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ON STRONGLY FLAT MODULES OVER INTEGRAL DOMAINS

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ABSTRACT. We investigate strongly flat modules over general integral domains. More detailed information is obtained for strongly flat modules over valuation domains, their pure submodules and on the existence of dense basic submodules.

1. Introduction. The notion of cotorsion abelian group, introduced by Harrison in 1959 [**6**], was generalized in various different ways for modules over any associative ring with unit.

A weak notion of cotorsion modules over commutative integral domains was introduced by Matlis [**7**]. By slightly modifying his definition, we say that a module C over a commutative domain R is *weakly cotorsion*, or Matlis cotorsion, provided that $\mathrm{Ext}^1_R(Q,C)=0$, where Q denotes the field of quotients of R. Let us denote by $\mathcal{WC}$ the class of weakly cotorsion modules.

An R-module M is *strongly flat* if $\mathrm{Ext}^1_R(M,C)=0$ for every weakly cotorsion module C. Let us denote by $\mathcal{SF}$ the class of strongly flat R-modules. In the terminology of cotorsion theories (see [**9**]) the pair $(\mathcal{SF},\mathcal{WC})$ is the cotorsion theory cogenerated by Q. Since Q is a flat module, the class $\mathcal{WC}$ contains the class $\mathcal{C}$ of the *cotorsion* modules in the sense of Enochs, namely, of those modules C such that $\mathrm{Ext}^1_R(F,C)=0$ for all flat modules F. It is well known that also the pair $(\mathcal{F},\mathcal{C})$ is a cotorsion theory, where $\mathcal{F}$ denotes the class of the flat modules; since the class $\mathcal{C}$ is contained in the class $\mathcal{WC}$, the class $\mathcal{F}$ contains the class $\mathcal{SF}$, i.e., every strongly flat module is flat.

The goal of this paper is to provide an initial contribution to the study of strongly flat modules.

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In Section 2 we give a characterization of strongly flat modules over general domains, which improves that given by Trlifaj in [**10**], with more satisfactory results for Matlis domains and valuation domains. Furthermore, we show that strongly flat ideals of coherent domains are necessarily projective, deriving that an ideal of a Prüfer domain is strongly flat exactly if it is finitely generated. This fact gives evidence to the failure of the closure property of the class of strongly flat modules with respect to direct limits. Finally we prove that all the localizations at maximal ideals of a strongly flat module are strongly flat modules over the corresponding localized rings and that the converse is true over h-local domains.

Section 3 is dedicated to study more deeply strongly flat modules over valuation domains. A satisfactory characterization is obtained also in the non-Matlis case. Then we prove that the class of strongly flat modules is closed under taking pure submodules of countable rank and that a Pontryagin type criterion holds. Counterexamples are also given to show that this closure property cannot be extended to pure submodules of higher rank. Finally the question of whether every strongly flat module has a dense basic submodule is explored and we show that this happens up to rank $\aleph_1$.

1. Preliminaries and notations. R will always denote a commutative domain which is not a field, Q its quotient field and K the divisible module Q/R. A torsion divisible module will be called K-free if it is isomorphic to a direct sum $K^{(\alpha)}$ of copies of K. The projective, respectively injective, dimension of a module M is denoted by p.d.M, respectively i.d.M. Following [**5**], we say that R is a *Matlis domain* if p.d.$Q = 1$.

We review now some well-known facts which will be used frequently in the sequel, see [**7, 4, 5**] for more details. An R-module M is said to be h-reduced if $\mathrm{Hom}_R(Q, M) = 0$. If M is torsion-free the sum $d(M)$ of all the divisible submodules of M is an injective submodule of M and $d(M) = 0$ if and only if $\mathrm{Hom}_R(Q, M) = 0$; in this case, M is said to be reduced. Any R-module M can be equipped with the R-topology, namely, the linear topology in which a subbasis of neighborhoods of 0 consists of the submodules rM of M for any nonzero element $r \in R$. The Hausdorff completion of M, endowed with the R-topology, will be denoted by $\widetilde{M}$. If M is a torsion-free module, then M is Hausdorff in the R-topology if and only if $d(M) = 0$; moreover, there exists a natural map $\varepsilon : M \to \widetilde{M}$ such that $\operatorname{Ker} \varepsilon = d(M)$, εM is RD-pure and dense in $\widetilde{M}$, i.e., $\widetilde{M}/\varepsilon M$ is torsion-free divisible. A torsion-free reduced module M is complete if and only if $\operatorname{Ext}^1_R(Q, M) = 0$, and $\widetilde{M} \cong \operatorname{Ext}^1_R(K, M)$; furthermore, $\widetilde{M}$ has a unique structure of $\widetilde{R}$-module.

The endomorphism ring of K is the completion $\widetilde{R}$ of R and it is a commutative ring. In case R is a valuation domain, then $\widetilde{R}$ is a valuation domain too and it is an immediate extension of R; hence, every element $\xi \in \widetilde{R}$ can be written in the form $r\eta$ for some $r \in R$ and some unit $\eta \in \widetilde{R}$.

Recall that a module is h-divisible if it is an epimorphic image of an injective module; for torsion-free modules the notion of h-divisibility is the same as the notion of divisibility. Moreover, the class of h-divisible modules coincides with the class of divisible modules exactly if R is a Matlis domain, see [**5**].

We will also make use of the classical Matlis category equivalence between torsion h-divisible modules and complete torsion-free modules, induced by the two functors $\operatorname{Hom}_R(K, -)$ and $K_R \otimes -$.

We recall that a domain R is h-local if every nonzero ideal is contained in only finitely many maximal ideals and every nonzero prime ideal is contained only in one maximal ideal, see [**7**] and [**5**].

2. Characterization and properties of strongly flat modules.

Let R be a commutative domain. Recall that a module M is strongly flat if $\operatorname{Ext}^1_R(M, C) = 0$ for every weakly cotorsion module, i.e., for every module C satisfying $\operatorname{Ext}^1_R(Q, C) = 0$. We have recalled in the introduction that strongly flat modules are flat, whence torsion-free. If a torsion-free module M decomposes as $M = D \oplus A$ with D divisible and A reduced, then M is strongly flat if and only if A is so. Thus, in order to characterize strongly flat modules, we will always consider reduced torsion-free modules.

In the next theorem we collect some characterizations of strongly flat modules. Two of them hold only over Matlis domains. The first one is an improvement of Trlifaj's characterization in Proposition 2.8 of [**10**], and the implication $4 \Rightarrow 1$, for Matlis domains, appears in Proposition 10.6 of [**7**].

Theorem 2.1. *Let R be a commutative domain and M a reduced torsion-free R-module. Consider the following statements.*

1. *M is strongly flat.*

2. *M is a summand of a reduced module N such that there is an exact sequence of the form*

$$0 \longrightarrow R^{(\alpha)} \longrightarrow N \longrightarrow Q^{(\beta)} \longrightarrow 0,$$

for suitable cardinals α and β.

3. *M is flat and $\mathrm{Ext}^1_R(M, C) = 0$ for every flat weakly cotorsion module C.*

4. *$\widetilde{M}$ is a summand of the completion of a free module.*

5. *$K \otimes_R M$ is a summand of a K-free module.*

Then $1 \Leftrightarrow 2 \Leftrightarrow 3 \Rightarrow 4 \Leftrightarrow 5$, and the five conditions are equivalent if R is a Matlis domain.

Proof. $1 \Rightarrow 2$. Let M be a strongly flat module and consider a free presentation of $M : 0 \to H \to R^{(\alpha)} \to M \to 0$. Let $\widetilde{H}$ be the completion of H and consider the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & H & \longrightarrow & R^{(\alpha)} & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \widetilde{H} & \longrightarrow & N & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \oplus Q & = & \oplus Q & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since M is strongly flat, the second row splits; hence M is a summand of N which is clearly a reduced module. Thus the second column of the above diagram satisfies condition 2.

$2 \Rightarrow 1$ is obvious since the class $\mathcal{SF}$ is closed under extensions and summands.

$1 \Rightarrow 3$ is obvious.

$3 \Rightarrow 1$. Let M be a flat R-module such that $\mathrm{Ext}^1_R(M, C) = 0$ for every flat weakly cotorsion module C, and let C_1 be any weakly cotorsion module. By the existence of flat covers, see [**2**], we can consider an exact sequence

$$0 \longrightarrow Y \longrightarrow G \longrightarrow C_1 \longrightarrow 0$$

where G is a flat module and Y is cotorsion. Since Y is weakly cotorsion we conclude that G is weakly cotorsion too, thus $\mathrm{Ext}^1_R(M, G) = 0$ by hypothesis. The above sequence gives rise to the exact sequence:

$$0 = \mathrm{Ext}^1_R(M, G) \longrightarrow \mathrm{Ext}^1_R(M, C_1) \longrightarrow \mathrm{Ext}^2_R(M, Y).$$

Thus the conclusion follows by the fact that $\mathrm{Ext}^n_R(M, Y) = 0$ for every n provided that M is flat and Y is cotorsion, see for instance [**11**, Proposition 3.1.2].

$2 \Rightarrow 5$. Assume that M is a summand of a torsion-free module N which fits in an exact sequence

$$0 \longrightarrow R^{(\alpha)} \longrightarrow N \longrightarrow Q^{(\beta)} \longrightarrow 0.$$

Note that the above sequence is pure, since $Q^{(\beta)}$ is a flat R-module. Tensoring by K we get $K^{(\alpha)} \cong K \otimes_R N$ and thus $K \otimes_R M$ is a summand of $K^{(\alpha)}$.

$4 \Leftrightarrow 5$. It is enough to recall that, for every torsion-free module X, $\widetilde{X} \cong \mathrm{Hom}_R(K, K \otimes_R X)$, hence $\mathrm{Hom}_R(K, K^{(\alpha)}) \cong \widetilde{R^{(\alpha)}}$ and moreover $K \otimes_R \widetilde{R^{(\alpha)}} \cong K^{(\alpha)}$.

Assuming that R is a Matlis domain, we prove $4 \Rightarrow 1$. Let M be such that $\widetilde{M}$ is a summand of $\widetilde{R^{(\alpha)}}$ for some cardinal α. Then $\widetilde{M}$ is strongly flat by the equivalence $1 \Leftrightarrow 2$. From the exact sequence

$$0 \longrightarrow M \longrightarrow \widetilde{M} \longrightarrow \oplus Q \longrightarrow 0$$

for any weakly cotorsion module C we obtain the exact sequence

$$\mathrm{Ext}^1_R(\oplus Q, C) \longrightarrow \mathrm{Ext}^1_R(\widetilde{M}, C) \longrightarrow \mathrm{Ext}^1_R(M, C) \longrightarrow \mathrm{Ext}^2_R(\oplus Q, C) \longrightarrow 0;$$

the first Ext vanishes, since $C \in Q^{\perp}$ and the last Ext vanishes, since p.d.$Q = 1$. Thus $\mathrm{Ext}^1_R(M,C) \cong \mathrm{Ext}^1_R(\widetilde{M},C)$ is zero, since $\widetilde{M}$ is strongly flat. □

In Theorem 2.1 condition 4 does not imply condition 1 if R is not a Matlis domain, even if it is a valuation domain as we will see in proving Proposition 3.6 (1). The obvious question whether condition 4 in Theorem 2.1 can be replaced by the stronger condition that $\widetilde{M}$ is the completion of a projective module has in general a negative answer. For instance, take the $\mathbf{Z}$-module $M = \mathbf{Z}_p$; then $\widetilde{M} = J_p$ is a summand of $\widetilde{\mathbf{Z}}$, but it is not the completion of a projective $\mathbf{Z}$-module. We will see in Corollary 2.7 that the answer to the above question is positive if R is a valuation domain or a local Matlis domain.

As applications of Theorem 2.1 we derive the next three results, which show that, under suitable hypotheses on the domain R, certain strongly flat R-modules are projectives.

Proposition 2.2. *Let R be a coherent domain. Then an ideal I of R is strongly flat if and only if it is projective.*

Proof. The sufficiency is obvious. Conversely, assume I is strongly flat; by Theorem 2.1 there exists an exact sequence

$$0 \longrightarrow R^{(\alpha)} \longrightarrow I \oplus X \longrightarrow Q^{(\beta)} \longrightarrow 0$$

that, tensored by K, gives the isomorphism $K^{(\alpha)} \cong (I \otimes_R K) \oplus (X \otimes_R K)$. But $I \otimes_R K$ is isomorphic to Q/I, hence $K^{(\alpha)}$ contains a cyclic module isomorphic to R/I. The annihilator of every nonzero element of $K^{(\alpha)}$ is a finite intersection of principal fractional ideals, hence, being R a coherent domain, it is finitely generated. Thus I is finitely generated flat, hence projective, see [**5**]. □

As an immediate consequence we obtain the following corollaries; the first one shows the abundance of flat modules which are not strongly flat in Prüfer domains, once one forsakes Dedekind domains.

Corollary 2.3. *Let R be a Prüfer domain. An ideal I of R is strongly flat if and only if I is finitely generated.*

Corollary 2.4. *Let R be a domain. Every ideal I of R is strongly flat if and only if R is a Dedekind domain.*

Proof. It is enough to recall that if every, finitely generated, ideal of a domain R is flat then R is a Prüfer domain and to apply Proposition 2.2. $\square$

Note that in general the class of strongly flat modules is not closed under direct limits; in fact, every flat module is a direct limit of projectives, hence strongly flat modules. Thus $\mathcal{SF}$ is closed under direct limits if and only if $\mathcal{SF} = \mathcal{F}$ and if the domain is a Prüfer domain this happens only if it is Dedekind.

Another application of Theorem 2.1 is the following

Proposition 2.5. *Let R be a Matlis domain. Let M be a module embeddable in a projective R-module with torsion cokernel*; *then M is strongly flat if and only if it is projective.*

Proof. Only the necessity needs to be proved. Assume M is strongly flat and let U be a projective module containing M and such that U/M is torsion. Tensoring by K the exact sequence $0 \to M \to U \to U/M \to 0$, since $\mathrm{Tor}_1^R(K, X)$ is isomorphic to the torsion submodule of X for every module X, we obtain

$$0 \longrightarrow U/M \longrightarrow K \otimes_R M \longrightarrow K \otimes_R U \longrightarrow 0.$$

Since M and U are strongly flat modules, $K \otimes_R M$ and $K \otimes_R U$ are summands of K-free modules, by Theorem 2.1; hence, $\mathrm{p.d}(K \otimes_R U) = \mathrm{p.d}(K \otimes_R M) = 1$. Kaplansky's lemma applied to the above sequence yields that $\mathrm{p.d.}(U/M)$ is at most 1. Thus M has to be projective, since U is projective. $\square$

The next result shows that the notion of strongly flat module is a local notion, provided the domain R is h-local.

Proposition 2.6. *Assume that M is a reduced strongly flat module. Then every localization of M at a maximal ideal P of R is a strongly* *flat module over the local domain R_P. The converse holds provided that R is a h-local domain.*

Proof. By Theorem 2.1, M is a strongly flat module if and only if it is a summand of a module N such that there is an exact sequence of the form

$$0 \longrightarrow R^{(\alpha)} \longrightarrow N \longrightarrow Q^{(\beta)} \longrightarrow 0.$$

Applying the functor $\otimes_R R_P$ we immediately get that M_P is a strongly flat R_P-module. Conversely, assume R is h-local and M is an R-module such that M_P is a strongly flat R_P-module for every maximal ideal P of R. Since flatness is a local property, M is a flat R-module. To conclude that M is strongly flat it is enough, by Theorem 2.1 (3), to verify that $\operatorname{Ext}^1_R(M, C) = 0$ for every torsion-free weakly cotorsion module C. Moreover we may clearly assume that C is reduced; thus by Theorem 3.1 in [**8**], C is isomorphic to $\prod_P C^P$ where $C^P = \operatorname{Hom}_R(R_P, C)$. C^P is a weakly cotorsion R_P-module, hence $\operatorname{Ext}^1_{R_P}(M_P, C^P) = 0$ for every P. Now the conclusion follows by the well-known canonical isomoprhisms:

$$\operatorname{Ext}^1_R(M, \prod_P C^P) \cong \prod_P \operatorname{Ext}^1_R(M, C^P) \cong \prod_P \operatorname{Ext}^1_{R_P}(M_P, C^P). \qquad \square$$

As already noted, one cannot hope to improve the characterizations of strongly flat modules given by Theorem 2.1 over general domains. But if we consider local domains, we get more satisfactory results. These results are quite different, depending on the projective dimension of Q as we will see more clearly in the next section dealing with valuation domains. This is not surprising, since strongly flat modules are intimately related with the completion process, and it is well known that this is heavily influenced by p.d.Q, see [**5**, Chapter VIII]. For valuation domains and local Matlis domains we get from Theorem 2.1:

Corollary 2.7. *Let R be a local domain. Consider the following conditions for a reduced torsion-free R-module M.*

1. *M is strongly flat.*
2. *$\widetilde{M}$ is the completion of a free module.*
3. *$K \otimes_R M$ is a K-free module.*

Then $2 \Leftrightarrow 3$ *and, if* R *is a valuation domain,* $1 \Rightarrow 2$*. If* R *is a Matlis domain, the three conditions are equivalent.*

Proof. The equivalence between 2 and 3 follows by Matlis equivalence. The implications $1 \Rightarrow 3$ follows by Theorem 2.1 under the assumption that R is either a valuation domain or a Matlis domain. In fact, in both cases K is countably small: by [**5**] if R is a valuation domain and by a result by Hamsher, see [**5**], if R is a local Matlis domain. Thus the Azumaya-Warfield's theorem, see [**5**], applies since the endomorphism of K is the local ring $\widetilde{R}$, so a direct summand of a K-free module is still K-free. The implication $2 \Rightarrow 1$ for a Matlis domain follows by Theorem 2.1. □

3. Strongly flat modules over valuation domains. We now study strongly flat modules over valuation domains; in this section, R denotes a valuation domain.

There are two facts that contribute to obtain in this case more satisfactory results. The first fact is that the completion $\widetilde{R}$ of a valuation domain R is still a domain, actually an immediate extension of R. The second one is that $K = Q/R$ is a uniserial R-module; this fact will play a central role in proving the next Proposition 3.7, which is crucial in the subsequent developments.

In case R is a Matlis valuation domain, we already obtained in Corollary 2.7 a characterization of strongly flat modules as modules whose completion is the completion of free modules. For strongly flat modules over valuation domains R such that p.d.$Q > 1$ we have another characterization, which makes use of the tensor product by $\widetilde{R}$ instead of the completion in the R-topology. We will make free use of the fact that for valuation domains R such that p.d.$Q > 1$, a free $\widetilde{R}$-module is complete in the R-topology, see [**5**].

Theorem 3.1. *Let* R *be a valuation domain such that* p.d.$Q > 1$.

1. *A reduced torsion-free* R*-module* M *is strongly flat if and only if* $M \otimes_R \widetilde{R}$ *is a direct sum of a free* $\widetilde{R}$*-module and of a torsion-free divisible* $\widetilde{R}$*-module.*

2. *A reduced complete strongly flat* R*-module is a free* $\widetilde{R}$*-module.*

3. *If R is complete, an R-module is strongly flat if and only if it is a direct sum of a free R-module and a torsion-free divisible R-module.*

Proof. 1. Let M be a reduced strongly flat R-module. By Theorem 2.1, there exists an exact sequence

$$0 \longrightarrow R^{(\alpha)} \longrightarrow M \oplus X \longrightarrow Q^{(\beta)} \longrightarrow 0$$

for some cardinal numbers α, β. Tensoring by the flat module $\widetilde{R}$ we obtain the exact sequence of $\widetilde{R}$-modules and $\widetilde{R}$-maps:

$$0 \longrightarrow \widetilde{R}^{(\alpha)} \longrightarrow (M \otimes_R \widetilde{R}) \oplus (X \otimes_R \widetilde{R}) \longrightarrow Q^{(\beta)} \otimes_R \widetilde{R} \longrightarrow 0.$$

$Q^{(\beta)} \otimes_R \widetilde{R}$ is a divisible torsion-free R-module and $\widetilde{R}^{(\alpha)}$ is complete, so the above exact sequence splits. Therefore $M \otimes_R \widetilde{R}$, as a direct summand of a free $\widetilde{R}$-module and of a torsion-free divisible $\widetilde{R}$-module, is of the same form. For the converse, it is clear that $M \otimes_R \widetilde{R}$ is a strongly flat R-module; to prove that also M is strongly flat, by Theorem 2.1, it is enough to show that $\mathrm{Ext}^1_R(M, C) = 0$ for every torsion-free reduced weakly cotorsion module C. C has a unique structure of $\widetilde{R}$-module, and $\mathrm{Ext}^1_R(M, C) = 0 \cong \mathrm{Ext}^1_{\widetilde{R}}(M \otimes_R \widetilde{R}, C)$, see [**7**, Proposition 5.7]. But $\mathrm{Ext}^1_{\widetilde{R}}(M \otimes_R \widetilde{R}, C) = 0$, since $M \otimes_R \widetilde{R}$ is a strongly flat $\widetilde{R}$-module, so we are done.

2. Assume that M is a reduced strongly flat complete R-module. Then the sequence $0 \to M \to M \otimes_R \widetilde{R} \to M \otimes_R (\widetilde{R}/R) \to 0$ splits, since $M \otimes_R (\widetilde{R}/R)$ is a torsion-free divisible R-module. Thus M is isomorphic to the reduced summand of $M \otimes_R \widetilde{R}$ so it is a free $\widetilde{R}$-module, by 1.

3. Obvious, in view of 1. □

Remarks. 1) If R is a Matlis valuation domain, then the statement 1 in Theorem 3.1 holds only in one direction: given a reduced torsion-free R-module M, if $M \otimes_R \widetilde{R}$ is a direct sum of a free $\widetilde{R}$-module F and a torsion-free divisible $\widetilde{R}$-module, then M is strongly flat, since M is pure and dense in $M \otimes_R \widetilde{R}$; hence $\widetilde{M} \cong \widetilde{M \otimes_R \widetilde{R}} \cong \widetilde{F}$. The converse does not hold. Pick the completion $\widetilde{B}$ of a free R-module B of infinite rank; then $\widetilde{B}$ is strongly flat, but $\widetilde{B} \otimes_R \widetilde{R}$ is not a direct sum of a free $\widetilde{R}$-module and a torsion-free divisible $\widetilde{R}$-module, since its reduced part is isomorphic to $\widetilde{B}$.

2) In an anti-symmetric way, if p.d.$Q > 1$, Corollary 2.7 holds only in one direction: if M is a reduced strongly flat module, then $\widetilde{M}$ is the completion of a free module, since $M \otimes_R \widetilde{R}$ is a direct sum of a free $\widetilde{R}$-module F and a torsion-free divisible $\widetilde{R}$-module, and $\widetilde{M} \cong \widetilde{M \otimes_R \widetilde{R}} \cong \widetilde{F}$. The converse does not hold, as the proof of the next Proposition 3.6(1) will show.

We investigate now pure submodules of strongly flat modules, starting with a particular kind of them, namely, basic submodules. Recall that a pre-basic submodule B of an R-module M is a direct sum of standard uniserial submodules, pure in M and maximal with respect to these properties. Every R-module has a pre-basic submodule, which is unique up to isomorphism, see [**5**]. A pre-basic submodule of M which is pure-essential in M is called a basic submodule.

When M is torsion-free, every pre-basic submodule B turns out to be basic. Since basic submodules remain basic in any pure-essential extension, see [**5**], given a torsion-free reduced module M and a basic submodule B of M, B is basic also in the completion $\widetilde{M}$. In general if B and B' are two basic submodules of M, M/B is not isomorphic to M/B'. For instance, if J is an $\aleph_0$-generated ideal of R, there exists an exact sequence $0 \to B' \to B \to J \to 0$, where B' and B are free modules, and both B and B' are basic in B. However, we have the following

Lemma 3.2. *Let R be a valuation domain and M a reduced torsion-free R-module. If B and B' are two basic submodules of finite rank of M and M/B is divisible, then $M/B \cong M/B'$.*

Proof. The rank of $\widetilde{M} = \widetilde{B}$ as an $\widetilde{R}$-module coincides with the rank of B as an R-module. This implies that $\widetilde{M} = \widetilde{B}'$, thus M/B' is divisible, being a pure submodule of $\widetilde{B}'/B'$, necessarily of the same rank as M/B. □

The first result is that basic submodules of reduced strongly flat modules are free.

Proposition 3.3. *Let R be a valuation domain and M a reduced strongly flat R-module. Then a basic submodule of M is free.*

Proof. It is enough to prove that a rank 1 pure submodule N of M is cyclic. The completion $\widetilde{N}$ is a rank 1 pure $\widetilde{R}$-submodule of $\widetilde{M}$, which is the completion of a free R-module, by Corollary 2.7. Hence $\widetilde{N} \cong \widetilde{I}$ for a nonzero ideal I of R, thus $K \otimes_R N \cong Q/I$. But $K \otimes_R M \cong K \otimes_R \widetilde{M}$ is K-free, hence $I \cong R$. To conclude, it is enough to note that a rank 1 pure R-submodule of $\widetilde{R}$ is cyclic. □

Before entering in the discussion of the existence of basic dense submodules in strongly flat modules, we give some example of strongly flat modules of finite rank; the situation appears quite different if the valuation domain R is complete or not, as the following two examples show.

Example 3.4. Let R be a noncomplete valuation domain such that $\mathrm{rk}\,(\widetilde{R}) \geq n \in \omega$. From results by Zanardo [**12**] we now that there exists a torsion-free indecomposable R-module M of rank $n+1$ with dense basic submodule isomorphic to R, and a torsion-free indecomposable R-module M' of rank $n+1$ with dense basic submodule isomorphic to R^n, see also [**5**]. Both M and M' are strongly flat modules.

Example 3.5. Let R be a complete valuation domain. Given any torsion-free reduced indecomposable R-module M of finite rank with free basic submodule B, M/B is reduced, see [**12**]. Assume that M is strongly flat. Then, by Corollary 2.7, $\widetilde{M} \cong \widetilde{F}$ for a free module F of finite rank, so $\widetilde{M} \cong \widetilde{B} \oplus (\widetilde{M/B})$, which is absurd unless $M \cong R$. Thus the only indecomposable strongly flat modules of finite rank are R and Q.

The next proposition characterizes different classes of valuation domains in terms of closure properties of the class of strongly flat modules under taking suitable submodules.

Proposition 3.6. *Let R be a valuation domain. The class of strongly flat R-modules is closed under taking*

1. *pure and dense submodules if and only if R is a Matlis domain*;

2. *pure submodules if and only if R is a Matlis domain with* gl.d.$R \leq 2$;

3. *arbitrary submodules if and only if R is a DVR.*

Proof. 1. If R is a Matlis domain, then the claim is a trivial consequence of Corollary 2.7. Conversely, assume that p.d.$Q > 1$, and let $0 \to H \to F \to Q \to 0$ be a free presentation of Q. Then $\widetilde{H} = \widetilde{F}$ is the completion of a free module. Assume, by way of contradiction, that H is strongly flat. By Theorem 3.1, $H \otimes_R \widetilde{R}$ is a free $\widetilde{R}$-module, since it is reduced. From the exact sequence

$$0 \longrightarrow H \otimes_R \widetilde{R} \longrightarrow F \otimes_R \widetilde{R} \longrightarrow Q \otimes_R \widetilde{R} \longrightarrow 0$$

we derive that the projective dimension of $Q \otimes_R \widetilde{R}$ as $\widetilde{R}$-module is 1. Thus $Q \otimes_R \widetilde{R}$ is countably generated as an $\widetilde{R}$-module. Using the fact that every element of $\widetilde{R}$ can be written in the form $r\eta$ for some $r \in R$ and some unit $\eta \in \widetilde{R}$, it is straightforward to check that Q is also countably generated as an R-module, a contradiction.

2. Assume that the class of strongly flat R-modules is closed under taking pure submodules. Then R is a Matlis domain by 1. If J is an uncountably generated ideal of R, let $0 \to H \to F \to J \to 0$ be a free presentation of J. Then H is not strongly flat, since otherwise from the exact sequence $0 \to H \otimes_R K \to F \otimes_R K \to J \otimes_R K \cong Q/J \to 0$ and, from Corollary 2.7, we deduce that p.d.$Q/J \leq 2$, absurd. Hence all the submodules of Q are countably generated, so gl.d.$R \leq 2$. Conversely, assume that R is a Matlis valuation domain such that gl.d.$R \leq 2$, and let N be a pure submodule of the strongly flat module M. Then $N \otimes_R K$ is a divisible submodule of $M \otimes_R K$, which is K-free, by Corollary 2.7; then p.d.$(M \otimes_R K) = 1$, and the hypothesis ensures that also p.d.$(N \otimes_R K) = 1$; consequently also $N \otimes_R K$ is K-free, so N is strongly flat.

3. If R is a DVR, then the class of strongly flat modules coincides with the class of torsion-free modules, which is closed under submodules. For the necessity it is enough to note that every ideal must be strongly flat, hence principal. □

We remark that the module H in the proof of Proposition 3.6 furnishes the announced example of a nonstrongly flat module over a valuation domain R with p.d.$Q > 1$, whose completion is the completion of a free module.

We give now two crucial results which will be used frequently in the sequel. The proof of the next proposition, as presented here, is due to Fuchs.

Proposition 3.7. *Let R be a valuation domain. Then, given any $n \in \omega$ and any cardinal α, every embedding of K^n into $K^{(\alpha)}$ is splitting.*

Proof. It is clearly enough to look at the case $n = 1$. Consider the exact sequence

$$0 \longrightarrow K \xrightarrow{\eta} K^{(\alpha)} \longrightarrow X \longrightarrow 0.$$

Pick a nonzero element $x \in K$; let $\eta x = x_{\alpha_1} + \cdots + x_{\alpha_n}$, where $0 \neq x_{\alpha_i}$ belongs to the ith component of K in $K^{(\alpha)}$ for all $1 \le i \le n$. If π_{α_i} denotes the canonical projection of $K^{(\alpha)}$ onto this component, $x \notin \cup_{1 \le i \le n} \mathrm{Ker}\, \pi_{\alpha_i}\eta$ and $x \in \mathrm{Ker}\, \pi_\alpha \eta$ for all $\alpha \neq \alpha_i$. We can assume, without loss of generality, that $\mathrm{Ker}\, \pi_{\alpha_1}\eta \le \mathrm{Ker}\, \pi_{\alpha_i}\eta$ for all $1 \le i \le n$; hence $\mathrm{Ker}\, \pi_{\alpha_1}\eta \le \cap_\alpha \mathrm{Ker}\, \pi_\alpha \eta$. But then, being $\cap_\alpha \mathrm{Ker}\, \pi_\alpha\eta = 0$, $\pi_{\alpha_1}\eta$ is an isomorphism, so we can substitute the α_1th copy of K by ηK, which turns out to be a summand of $K^{(\alpha)}$. □

Using the Matlis equivalence, we can derive from Proposition 3.7 the following

Corollary 3.8. *Let R be a valuation domain. Then every pure embedding of $\widetilde{R}^n$, $n \in \omega$, into the completion of a free R-module splits.*

Proof. Consider an exact sequence $0 \to \widetilde{R}^n \to \widetilde{F} \to X \to 0$, where F is a free module, and X is torsion-free. If R is a Matlis domain, then X is also complete, as one easily checks by applying the functor $\mathrm{Ext}^1_R(Q, -)$ to the exact sequence. The Matlis equivalent exact sequence splits, by Proposition 3.7, whence the original sequence splits too.

If p.d.$Q > 1$, then every free $\widetilde{R}$-module is complete, see [**5**]. Every pure embedding of $\widetilde{R}^n$ in a free $\widetilde{R}$-module is a pure embedding of $\widetilde{R}$-modules, so it splits by [**4**]. ◻

Recall that, given a torsion-free R-module H, its completion $\widetilde{H}$ is in a natural way an $\widetilde{R}$-module; we shall denote by $\operatorname{rank}_{\widetilde{R}}(\widetilde{H})$ its rank as $\widetilde{R}$-module, while $\operatorname{rank}(\widetilde{H})$ denotes, as usual, its rank as R-module.

The next two results are instrumental in proving the subsequent proposition.

Lemma 3.9. *Let R be a valuation domain. Let H be a torsion-free R-module of finite rank n. Then* $\operatorname{rank}_{\widetilde{R}}(\widetilde{H}) \leq n$.

Proof. By induction on n. If $n = 1$, then $\widetilde{H} = 0$ if $H \cong Q$, otherwise $\widetilde{H}$ is isomorphic to an ideal of $\widetilde{R}$, so $\operatorname{rank}_{\widetilde{R}}(\widetilde{H}) \leq 1$. If $n > 1$, consider an exact sequence $0 \to A \to H \to J \to 0$, where A is a pure submodule of H of rank $n-1$ and J is a rank 1 torsion-free module. Passing to the completions, we obtain the exact sequence of $\widetilde{R}$-modules $0 \to \widetilde{A} \to \widetilde{H} \to \widetilde{J}$. Thus $\operatorname{rank}_{\widetilde{R}}(\widetilde{H}) \leq \operatorname{rank}_{\widetilde{R}}(\widetilde{A}) + \operatorname{rank}_{\widetilde{R}}(\widetilde{J}) \leq n-1+1 = n$ by the inductive hypothesis. ◻

Lemma 3.10. *Let R be a valuation domain and C the completion of a free R-module. If N is a pure submodule of finite rank of C, then $\widetilde{N}$ is a free $\widetilde{R}$-module which is a summand of C.*

Proof. Note that $\widetilde{N}$ is a pure $\widetilde{R}$-submodule of C of finite $\widetilde{R}$-rank, by Lemma 3.9. By Corollary 3.8, it is enough to prove that $\widetilde{N}$ is a free $\widetilde{R}$-module. We induct on $n = \operatorname{rank}_{\widetilde{R}}\widetilde{N}$. If $n = 1$, then $\widetilde{N} \cong \widetilde{R}$ by Proposition 3.3. If $n > 1$, then a rank 1 pure $\widetilde{R}$-submodule of $\widetilde{N}$ is a summand in C, by Corollary 3.8, hence, in $\widetilde{N}$; an easy induction concludes the proof. ◻

The following two results deal with closure properties of the class of strongly flat modules over arbitrary valuation domains with respect to pure submodules, under suitable additional conditions.

Proposition 3.11. *Let R be an arbitrary valuation domain.*

1. *The class of strongly flat R-modules is closed under taking pure submodules of finite rank.*

2. *A torsion-free module of countable rank M is strongly flat if and only if every pure submodule of finite rank is strongly flat.*

Proof. 1. Let M be a strongly flat module and N a pure submodule of finite rank. Without loss of generality, we can assume M to be reduced. Then $\widetilde{N}$ is a pure $\widetilde{R}$-submodule of $\widetilde{M}$. If R is a Matlis domain, then $\widetilde{M}$ is the completion of a free R-module, so $\widetilde{N}$ is a free $\widetilde{R}$-summand of $\widetilde{M}$ of finite rank, by Lemma 3.10. There follows that N is strongly flat. If p.d.$Q > 1$, $N \otimes_R \widetilde{R}$ is a pure $\widetilde{R}$-submodule of $M \otimes_R \widetilde{R}$, which is a direct sum of a free $\widetilde{R}$-module and of a divisible torsion-free $\widetilde{R}$-module, by Theorem 3.1. Hence $N \otimes_R \widetilde{R}$ is of the same shape, so N is strongly flat.

2. Necessity is an immediate consequence of part 1. For the sufficiency, write M (that we can assume to be reduced) as the union of a countable ascending chain of pure submodules M_i of finite rank. Each M_i is strongly flat, by hypothesis. If R is a Matlis domain, from Corollary 2.7 and Corollary 3.8 we deduce that $\widetilde{M_{i+1}} = \widetilde{M_i} \oplus \widetilde{B_i}$ for all i, where B_i is a free R-module of finite rank. Thus $\widetilde{M} = \widetilde{\oplus_i B_i}$ is the completion of a free R-module, so it is strongly flat by Corollary 2.7. If p.d.$Q > 1$, from Theorem 3.1 we know that $M_{i+1} \otimes_R \widetilde{R} = (M_i \otimes_R \widetilde{R}) \oplus C_i$ for all i, where C_i is a direct sum of a free $\widetilde{R}$-module and a divisible torsion-free $\widetilde{R}$-module of finite rank. Thus $M \otimes_R \widetilde{R} = \oplus_i C_i$ is strongly flat, by Theorem 3.1. □

We remark that the proof of Proposition 3.6 shows that we cannot extend Proposition 3.11 to submodules of uncountable rank unless R is a Matlis domain with gl.d.$R \leq 2$. Furthermore, part 2 in Proposition 3.11, that can be viewed as a Pontryagin criterion for strongly flat modules over valuation domains, has as an immediate consequence the following extension of part 1 to countable rank.

Corollary 3.12. *The class of strongly flat modules over arbitrary valuation domains is closed under taking pure submodules of countable rank.*

The rest of this section is devoted to investigating the following question: does every reduced strongly flat module M over a valuation domain admit a (free) dense basic submodule or, equivalently, does M fit in an exact sequence of the form $0 \to R^{(\alpha)} \to M \to Q^{(\beta)} \to 0$ for suitable cardinals α and β?

In view of the results proved up to now, this question has a positive answer in each one of the following cases: M is complete in the R-topology, Corollary 2.7, or R is complete and p.d.$Q > 1$, Theorem 3.1. The next result furnishes two more sufficient conditions for the existence of dense basic submodules.

Theorem 3.13. *A strongly flat module M over a valuation domain R has a* (*free*) *dense basic submodule provided that one of the following conditions hold*:

1. *basic submodules of M have finite rank*;
2. $\operatorname{rank}(M) \leq \aleph_0$.

Proof. 1. We can assume that M is reduced. Consider the exact sequence

$$0 \longrightarrow B \longrightarrow M \longrightarrow M/B \longrightarrow 0 \tag{1}$$

where B is a basic submodule of finite rank of M. B is free, by Proposition 3.3. If R is a Matlis domain, we have the exact sequence $0 \to \widetilde{B} \to \widetilde{M} \to \widetilde{M/B} \to 0$. Since $\widetilde{M}$ is the completion of a free R-module, from Corollary 3.8 we obtain that $\widetilde{M} \cong \widetilde{B} \oplus (\widetilde{M/B})$ so M/B is strongly flat. The same conclusion holds in case p.d.$Q > 1$, tensoring the exact sequence (1) by $\widetilde{R}$, instead of passing to completions, and using Theorem 3.1.

Now if M/B is not divisible, it contains a pure submodule $N/B \cong R$ by Proposition 3.3; therefore, $N \cong B \oplus R$ is pure in M, contradicting the fact that B is basic in M. Thus M/B must be divisible. Note that, according to Lemma 3.2, this happens for every basic submodule B of M.

2. If M is of finite rank the claim follows trivially from point 1. Thus, let us assume that the $\operatorname{rank}(M) = \aleph_0$. There exists a countable

ascending chain of pure submodules of M:

$$0 < M_1 < M_2 < \cdots < M_n < \cdots$$

such that $\cup M_n = M$ and M_{n+1}/M_n has rank 1 for all n. Each submodule M_n is strongly flat, by Proposition 3.11 and, given any basic submodule B_n of M_n, M_n/B_n is divisible, by what we have seen before. Furthermore, passing to M_{n+1}, B_n extends to a basic submodule B_{n+1} of $M_{n+1} : B_{n+1} = B_n \oplus C_n$, where C_n is either isomorphic to R or equal to 0.

Clearly, setting $B_1 = C_1$ and $B = \cup B_n$, $B = \oplus_n C_n$ is free and pure in M. In order to conclude, it is enough to prove that M/B is divisible. Given any $0 \neq r \in R$, for every $x \in M$ there exists an index n such that $x \in M_n$; from the divisibility of M_n/B_n it follows that $x \in rM_n + B_n$, thus $x \in rM + B$. Consequently, $M = rM + B$ and $M/B = r(M/B)$ so M/B is divisible. □

Our next goal is to extend Theorem 3.13 to modules of rank $\aleph_1$. The next lemma is crucial.

Lemma 3.14. *Let M be a reduced torsion-free module which is the union of a smooth ascending chain of pure submodules M_σ*

$$0 = M_0 < M_1 < M_2 < \cdots < M_\sigma < \cdots \quad (\sigma < \kappa)$$

for some infinite cardinal κ, such that $M_{\sigma+1}/M_\sigma$ has a free dense basic submodule for all $\sigma < \kappa$. Then M has a free dense basic submodule.

Proof. We will define, by transfinite induction on σ, a family $\{X_\sigma\}_{\sigma<\kappa}$ of free modules such that each submodule M_σ has a free dense basic submodule of the form $B_\sigma = \oplus_{\rho<\sigma} X_\rho$.

The first nonzero submodule M_1 has a free dense basic submodule X_0 by hypothesis. Let now $\sigma < \kappa$ and X_ρ, $\rho < \sigma$, already defined, and assume that M_σ satisfies the desired condition. Since $M_{\sigma+1}/M_\sigma$ is an extension of a free module F_σ by a torsion-free divisible module D_σ, we get an exact sequence

$$0 \longrightarrow M_\sigma \longrightarrow C_{\sigma+1} \longrightarrow F_\sigma \longrightarrow 0$$

where $C_{\sigma+1} \leq M_{\sigma+1}$. This sequence clearly splits, thus $C_{\sigma+1} = M_\sigma \oplus X_\sigma$ with $X_\sigma \cong F_\sigma$ free, and $M_{\sigma+1}/C_{\sigma+1} \cong D_\sigma$. There follows that $B_{\sigma+1} = B_\sigma \oplus X_\sigma$ is a free submodule of $M_{\sigma+1}$, and we have the exact sequence

$$0 \longrightarrow C_{\sigma+1}/B_{\sigma+1} \longrightarrow M_{\sigma+1}/B_{\sigma+1} \longrightarrow M_{\sigma+1}/C_{\sigma+1} \longrightarrow 0.$$

Since $C_{\sigma+1}/B_{\sigma+1} \cong M_\sigma/B_\sigma$ is divisible torsion-free, we can conclude that such is $M_{\sigma+1}/B_{\sigma+1}$.

Let us assume now that $\lambda < \kappa$ is a limit ordinal and that, for all ordinals $\sigma < \lambda$, the submodule M_σ has a free dense pure submodule $B_\sigma = \oplus_{\rho<\sigma} X_\rho$. Let $B_\lambda = \oplus_{\rho<\lambda} X_\rho$. Clearly B_λ is a free and pure submodule of M_λ, and we must only show that M_λ/B_λ is divisible, i.e., that for every $0 \neq r \in R$ the inclusion $M_\lambda \leq rM_\lambda + B_\lambda$ holds. Pick $a \in M_\lambda$; then $a \in M_\rho$ for some $\rho < \lambda$; since B_ρ is dense in M_ρ, $a \in rM_\rho + B_\rho \leq rM_\lambda + B_\lambda$, so we are done. □

With the aid of Lemma 3.14 we can extend Theorem 3.13 to the case of rank $\aleph_1$. The proof of the next theorem reminds the proof of a classical criterion for projectivity, see [**5**, XVI.1.2], but it requires a more delicate modification of the involved filtrations of submodules.

Theorem 3.15. *Let R be a valuation domain and M a reduced strongly flat R-module of rank $\aleph_1$. Then M has a (free) dense basic submodule.*

Proof. Pick a continuous well-ordered ascending chain of pure submodules of M

$$0 = M_0 < M_1 < M_2 < \cdots < M_0 < \cdots \quad (\sigma < \aleph_1),$$

such that $M_{\sigma+1}/M_\sigma$ has rank 1 for all σ. Consider the subset of $\aleph_1$:

$$E = \{\sigma < \aleph_1 \mid M_\tau/M_\sigma \text{ is not strongly flat for some } \tau\}.$$

If E is not stationary in $\aleph_1$, there is a cub C in $\aleph_1$ such that M_τ/M_σ is strongly flat for all $\sigma < \tau \in C$. Relabelling indexes, we can assume that in the initial ascending chain all the quotients $M_{\sigma+1}/M_\sigma$ are of at most countable rank and strongly flat; from Theorem 3.13 we know that each quotient $M_{\sigma+1}/M_\sigma$ has a dense basic submodule, hence Lemma 3.14 ensures that M has a dense basic submodule. To conclude, it is enough to prove that the subset E cannot be stationary in $\aleph_1$.

Assume, by way of contradiction, that E is stationary in $\aleph_1$. From the initial ascending chain we can derive a cofinal continuous subchain

$$0 = M'_0 < M'_1 < M'_2 < \cdots < M'_\sigma < \cdots \quad (\sigma < \aleph_1)$$

such that $M'_{\sigma+1}/M'_\sigma$ is countably generated and not strongly flat for all $\sigma \geq 1$.

Assume first that R is a Matlis domain. Tensoring by K, we obtain the strictly ascending chain

$$0 < M'_1 \otimes_R K < M'_2 \otimes_R K < \cdots < M'_\sigma \otimes_R K < \cdots$$

whose union is $M \otimes_R K \cong \widetilde{M} \otimes_R K$. The module $\widetilde{M}$ is the completion of a free R-module F, which clearly has rank $\aleph_1$, say $F = \oplus_{\rho<\aleph_1} R_\rho$, ($R_\rho \cong R$ for all ρ). Thus $F = \cup_{\sigma<\aleph_1} F_\sigma$ with $F_\sigma = \oplus_{\rho<\sigma} R_\rho$ for all σ. We derive a continuous ascending chain of pure $\widetilde{R}$-submodules of $\widetilde{M}$:

$$0 = Y_0 < Y_1 < Y_2 < \cdots < Y_\sigma < \cdots \quad (\sigma \leq \aleph_1),$$

where, for λ limit ordinal, $Y_{\lambda+1} = \widetilde{Y}_\lambda$, and for σ nonlimit $Y_{\sigma+1} = \widetilde{F}_\sigma$. Notice that $\widetilde{M} = \widetilde{\cup_\sigma Y_\sigma}$, $Y_{\sigma+1}/Y_\sigma \cong \widetilde{R}$ for all σ nonlimit, and Y_τ/Y_σ is strongly flat for all $\sigma < \tau < \aleph_1$. Tensoring also this chain by K, we obtain the ascending chain

$$0 < Y_1 \otimes_R K < Y_2 \otimes_R K < \cdots \leq Y_\sigma \otimes_R K \leq \cdots$$

where $Y_\lambda \otimes_R K = Y_{\lambda+1} \otimes_R K$ for all λ limit and $Y_\sigma \otimes_R K < Y_{\sigma+1} \otimes_R K$ for all σ nonlimit. The union of this chain is $\widetilde{M} \otimes_R K$. Comparing the two chains tensorized by K whose union is $\widetilde{M} \otimes_R K$, since all the submodules $M'_\sigma \otimes_R K$ and $Y_\sigma \otimes_R K$ are countably generated (here we use the hypothesis that p.d.$K = 1$), and using the usual zig-zag argument (see [**5**, Lemma A.2]) we find a cub C' in $\aleph_1$ such that $M'_\alpha \otimes_R K$ appears in the chain of the $Y_\sigma \otimes_R K$ for all $\alpha \in C'$. Therefore, for all $\alpha < \beta \in C'$, $(M'_\beta \otimes_R K)/(M'_\alpha \otimes_R K) \cong (M'_\beta/M'_\alpha) \otimes_R K$ equals

$(Y_\tau \otimes_R K)/(Y_\sigma \otimes_R K) \cong (Y_\tau/Y_\sigma) \otimes_R K$ for suitable $\sigma < \tau \in \aleph_1$. But $(Y_\tau/Y_\sigma) \otimes_R K$ is K-free, whence M'_β/M'_α is strongly flat, contradicting our assumption.

Assume now that p.d.$Q > 1$. Tensoring by $\widetilde{R}$ the chain of the submodules M'_σ we obtain the pure chain of $\widetilde{R}$-submodules of $M \otimes_R \widetilde{R}$

$$0 < M'_1 \otimes_R \widetilde{R} < M'_2 \otimes_R \widetilde{R} < \cdots < M'_\sigma \otimes_R \widetilde{R} < \cdots$$

whose union is $M \otimes_R \widetilde{R}$. Note that, as an immediate consequence of Lemma 3.9, for every $\sigma < \kappa$, $\operatorname{rank}_{\widetilde{R}}(M'_\sigma \otimes_R \widetilde{R}) \leq \aleph_0$. From Theorem 3.1 we know that $M \otimes_R \widetilde{R} \cong (\oplus_\alpha \widetilde{R}) \oplus (\oplus_\beta \widetilde{Q})$ for certain cardinals α, β; clearly $\alpha + \beta = \aleph_1$. Therefore $M \otimes_R \widetilde{R}$ is the union of a continuous well-ordered ascending chain of $\widetilde{R}$-submodules

$$0 < Z_1 < Z_2 < \cdots < Z_\sigma < \cdots \quad (\sigma < \aleph_1)$$

such that every module Z_σ and every factor Z_τ/Z_σ for all $\sigma < \tau < \aleph_1$ is the direct sum of a free and a divisible $\widetilde{R}$-module of countable rank. Comparing the two above ascending chains whose union is $M \otimes_R \widetilde{R}$, we find that there is a cub C'' in $\aleph_1$ such that $M'_\alpha \otimes_R \widetilde{R}$ appears in the chain of the Z_σ for all $\alpha \in C''$. Therefore, for all $\alpha < \beta \in C''$, $(M'_\beta \otimes_R \widetilde{R})/(M'_\alpha \otimes_R \widetilde{R}) \cong (M'_\beta/M'_\alpha) \otimes_R \widetilde{R}$ equals Z_r/Z_σ for suitable $\sigma < \tau \in \aleph_1$. Thus $(M'_\beta/M'_\alpha) \otimes_R \widetilde{R}$ is a direct sum of a free and a divisible $\widetilde{R}$-module, whence M'_β/M'_α is strongly flat, again contradicting our assumption. Thus the subset E of $\aleph_1$ is not stationary and the claim follows. □

The problem of extending Theorem 3.15 to modules of rank $> \aleph_1$ arguing by transfinite induction seems to require the introduction of a notion of κ-strongly flat modules, which is delicate as the definitions of the similar notions of κ-free (see [**3**, p. 83]) or κ-projective (see [**5**, p. 537]) modules show. Furthermore, a singular compactness theorem is needed, analogous to the Shelah's result for λ-free modules (λ a singular cardinal; see [**3**, p. 107]) or its analogous for λ-projective modules, see [**5**, p. 536].

We remark that the notion equivalent in Matlis's sense to κ-strongly flat modules is that of $\kappa - K$-free modules; this notion and the relative singular compactness theorem have been already developed in [**1**] for Matlis valuation domains R of global dimension 2.

We close this section with an example of a reduced torsion-free module of rank $\aleph_1$ which is not strongly flat, but which is very close to being so.

Example 3.16. In [**5**, XVI.2.2] a module M is constructed over any valuation domain admitting a non principal countably generated ideal J, which is the union of a continuous well-ordered ascending chain $\{M_\sigma\}_{\sigma<\aleph_1}$ of countably generated projective pure submodules, and such that M is not projective. M turns out to be $\aleph_1$-projective, in the sense defined in [**5**, XVI.2]. For any limit ordinal $\lambda < \aleph_1$, the factor module $M_{\lambda+1}/M_\lambda$ is isomorphic to J, whence it is not strongly flat. It follows that the set $E = \{\sigma < \aleph_1 \mid M_\tau/M_\sigma$ is not strongly flat for some $\tau\}$ is stationary in $\aleph_1$. From the proof of Theorem 3.15, it follows that M cannot be strongly flat.

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