimes P_1 \otimes s_1^{\lambda} = s_2^{-\lambda} \otimes P_2 \otimes s_2^{\lambda}$ if and only if $P_2 = (s_1/s_2)^{-\lambda} P_1 (s_1/s_2)^{\lambda}$. If

$\lambda - \mu \in \mathbb{Z}$, the functor $\mathcal{O}_{\mathbb{P}}(\mu - \lambda) \otimes_{\mathcal{O}} (\bullet)$ gives an equivalence of categories from $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_{\mathbb{P},\lambda})$ to $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_{\mathbb{P},\mu})$, so that classical $\mathscr{D}$-modules correspond to the case $\lambda \in \mathbb{Z}$.

We do not recall here the theory of TDO-modules, referring instead to [1,11,15]. We just point out that this allows one to consider for $\lambda \in \mathbb{C} \backslash \mathbb{Z}$ the twisted Radon kernel (see [6,15]),

$$\mathscr{R}_{t^{\lambda}} : \ \mathsf{D}^{\mathrm{b}}(\mathscr{D}_{\mathbb{P},\lambda^*}) \to \mathsf{D}^{\mathrm{b}}(\mathscr{D}_{\mathbb{P}^*,\lambda}),$$

where $\lambda^* = -n - 1 - \lambda$, as well as a blow-up kernel

$$\mathscr{S}_{t^{-\lambda-1}} : \ \mathsf{D}^{\mathrm{b}}(\mathscr{D}_{\mathbb{P},\lambda^*}) \to \mathsf{D}^{\mathrm{b}}(\mathscr{D}_{\mathbb{V}}).$$

The following analogue of Theorem 3 is then obtained by much the same proof.

Theorem 4. *Let $\lambda \in \mathbb{C} \backslash \mathbb{Z}$ and $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\text{q-coh}}(\mathscr{D}_{\mathbb{P},\lambda^*})$. Then there is a natural isomorphism in* $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_{\dot{\mathbb{V}}^*})$:

$$\mathbb{D}\pi^*(\mathscr{M} \overset{\mathbb{D}}{\circ} \mathscr{R}_{t^{\lambda}}) \simeq \mathbb{D}j^*(\mathscr{M} \overset{\mathbb{D}}{\circ} \mathbb{D}\tilde{\imath}_* \mathscr{S}_{t^{-\lambda-1}} \overset{\mathbb{D}}{\circ} \mathscr{L}).$$

2. Radon and Fourier transforms for sheaves

2.1. Review on sheaves

Mainly to fix the notations, we recall here some definitions from the theory of sheaves. Refer to [13] for details. In this section, we will take $\mathbf{k} = \mathbb{C}$ and work in the analytic topology.

Let X be a locally compact topological space. Let $\mathbf{k}_X$ be the constant sheaf with fiber $\mathbf{k} = \mathbb{C}$, and for a locally closed subset $A \subset X$, let $\mathbf{k}_{A|X}$ be the sheaf on X characterized by $(\mathbf{k}_{A|X})|_A = \mathbf{k}_A$, $(\mathbf{k}_{A|X})|_{X \backslash A} = 0$. Denote by $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)$ the bounded derived category of sheaves of $\mathbf{k}$-vector spaces on X, and by $\otimes$, f^{-1}, $Rf_!$, $R\mathscr{H}om$, Rf_* and $f^!$ the usual six operations, where $f : X \to Y$ is a continuous map with finite c-soft dimension. For $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)$, we set

$$D'_X F = R\mathscr{H}om(F, \mathbf{k}_X).$$

Let Y and Z be locally compact topological spaces, and let $K \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{X \times Y})$, $L \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{Y \times Z})$. As for $\mathscr{D}$-modules, one sets

$$K \circ L = Rq_{13!}(q_{12}^{-1} K \otimes q_{23}^{-1} L).$$

In particular, the integral transform with kernel K is the functor

$$(\bullet) \circ K : \ \mathsf{D}^{\mathrm{b}}(\mathbf{k}_X) \to \mathsf{D}^{\mathrm{b}}(\mathbf{k}_Y),$$

$$F \mapsto F \circ K = Rq_!(p^{-1} F \otimes K).$$

The operation $\circ$ is associative, and the identity is associated with the kernel $\mathbf{k}_{X|X\times X}$, where X is diagonally embedded in $X \times X$.

Assume that X is a real analytic manifold. To $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_X)$ one associates its microsupport $\mathrm{SS}(F)$, a closed involutive subset of T^*X whose complement describes the codirections along which F propagates. One says that K and L are transversal if

$$(\mathrm{SS}(K) \times T^*_Z Z) \cap (T^*_X X \times \mathrm{SS}(L)) \subset T^*_{X\times Y\times Z}(X \times Y \times Z).$$

2.2. *Radon and Fourier transforms for sheaves*

Let us use the same notations as in Section 1, summarized in (1.6). Note that here we consider all spaces $\mathbb{V}$, $\mathbb{V}^*$, $\mathbb{P}$, ... as well as the maps between them, in the category of real analytic manifolds.

Denote by $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_{\mathbb{V}})$ the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{V}})$ whose objects have conic cohomologies, i.e. cohomologies which are locally constant along the orbits of the multiplicative group $\mathbb{R}^+$ of positive real numbers. The Fourier–Sato transform for sheaves is the equivalence of categories

$$(\bullet) \circ \mathsf{L} \; : \; \mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_{\mathbb{V}}) \to \mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_{\mathbb{V}^*}),$$

where $\mathsf{L} = \mathbf{k}_{Q|\mathbb{V}\times\mathbb{V}^*}$ for $Q = \{(x,y) \in \mathbb{V} \times \mathbb{V}^* : \; \mathrm{Re}\langle x,y\rangle \leqslant 0\}$ (cf. e.g. [13]).

For the Radon and blow-up transforms, one considers the solution complexes of the corresponding kernels for $\mathscr{D}$-modules in (1.12) and (1.13), i.e. one considers

$u =$	1	Y	$1/t$	δ
$\mathsf{R}_u =$	$\mathbf{k}_{\mathbb{P}\times\mathbb{P}^*}$	$D'_{\mathbb{P}\times\mathbb{P}^*}\mathbf{k}_{\mathbb{U}\|\mathbb{P}\times\mathbb{P}^*}$	$\mathbf{k}_{\mathbb{U}\|\mathbb{P}\times\mathbb{P}^*}$	$\mathbf{k}_{\mathbb{S}\|\mathbb{P}\times\mathbb{P}^*}[-1]$
$(\bullet) \circ \mathsf{R}_u \simeq$	$\mathbf{k}_{\mathbb{P}^*} \otimes \mathrm{R}\Gamma(\mathbb{P};\bullet)$	$Rq_{\mathbb{U}*}p_{\mathbb{U}}^{-1}$	$Rq_{\mathbb{U}!}p_{\mathbb{U}}^{-1}$	$Rq_{\mathbb{S}!}p_{\mathbb{S}}^{-1}$
$\mathsf{S}_u =$	$\mathbf{k}_{\widetilde{\mathbb{V}_0}}$	$D'_{\widetilde{\mathbb{V}_0}}\mathbf{k}_{\dot{\mathbb{V}}\|\widetilde{\mathbb{V}_0}}$	$\mathbf{k}_{\dot{\mathbb{V}}\|\widetilde{\mathbb{V}_0}}$	$\mathbf{k}_{\mathbb{E}\|\widetilde{\mathbb{V}_0}}[-1]$
$(\bullet) \circ R\tilde{\imath}_!\mathsf{S}_u \simeq$	$R\tilde{\jmath}_!\tilde{\pi}^{-1}$	$Rj_*\pi^{-1}$	$Rj_!\pi^{-1}$	$\mathbf{k}_{0\|\mathbb{V}} \otimes \mathrm{R}\Gamma(\mathbb{P};\bullet)$

where, as in the $\mathscr{D}$-module case, one uses transversality in order to get the above isomorphisms of functors. Consider the maps

$$\widetilde{\mathbb{V}_0} \overset{\tilde{\imath}}{\hookrightarrow} \mathbb{P} \times \mathbb{V}, \quad \mathbb{P}^* \overset{\pi}{\leftarrow} \dot{\mathbb{V}}^* \overset{j}{\to} \mathbb{V}^*.$$

Theorem 5. *Let* $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{P}})$, *and let* u *be one of the four generators in* (1.10), *so that*

$$u = 1, Y, 1/t, \delta, \quad \hat{u} = \delta, 1/t, Y, 1,$$

respectively. Then there is a natural isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{\dot{\mathbb{V}}^*})$

$$\pi^{-1}(F \circ \mathsf{R}_{\hat{u}}) \simeq j^{-1}(F \circ R\tilde{\imath}_! \mathsf{S}_u \circ \mathsf{L})[1].$$

The proof is a line by line analogue of the one for $\mathscr{D}$-modules, making use of the isomorphisms

$$\mathbf{k}_{\{0\}|\mathbb{C}} \circ \mathsf{L}_1 \simeq \mathbf{k}_{\mathbb{C}}, \quad \mathbf{k}_{\mathbb{C}} \circ \mathsf{L}_1 \simeq \mathbf{k}_{\{0\}|\mathbb{C}}[-2],$$

$$\mathbf{k}_{\dot{\mathbb{C}}|\mathbb{C}} \circ \mathsf{L}_1 \simeq D'_{\mathbb{C}} \mathbf{k}_{\dot{\mathbb{C}}|\mathbb{C}}[-1], \quad D'_{\mathbb{C}} \mathbf{k}_{\dot{\mathbb{C}}|\mathbb{C}} \circ \mathsf{L}_1 \simeq \mathbf{k}_{\dot{\mathbb{C}}|\mathbb{C}}[-1].$$

Here, $\mathsf{L}_1 = \mathbf{k}_{\{\mathrm{Re}\langle t,u\rangle \leqslant 0\}|\mathbb{C}\times\mathbb{C}}$ is the kernel of the Fourier–Sato transform on $\mathbb{C}$.

Remark 1. Let $\mathscr{M}$ be a coherent algebraic $\mathscr{D}$-module on $\mathbb{P}$, denote by $\mathscr{M}^{\mathrm{an}}$ the associated analytic $\mathscr{D}$-module on $\mathbb{P}$, considered as a complex analytic manifold, and set $\mathscr{S}ol(\mathscr{M}) = R\mathscr{H}om_{\mathscr{D}^{\mathrm{an}}_{\mathbb{P}}}(\mathscr{M}^{\mathrm{an}}, \mathcal{O}^{\mathrm{an}}_{\mathbb{P}})$. Using the Riemann–Hilbert correspondence and the compatibility between Fourier and the solution functor (see e.g. [14]), one can recover the isomorphism in Theorem 5 for $F = \mathscr{S}ol(\mathscr{M})$ from the one in Theorem 3.

Remark 2. As for $\mathscr{D}$-modules and TDOs, one has a statement analogue to Theorem 5 in the framework of twisted sheaves.

2.3. Link with the real blow-up

The Fourier–Sato kernel is related to the real analytic space structure underlying the complex vector space $\mathbb{V}$. We give here an alternative description of the blow-up transform, using such a real structure. Although addressing a natural question, this subsection is independent from the rest of this paper. The reader in a hurry may prefer to skip to Section 3.

Let $\mathbb{P}_{\mathbb{R}} = \mathbb{P}_{\mathbb{R}}(\mathbb{V})$ be the real projective space of lines in the $2(n+1)$-dimensional real vector space underlying $\mathbb{V}$. Note that $\mathbb{P}_{\mathbb{R}}$ is orientable, and recall that for $n>1$ one has $\pi_1(\mathbb{P}_{\mathbb{R}}) = \mathbb{Z}/2\mathbb{Z}$. Thus, up to isomorphism, there are only two locally constant sheaves of rank one on $\mathbb{P}_{\mathbb{R}}$. We denote them by $\mathbf{k}_{\mathbb{P}_{\mathbb{R}}}(\varepsilon)$ for $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$, assuming that $\mathbf{k}_{\mathbb{P}_{\mathbb{R}}}(0)$ is the constant sheaf. There is a natural fibration with fiber $\mathbb{P}_{\mathbb{R}}(\mathbb{C}) \simeq S^1$:

$$\rho : \mathbb{P}_{\mathbb{R}} \to \mathbb{P}$$

associating to a real line $\mathbb{R}x$ in $\mathbb{V}$ its complexification $\mathbb{C}x$. Recall that $R\rho_! \mathbf{k}_{\mathbb{P}_{\mathbb{R}}}(1) = 0$.

As in the complex case, the natural maps

$$\mathbb{P}_{\mathbb{R}} \overset{\pi_{\mathbb{R}}}{\leftarrow} \dot{\mathbb{V}} \overset{j}{\hookrightarrow} \mathbb{V}$$

induce an embedding of $\dot{\mathbb{V}}$ as a locally closed subset in $\mathbb{P}_{\mathbb{R}} \times \mathbb{V}$. We denote by $\widetilde{\mathbb{V}_0^{\mathbb{R}}}$ the closure of $\dot{\mathbb{V}}$ in $\mathbb{P}_{\mathbb{R}} \times \mathbb{V}$, and set $\mathbb{E}_{\mathbb{R}} = \widetilde{\mathbb{V}_0^{\mathbb{R}}} \setminus \dot{\mathbb{V}}$. These are, respectively, the real blow-up of 0 in $\mathbb{V}$, and its exceptional divisor. The natural projections from $\mathbb{P}_{\mathbb{R}} \times \mathbb{V}$ induce maps

$$\mathbb{P}_{\mathbb{R}} \xleftarrow{\widetilde{\pi_{\mathbb{R}}}} \widetilde{\mathbb{V}_0^{\mathbb{R}}} \xrightarrow{\widetilde{j_{\mathbb{R}}}} \mathbb{V}.$$

Since $\widetilde{\pi_{\mathbb{R}}}$ is a line bundle, one has $\pi_1(\widetilde{\mathbb{V}_0^{\mathbb{R}}}) = \mathbb{Z}/2\mathbb{Z}$. For $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$, we denote by $\mathbf{k}_{\widetilde{\mathbb{V}_0^{\mathbb{R}}}}(\varepsilon)$ the two locally constant sheaves of rank one on $\widetilde{\mathbb{V}_0^{\mathbb{R}}}$. Note that the relative orientation sheaf $\mathrm{or}_{\mathbb{E}_{\mathbb{R}}/\widetilde{\mathbb{V}_0^{\mathbb{R}}}}$ is non-trivial, and hence $\mathrm{or}_{\widetilde{\mathbb{V}_0^{\mathbb{R}}}} \simeq \mathrm{or}_{\widetilde{\mathbb{V}_0^{\mathbb{R}}}/\mathbb{P}_{\mathbb{R}}} \simeq \mathrm{or}_{\mathbb{E}_{\mathbb{R}}/\widetilde{\mathbb{V}_0^{\mathbb{R}}}}$ is non-trivial. Consider the diagram

$$\begin{tikzcd}
\dot{\mathbb{V}} \arrow[r, hook, "\widetilde{j_{\mathbb{R}}}"] \arrow[dr, hook, "\widetilde{j}"'] & \widetilde{\mathbb{V}_0^{\mathbb{R}}} \arrow[d, "\rho''"] & \mathbb{E}_{\mathbb{R}} \simeq \mathbb{P}_{\mathbb{R}} \arrow[l, hook', "l_{\mathbb{R}}"'] \arrow[d, "\rho"] \\
& \widetilde{\mathbb{V}_0} & \mathbb{E} \simeq \mathbb{P} \arrow[l, hook', "l"]
\end{tikzcd}$$

where $\rho'' = (\rho \times \mathrm{id}_{\mathbb{V}})|_{\widetilde{\mathbb{V}_0^{\mathbb{R}}}}$.

Proposition 2. *There are natural isomorphisms in* $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{\widetilde{\mathbb{V}_0}})$

$$D'_{\widetilde{\mathbb{V}_0}} \mathbf{k}_{\dot{\mathbb{V}}|\widetilde{\mathbb{V}_0}} \simeq R\rho''_! \mathbf{k}_{\widetilde{\mathbb{V}_0^{\mathbb{R}}}}, \quad \mathbf{k}_{\dot{\mathbb{V}}|\widetilde{\mathbb{V}_0}} \simeq R\rho''_! \mathbf{k}_{\widetilde{\mathbb{V}_0^{\mathbb{R}}}}(1).$$

Proof. Note that $\rho''^{!}\mathbf{k}_{\widetilde{\mathbb{V}_0}} \simeq \rho''^{!}\omega_{\widetilde{\mathbb{V}_0}}[-\dim^{\mathbb{R}} \widetilde{\mathbb{V}_0}] \simeq \omega_{\widetilde{\mathbb{V}_0^{\mathbb{R}}}}[-\dim \widetilde{\mathbb{V}_0^{\mathbb{R}}}] \simeq \mathbf{k}_{\widetilde{\mathbb{V}_0^{\mathbb{R}}}}(1)$, where $\omega_{\widetilde{\mathbb{V}_0}}$ denotes the dualizing complex. Hence, for $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{\widetilde{\mathbb{V}_0^{\mathbb{R}}}})$ one has

$$\begin{aligned}
D'_{\widetilde{\mathbb{V}_0}} R\rho''_! F &\simeq R\mathscr{H}om(R\rho''_! F, \mathbf{k}_{\widetilde{\mathbb{V}_0}}) \\
&\simeq R\rho''_* R\mathscr{H}om(F, \rho''^{!}\mathbf{k}_{\widetilde{\mathbb{V}_0}}) \\
&\simeq R\rho''_* D'_{\widetilde{\mathbb{V}_0^{\mathbb{R}}}}(F \otimes \mathbf{k}_{\widetilde{\mathbb{V}_0^{\mathbb{R}}}}(1)).
\end{aligned}$$

The second isomorphism in the statement thus follows from the first one. To prove the first isomorphism, note that

$$\begin{aligned}
D'_{\widetilde{\mathbb{V}_0}} \mathbf{k}_{\dot{\mathbb{V}}|\widetilde{\mathbb{V}_0}} &\simeq D'_{\widetilde{\mathbb{V}_0}} R\rho''_! \mathbf{k}_{\dot{\mathbb{V}}|\widetilde{\mathbb{V}_0^{\mathbb{R}}}} \\
&\simeq R\rho''_! D'_{\widetilde{\mathbb{V}_0^{\mathbb{R}}}}(\mathbf{k}_{\dot{\mathbb{V}}|\widetilde{\mathbb{V}_0^{\mathbb{R}}}} \otimes \mathbf{k}_{\widetilde{\mathbb{V}_0^{\mathbb{R}}}}(1)) \\
&\simeq R\rho''_! D'_{\widetilde{\mathbb{V}_0^{\mathbb{R}}}} \mathbf{k}_{\dot{\mathbb{V}}|\widetilde{\mathbb{V}_0^{\mathbb{R}}}}.
\end{aligned}$$

Using the distinguished triangle

$$D'_{\widetilde{\mathbb{V}_0^{\mathbb{R}}}} \mathbf{k}_{\mathbb{E}_{\mathbb{R}}|\widetilde{\mathbb{V}_0^{\mathbb{R}}}} \to \mathbf{k}_{\widetilde{\mathbb{V}_0^{\mathbb{R}}}} \to D'_{\widetilde{\mathbb{V}_0^{\mathbb{R}}}} \mathbf{k}_{\dot{\mathbb{V}}|\widetilde{\mathbb{V}_0^{\mathbb{R}}}} \xrightarrow{+1},$$

it is then enough to prove that $R\rho''_! D'_{\widetilde{\mathbb{V}_0^{\mathbb{R}}}} \mathbf{k}_{\mathbb{E}_{\mathbb{R}}|\widetilde{\mathbb{V}_0^{\mathbb{R}}}} = 0$. Since $\mathbb{E}_{\mathbb{R}}$ is not relatively orientable in $\widetilde{\mathbb{V}_0^{\mathbb{R}}}$, one has $D'_{\widetilde{\mathbb{V}_0^{\mathbb{R}}}} \mathbf{k}_{\mathbb{E}_{\mathbb{R}}|\widetilde{\mathbb{V}_0^{\mathbb{R}}}} \simeq \mathbf{k}_{\mathbb{E}_{\mathbb{R}}|\widetilde{\mathbb{V}_0^{\mathbb{R}}}} \otimes \mathbf{k}_{\widetilde{\mathbb{V}_0^{\mathbb{R}}}}(1)[-1]$, and hence

$$\begin{aligned} R\rho''_! D'_{\widetilde{\mathbb{V}_0^{\mathbb{R}}}} \mathbf{k}_{\mathbb{E}_{\mathbb{R}}|\widetilde{\mathbb{V}_0^{\mathbb{R}}}} &\simeq R\rho''_! (\mathbf{k}_{\mathbb{E}_{\mathbb{R}}|\widetilde{\mathbb{V}_0^{\mathbb{R}}}} \otimes \mathbf{k}_{\widetilde{\mathbb{V}_0^{\mathbb{R}}}}(1))[-1] \\ &\simeq R\rho''_! R l_{\mathbb{R}!} \mathbf{k}_{\mathbb{E}_{\mathbb{R}}}(1)[-1] \\ &\simeq R l_! R\rho_! \mathbf{k}_{\mathbb{E}_{\mathbb{R}}}(1)[-1] = 0. \qquad \square \end{aligned}$$

3. Applications

For the remainder of this paper we consider the case $\mathbf{k} = \mathbb{C}$, and we concentrate on the Radon transform $\mathscr{R}_{1/t} = \mathscr{B}_{\mathbb{U}|\mathbb{P}\times\mathbb{P}^*}$. From now on we thus simply set

$$\mathscr{R} = \mathscr{B}_{\mathbb{U}|\mathbb{P}\times\mathbb{P}^*}, \quad \mathsf{R} = \mathbf{k}_{\mathbb{U}|\mathbb{P}\times\mathbb{P}^*},$$

so that

$$(\bullet) \overset{\mathbb{D}}{\circ} \mathscr{R} \simeq \mathbb{D}q_{\mathbb{U}*}\mathbb{D}p^*_{\mathbb{U}}, \quad (\bullet) \circ \mathsf{R} \simeq Rq_{\mathbb{U}!}p^{-1}_{\mathbb{U}}.$$

3.1. Radon transform of line bundles

For $m \in \mathbb{Z}$, let $\mathscr{O}_{\mathbb{P}}(m)$ denote the $-m$th tensor power of the tautological line bundle $\mathscr{O}_{\mathbb{P}}(-1)$. The Leray form on $\mathbb{P}$ is defined in homogeneous coordinates by

$$\theta \lrcorner dx_0 \wedge \cdots \wedge dx_n = \sum_{j=0}^{n} (-1)^j x_j \, dx_0 \wedge \cdots \widehat{dx_j} \cdots \wedge dx_n,$$

where $\lrcorner$ denotes the interior product and θ the Euler vector field. It is thus a global section of $\Omega_{\mathbb{P}} \otimes_{\mathscr{O}} \mathscr{O}_{\mathbb{P}}(n+1)$ which only depends on the choice of a volume element in $\det \mathbb{V}^*$. Removing this dependency, we get a canonical section

$$\omega(x) \in \Gamma(\mathbb{P}; \Omega_{\mathbb{P}} \otimes_{\mathscr{O}} \mathscr{O}_{\mathbb{P}}(n+1) \otimes \det \mathbb{V}).$$

Set

$$\mathscr{D}_{\mathbb{P}}(m) = \mathscr{D}_{\mathbb{P}} \otimes_{\mathscr{O}_{\mathbb{P}}} \mathscr{O}_{\mathbb{P}}(m), \quad m^* = -m-n-1,$$

and note that, using the identification

$$\Omega_{\mathbb{P}} \simeq \mathscr{O}_{\mathbb{P}}(-n-1) \otimes \det \mathbb{V}^* \tag{3.1}$$

induced by $\omega(x)$, we get an identification

$$\mathbb{D}_{\mathbb{P}}(\mathscr{D}_{\mathbb{P}}(-m))[-n] \simeq \mathscr{D}_{\mathbb{P}}(-m^*) \otimes \det \mathbb{V}. \tag{3.2}$$

It was shown in [7] that for $m<0$ the integral kernel

$$\langle x, y \rangle^m \omega(x) \in \Gamma(\mathbb{P} \times \mathbb{P}^*; (\Omega_{\mathbb{P}}(-m^*) \boxtimes \mathscr{O}_{\mathbb{P}^*}(m)) \otimes_{\mathscr{O}_{\mathbb{P}} \boxtimes \mathscr{O}_{\mathbb{P}^*}} \mathscr{R} \otimes \det \mathbb{V})$$

induces an isomorphism

$$\mathscr{D}_{\mathbb{P}}(-m^*) \overset{\mathbb{D}}{\circ} \mathscr{R} \otimes \det \mathbb{V} \xleftarrow{\sim} \mathscr{D}_{\mathbb{P}^*}(-m). \tag{3.3}$$

The integral kernel

$$\langle x, y \rangle^{m^*} Y(\langle x, y \rangle) \omega(y) \in \Gamma(\mathbb{P} \times \mathbb{P}^*; (\mathscr{O}_{\mathbb{P}}(m^*) \boxtimes \Omega^*_{\mathbb{P}}(-m))$$
$$\otimes_{\mathscr{O}_{\mathbb{P}} \boxtimes \mathscr{O}_{\mathbb{P}^*}} \mathbb{D}_{\mathbb{P} \times \mathbb{P}^*} \mathscr{R} \otimes \det \mathbb{V}^*)$$

gives a morphism

$$\mathscr{D}_{\mathbb{P}}(-m^*) \overset{\mathbb{D}}{\circ} \mathscr{R} \otimes \det \mathbb{V} \to \mathscr{D}_{\mathbb{P}^*}(-m)$$

which is an inverse to (3.3) for $m<0$. The following statement describes its kernel and cokernel for $m \geqslant 0$ (this should be compared with the topological results in [5]), and recovers the case $m<0$ by different methods, using the results from Section 1.

Let us denote by $\mathsf{S}^m\mathbb{V}$ the mth symmetric tensor power of $\mathbb{V}$.

Theorem 6. *For any $m \in \mathbb{Z}$ there is a long exact sequence of $\mathscr{D}_{\mathbb{P}^*}$-modules*

$$0 \to \mathscr{O}_{\mathbb{P}^*} \otimes \mathsf{S}^m\mathbb{V}^* \to \mathscr{D}_{\mathbb{P}}(-m^*) \overset{\mathbb{D}}{\circ} \mathscr{R} \otimes \det \mathbb{V} \to \mathscr{D}_{\mathbb{P}^*}(-m) \to \mathscr{O}_{\mathbb{P}^*} \otimes \mathsf{S}^m\mathbb{V}^* \to 0.$$

Before starting the proof, let us explicitly describe the morphisms entering the above long exact sequence. The natural identification $\mathsf{S}^m\mathbb{V}^* \simeq \Gamma(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(m))$ gives a canonical monomorphism

$$\mathbf{k}_{\mathbb{P}} \otimes \mathsf{S}^m\mathbb{V}^* \to \mathscr{O}_{\mathbb{P}}(m),$$

which in turn corresponds to a surjective $\mathscr{D}_{\mathbb{P}}$-linear morphism

$$\mathscr{D}_{\mathbb{P}}(-m) \to \mathscr{O}_{\mathbb{P}} \otimes \mathsf{S}^m\mathbb{V}$$

(for $m = 0$ this is but the beginning of the Spencer resolution of $\mathscr{O}_{\mathbb{P}}$). Consider its kernel

$$\mathscr{D}'_{\mathbb{P}}(-m) = \ker(\mathscr{D}_{\mathbb{P}}(-m) \to \mathscr{O}_{\mathbb{P}} \otimes \mathsf{S}^m\mathbb{V}),$$

and set

$$\mathscr{D}'_{\mathbb{P}}(-m)^* = \mathbb{D}_{\mathbb{P}}(\mathscr{D}'_{\mathbb{P}}(-m))[-n].$$

Note that by (3.2) there is a distinguished triangle

$$\mathscr{O}_{\mathbb{P}}[-n] \otimes \mathsf{S}^m \mathbb{V}^* \to \mathscr{D}_{\mathbb{P}}(-m^*) \otimes \det \mathbb{V} \to \mathscr{D}'_{\mathbb{P}}(-m)^* \xrightarrow{+1}.$$

Then the statement of Theorem 6 is equivalent to the isomorphism

$$\mathscr{D}'_{\mathbb{P}}(-m)^* \overset{\mathbb{D}}{\circ} \mathscr{R} \xrightarrow{\sim} \mathscr{D}'_{\mathbb{P}^*}(-m). \tag{3.4}$$

Proof. By Lemma 2, it suffices to prove that there is a distinguished triangle in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_{\dot{\mathbb{V}}^*})$

$$\mathbb{D}\pi^*(\mathscr{D}_{\mathbb{P}}(-m^*) \overset{\mathbb{D}}{\circ} \mathscr{R}) \otimes \det \mathbb{V} \to \mathbb{D}\pi^* \mathscr{D}_{\mathbb{P}^*}(-m)$$
$$\to \mathbb{D}\pi^*(\mathscr{O}_{\mathbb{P}^*} \oplus \mathscr{O}_{\mathbb{P}^*}[1]) \otimes \mathsf{S}^m \mathbb{V}^* \xrightarrow{+1}.$$

Consider the cyclic $\mathscr{D}_{\mathbb{V}}$-module

$$\mathscr{D}_{\mathbb{V}}(m) = \mathscr{D}_{\mathbb{V}} / \langle \theta + m \rangle,$$

and note that $\mathbb{D}\pi^* \mathscr{D}_{\mathbb{P}}(m) \simeq \mathbb{D}j^* \mathscr{D}_{\mathbb{V}}(m)$. Since one also has $\mathbb{D}\pi^* \mathscr{O}_{\mathbb{P}} \simeq \mathbb{D}j^* \mathscr{O}_{\mathbb{V}}$, the above distinguished triangle is equivalent to

$$\mathbb{D}\pi^*(\mathscr{D}_{\mathbb{P}}(-m^*) \overset{\mathbb{D}}{\circ} \mathscr{R}) \to \mathbb{D}j^* \mathscr{D}_{\mathbb{V}^*}(-m) \otimes \det \mathbb{V}^*$$
$$\to \mathbb{D}j^*(\mathscr{O}_{\mathbb{V}^*} \oplus \mathscr{O}_{\mathbb{V}^*}[1]) \otimes \mathsf{S}^m \mathbb{V}^* \otimes \det \mathbb{V}^* \xrightarrow{+1}.$$

By Theorem 3, it is enough to prove that there is a distinguished triangle in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_{\mathbb{V}^*})$

$$\mathscr{D}_{\mathbb{P}}(-m^*) \overset{\mathbb{D}}{\circ} \mathbb{D}\tilde{\imath}_* \mathbb{D}_{\widetilde{\mathbb{V}}_0} \mathscr{B}_{\dot{\mathbb{V}}|\widetilde{\mathbb{V}}_0} \overset{\mathbb{D}}{\circ} \mathscr{L} \to \mathscr{D}_{\mathbb{V}^*}(-m) \otimes \det \mathbb{V}^*$$
$$\to (\mathscr{O}_{\mathbb{V}^*} \oplus \mathscr{O}_{\mathbb{V}^*}[1]) \otimes \mathsf{S}^m \mathbb{V}^* \otimes \det \mathbb{V}^* \xrightarrow{+1}.$$

This is obtained by Fourier transform if we prove that there is a distinguished triangle in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_{\mathbb{V}})$:

$$\mathscr{D}_{\mathbb{P}}(-m^*) \overset{\mathbb{D}}{\circ} \mathbb{D}\tilde{\imath}_* \mathbb{D}_{\widetilde{\mathbb{V}}_0} \mathscr{B}_{\dot{\mathbb{V}}|\widetilde{\mathbb{V}}_0} \to \mathscr{D}_{\mathbb{V}}(-m^*) \to (\mathscr{B}_{0|V} \oplus \mathscr{B}_{0|V}[1]) \otimes \mathsf{S}^m \mathbb{V}^* \otimes \det \mathbb{V}^* \xrightarrow{+1}.$$

Since $\mathscr{D}_{\mathbb{P}}(-m^*) \overset{\mathbb{D}}{\circ} \mathbb{D}\tilde{\imath}_* \mathbb{D}_{\widetilde{\mathbb{V}}_0} \mathscr{B}_{\dot{\mathbb{V}}|\widetilde{\mathbb{V}}_0} \simeq \mathbb{D}j_! \mathbb{D}\pi^* \mathscr{D}_{\mathbb{P}}(-m^*) \simeq \mathbb{D}j_! \mathbb{D}j^* \mathscr{D}_{\mathbb{V}}(-m^*)$, this is exactly what is claimed in Proposition 3 below. $\square$

Recall that on a smooth variety X there is a natural isomorphism of left $\mathscr{D}_X \otimes \mathscr{D}_X$-modules:

$$\mathscr{B}_{X|X\times X} \simeq \mathscr{D}_X \otimes_{\mathscr{O}_X} \Omega_X^{\otimes -1},$$

where $\mathscr{D}_X$ acts on $\mathscr{B}_{X|X\times X}$ via the first and second projections. Concerning $\mathscr{B}_{0|\mathbb{V}}$, recall that $\mathscr{B}_{0|\mathbb{V}} \otimes \det \mathbb{V}^*$ has a generator $\delta_{0|\mathbb{V}}$ and relations $x_i\,\delta_{0|\mathbb{V}} = 0$ for $i = 0, \ldots, n$. One then has an identification of $\mathbf{k}$-vector spaces $\mathscr{B}_{0|\mathbb{V}} \otimes \det \mathbb{V}^* \simeq \bigoplus_\alpha \mathbf{k} \cdot \partial_x^\alpha \delta_{0|\mathbb{V}}$ or, more intrinsically,

$$\mathscr{B}_{0|\mathbb{V}} \simeq \mathsf{S}^\bullet \mathbb{V} \otimes \det \mathbb{V}.$$

Proposition 3. *For any $m \in \mathbb{Z}$ there is a distinguished triangle in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_\mathbb{V})$:*

$$\mathbb{D}j_!\mathbb{D}j^*\mathscr{D}_\mathbb{V}(-m^*) \to \mathscr{D}_\mathbb{V}(-m^*) \to (\mathscr{B}_{0|\mathbb{V}} \oplus \mathscr{B}_{0|\mathbb{V}}[1]) \otimes \mathsf{S}^m\mathbb{V}^* \otimes \det \mathbb{V}^* \xrightarrow{+1}.$$

Proof. One has $\mathbb{D}j_!\mathbb{D}j^*\mathscr{D}_\mathbb{V}(-m^*) \simeq \mathbb{D}_\mathbb{V} \mathrm{R}\Gamma_{[\dot{\mathbb{V}}]} \mathbb{D}_\mathbb{V} \mathscr{D}_\mathbb{V}(-m^*)$. Using the distinguished triangle deduced from (1.9),

$$\mathbb{D}_\mathbb{V} \mathrm{R}\Gamma_{[\dot{\mathbb{V}}]} \mathbb{D}_\mathbb{V} \mathscr{D}_\mathbb{V}(-m^*) \to \mathscr{D}_\mathbb{V}(-m^*) \to \mathbb{D}_\mathbb{V} \mathrm{R}\Gamma_{[0]} \mathbb{D}_\mathbb{V} \mathscr{D}_\mathbb{V}(-m^*) \xrightarrow{+1}$$

it is then enough to prove the isomorphism

$$\mathbb{D}_\mathbb{V} \mathrm{R}\Gamma_{[0]} \mathbb{D}_\mathbb{V} \mathscr{D}_\mathbb{V}(-m^*) \simeq (\mathscr{B}_{0|\mathbb{V}} \oplus \mathscr{B}_{0|\mathbb{V}}[1]) \otimes \mathsf{S}^m\mathbb{V}^* \otimes \det \mathbb{V}^*. \tag{3.5}$$

Consider the short exact sequence

$$0 \to \mathscr{D}_\mathbb{V} \xrightarrow{\cdot(\theta - m^*)} \mathscr{D}_\mathbb{V} \to \mathscr{D}_\mathbb{V}(-m^*) \to 0.$$

Using the identification $\mathscr{D}_\mathbb{V} \otimes_{\mathscr{O}_\mathbb{V}} \Omega_\mathbb{V}^{\otimes -1} \simeq \mathscr{B}_{\mathbb{V}|\mathbb{V}\times\mathbb{V}}$, we get a distinguished triangle

$$\mathbb{D}_\mathbb{V}\mathscr{D}_\mathbb{V}(-m^*) \to \mathscr{B}_{\mathbb{V}|\mathbb{V}\times\mathbb{V}}[n+1] \xrightarrow{(\theta - m^*)_2} \mathscr{B}_{\mathbb{V}|\mathbb{V}\times\mathbb{V}}[n+1] \xrightarrow{+1},$$

where $(\theta - m^*)_2$ means that $\theta - m^*$ acts on $\mathscr{B}_{\mathbb{V}|\mathbb{V}\times\mathbb{V}}$ via the second projection. Using the identifications

$$\begin{aligned}\mathrm{R}\Gamma_{[0]}\mathscr{B}_{\mathbb{V}|\mathbb{V}\times\mathbb{V}}[n+1] &\simeq \mathrm{R}\Gamma_{[0]}\mathrm{R}\Gamma_{[\mathbb{V}]}\mathscr{O}_{\mathbb{V}\times\mathbb{V}}[2(n+1)]\\ &\simeq \mathrm{R}\Gamma_{[0]}\mathscr{O}_{\mathbb{V}\times\mathbb{V}}[2(n+1)]\\ &\simeq \mathscr{B}_{0|\mathbb{V}\times\mathbb{V}},\end{aligned}$$

we get a distinguished triangle

$$\mathrm{R}\Gamma_{[0]}\mathbb{D}_{\mathbb{V}}\mathscr{D}_{\mathbb{V}}(-m^*) \to \mathscr{B}_{0|\mathbb{V}\times\mathbb{V}} \xrightarrow{(\theta-m^*)_2} \mathscr{B}_{0|\mathbb{V}\times\mathbb{V}} \xrightarrow{+1}. \tag{3.6}$$

As we recalled before entering the proof, $\mathscr{B}_{0|\mathbb{V}\times\mathbb{V}} \otimes \det^2 \mathbb{V}^*$ is generated as a $\mathbf{k}$-vector space by $\partial_x^\alpha \partial_{\tilde{x}}^\beta \delta_{0|\mathbb{V}\times\mathbb{V}}$. Using the commutation relation $[\partial_{\tilde{x}_i}, \tilde{x}_i] = 1$, one gets

$$\theta_2\, \partial_x^\alpha \partial_{\tilde{x}}^\beta \delta_{0|\mathbb{V}\times\mathbb{V}} = \left(\sum_{i=0}^{n} \tilde{x}_i \partial_{\tilde{x}_i}\right) \partial_x^\alpha \partial_{\tilde{x}}^\beta \delta_{0|\mathbb{V}\times\mathbb{V}} = (-n-1-|\beta|)\, \partial_x^\alpha \partial_{\tilde{x}}^\beta \delta_{0|\mathbb{V}\times\mathbb{V}}.$$

In particular, $(\theta - m^*)_2$ acts diagonally sending to zero only the base elements $\partial_x^\alpha \partial_{\tilde{x}}^\beta \delta_{0|\mathbb{V}\times\mathbb{V}}$ with $|\beta| = -m^* - n - 1 = m$. We thus get an isomorphism of $\mathscr{D}_{\mathbb{V}}$-modules:

$$\ker(\theta - m^*)_2 \simeq \operatorname{coker}\,(\theta - m^*)_2 \simeq \mathscr{B}_{0|\mathbb{V}} \otimes \mathsf{S}^m\mathbb{V} \otimes \det \mathbb{V}.$$

It follows from (3.6) that

$$H^i\mathrm{R}\Gamma_{[0]}\mathbb{D}_{\mathbb{V}}\mathscr{D}_{\mathbb{V}}(-m^*) = \begin{cases} \mathscr{B}_{0|\mathbb{V}} \otimes \mathsf{S}^m\mathbb{V} \otimes \det \mathbb{V} & \text{for } i = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, there is a distinguished triangle

$$\mathscr{B}_{0|\mathbb{V}} \otimes \mathsf{S}^m\mathbb{V} \otimes \det \mathbb{V} \to \mathrm{R}\Gamma_{[0]}\mathbb{D}_{\mathbb{V}}\mathscr{D}_{\mathbb{V}}(-m^*) \to \mathscr{B}_{0|\mathbb{V}}[-1] \otimes \mathsf{S}^m\mathbb{V} \otimes \det \mathbb{V} \xrightarrow{+1}.$$

Since $\mathrm{Hom}_{\mathscr{D}_{\mathbb{V}}}(\mathscr{B}_{0|\mathbb{V}}[-1], \mathscr{B}_{0|\mathbb{V}}[1]) = 0$, one has

$$\mathrm{R}\Gamma_{[0]}\mathbb{D}_{\mathbb{V}}\mathscr{D}_{\mathbb{V}}(-m^*) \simeq (\mathscr{B}_{0|\mathbb{V}} \oplus \mathscr{B}_{0|\mathbb{V}}[-1]) \otimes \mathsf{S}^m\mathbb{V} \otimes \det \mathbb{V},$$

and (3.5) follows by duality. $\square$

3.2. Radon transform of closed forms

Let X be a smooth n-dimensional algebraic variety. Recall that if $\mathscr{F}$ and $\mathscr{G}$ are locally free $\mathscr{O}_X$-modules of finite rank there is a natural isomorphism

$$\mathscr{D}\!\mathit{iff}(\mathscr{F}, \mathscr{G}) \simeq \mathscr{H}\!\mathit{om}_{\mathscr{D}_X}(\mathscr{D}_X \otimes_{\mathscr{O}} \mathscr{G}^*, \mathscr{D}_X \otimes_{\mathscr{O}} \mathscr{F}^*), \tag{3.7}$$

where $\mathscr{F}^* = \mathscr{H}\!\mathit{om}_{\mathscr{O}_{\mathbb{P}}}(\mathscr{F}, \mathscr{O}_{\mathbb{P}})$, and where $\mathscr{D}\!\mathit{iff}$ denotes the sheaf of differential homomorphisms. The de Rham complex

$$\Omega_X^\bullet = (\Omega_X^0 \xrightarrow{d_X^0} \Omega_X^1 \to \cdots \to \Omega_X^{n-1} \xrightarrow{d_X^{n-1}} \Omega_X^n),$$

thus corresponds to a complex of $\mathscr{D}_X$-modules, called the Spencer complex,

$$\mathscr{S}p^X_\bullet = (\mathscr{S}p^X_0 \overset{d^X_0}{\leftarrow} \mathscr{S}p^X_1 \leftarrow \cdots \leftarrow \mathscr{S}p^X_{n-1} \overset{d^X_{n-1}}{\leftarrow} \mathscr{S}p^X_n),$$

where we set $\mathscr{S}p^X_q = \mathscr{D}_X \otimes_{\mathcal{O}} \bigwedge^q_{\mathcal{O}} \Theta_X$, denoting by Θ_X the sheaf of holomorphic vector fields. Recall that the map $P \mapsto P1$ gives a quasi-isomorphism

$$\mathcal{O}_X \overset{\text{qis}}{\leftarrow} \mathscr{S}p^X_\bullet. \tag{3.8}$$

Moreover, one checks that

$$\begin{aligned} d^X_{q-1}(P \otimes \theta_1 \wedge \cdots \wedge \theta_q) = & \sum_{i=1}^q (-1)^{i-1} P\theta_i \otimes \theta_1 \wedge \cdots \widehat{\theta}_i \cdots \wedge \theta_q \\ & + \sum_{1 \leqslant i<j \leqslant q} (-1)^{i+j} P \otimes [\theta_i, \theta_j] \wedge \theta_1 \wedge \cdots \widehat{\theta}_i \cdots \widehat{\theta}_j \cdots \wedge \theta_q. \end{aligned}$$

(See [12] for a detailed exposition.)

Let us denote by $\mathscr{S}p^X_{\geqslant q}$ the subcomplex obtained from $\mathscr{S}p^X_\bullet$ by replacing $\mathscr{S}p^X_j$ with 0 when $j<q$. Thus,

$$\mathscr{S}p^X_{\geqslant q}[q] = (0 \leftarrow \mathscr{S}p^X_q \overset{d^X_q}{\leftarrow} \cdots \leftarrow \mathscr{S}p^X_{n-1} \overset{d^X_{n-1}}{\leftarrow} \mathscr{S}p^X_n) \overset{\text{qis}}{\rightarrow} \operatorname{coker} d^X_q$$

is concentrated in degree zero, and for $0 \leqslant q \leqslant n$ it has the sheaf of closed q-forms as solutions. We similarly define $\mathscr{S}p^X_{\leqslant q}$. Note that $\mathscr{S}p^X_{\leqslant q}[q]$ is isomorphic to $\mathscr{S}p^X_{\geqslant q+1}[q+1]$, up to flat connections, and that one has isomorphisms

$$\mathbb{D}_X \mathscr{S}p^X_{\leqslant q} \simeq \mathscr{S}p^X_{\geqslant n-q}.$$

Finally, note that $\mathscr{S}p^X_{\geqslant q}[q]$ and $\mathscr{S}p^X_{\leqslant q}[q]$ are microlocally free outside of the zero section.

Theorem 7. *There are natural isomorphisms in* $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_{\mathbb{P}^*})$:

$$\mathscr{S}p^{\mathbb{P}}_{\leqslant q}[q] \overset{\mathbb{D}}{\circ} \mathscr{R} \overset{\sim}{\leftarrow} \mathscr{S}p^{\mathbb{P}^*}_{\geqslant n-q}[n-q].$$

In fact, the more general statement obtained by replacing the Spencer complex with a "BGG sequence" also holds, but we will discuss this matter elsewhere. Here, we will obtain Theorem 7 as a corollary of Theorem 8 below, which computes the Radon transforms of $\mathscr{S}p^{\mathbb{P}}_q$ itself.

Note that for $q=n$ the above statement gives the isomorphism

$$\mathcal{O}_{\mathbb{P}}[-n] \overset{\mathbb{D}}{\circ} \mathscr{R} \overset{\sim}{\leftarrow} \mathcal{O}_{\mathbb{P}^*}.$$

For $q = 0$ and $n - 1$ one recovers the isomorphisms (3.4) for $m = n + 1$ and $m = 0$, respectively. In fact, using the identification (3.1) one has an identification

$$\mathcal{S}p_n^{\mathbb{P}} \simeq \mathcal{D}_{\mathbb{P}}(n+1) \otimes \det \mathbb{V}.$$

The case $q = n - 1$ is related to the so-called Andreotti–Norguet correspondence, of which a $\mathcal{D}$-module interpretation was given in [9]. Finally, note that taking holomorphic solutions in the analytic category we get the isomorphisms in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{P}^*})$

$$\Omega_{\mathbb{P}}^{\leqslant q}[q] \circ \mathsf{R} \xrightarrow{\sim} \Omega_{\mathbb{P}^*}^{\geqslant n-q}[-q],$$

describing the Radon transform of the sheaf of closed q-forms.

3.3. Euler complex

Denote by θ the Euler vector field on the vector space $\mathbb{V}$, which is the infinitesimal generator of the action of the multiplicative group $\mathbf{k}^{\times}$. As any vector field, θ acts on differential forms in two ways, by interior product and Lie derivative:

$$e_{q-1}^{\mathbb{V}} = \theta \lrcorner \bullet : \Omega_{\mathbb{V}}^{q} \to \Omega_{\mathbb{V}}^{q-1},$$
$$h_q^{\mathbb{V}} = L_\theta : \Omega_{\mathbb{V}}^{q} \to \Omega_{\mathbb{V}}^{q}.$$

Recall that there is a long exact sequence

$$0 \to \Omega_{\mathbb{V}}^{n+1} \xrightarrow{e_n^{\mathbb{V}}} \cdots \to \Omega_{\mathbb{V}}^{1} \xrightarrow{e_0^{\mathbb{V}}} \Omega_{\mathbb{V}}^{0} \to \mathbf{k}_{\{0\}|\mathbb{V}} \to 0,$$

and that $e_q^{\mathbb{V}}$, $h_q^{\mathbb{V}}$, and the exterior differential $d_{\mathbb{V}}^{q}$ are related by the homotopy formula

$$h_q^{\mathbb{V}} = e_q^{\mathbb{V}} \circ d_{\mathbb{V}}^{q} + d_{\mathbb{V}}^{q-1} \circ e_{q-1}^{\mathbb{V}}. \tag{3.9}$$

By (3.7), to $e_{q-1}^{\mathbb{V}}$ and $h_q^{\mathbb{V}}$ correspond $\mathcal{D}_{\mathbb{V}}$-linear morphisms

$$e_{\mathbb{V}}^{q-1} : \mathcal{S}p_{q-1}^{\mathbb{V}} \to \mathcal{S}p_q^{\mathbb{V}},$$
$$h_{\mathbb{V}}^{q} : \mathcal{S}p_q^{\mathbb{V}} \to \mathcal{S}p_q^{\mathbb{V}},$$

and we consider the Euler complex defined by

$$\mathcal{E}u_{\mathbb{V}}^{\bullet} = (\mathcal{S}p_0^{\mathbb{V}} \xrightarrow{e_{\mathbb{V}}^{0}} \mathcal{S}p_1^{\mathbb{V}} \to \cdots \to \mathcal{S}p_n^{\mathbb{V}} \xrightarrow{e_{\mathbb{V}}^{n}} \mathcal{S}p_{n+1}^{\mathbb{V}}).$$

Recall that there is a quasi-isomorphism

$$\mathscr{E}u_{\mathbb{V}}^{\bullet}[n+1] \xrightarrow{\text{qis}} \mathscr{B}_{\{0\}|\mathbb{V}}. \tag{3.10}$$

Note also that on $\mathbb{V}$ there is a natural identification

$$\mathscr{S}p_q^{\mathbb{V}} = \mathscr{E}u_{\mathbb{V}}^q = \mathscr{D}_{\mathbb{V}} \underset{\mathcal{O}}{\otimes} \bigwedge_{\mathcal{O}}^{q} \Theta_{\mathbb{V}} \simeq \mathscr{D}_{\mathbb{V}} \otimes \bigwedge^{q} \mathbb{V}. \tag{3.11}$$

Remark 3. The Euler vector field is written $\theta = \sum_{j=0}^{n} x_j \partial_{x_j}$ in the system of coordinates $(x_0, \ldots, x_n)$, and using the identification (3.11) one checks the equalities

$$d_{q-1}^{\mathbb{V}}(P \otimes \partial^{\alpha}) = \sum_{j=0}^{n} P\partial_{x_j} \otimes \widehat{\partial_{x_j}} \wedge \partial^{\alpha}, \quad e_{\mathbb{V}}^{q}(P \otimes \partial^{\alpha}) = \sum_{j=0}^{n} Px_j \otimes \partial_{x_j} \wedge \partial^{\alpha},$$
$$h_{\mathbb{V}}^{q}(P \otimes \partial^{\alpha}) = P(\theta + q) \otimes \partial^{\alpha},$$

where we set $\alpha = (\alpha_1, \ldots, \alpha_q)$ with $0 \leqslant \alpha_1 < \cdots < \alpha_q \leqslant n$, $\partial^{\alpha} = \partial_{x_{\alpha_1}} \wedge \cdots \wedge \partial_{x_{\alpha_q}}$, and we used the notation

$$\widehat{\partial_{x_j}} \wedge \partial^{\alpha} = \begin{cases} 0, & \text{if } j \neq \alpha_i \text{ for any } i, \\ (-1)^{i-1} \partial_{x_{\alpha_1}} \wedge \cdots \widehat{\partial_{x_{\alpha_j}}} \cdots \wedge \partial_{x_{\alpha_q}}, & \text{if } j = \alpha_i. \end{cases}$$

From (3.9), it follows that $h_{\mathbb{V}}^{\bullet}$ induces endomorphisms of the complexes $\mathscr{S}p_{\bullet}^{\mathbb{V}}$ and $\mathscr{E}u_{\mathbb{V}}^{\bullet}$, and we can consider the complexes $\mathscr{S}p_{\bullet}^{\mathbb{V},\theta}$ and $\mathscr{E}u_{\mathbb{V},\theta}^{\bullet}$ defined by the short exact sequences

$$0 \to \mathscr{S}p_{\bullet}^{\mathbb{V}} \xrightarrow{h_{\mathbb{V}}^{\bullet}} \mathscr{S}p_{\bullet}^{\mathbb{V}} \to \mathscr{S}p_{\bullet}^{\mathbb{V},\theta} \to 0,$$

$$0 \to \mathscr{E}u_{\mathbb{V}}^{\bullet} \xrightarrow{h_{\mathbb{V}}^{\bullet}} \mathscr{E}u_{\mathbb{V}}^{\bullet} \to \mathscr{E}u_{\mathbb{V},\theta}^{\bullet} \to 0.$$

Lemma 3. *In* $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_{\mathbb{V}})$ *one has the isomorphisms* $\mathscr{E}u_{\mathbb{V},\theta}^{\bullet} \simeq \mathscr{B}_{\{0\}|\mathbb{V}}[-n] \oplus \mathscr{B}_{\{0\}|\mathbb{V}}[-n-1]$ *and* $\mathscr{S}p_{\bullet}^{\mathbb{V},\theta} \simeq \mathcal{O}_{\mathbb{V}} \oplus \mathcal{O}_{\mathbb{V}}[1]$.

Proof. By (3.10) there is a distinguished triangle

$$\mathscr{B}_{\{0\}|\mathbb{V}} \xrightarrow{h} \mathscr{B}_{\{0\}|\mathbb{V}} \to \mathscr{E}u_{\mathbb{V},\theta}^{\bullet}[n+1] \xrightarrow{+1},$$

where h is defined by the commutative diagram with exact rows

$$\begin{tikzcd}
\mathscr{S}p_n^{\mathbb{V}} \arrow[r, "e_{\mathbb{V}}^n"] \arrow[d, "h_{\mathbb{V}}^n"] & \mathscr{S}p_{n+1}^{\mathbb{V}} \arrow[r, "q"] \arrow[d, "h_{\mathbb{V}}^{n+1}"] & \mathscr{B}_{\{0\}|\mathbb{V}} \arrow[r] \arrow[d, "h"] & 0 \\
\mathscr{S}p_n^{\mathbb{V}} \arrow[r, "e_{\mathbb{V}}^n"] & \mathscr{S}p_{n+1}^{\mathbb{V}} \arrow[r, "q"] & \mathscr{B}_{\{0\}|\mathbb{V}} \arrow[r] & 0
\end{tikzcd}$$

Let us use the notations in Remark 3. For $\alpha = (0, 1, \ldots, n)$ one has

$$\begin{aligned}
h(q(P \otimes \partial^\alpha)) &= q(h_{\mathbb{V}}^{n+1}(P \otimes \partial^\alpha)) \\
&= q\left(P\left(\sum_{j=0}^{n} x_j \partial_{x_j} + n + 1\right) \otimes \partial^\alpha\right) \\
&= q\left(P\left(\sum_{j=0}^{n} \partial_{x_j} x_j\right) \otimes \partial^\alpha\right) = q\left(e_{\mathbb{V}}^n\left(\sum_{i=0}^{n} P\partial_{x_i} \otimes \widehat{\partial_{x_i}} \wedge \partial^\alpha\right)\right) = 0.
\end{aligned}$$

So $h = 0$, and the first isomorphism is proved. The proof of the second isomorphism is like the one above, using (3.8) instead of (3.10). $\square$

Consider the maps $\mathbb{P} \overset{\pi}{\leftarrow} \dot{\mathbb{V}} \overset{j}{\rightarrow} \mathbb{V}$. By (3.11) one has identifications

$$\mathscr{S}p_q^{\mathbb{V},\theta} = \mathscr{E}u_{\mathbb{V},\theta}^q \simeq \mathscr{D}_{\mathbb{V}}(q) \otimes \bigwedge^q \mathbb{V}, \tag{3.12}$$

so that

$$\mathbb{D}j^* \mathscr{S}p_q^{\mathbb{V},\theta} = \mathbb{D}j^* \mathscr{E}u_{\mathbb{V},\theta}^q \simeq \mathbb{D}\pi^* \mathscr{D}_{\mathbb{P}}(q) \otimes \bigwedge^q \mathbb{V}.$$

We can then consider the complexes

$$\widetilde{\mathscr{S}p}_{\bullet}^{\mathbb{P}} = (\mathscr{D}_{\mathbb{P}}(0) \overset{d_0^{\mathbb{V}}}{\leftarrow} \mathscr{D}_{\mathbb{P}}(1) \otimes \mathbb{V} \leftarrow \cdots \overset{d_n^{\mathbb{V}}}{\leftarrow} \mathscr{D}_{\mathbb{P}}(n+1) \otimes \det \mathbb{V}),$$

$$\widetilde{\mathscr{E}u}_{\mathbb{P}}^{\bullet} = (\mathscr{D}_{\mathbb{P}}(0) \overset{e_{\mathbb{V}}^0}{\rightarrow} \mathscr{D}_{\mathbb{P}}(1) \otimes \mathbb{V} \rightarrow \cdots \overset{e_{\mathbb{V}}^n}{\rightarrow} \mathscr{D}_{\mathbb{P}}(n+1) \otimes \det \mathbb{V}),$$

whose differentials are induced, via Lemma 2, by those of $\mathscr{S}p_{\bullet}^{\mathbb{V},\theta}$ and $\mathscr{E}u_{\mathbb{V},\theta}^{\bullet}$, respectively.

Lemma 4. *The complex $\widetilde{\mathscr{E}u}_{\mathbb{P}}^{\bullet}$ is exact, and there is a distinguished triangle in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_{\mathbb{P}})$*

$$\mathscr{O}_{\mathbb{P}}[1] \rightarrow \widetilde{\mathscr{S}p}_{\bullet}^{\mathbb{P}} \rightarrow \mathscr{O}_{\mathbb{P}} \overset{+1}{\rightarrow}.$$

Proof. By Lemma 3 one has the isomorphisms in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_{\dot{\mathbb{V}}})$:

$$\mathbb{D}\pi^*\widetilde{\mathscr{E}u_{\mathbb{P}}^{\bullet}} \simeq \mathbb{D}j^*\mathscr{E}u_{\mathbb{V},\theta}^{\bullet} \simeq \mathbb{D}j^*(\mathscr{B}_{\{0\}|\mathbb{V}}[-n] \oplus \mathscr{B}_{\{0\}|\mathbb{V}}[-n-1]) \simeq 0,$$

hence $\widetilde{\mathscr{E}u_{\mathbb{P}}^{\bullet}}$ is exact by Lemma 2. Again by Lemma 3, one has the isomorphisms in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_{\dot{\mathbb{V}}})$:

$$\mathbb{D}\pi^*\widetilde{\mathscr{S}p_{\bullet}^{\mathbb{P}}} \simeq \mathbb{D}j^*\mathscr{S}p_{\bullet}^{\mathbb{V},\theta} \simeq \mathbb{D}j^*(\mathscr{O}_{\mathbb{V}} \oplus \mathscr{O}_{\mathbb{V}}[1]) \simeq \mathbb{D}\pi^*(\mathscr{O}_{\mathbb{P}} \oplus \mathscr{O}_{\mathbb{P}}[1]).$$

It follows from Lemma 2 that

$$H^j(\widetilde{\mathscr{S}p_{\bullet}^{\mathbb{P}}}) \simeq \begin{cases} 0, & \text{for } j \neq 0, -1, \\ \mathscr{O}_{\mathbb{P}}, & \text{for } j = 0, -1, \end{cases} \tag{3.13}$$

and hence there is a distinguished triangle as stated. $\square$

Recall that a form $\omega \in j^{-1}\Omega_{\mathbb{V}}^q$ is the pull-back $\omega = \pi^*\alpha$ of a form $\alpha \in \Omega_{\mathbb{P}}^q$ if and only if

$$\begin{cases} h_q^{\mathbb{V}}\omega = 0, \\ e_{q-1}^{\mathbb{V}}\omega = 0. \end{cases}$$

In other words, there is a quasi-isomorphism

$$\mathscr{S}p_q^{\mathbb{P}} \overset{\text{qis}}{\leftarrow} \widetilde{\mathscr{E}u_{\mathbb{P}}^{\leqslant q}}[q], \tag{3.14}$$

and moreover the Spencer differentials $d_q^{\mathbb{P}}$ correspond to the morphisms of complexes

$$d_{\bullet}^{\mathbb{V}} : \widetilde{\mathscr{E}u_{\mathbb{P}}^{\leqslant q-1}}[q-1] \leftarrow \widetilde{\mathscr{E}u_{\mathbb{P}}^{\leqslant q}}[q].$$

Note also that by Lemma 4 there is a quasi-isomorphism $\widetilde{\mathscr{E}u_{\mathbb{P}}^{\leqslant q}}[q] \overset{\text{qis}}{\rightarrow} \widetilde{\mathscr{E}u_{\mathbb{P}}^{\geqslant q+1}}[q+1]$.

Interchanging the role of Spencer and Euler, let us set the following definition.

Definition 1. For $0 \leqslant q \leqslant n$ set

$$\mathscr{E}u_{\mathbb{P}}^q = H^0(\widetilde{\mathscr{S}p_{\geqslant q+1}^{\mathbb{P}}}[q+1]),$$

and consider the complex

$$\mathscr{E}u_{\mathbb{P}}^{\bullet} = (\mathscr{E}u_{\mathbb{P}}^0 \xrightarrow{e_{\mathbb{P}}^0} \mathscr{E}u_{\mathbb{P}}^1 \rightarrow \cdots \xrightarrow{e_{\mathbb{P}}^{n-1}} \mathscr{E}u_{\mathbb{P}}^n)$$

whose differentials are induced by the morphisms of complexes

$$e_{\mathbb{V}}^{\bullet} : \widetilde{\mathscr{S}p_{\geqslant q+1}^{\mathbb{P}}}[q+1] \rightarrow \widetilde{\mathscr{S}p_{\geqslant q+2}^{\mathbb{P}}}[q+2].$$

Note that by Lemma 4 there is a quasi-isomorphism

$$\mathscr{E}u^q_{\mathbb{P}} \overset{\text{qis}}{\leftarrow} \widetilde{\mathscr{S}p}^{\mathbb{P}}_{\geqslant q+1}[q+1],$$

but one should beware that $\widetilde{\mathscr{S}p}^{\mathbb{P}}_{\leqslant q}[q] \not\simeq \widetilde{\mathscr{S}p}^{\mathbb{P}}_{\geqslant q+1}[q+1]$. Note also that, by definition

$$\mathscr{E}u^n_{\mathbb{P}} = \mathscr{S}p^{\mathbb{P}}_n.$$

Lemma 5. *For* $0 \leqslant q \leqslant n$ *there are isomorphisms in* $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_{\mathbb{P}})$

$$\mathscr{E}u^{\geqslant q}_{\mathbb{P}}[q] \simeq \mathscr{S}p^{\mathbb{P}}_{\geqslant q}[q].$$

Proof. Denoting by $s^\bullet$ and $s_\bullet$ the simple complexes associated with a double complex, one has

$$\mathscr{E}u^{\geqslant q}_{\mathbb{P}}[q] \simeq s^\bullet(\widetilde{\mathscr{S}p}^{\mathbb{P}}_{\geqslant q+1}[q+1] \xrightarrow{e_\vee} \widetilde{\mathscr{S}p}^{\mathbb{P}}_{\geqslant q+2}[q+2] \xrightarrow{e_\vee} \cdots \xrightarrow{e_\vee} \widetilde{\mathscr{S}p}^{\mathbb{P}}_{\geqslant n+1}[n+1]),$$

$$\mathscr{S}p^{\mathbb{P}}_{\geqslant q}[q] \simeq s_\bullet(\widetilde{\mathscr{E}u}^{\geqslant q+1}_{\mathbb{P}}[q+1] \xleftarrow{d^\vee} \widetilde{\mathscr{E}u}^{\geqslant q+2}_{\mathbb{P}}[q+2] \xleftarrow{d^\vee} \cdots \xleftarrow{d^\vee} \widetilde{\mathscr{E}u}^{\geqslant n+1}_{\mathbb{P}}[n+1]).$$

One concludes by noticing that the first double complex coincides with the second one after interchanging the roles of rows and columns. $\square$

In particular, for $q=0$ we get a quasi-isomorphism

$$\mathscr{O}_{\mathbb{P}} \xrightarrow{\sim} \mathscr{E}u^\bullet_{\mathbb{P}}.$$

Moreover, using the distinguished triangle

$$\mathscr{E}u^{\geqslant q+1}_{\mathbb{P}} \to \mathscr{E}u^{\geqslant q}_{\mathbb{P}} \to \mathscr{E}u^q_{\mathbb{P}}[-q] \xrightarrow{+1},$$

one gets short exact sequences

$$0 \to \operatorname{coker} d^{\mathbb{P}}_q \to \mathscr{E}u^q_{\mathbb{P}} \to \operatorname{coker} d^{\mathbb{P}}_{q+1} \to 0 \tag{3.15}$$

which should be compared with the usual

$$0 \to \operatorname{coker} d^{\mathbb{P}}_{q+1} \to \mathscr{S}p^{\mathbb{P}}_q \to \operatorname{coker} d^{\mathbb{P}}_q \to 0. \tag{3.16}$$

To end this section, it is interesting to note that the distinguished triangle in Lemma 4 does not split. In other words, the complex $\widetilde{\mathscr{S}p}^{\mathbb{P}}_\bullet$ is not isomorphic to the direct sum $\mathscr{O}_{\mathbb{P}} \oplus \mathscr{O}_{\mathbb{P}}[1]$.

Proposition 4. *The morphism* $\alpha: \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}}[2]$ *induced by the distinguished triangle in Lemma* 4 *is not zero in* $\mathrm{Hom}_{\mathcal{D}_{\mathbb{P}}}(\mathcal{O}_{\mathbb{P}}, \mathcal{O}_{\mathbb{P}}[2]) \simeq \mathbf{k}$.

Proof. Using (3.13), from the distinguished triangle

$$\widetilde{\mathcal{S}p}_0^{\mathbb{P}} \xrightarrow{d^{\mathbb{V}}} \widetilde{\mathcal{S}p}_{\bullet}^{\mathbb{P}} \xrightarrow{d^{\mathbb{V}}} \widetilde{\mathcal{S}p}_{\geqslant 1}^{\mathbb{P}} \xrightarrow{+1}$$

we get the long exact cohomology sequence

$$0 \to \mathcal{O}_{\mathbb{P}} \to \mathcal{E}u_{\mathbb{P}}^0 \xrightarrow{d^{\mathbb{V}}} \mathcal{D}_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}} \to 0,$$

which describes α as a Yoneda extension. Since $\mathrm{im}\, d^{\mathbb{V}} = \mathcal{D}_{\mathbb{P}}\Theta_{\mathbb{P}} \subset \mathcal{D}_{\mathbb{P}}$, this sequence decomposes into the short exact sequences

$$0 \to \mathcal{O}_{\mathbb{P}} \to \mathcal{E}u_{\mathbb{P}}^0 \to \mathcal{D}_{\mathbb{P}}\Theta_{\mathbb{P}} \to 0,$$

$$0 \to \mathcal{D}_{\mathbb{P}}\Theta_{\mathbb{P}} \to \mathcal{D}_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}} \to 0, \tag{3.17}$$

which are but (3.15) and (3.16) for $q = 0$. These sequences describe, as Yoneda extensions, the morphisms $\beta: \mathcal{D}_{\mathbb{P}}\Theta_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}}[1]$ and $\gamma: \mathcal{O}_{\mathbb{P}} \to \mathcal{D}_{\mathbb{P}}\Theta_{\mathbb{P}}[1]$, respectively, and one has $\alpha = \beta[1] \circ \gamma$. Note that β and γ are essentially unique, since

$$\mathrm{Hom}_{\mathcal{D}_{\mathbb{P}}}(\mathcal{D}_{\mathbb{P}}\Theta_{\mathbb{P}}, \mathcal{O}_{\mathbb{P}}[1]) \simeq \mathbf{k} \simeq \mathrm{Hom}_{\mathcal{D}_{\mathbb{P}}}(\mathcal{O}_{\mathbb{P}}, \mathcal{D}_{\mathbb{P}}\Theta_{\mathbb{P}}[1]),$$

as follows by applying the functors $\mathrm{RHom}_{\mathcal{D}_{\mathbb{P}}}(\bullet, \mathcal{O}_{\mathbb{P}})$ and $\mathrm{RHom}_{\mathcal{D}_{\mathbb{P}}}(\mathcal{O}_{\mathbb{P}}, \bullet)$ to the exact sequence (3.17). Note also that $\beta \neq 0 \neq \gamma$ since

$$\mathrm{Hom}_{\mathcal{D}_{\mathbb{P}}}(\mathcal{E}u_{\mathbb{P}}^0, \mathcal{O}_{\mathbb{P}}) = 0 = \mathrm{Hom}_{\mathcal{D}_{\mathbb{P}}}(\mathcal{O}_{\mathbb{P}}, \mathcal{D}_{\mathbb{P}}),$$

where the second equality is obvious, and the first one follows from the exact sequence $0 \to \mathrm{Hom}_{\mathcal{D}_{\mathbb{P}}}(\mathcal{E}u_{\mathbb{P}}^0, \mathcal{O}_{\mathbb{P}}) \to \Gamma(\mathbb{P}; \mathcal{O}_{\mathbb{P}}(-1)) \otimes \mathbb{V}^* = 0$ obtained by applying the functor $\mathrm{Hom}_{\mathcal{D}_{\mathbb{P}}}(\bullet, \mathcal{O}_{\mathbb{P}})$ to the exact sequence $\mathcal{D}_{\mathbb{P}}(1) \otimes \mathbb{V} \to \mathcal{E}u_{\mathbb{P}}^0 \to 0$. To conclude, consider the morphism of distinguished triangles:

$$\begin{tikzcd}
\mathcal{D}_{\mathbb{P}} \arrow[r] \arrow[d, "0"] & \mathcal{O}_{\mathbb{P}} \arrow[r, "\gamma"] \arrow[d, "\alpha"] & \mathcal{D}_{\mathbb{P}}\Theta_{\mathbb{P}}[1] \arrow[r, "+1"] \arrow[d, "\beta[1]"] & {} \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}}[2] \arrow[r, "id"] & \mathcal{O}_{\mathbb{P}}[2] \arrow[r, "+1"] & {}
\end{tikzcd}$$

If α were zero, then β also would be zero, which is a contradiction. $\square$

3.4. *Radon transform of differential forms*

Theorem 8. *There are natural isomorphisms*

$$\mathcal{S}p_q^{\mathbb{P}} \overset{\mathbb{D}}{\circ} \mathcal{R} \xleftarrow{\sim} \mathcal{E}u_{\mathbb{P}^*}^{n-q}.$$

Taking holomorphic solutions one get a description of the Radon transform of the sheaf of differential forms which, using (3.15), should be compared with the results in [8].

Proof. First, note that using the identification (3.12) one has

$$\mathscr{E}u^q_{\mathbb{V},\theta} \overset{\mathbb{D}}{\circ} \mathscr{L} \simeq \mathscr{D}_{V^*}(n+1-q) \otimes \det \mathbb{V}^* \otimes \bigwedge^q \mathbb{V} \simeq \mathscr{S}p^{\mathbb{V}^*,\theta}_{n+1-q}. \tag{3.18}$$

Since $d^{\mathbb{V}^*}$ and $e_{\mathbb{V}}$ are interchanged by Fourier, one gets the following isomorphisms

$$\mathscr{E}u^{\leqslant q}_{\mathbb{V},\theta}[q] \overset{\mathbb{D}}{\circ} \mathscr{L} \simeq \mathscr{S}p^{\mathbb{V}^*,\theta}_{\geqslant n+1-q}[n+1-q]. \tag{3.19}$$

One has the chain of isomorphisms

$$\begin{aligned}
\mathbb{D}\pi^*(\mathscr{S}p^{\mathbb{P}}_q \overset{\mathbb{D}}{\circ} \mathscr{R}) &\simeq \mathbb{D}j^*[(\mathbb{D}j_!\mathbb{D}\pi^*\mathscr{S}p^{\mathbb{P}}_q) \overset{\mathbb{D}}{\circ} \mathscr{L}] \\
&\simeq \mathbb{D}j^*[(\mathbb{D}j_!\mathbb{D}j^*\mathscr{E}u^{\leqslant q}_{\mathbb{V},\theta}[q]) \overset{\mathbb{D}}{\circ} \mathscr{L}] \\
&\simeq \mathbb{D}j^*(\mathscr{E}u^{\leqslant q}_{\mathbb{V},\theta}[q] \overset{\mathbb{D}}{\circ} \mathscr{L}) \\
&\simeq \mathbb{D}j^*\mathscr{S}p^{\mathbb{V}^*,\theta}_{\geqslant n+1-q}[n+1-q] \\
&\simeq \mathbb{D}\pi^*\mathscr{E}u^{n-q}_{\mathbb{P}^*},
\end{aligned}$$

where the first isomorphism follows from Theorem 3, the second by (3.14), the fourth by (3.19), and the last by the definition of $\mathscr{E}u^{n-q}_{\mathbb{P}}$. The third isomorphism follows from Proposition 3, using the identification (3.12). $\square$

Proof of Theorem 7. The proof goes as the one above, considering the chain of isomorphisms:

$$\begin{aligned}
\mathbb{D}\pi^*(\mathscr{S}p^{\mathbb{P}}_{\leqslant p}[q] \overset{\mathbb{D}}{\circ} \mathscr{R}) &\simeq \mathbb{D}j^*[(\mathbb{D}j_!\mathbb{D}\pi^*\mathscr{S}p^{\mathbb{P}}_{\leqslant q}[q]) \overset{\mathbb{D}}{\circ} \mathscr{L}] \\
&\simeq \mathbb{D}j^*[(\mathbb{D}j_!\mathbb{D}j^*s_\bullet(\mathscr{E}u^{\leqslant 0}_{\mathbb{V},\theta}[0] \overset{d^{\mathbb{V}}}{\leftarrow} \cdots \overset{d^{\mathbb{V}}}{\leftarrow} \mathscr{E}u^{\leqslant q}_{\mathbb{V},\theta}[q])[q]) \overset{\mathbb{D}}{\circ} \mathscr{L}] \\
&\simeq \mathbb{D}j^*[s_\bullet(\mathscr{E}u^{\leqslant 0}_{\mathbb{V},\theta}[0] \overset{d^{\mathbb{V}}}{\leftarrow} \cdots \overset{d^{\mathbb{V}}}{\leftarrow} \mathscr{E}u^{\leqslant q}_{\mathbb{V},\theta}[q])[q] \overset{\mathbb{D}}{\circ} \mathscr{L}] \\
&\simeq \mathbb{D}j^*s_\bullet(\mathscr{S}p^{\mathbb{V}^*,\theta}_{\geqslant n+1}[n+1] \overset{e_{\mathbb{V}^*}}{\leftarrow} \cdots \overset{e_{\mathbb{V}^*}}{\leftarrow} \mathscr{S}p^{\mathbb{V}^*,\theta}_{\geqslant n+1-q}[n+1-q])[q]
\end{aligned}$$

$$
\begin{aligned}
&\simeq \mathbb{D}j^* s^\bullet (\mathscr{E}u^{\geqslant n+1}_{\mathbb{V}^*,\theta}[n+1] \xrightarrow{d^{\mathbb{V}^*}} \cdots \xrightarrow{d^{\mathbb{V}^*}} \mathscr{E}u^{\geqslant n+1-q}_{\mathbb{V}^*,\theta}[n+1-q])[q] \\
&\simeq \mathbb{D}j^* s_\bullet (\mathscr{E}u^{\leqslant n-q}_{\mathbb{V}^*,\theta}[n-q] \xleftarrow{d^{\mathbb{V}^*}} \cdots \xleftarrow{d^{\mathbb{V}^*}} \mathscr{E}u^{\leqslant n}_{\mathbb{V}^*,\theta}[n]) \\
&\simeq \mathbb{D}\pi^* \mathscr{S}p^{\mathbb{P}^*}_{\geqslant n-q}[n-q],
\end{aligned}
$$

where the sixth isomorphism is due to Lemma 3, and the fifth uses the same argument as in Lemma 5. $\square$

3.5. Quantization of the Radon transform for differential forms

According to [7], the integral kernel of the morphism

$$\mathscr{S}p^{\mathbb{P}^*}_{\geqslant n-q}[n-q] \to \mathscr{S}p^{\mathbb{P}}_{\leqslant q}[q] \overset{\mathbb{D}}{\circ} \mathscr{R}$$

in Theorem 7 is given by a section

$$s_{n-q}(x,y) \in \mathrm{Hom}_{\mathscr{D}_{\mathbb{P}\times\mathbb{P}^*}}(\mathscr{S}p^{\mathbb{P}}_{\geqslant n-q}[n-q] \overset{\mathbb{D}}{\boxtimes} \mathscr{S}p^{\mathbb{P}^*}_{\geqslant n-q}[n-q], \mathscr{B}_{\mathbb{U}|\mathbb{P}\times\mathbb{P}^*}).$$

Similarly, the integral kernel of the morphism

$$\mathscr{E}u^{n-q}_{\mathbb{P}^*} \to \mathscr{S}p^{\mathbb{P}}_q \overset{\mathbb{D}}{\circ} \mathscr{R}$$

in Theorem 8 is given by a section

$$t_{n-q}(x,y) \in \mathrm{Hom}_{\mathscr{D}_{\mathbb{P}\times\mathbb{P}^*}}(\mathscr{S}p^{\mathbb{P}}_{n-q} \overset{\mathbb{D}}{\boxtimes} \mathscr{E}u^{n-q}_{\mathbb{P}^*}, \mathscr{B}_{\mathbb{U}|\mathbb{P}\times\mathbb{P}^*}).$$

Let us describe them.

The canonical map $\mathbf{k} \to \bigwedge^q \mathbb{V}^* \otimes \bigwedge^q \mathbb{V}$ induces a monomorphism

$$\mathscr{O}_{\mathbb{V}\times\mathbb{V}^*} \hookrightarrow \Omega^q_{\mathbb{V}} \overset{\mathscr{O}}{\boxtimes} \Omega^q_{\mathbb{V}^*},$$

and we denote by $\sigma_q(x,y)$ the image of 1. Equivalently, consider the maps

$$\Omega^1_{\mathbb{V}} \overset{\mathscr{O}}{\boxtimes} \Omega^1_{\mathbb{V}^*} \hookrightarrow \Omega^2_{\mathbb{V}\times\mathbb{V}^*} \xrightarrow{\bigwedge^q} \Omega^{2q}_{\mathbb{V}\times\mathbb{V}^*} \xrightarrow{p} \Omega^q_{\mathbb{V}} \overset{\mathscr{O}}{\boxtimes} \Omega^q_{\mathbb{V}^*},$$

where p is the projector to the (q,q) component. Then σ_1 is the symplectic form of $\mathbb{V} \times \mathbb{V}^*$, and σ_q coincides, suitably normalized, with $p(\bigwedge^q \sigma_1)$.

Setting

$$u_q(x,y) = \frac{\sigma_q(x,y)}{\langle x,y \rangle^q},$$

one checks that

$$\begin{cases} h^{\mathbb{V}} u_q(x,y) = h^{\mathbb{V}^*} u_q(x,y) = 0, \\ d_{\mathbb{V}^*} e^{\mathbb{V}} u_q(x,y) = d_{\mathbb{V}} e^{\mathbb{V}^*} u_q(x,y) = 0, \\ e^{\mathbb{V}^*} e^{\mathbb{V}} u_{q+1}(x,y) = d_{\mathbb{V}} d_{\mathbb{V}^*} u_{q-1}(x,y). \end{cases}$$

Then, one has

$$t_{n-q}(x,y) = e^{\mathbb{V}} u_{n+1-q}(x,y),$$

$$s_{n-q}(x,y) = e^{\mathbb{V}^*} e^{\mathbb{V}} u_{n+1-q}(x,y).$$

Using homogeneous coordinates,

$$s_{n-q}(x,y) = \langle x,y \rangle^{-n-1+q} \det\left(y, \overbrace{dy, \ldots, dy}^{n-q}, \overbrace{\partial_x, \ldots, \partial_x}^{q} \right) \lrcorner\, \omega(x),$$

where $\lrcorner$ denotes the interior product, and ω the Leray form. In particular, one has

$$s_1(x,y) = -d_{\mathbb{V}} d_{\mathbb{V}^*} \log\langle x,y \rangle = -d_{\mathbb{V}^*} \frac{\langle y, dx \rangle}{\langle x,y \rangle},$$

and

$$s_n(x,y) = \frac{\omega(x) \wedge \omega(y)}{\langle x,y \rangle^{n+1}}.$$

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References

[1] A. Beilinson, J. Bernstein, A proof of Jantzen conjectures, in: S.I. Gelfand, S.G. Gindikin (Eds.), I.M. Gelfand Seminar, Advances in Soviet Mathematics, Vol. 16 (Part 1), American Mathematical Society, Providence, RI, 1993, pp. 1–50.

[2] J. Bernstein, Lectures on Algebraic $\mathscr{D}$-modules at Berkeley, 2001.
[3] A. Borel, Algebraic $\mathscr{D}$-modules, in: Perspectives in Mathematics, Vol. 2, Academic Press, New York, 1987.
[4] J.-L. Brylinski, Transformations canoniques, dualité projective, théorie de Lefschetz, transformations de Fourier et sommes trigonométriques, Géométrie et analyse microlocales, Astérisque 140–141 (1986) 3–134.
[5] A. D'Agnolo, C. Marastoni, A topological obstruction for the real Radon transform, in: M. Picardello (Ed.), Harmonic Analysis and Integral Geometry (Safi, 1998), CRC Research: Notes in Mathematics, Vol. 422, Chapman & Hall, London, 2001, pp. 127–133.
[6] A. D'Agnolo, P. Polesello, Quantization of twisted projective duality, in preparation.
[7] A. D'Agnolo, P. Schapira, Leray's quantization of projective duality, Duke Math. J. 84 (2) (1996) 453–496.
[8] M.G. Eastwood, A duality for homogeneous vector bundles on twistor space, J. London Math. Soc. 31 (1985) 349–356.
[9] J.-L. Frot, Correspondance d'Andreotti–Norguet et $\mathscr{D}$-modules, Publ. Res. Inst. Math. Sci. 35 (4) (1999) 637–677.
[10] M. Kashiwara, Algebraic study of systems of partial differential equations, Mém. Soc. Math. France (N.S.) 63 (1995) xiv + 72; Kashiwara's Master's Thesis, Tokyo University, 1970 (A. D'Agnolo, J.-P. Schneiders, Transl. from Japanese).
[11] M. Kashiwara, Representation theory and $\mathscr{D}$-modules on flag varieties, Orbites unipotentes et représentations, III, Astérisque 173–174 (9) (1989) 55–109.
[12] M. Kashiwara, $\mathscr{D}$-modules and microlocal calculus (translated from the 2000 Japanese original by M. Saito), Transl. of Math. Monographs 217 AMS (2003).
[13] M. Kashiwara, P. Schapira, Sheaves on manifolds, Grundlehren der Mathematischen Wissenschaften, Vol. 292, Springer, Berlin, 1990.
[14] M. Kashiwara, P. Schapira, Integral transforms with exponential kernels and Laplace transform, J. Amer. Math. Soc. 10 (1997) 939–972.
[15] M. Kashiwara, T. Tanisaki, Kazhdan–Lusztig conjecture for affine Lie algebras with negative level. II. Nonintegral case, Duke Math. J. 84 (3) (1996) 771–813.
[16] N.M. Katz, G. Laumon, Transformation de Fourier et majoration de sommes exponentielles, Inst. Hautes Études Sci. Publ. Math. 62 (1985) 361–418.
[17] M. Sato, T. Kawai, M. Kashiwara, Microfunctions and pseudo-differential equations, in: H. Komatsu (Ed.), Hyperfunctions and Pseudo-differential Equations (Katata, 1971), Lecture Notes in Mathematics, Vol. 287, Springer, New York, 1973, pp. 265–529.