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On the use of configuration variables as control variables in mechanical systems

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Abstract

The branch of Control Theory for Lagrangian mechanical systems where configuration variables are used as control variables is called here Hyper–impulsive control theory. We suppose that part of system's coordinates are 'controllized' i.e. identified with a control function u that can be discontinuous. Due to the presence of the derivative of u in the dynamic equation, the resulting motion is discontinuous both in the configurations and momenta in the general case (hyper–impulsive motion). In this paper we briefly review the theory of hyper–impulsive motion, we study in detail the robustness of hyper–impulsive control and we apply the theory to test mechanical systems: the rigid body, a planar space robot and the ball in the hoop system.

1 Introduction

Some authors ([3],[5],[6]) have considered the possibility of realizing a control over (some of) the mechanical system' configuration variables by using the reaction forces internal to the system as control forces. To mathematically describe this, we suppose that the manifold M representing the possible configurations of the system is fibered over a manifold N that represents the state of the constraints. Then we assign a path $u(t)$ in the base space N, that is an open loop control, and we assume that workless reaction forces can steer the whole system along a path $q(t)$ in M that projects on $u(t)$. The (kinematic) control thus realized by assigning the value of some of the system' configuration variables (in some coordinate chart) is generally more flexible and accurate with respect to the dynamic control prescribing force values; moreover it is widely used in presence of nonholonomic constraints as the ones arising in the dynamics of free-flying space robots.

In this paper we focus on the case where the control function to be implemented presents (finite) discontinuities; this is frequently the case for optimal trajectories related to optimal control problems. On a local trivialization of the fibration, with fibered coordinates $q = (x, u)$, the dynamic equations of the system in phase space $z = (x, p)$, when the control is $u(t)$ read

$$\dot{z} = F(z, u) + G(z, u)\dot{u} + \dot{u}^t R(z, u)\dot{u}, \qquad (1.1)$$

where dot denotes time derivative and ^{t} denotes transposition. Here F is a smooth drift vector field and G and R are smooth vector fields respectively linear and quadratic on $\dot{u}$.

Note that when the control $u(t)$ is discontinuous, defining the solution of (1.1) is a nontrivial problem. A key result in [2] is that a measurable control u can be successfully implemented only if equation (1.1) is *linear* on $\dot{u}$. This allows to define the solution $z(u(\cdot), \cdot)$ of (1.1) for a nonsmooth control u in a weak sense, i.e. as the limit of the sequence $z_n(u_n, \cdot)$ of solutions corresponding to a sequence u_n of more regular controls approaching u. Note that, in the general case, a discontinuity on the control u induces a discontinuity on the solution $z(u, \cdot)$. These results are used in [3] to construct a theory of *hyper-impulsive motions* in which discontinuities of the configurations (x, u) are allowed, thereby extending the classical theory of impulsive motion for mechanical systems (which only allows for discontinuity of momenta p).

One of the major difficulties of the theory is that, when dealing with multi-dimensional controls, the *hyper-impulsive* motion depends ([2]) on the choice of the approximating sequence in the general case. In [4] we characterized completely this dependence using the *dynamic connection* in [6] which arises naturally in this context.

The main result of the paper [4] is that the state $z(\tau^+)$ of the system in phase space $z = (x, p)$ after a *jump* of the control $u(t)$ at $t = \tau$ can be determined by *parallel transport of* $z(\tau^-)$ along a path γ joining $u(\tau^+)$ with $u(\tau^-)$ relatively to the dynamic connection on the fibration $\pi : V^*M \longrightarrow N$.

Note that, in the general case, the state $z(\tau^+)$ depends on the path γ; each choice of γ is called here a *control–completion* of the discontinuous control u. In particular, when the dynamic connection is flat, the parallel transport depends only on the homotopy class of γ.

In this paper we briefly review the theory of hyper-impulsive motion, we study in detail the robustness of hyper-impulsive control and we apply the theory to test mechanical systems: the rigid body, a planar space robot and the ball in the hoop system.

2 Control equations

Let $q = (x, u)$ be local coordinates on M adapted to the fibration $\pi : M \longrightarrow N$; we make the fundamental hypothesis that,

(H) for every t, the reaction forces that implement the constraint $y \equiv u(t)$ are ideal (workless) with respect to the set $V_{q(t)}M = \ker T_{q(t)}\pi$ of the virtual displacements compatible with the constraint $y \equiv u(t)$.

Let

$$2T(q,\dot q) = \dot q^t g(q)\dot q = \dot x^t \mathcal{A}\dot x + \dot x^t \mathcal{M}\dot y + \dot y^t \mathcal{M}^t \dot x + \dot y^t \mathcal{B}\dot y \tag{2.1}$$

be the local block representation of the kinetic energy, where $\mathcal{A}, \mathcal{B}$ are symmetric and invertible respectively $k \times k$ and $(n-k)\times(n-k)$ matrices. If $T(q,\dot q)$ is the kinetic energy of the unconstrained system (M, g), then the kinetic energy of the system subject to the time dependent constraint $y \equiv u(t)$ is $T(x, u(t), \dot x, \dot u(t))$. The dynamic equation can be put in Hamiltonian form; introducing the quantities A and K below:

$$A(q) = \mathcal{A}^{-1}\mathcal{M}, \qquad K(q) = \mathcal{B} - \mathcal{M}^t\mathcal{A}^{-1}\mathcal{M},$$

whose geometric meaning is explained in Sect.2.1 –see (2.4) and (2.5)–, the control equations are

$$\begin{aligned} \dot x &= \mathcal{A}^{-1}p - A\dot u, \\ \dot p &= -\frac{1}{2}p^t\frac{\partial \mathcal{A}^{-1}}{\partial x}p + p^t\frac{\partial A}{\partial x}\dot u + \frac{1}{2}\dot u^t\frac{\partial K}{\partial x}\dot u. \end{aligned} \tag{2.2}$$

Note that, by the above hypothesis, the reaction forces that implement the constraint do not appear in the above equation. To our analysis, it is necessary to dispose of the dynamic equations in their global (i.e. chart–independent) form. These are provided by the following Theorem in [6]. Let $VM = \ker T\pi$ be the vertical subbundle and V^*M the dual of VM. Denote with $p_M : T^*M \longrightarrow M$ the cotangent projection and set $\tilde\pi := \pi \circ p_M$, $\tilde\pi : V^*M \longrightarrow N$. Now, to every $y \in N$, the fiber $\tilde\pi^{-1}(y)$, canonically simplectomorphic to $T^*(\pi^{-1}(y))$, represents the phase space of the system constrained to the π– fiber over y.

Suppose that a control vector field Y is given on N and that the path $u(t)$ is an integral curve of Y. Then (see [6], [4]) the dynamic equations (2.2) are the local expression of a vector field D_Y over V^*M that projects on Y by $\tilde\pi$. Moreover, the field D_Y is tangent to the fiber of $\tilde\pi$ only if the control is vanishing.

Theorem 2.1 *To every control vector field Y on N, the corresponding dynamic vector field D_Y can be expressed as the sum of three terms:*

$$D_Y = X_{H_0} - X_{K_Y} + \mathrm{hor}(Y)$$

where X_{H_0} is the Hamiltonian vector field corresponding to the case of locked control, X_{K_Y} is the Hamiltonian vector field on V^*M associated to K_Y and hor *is the horizontal lift of an Ehresmann connection on $\tilde\pi : V^*M \longrightarrow N$ entirely determined by π and the metric.*

The horizontal lift of a vector $Y \in TN$ has the local expression

$$\mathrm{hor}(Y) = (\frac{\partial}{\partial y} - A\frac{\partial}{\partial x} + p^t\frac{\partial A}{\partial x}\frac{\partial}{\partial p})Y. \tag{2.3}$$

2.1 Bundle–like metrics

The first requirement to perform hyper–impulsive control is that the dynamic equations (2.2) are at most linear on $\dot u$, i.e. that $K = K(y)$. Note that this condition on the form of the kinetic energy metric is independent of the coordinate system. To see this, call HM the subspace orthogonal to $VM = \ker T\pi$ with respect to g. Therefore HM can be equivalently assigned through the following connection one-form, called *mechanical* connection

$$\omega(q) = (dx + A(q)dy) \otimes \frac{\partial}{\partial x}, \tag{2.4}$$

whose kernel and range are respectively HM and VM. The orthogonal splitting of a vector into its horizontal and vertical components (i.e. along HM and VM respectively) is intrinsic; using the above decomposition, we get the induced splitting of the kinetic energy metric tensor into its vertical and horizontal part:

$$g(q)dq \otimes dq = \mathcal{A}(q)\omega(q) \otimes \omega(q) + K(q)dy \otimes dy, \tag{2.5}$$

where $K(q) = \mathcal{B} - \mathcal{M}^t\mathcal{A}^{-1}\mathcal{M}$. Now we say that

Definition 2.1 *The kinetic energy metric g is a* bundle–like metric *for the fibration $\pi : M \longrightarrow N$ iff $K = K(y)$ in the above orthogonal splitting of g.*

As a straightforward consequence, the control equation (1.1), whose local expression is (2.2), is linear in $\dot u$ for every $\dot u$ (i.e. $R \equiv 0$) if and only if g is bundle–like .

The following example will clarify the physical meaning of bundle–like metrics. Suppose that the mechanical system is a collection of rigid bodies linked by ideal joints and with a fixed point. Then the configuration manifold is a fiber bundle with $SO(3)$ as fiber; the path $x(t)$ on the fiber corresponds to a 'rigid motion' of the system, i.e. a motion with all the joints locked, while a motion $y(t)$ on the quotient (shape) space is a 'pure deformation'. The term $\omega(q)\dot{q} = \dot{x} + A\dot{y}$ is the *locked spatial angular velocity*, i.e. the angular velocity of the rigid body instantaneously associated to the mechanical system by locking all the joints.

The term $K(q)$ is the kinetic energy corresponding to a 'pure deformation' and it defines a metric on the shape (joint) space. The condition $K = K(y)$ then states that the deformation kinetic energy is independent of the orientation of the system in space and it is equivalent to the invariance of the horizontal kinetic energy for translations orthogonal to the fibers.

2.2 Hyper–impulsive motions

Let $c > 0$, $u : [-c, c] \longrightarrow N$ be a smooth path on N presenting a first order discontinuity at $t = 0$, and set

$$\lim_{t\to 0^-} u(t) = P_1, \qquad \lim_{t\to 0^+} u(t) = P_2,$$

where P_1 and P_2 are points of N. The discontinuous control u can be physically realized as follows: first we define the sequence of smooth paths that run from P_1 to P_2 at increasing speed; given a path $\gamma : [0, T] \longrightarrow N$ with $\gamma(0) = P_1$, $\gamma(T) = P_2$, let $\lambda \in [1, +\infty)$ and set:

$$\gamma_\lambda(t) = \gamma(\lambda t), \qquad 0 \le t \le \frac{T}{\lambda}.$$

Hence, for $\lambda \ge 1$,

$$\gamma_\lambda(0) = \gamma(0) = P_1,\ \gamma_\lambda(\frac{T}{\lambda}) = \gamma(T) = P_2,\ \dot{\gamma}_\lambda(t) = \lambda\dot{\gamma}(\lambda t).$$

Then the discontinuous control u can be realized as the limit for $\lambda \to +\infty$ of the continuous piecewise–smooth control u_γ:

$$u_\gamma(t) = \begin{cases} u(t) & -c \le t < 0, \\ \gamma_\lambda(t) & 0 \le t \le \frac{T}{\lambda}, \\ u(\frac{T}{\lambda} + t) & \frac{T}{\lambda} < t \le c. \end{cases} \qquad (2.6)$$

Let $z_\lambda(t) = z(\gamma_\lambda(t), t)$ be the maximal solution of the following family of control problems with $X_K \equiv 0$ and $\gamma = \gamma_\lambda(t)$, that we rewrite as

$$\dot{z} = X_{H_0} + \mathrm{hor}(\dot{\gamma}), \quad z(0) = z(0^-).$$

Theorem 2.2 ([4]) *The state of the system immediately after the jump of the control u_γ, defined as the limit*

$$z(0^+) := \lim_{\lambda\to+\infty} z_\lambda(\frac{T}{\lambda}),$$

coincides with the value at $t = T$ of the solution $z(\cdot)$ of the parallel transport equation

$$\dot{z} = \mathrm{hor}(\dot{\gamma}), \quad z(0) = z(0^-) \qquad (2.7)$$

relatively to the dynamic connection.

By Theorem 2.2, at each time where the control has a jump (a finite discontinuity), i) the phase–space state of the system immediately after the jump is obtained by parallel translating the initial phase–space state along a path that 'bridges' the discontinuity; ii) by (2.3), the *configuration* $x(0^+)$ of the system after the jump can be obtained by parallel transport relatively to the *mechanical* connection (2.4) on the fibration $\pi : M \longrightarrow N$.

The state of the system after the jump is unaffected by the drift field X_{H_0}. This implies that the result of the above Theorem remains true when the system is acted on by external *positional* forces, independent of $\dot{\gamma}$. The state of the system after a jump of the control has finite amplitude but the above formula no longer holds when dissipating forces depending linearly or quadratically on $\dot{\gamma}$ (e.g. in underwater systems) are taken into account for the physical device realizing the jump of the controlled coordinates.

3 Robustness of hyper–impulsive control

Given two vector fields X, Y on N, the *curvature* Ω of the dynamic connection is the vertical–valued two form defined on N as

$$\Omega(X, Y) = \mathrm{hor}([X, Y]) - [\mathrm{hor}X, \mathrm{hor}Y], \qquad (3.1)$$

where the map hor –see (2.3)–is the inverse of $T\tilde{\pi}$ restricted to $H(V^*M)$. The curvature of the connection relates the Lie brackets in N and $H(V^*M)$ and is a measure of the failure of the horizontal distribution to be integrable. Indeed, by Frobenius' Theorem, the horizontal distribution is integrable iff $\Omega \equiv 0$. Now we investigate the dependence of the parallel transport upon the path. This is essential to enquiry about the robustness of the control. Given a path u, let $z(t) = (x(t), p(t))$ be the solution of the parallel transport equation (2.7), whose local form is —see (2.3)—

$$\dot{x} = -A(x, u)\dot{u},\ \dot{p} = p^t\frac{\partial A(x, u)}{\partial x}\dot{u},\ z(0) = (x_0, p_0). \qquad (3.2)$$

Given a smooth homotopy $\gamma_s(t) = \gamma(t, s)$ of u with $\gamma_0 = u$ and fixed endpoints, we denote with $z_s(t)$ the solution of (3.2) for $u = \gamma_s$. The sensitivity of the system with respect to variations of the path around $u(t)$ is expressed by

$$\delta z(t) = (\delta x(t), \delta p(t)) = \frac{\partial z}{\partial s}(t, 0)$$

Proposition 3.1 *The variation δz of the parallel transport at $t = T$ is given by*

$$a)\ \delta x(T) = \int_0^T R(T, \tau)\Omega^{(M)}(x, u)(\delta\gamma, \dot{u})d\tau$$

$$b)\ \delta p(T) = \int_0^T R^{-1}(T, \tau)\Omega^{(D)}(x, p, u)(\delta\gamma, \dot{u})d\tau$$

where $\delta\gamma = \frac{\partial\gamma}{\partial s}(t,0)$, $R(t,\tau)$ *is the fundamental matrix solution of the linear system*

$$\dot{\Psi}^\alpha = -\frac{\partial A_l^\alpha}{\partial x_\beta}(x(t),u(t))\dot{u}^l(t)\Psi^\beta = D_\beta^\alpha \Psi^\beta$$

and $\Omega = (\Omega^{(M)}, \Omega^{(D)})$ *is the curvature of the dynamic connection.*

Proof. We prove a) first. The proof of b) goes exactly the same way. We consider equation $(3.2)_1$ for $u = \gamma_s$. Since

$$\frac{\partial \dot{x}}{\partial s}(t,0) = \dot{\delta x} = (-\frac{\partial A}{\partial x}\dot{u})\delta x - \frac{\partial}{\partial s}(A(x,\gamma_s)\dot{\gamma}_s),$$

the variation δx is the solution of the linear non homogeneous equation

$$(*)\ \dot{\delta x} = D(t)\delta x + b(t), \text{ where } b = -\frac{\partial}{\partial s}A(x,\gamma_s)\dot{\gamma}_s.$$

Let $R(t,\tau) = R(t,0)R^{-1}(\tau,0)$ be the fundamental matrix solution with initial point 0 of the associated homogeneous system. The general solution of $(*)$ is then

$$\delta x = \int_0^T R(t,\tau)b(\tau)d\tau.$$

Therefore ($\delta\gamma(0) = \delta\gamma(T) = 0$),

$$\begin{aligned}
\delta x(T) &= -\int_0^T R(\frac{\partial A}{\partial u}\delta\gamma\dot{u} + A\frac{d}{ds}\dot{\gamma}_s)d\tau \\
&= -\int_0^T R(\frac{\partial A}{\partial u}\delta\gamma\dot{u} + A\dot{\delta\gamma})d\tau \\
&= -\int_0^T [R\frac{\partial A}{\partial u}\delta\gamma\dot{u} - \frac{d}{dt}(RA)\delta\gamma]d\tau \\
&= -\int_0^T \{R\frac{\partial A}{\partial u}\delta\gamma\dot{u} - [\frac{dR}{d\tau}A + R(\frac{dA}{d\tau})]\delta\gamma\}d\tau \\
&= -\int_0^T \{R\frac{\partial A}{\partial u}\delta\gamma\dot{u} - [-R(-\frac{\partial A}{\partial x}\dot{u})A + \\
&\qquad R(\frac{\partial A}{\partial x}(-A\dot{u}) + \frac{\partial A}{\partial u}\dot{u})]\delta\gamma\}d\tau
\end{aligned}$$

since $\frac{d}{d\tau}R(t,\tau) = -R(t,\tau)D(\tau)$ and $\dot{x} = -A\dot{u}$. Therefore

$$\begin{aligned}
\delta x^\gamma &= -\int_0^T R_\alpha^\gamma(\frac{\partial A_m^\alpha}{\partial y^l} - \frac{\partial A_l^\alpha}{\partial u^m} - \frac{\partial A_m^\alpha}{\partial x^\beta}A_l^\beta + \\
&\qquad \frac{\partial A_l^\alpha}{\partial x^\beta}A_m^\beta)\delta\gamma^l\dot{u}^m d\tau \\
&= \int_0^T R(T,\tau)\Omega^{(M)}(x,u)(\delta\gamma,\dot{u})d\tau.
\end{aligned}$$

To prove b), note that

$$(**)\ \dot{\delta p} = \delta p^t\frac{\partial A}{\partial x}\dot{u} + p^t\frac{\partial}{\partial s}(\frac{\partial A}{\partial x}\dot{\gamma}_s) = -\delta p^t D(t) + c(t).$$

Hence the homogeneous system associated to $(**)$ is the adjoint system of the homogeneous system associated to $(*)$. Therefore, its fundamental matrix is simply $R^{-1}(t,\tau)$. The general solution of $(3.2)_2$ is

$$\delta p = -\int_0^T R^{-1}(t,\tau)c(t)d\tau.$$

From now on, the proof follows the same pattern used above.

□

When the curvature Ω is zero, the horizontal distribution is integrable and the connection is called *flat*. In this case the horizontal lift of a path lies on a single leaf of the horizontal foliation and, by the above Proposition, $\delta z = 0$; therefore *two paths that can be continuously deformed one into the other give the same state for the system after the jump*, that is the parallel transport depends only on the *homotopy* class of the path. As a straightforward consequence, if N is *simply connected*, that is every closed path can be continuously shrunk to a point, the parallel transport is *independent* of (the homotopy class of) the path in the base N.

4 Mechanical examples

We can resume the results of the previous Sections as follows:

a) discontinuous control laws can be safely implemented provided that the dynamic equations are linear on $\dot{u}$, that is the kinetic energy is bundle–like; the resulting motion is hyper–impulsive,

b) the point in phase–space, describing the state of the system after a jump of the (multi–dimensional) control, depends on:

b.1) the control completion γ if the dynamic connection has nonvanishing curvature,

b.2) the homotopy class of γ if the dynamic connection is flat,

b.3) is independent of the path γ if the dynamic connection is flat and base manifold N of $\pi : M \longrightarrow N$ is simply connected,

c) two sufficiently close paths are homotopic.

Now we apply our analysis to systems which may be interesting for practical applications:

1) The rigid body fibration.

Let G be a Lie group and let H be a *closed* subgroup. Then (see [1]) the coset space fibration $\pi : G \longrightarrow G/H$ is a principal bundle fibration with structure group H. Every left–invariant metric on G defines a bundle–like metric.

For our study, take $G = SO(3)$, the configuration space of a force–free rigid body with a fixed point. As it is well known, an $SO(3)$–invariant metric is defined by the inertia tensor scalar product on $so(3)$, which defines

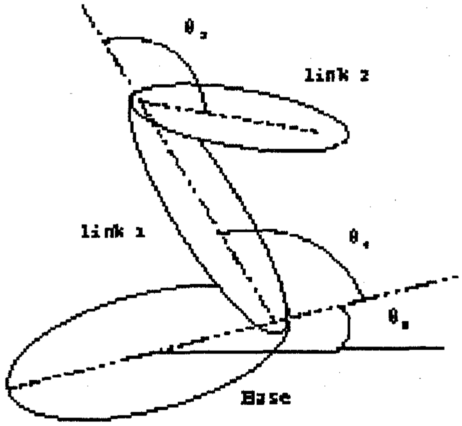

Figure 1: Planar space robot

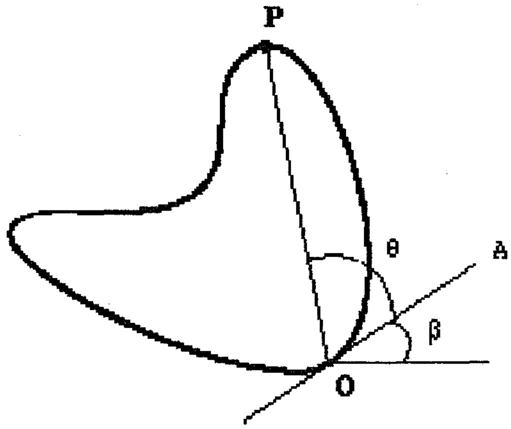

Figure 2: Ball in the hoop system

a bi–invariant metric. Take $H = SO(2)$, the subgroup of rotations around an axis fixed in space. It is interesting to notice that the mechanical connection, defined by taking as the horizontal sub bundle the orthogonal complement to the vertical sub bundle, *cannot be flat*. Indeed, since the base $S^2 = SO(3)/SO(2)$ is simply connected, if the connection is flat, then it is globally flat (see [4]) i.e. the bundle admits a global section and hence it is a trivial bundle, which is obviously false for the case at hand.

2) **Planar space robot**

We consider the planar space robot with two links depicted in Fig.1 for simplicity, but the results apply equally well to a robot with n links.

It is easy to see that the configuration of the system is the trivial fibration $\pi : T^2 \times S^1 \longrightarrow T^2$, $\pi(\theta_1, \theta_2, \theta_0) = (\theta_1, \theta_2)$, where T^2 is the two-torus, an homotopically non trivial manifold. Since the kinetic energy of the system is defined by the inertia tensor $I(\theta_1, \theta_2)$, which is independent of the base attitude θ_0, the kinetic energy metric g is bundle–like for the fibration. Therefore the system can undergo an hyper–impulsive (i.e. discontinuous) control in the link variables $\theta_1 = u_1(t), \theta_2 = u_2(t)$. The control equations are (dropping the subscript 0 for simplicity)

$$\begin{cases} \dot\theta &= \frac{p}{g_0(\theta_1,\theta_2)} - \frac{g_1(\theta_1,\theta_2)\dot\theta_1 + g_2(\theta_1,\theta_2)\dot\theta_2}{g_0(\theta_1,\theta_2)}, \\ \dot p &= Q \end{cases} \tag{4.1}$$

where Q represents external forces. The mechanical connection coincides with the constraint that the component of the system' angular momentum normal to the plane be equal to zero:

$$g_0(\theta_1, \theta_2)\dot\theta_0 + g_1(\theta_1, \theta_2)\dot\theta_1 + g_2(\theta_1, \theta_2)\dot\theta_2 = 0.$$

The associated curvature is (see e.g. [9] for the explicit formulae for connection and curvature)

$$\Omega = \mathrm{d}\omega = \left(\frac{\partial}{\partial\theta_1}(\frac{g_2}{g_0}) - \frac{\partial}{\partial\theta_2}(\frac{g_1}{g_0})\right) \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2$$

The curvature Ω is generally non zero. Therefore, the state of the system after a jump of the control $t \to u(t) = (\theta_1(t), \theta_2(t))$ between states u_1 and u_2 depends on the path γ bringing $u_i(t^-)$ into $u_i(t^+)$, $i = 1, 2$.

3) **The ball in the hoop**

This mechanical system has been studied [7] in the context of Geometric Phase Theory and in [6] as a control system. In our scheme, it represents a mechanical system to which hyper–impulsive control is not applicable.

The system is formed by a rigid planar closed guide (the hoop) not necessarily circular, but star–shaped with respect to a point O on it, and a material point P that can slide without friction along the guide. Suppose that the hoop can rotate around the point O fixed in the plane with an assigned control law $\beta = \beta(t)$. The position of the point P on the hoop is defined by the angle θ formed by the vector $OP = r(\theta)$ and the tangent in O to the guide OA. Therefore the system is described by the trivial fibration $\pi : T^2 \longrightarrow S^1$, $\pi(\theta, \beta) = \beta$. The kinetic energy of the point P with respect to the inertial frame is

$$2T = m[\dot r^2 + r^2(\dot\theta + \dot\beta)^2] = m[(r'^2 + r^2)\dot\theta^2 + 2r^2\dot\theta\dot\beta + r^2\dot\beta^2]$$

where $r' = \frac{dr}{d\theta}$. Therefore

$$K(\theta) = \mathcal{B} - \mathcal{M}^t\mathcal{A}^{-1}\mathcal{M} = m\frac{r^2(\theta)r'^2(\theta)}{r^2(\theta) + r'^2(\theta)}$$

and the kinetic energy metric does not fulfill the requirement of Definition 2.1, hence hyper–impulsive control theory is not applicable to this system.

5 Conclusions

We have presented a theory of hyper–impulsive control for mechanical systems. The theory tells if a mechanical system can safely undergo a discontinuity in the *configuration* variables and it determines uniquely the phase–space state of the system after the jump when a control law 'bridging' the discontinuity is provided. We have discussed robustness of the control system with respect to changes of the control law and we have applied the theory to test mechanical examples.

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