The Internal Type Theory of a Heyting Pretopos

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Abstract. We present the internal type theory of a Heyting pretopos with a natural numbers object. The resulting theory is based on dependent types and proof-terms. We prove that there is a sort of equivalence between such type theories and the category of Heyting pretoposes. By using the type theory we also build the free Heyting pretopos generated by a category.

Introduction

An elementary topos can be viewed as a generalized universe of sets to develop mathematics. From a logical point of view, topos theory corresponds to an intuitionistic higher order logic with typed variables [LS86]. Suitable toposes provide models of restricted Zermelo set theory [MM92]. Recently, Joyal and Moerdijk built a model of the whole intuitionistic set theory by using the notion of small map and by taking a Heyting pretopos with a natural numbers object as the categorical universe [JM95]. The main difference w.r.t. a topos is that a Heyting pretopos (H-pretopos, for short) correspond to a first order logical framework, precisely to a first order dependent type theory with quotients, called the *HP* typed calculus in [Mai97]. This calculus is valid and complete w.r.t. the class of H-pretoposes. It consists of the following dependent types: the terminal type, the falsum type, the indexed sum type, the extensional equality type, the disjoint sum type with the disjointness axiom, the natural numbers type, as in the extensional version of Martin-Löf's type theory [Mar84], the forall type, i.e. the product type restricted to types with at most one proof, and finally a particular quotient type satisfying an effectiveness axiom.

This type theory may be of interest to computer scientists expecially for the proposal of effective quotient types based only on proof-irrelevance equivalence relations. These particular quotient types are different from the extensional quotient types of Nuprl, where effectiveness is not always available (it holds certainly for decidable equivalence relations).

In this paper, we show that the internal type theory of a given H-pretopos is a particular *HP* type theory plus specific dependent types and terms. To this purpose, we prove that there is a sort of equivalence between such type theories and H-pretoposes. The type theory of a given H-pretopos $\mathcal{P}$

is described by a categorical semantics: it combines together the notion of model given by display maps [HP89], [See84] with the tools provided by contextual categories to interpret substitution correctly [Car86]. We shall emphasise context formation. Indeed, the judgement $B\ [\Gamma]$, asserting that B is a dependent type under the context Γ, is interpreted as a suitable sequence of morphisms of $\mathcal{P}$ to the terminal object. Moreover, the judgement $b \in B[\Gamma]$, asserting that b is a term of type B under the context Γ, is interpreted as a section of the last morphism of the sequence representing the dependent type B. Since we want to express substitution by means of pullback, which is determined up to isomorphisms, we use fibred functors, as in [Hof94] to interpret substitution correctly. But in our semantics, a type judgement corresponds to a sequence of fibred functors, which represents the type under a context with all its possible substitutions, and a term judgement corresponds to a natural transformation, which also represents the term under a context with all its possible substitutions. The specific dependent types of $\mathcal{P}$ correspond to sequences of fibred functors obtained by pullback from a sequence of morphisms of $\mathcal{P}$.

By means of the type theory, we can build the free H-pretopos generated by a category: it is sufficient to consider the objects of the category as closed types and its morphisms as dependent terms. This is the free construction for categories corresponding to dependent typed lambda calculi. For example, in the same way, one can build both the free category with finite limits - corresponding to the type theory with the terminal type, the extensional equality type, the indexed sum type- and the free locally cartesian closed category with finite coproducts and a natural numbers object -corresponding to Martin-Löf's type theory with extensional equality but without universes and well founded sets [Mar84].

The type theory of a H-pretopos constitutes a new tool to analyze, from a type theoretical point of view, the notion of small map used in [JM95] to provide a model of intuitionistic set theory. Moreover, it seems interesting to investigate how much of the type theory of H-pretoposes can be formalized within Martin-Löf's type theory.

The paper is divided as follows: in section 1 we describe the categorical preliminaries to interpret the HP dependent typed calculus in a H-pretopos; in section 2 we show the HP type theory arising from a given H-pretopos, in section 3 we prove a sort of equivalence between the HP type theories and the category of H-pretoposes, in section 4 we build the free H-pretopos and finally we draw the conclusions.

1 Preliminaries

First of all we recall the categorical definition of Heyting pretopos [JM95]. The notion of pretopos was introduced by Grothendieck [MR77].

Definition 1. A *pretopos* is a category equipped with finite limits, stable finite disjoint coproducts and stable effective quotients of equivalence relations. A *Heyting pretopos* is a pretopos where pullback functors on subobjects have right adjoints.

In the following, by H-pretopos we shall mean a Heyting pretopos also with a natural numbers object. Since we intend to describe syntactically categorical properties, we shall assume that, with a H-pretopos, a given choice of categorical constructors is made, i.e. fixed choices of finite limits, initial object, coproducts, quotients of equivalence relations and right adjoints on subobjects.

An essential feature for the interpretation of a dependent typed calculus is the local property of a H-pretopos: for every object A of the H-pretopos $\mathcal{P}$, the comma category $\mathcal{P}/A$ is a H-pretopos. Indeed, constructing a type, which depends on a context Γ, from other types corresponds to a categorical property of $\mathcal{P}/A$, where A is determined by Γ. Moreover, since substitution corresponds to pullback, the various categorical properties must be stable under pullback. By the way, in a H-pretopos also Beck-Chevalley conditions for right adjoints are satisfied.

Given a H-pretopos $\mathcal{P}$ we want to describe its internal dependent type theory $T(\mathcal{P})$. The type theory is based on the HP typed calculus for H-pretoposes (see the appendix), augmented with the specific type and term judgements of $\mathcal{P}$. As in [Mai97], the idea is to consider a dependent type as a sequence of morphisms of $\mathcal{P}$, ending with the terminal object 1, whereas the terms are sections of the last morphism of the type to which they belong. Thus, we consider the algebraic development of the fibration *cod* of $\mathcal{P}$: it is the category $Pgr(\mathcal{P})$.

Definition 2. The objects of the category $\boldsymbol{Pgr(\mathcal{P})}$ are finite sequences $a_1, a_2, ..., a_n$ of morphisms of $\mathcal{P}$

$$A_n \xrightarrow{a_n} \cdots\cdots A_2 \xrightarrow{a_2} A_1 \xrightarrow{a_1} 1$$

and a morphism from $a_1, a_2, ..., a_n$ to $b_1, b_2, ..., b_m$ is a morphism b of $\mathcal{P}$ such that $b_n \cdot b = a_n$

$$\begin{array}{ccccccc} & & & & A_n & \xrightarrow{\;b\;} & B_n \\ & & & & & {\scriptstyle a_n}\searrow \quad \swarrow{\scriptstyle b_n} & \\ 1 & \xleftarrow[!_{A_1}]{} & A_1 & \cdots\cdots & \xleftarrow[a_{n-1}]{} & A_{n-1} & \end{array}$$

provided $m = n$ and $a_i = b_i$ for $i = 1, ..., n-1$.

Remark 3. We would like to interpret substitution by means of pullback, using the indexed pseudofunctor $F : \mathcal{P}^{OP} \longrightarrow Cat$ defined as follows: F associates to every $A \in Ob\mathcal{P}$ the category $\mathcal{P}/A$ and to every morphism $f : B \to A$ of $\mathcal{P}$ the pullback pseudofunctor $f^* : \mathcal{P}/A \to \mathcal{P}/B$. But, in general, for an arbitrary choice of pullbacks, F would not be a functor: for instance, even $F(id)$ may not be an identity. Therefore, if substitution were interpreted by F then it would not be well defined. The solution is to replace F by an equivalent pseudofunctor $S : \mathcal{P}^{OP} \longrightarrow Cat$ [Ben85], [Jac91]. S is defined as follows. For every object A in $\mathcal{P}$, $S(A) \equiv Fib(\mathcal{P}/A, \mathcal{P}^{\rightarrow})$, where $Fib(\mathcal{P}/A, \mathcal{P}^{\rightarrow})$ is the class of functors $\sigma : \mathcal{P}/A \rightarrow \mathcal{P}^{\rightarrow}$, fibred from dom_A to cod_A (they send cartesian morphisms of dom_A to cartesian morphisms of cod_A). A fibred functor $\sigma : \mathcal{P}/A \rightarrow \mathcal{P}^{\rightarrow}$ associates to every triangle $\begin{array}{ccc} C & \xrightarrow{t} & B \\ & {\scriptstyle b'}\searrow \quad \swarrow{\scriptstyle b} & \\ & A & \end{array}$ a pullback diagram $\begin{array}{ccc} C' & \xrightarrow{q(t,\sigma(b))} & B' \\ {\scriptstyle \sigma(b\cdot t)}\downarrow & & \downarrow{\scriptstyle \sigma(b)} \\ C & \xrightarrow[t]{} & B \end{array}$. Moreover, for a morphism $f : B \rightarrow A$ of $\mathcal{P}$, the functor $S(f) : Fib(\mathcal{P}/A, \mathcal{P}^{\rightarrow}) \rightarrow Fib(\mathcal{P}/B, \mathcal{P}^{\rightarrow})$ associates to every fibred functor σ a fibred functor $\sigma[f]$. $\sigma[f]$ is defined as follows: for every $t : C \rightarrow B$, $\sigma[f](t) \equiv \sigma(f \cdot t)$. Besides, for every natural transformation ρ, $S(f)(\rho) \equiv \rho[f]$, where $\rho[f](t) \equiv \rho(f \cdot t)$ for every $t : C \rightarrow B$. Note that the pseudofunctor $F : \mathcal{P}^{OP} \longrightarrow Cat$ is equivalent to the functor S in the appropriate 2-category of pseudofunctors. The functors establishing such an equivalence can be described as follows. To define the functor from $\mathcal{P}/A$ to $Fib(\mathcal{P}/A, \mathcal{P}^{\rightarrow})$, first we choose pullbacks, then we associate to every object $b : B \rightarrow A$ the fibred functor $\widehat{b}$, defined as $\widehat{b}(t) \equiv t^*(b)$ in the pullback[1] $\begin{array}{ccc} B_\Sigma & \xrightarrow{b^*(t)} & B \\ {\scriptstyle t^*(b)}\downarrow & & \downarrow{\scriptstyle b} \\ D & \xrightarrow[t]{} & A \end{array}$ and extended to morphisms by the universal property of pullback. In order to define a functor from $Fib(\mathcal{P}/A, \mathcal{P}^{\rightarrow})$ to $\mathcal{P}/A$, we associate to every fibred functor its evaluation on the identity of the object A.

We use fibred functors to interpret the dependent types with all its possible substitutions, as in [Hof94]. Moreover, we use natural transformations to represent terms with all its possible substitutions. We call preinterpretation an assignment of fibred functors to type judgements and of natural transformations to term judgements. To this purpose, we consider the category $Pgf(\mathcal{P})$, where the judgements of the typed calculus for H-pretoposes are preinterpreted. We put $I(\sigma) = A$ if $\sigma \in [\mathcal{P}/A, \mathcal{P}^{\rightarrow}]$.

Definition 4. The objects of the category $\boldsymbol{Pgf(\mathcal{P})}$ are finite sequences $\sigma_1, \sigma_2, ..., \sigma_n$ of fibred functors such that $\sigma_1(id_{A_1}), \sigma_2(id_{A_2}), ..., \sigma_n(id_{A_n})$ is an object of $Pgr(\mathcal{P})$, where $A_i = I(\sigma_i)$ for $i = 1, ..., n$. The morphisms of $Pgf(\mathcal{P})$ from $\sigma_1, \sigma_2, ..., \sigma_m$ to $\tau_1, \tau_2, ..., \tau_n$ are defined only if $m = n$ and $\sigma_i = \tau_i$ for $i = 1, ..., n-1$, and they are natural transformations from the functor σ_n to τ_n such that, if $A_n = I(\sigma_n) = I(\tau_n)$, then for every $b : B \rightarrow A_n$

[1] We call the second projection $b^*(t)$, since the notation $q(t, b)$ is reserved for projections of functorial choices and not any choice of pullback is functorial.

the second member of $\rho(b)$ is the identity (recall that $\rho(b)$ is a morphism of $\mathcal{P}^{\rightarrow}$), that is the triangle $\begin{array}{ccc} & \xrightarrow{\rho_1(b)} & \\ {\scriptstyle\sigma_n(b)}\searrow & & \swarrow{\scriptstyle\tau_n(b)} \\ & B & \end{array}$ commutes.

In the following, since the second member of $\rho(b)$ is always the identity, we confuse $\rho(b)$ with the first member $\rho_1(b)$.

Besides, notice that by naturality any component $\rho(b)$ of a morphism ρ of $Pgf(\mathcal{P})$ is determined by the properties of pullback from $\rho(id_{A_n})$. Indeed, if we consider $\begin{array}{ccc} B & \xrightarrow{b} & A_n \\ {\scriptstyle b}\searrow & & \swarrow{\scriptstyle id} \\ & A_n & \end{array}$, we get that $\rho(b)$ is equal to $b^*(\rho(id_{A_n}))$, that is the unique morphism from $\sigma_n(b)$ to $\tau_n(b)$, which are obtained respectively by the pullbacks of b and $\sigma_n(id)$ and of b and $\tau_n(id)$.

Finally, for every $A \in Ob\mathcal{P}$, we define the fibred functor $i_A : \mathcal{P}/A \rightarrow \mathcal{P}^{\rightarrow}$ which associates to every triangle $\begin{array}{ccc} C & \xrightarrow{t} & B \\ {\scriptstyle b'}\searrow & & \swarrow{\scriptstyle b} \\ & A & \end{array}$ the following pullback diagram $\begin{array}{ccc} C & \xrightarrow{t} & B \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle id} \\ C & \xrightarrow[t]{} & B \end{array}$.

2 The type theory of $\mathcal{P}$

The type theory $T(\mathcal{P})$ of a H-pretopos $\mathcal{P}$, with a fixed choice of its categorical structure, is a particular HP calculus plus type judgements and term judgements that are specific to $\mathcal{P}$. It is formulated in the style of Martin-Löf's type theory with four kinds of judgements [NPS90]. There are the type judgements and the judgements about equality between types, which are given by formation rules, the term judgements given by introduction and elimination rules, and the judgements about equality between terms of the same type given by conversion rules. Since the types are allowed to depend on variables of other types, the contexts are telescopic [dB91]. We assume all the inference rules about the formation of contexts, declarations of typed variables, about reflexivity, symmetry and transitivity of the equality between types and terms and finally, the substitution rules for all the four kinds of judgements [NPS90]. The dependent types are introduced under a context. A type judgement arises from a object of $Pgf(\mathcal{P})$, which represents a dependent type with all its possible substitutions. More precisely, a type judgement corresponds to the evaluation of a finite sequence of fibred functors on the identity. Indeed, for a sequence of fibred functors $\alpha_1, \alpha_2, ..., \alpha_n, \beta$ of $Pgf(\mathcal{P})$, we define

$$\beta^{-1}(x_1, ..., x_n)[x_1 \in \alpha_1^{-1}, ..., x_n \in \alpha_n^{-1}(x_1, ..., x_{n-1})]$$

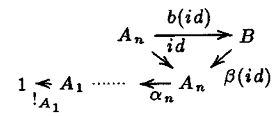
as the type judgement corresponding to

$$B \xrightarrow{\beta(id)} A_n \xrightarrow{\alpha_n(id)} \cdots\cdots A_1 \xrightarrow{\alpha_1(id)} 1$$

by thinking of the fibers of the morphism $\beta(id)$. This notation turns out to be very clear when we look at the category of paths built on any syntactic H-pretopos. The equality between types corresponds to the equality between objects of $Pgf(\mathcal{P})$, which implies the equality between objects of $Pgr(\mathcal{P})$. For short, we use the abbreviation $\Gamma_n \equiv x_1 \in \alpha_1^{-1}, ..., x_n \in \alpha_n^{-1}(x_1, ..., x_{n-1})$ in the contexts. On the other hand, a term judgement arises from a morphism of $Pgf(\mathcal{P})$, which is a natural transformation representing a term with all its possible substitutions. The evaluation of a natural transformation on the identical substitution is a term judgement. Indeed, for a suitable morphism b of $Pgf(\mathcal{P})$ from $\alpha_1, \alpha_1, ..., \alpha_n, i_{A_n}$ to $\alpha_1, \alpha_2, ..., \alpha_n, \beta$, the term judgement

$$b \in \beta^{-1}(x_1, ..., x_n)[\Gamma_n]$$

corresponds to a section of $\beta(id)$



by choosing the identity as the terminal object in $\mathcal{P}/A_n$.

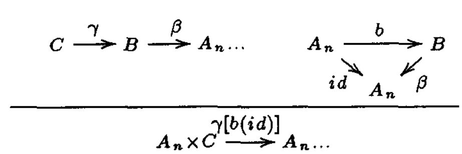
The equality between terms corresponds to the equality between morphisms of $Pgf(\mathcal{P})$. The contexts are generated from the following formation rules:

$$1C)\quad \emptyset \quad cont \qquad 2C)\quad \frac{\Gamma \quad cont \qquad A\ type\ [\Gamma]}{\Gamma, x \in A \quad cont}\ (x \in A \notin \Gamma)$$

In the following, to make formulas more readable in type judgements, we will write $\beta[\Gamma_n]$ instead of $\beta^{-1}[\Gamma_n]$. In the diagrams we will often write σ_i instead of $\sigma_i(id_{A_i})$ for fibred functors and b instead of $b(id)$ for natural transformations.

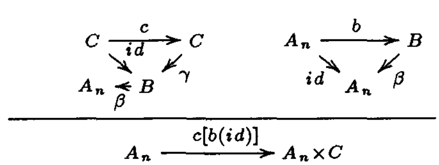
The rules for *substitution* of variables in a type and in a term and for *weakening* of a variable w.r.t type and term judgements are the usual ones. We only show how they work in these particular cases:

$$sT\ \frac{\gamma[\Gamma_n, y \in \beta] \quad b \in \beta[\Gamma_n]}{\gamma[b(id)][\Gamma_n]}$$ is



where we put $\gamma[b(id)](id) \equiv \gamma(b(id))$

$$st\ \frac{c \in \gamma[\Gamma_n, y \in \beta] \quad b \in \beta[\Gamma_n]}{c[b(id)] \in \gamma[b(id)][\Gamma_n]}$$ is

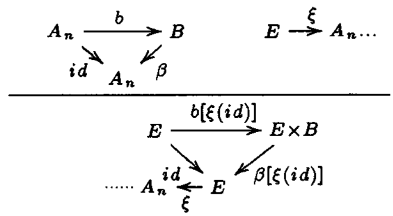


where we put $c[b(id)](id) \equiv c(b(id))$

$$wT\ \frac{\beta[\Gamma_n] \quad \delta[\Gamma_n]}{\beta[\Gamma_n, y \in \delta]} \quad \text{is} \quad \frac{B \xrightarrow{\beta} A_n \cdots \qquad D \xrightarrow{\delta} A_n \cdots}{D \times B \xrightarrow{\beta[\delta(id)]} D \xrightarrow{\delta} A_n \cdots}$$

where we put $\beta[\delta(id)](id) \equiv \beta(\delta(id))$

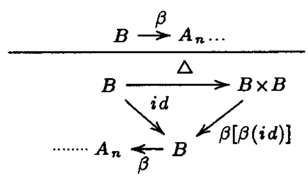
$$wt\ \frac{b \in \beta[\Gamma_n] \quad \xi[\Gamma_n]}{b \in \beta[\Gamma_n, w \in \xi]} \quad \text{is}$$



where we put $b[\xi(id)](id) \equiv (\xi(id))^*(b(id))$, that is the unique morphism of $\mathcal{P}/E$ from $i_{A_n}(\xi(id))$ to $\beta(\xi(id))$, obtained from $b(id)$ by the properties of pullback.

The rule expressing the *assumption of variable* is the following:

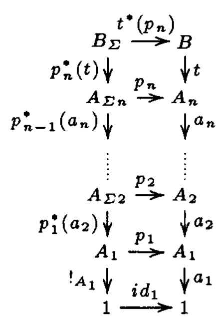
$$var\ \frac{\beta\ [\Gamma_n]}{x \in \beta[\Gamma_n, x \in \beta]} \quad \text{is}$$



where $x(id) \equiv \Delta_B \equiv \langle id_B, id_B \rangle$.

Now, we show the formation rules for types and then the introduction, elimination and conversion rules for their terms.

The *proper types* and *terms* of $T(\mathcal{P})$ are described as follows. Proper type judgements arise from objects of $Pgr(\mathcal{P})$ and proper term judgements arise from morphisms of $Pgr(\mathcal{P})$. For every object of $Pgr(\mathcal{P})$ $\boldsymbol{a_1, a_2, ..., a_n, t}$ we consider the sequence obtained by making the pullback of a_1 along the identity, then by making the pullback of a_2 along the second projection p_1 of the previous pullback, and so on, that is we obtain the following sequence of pullbacks:



where p_i is the second projection of the pullback of a_i and p_{i-1}, for $i = 1, \ldots, n$. Finally, we consider the associate sequence of fibred functors

$$\widehat{A_1},\ \widehat{a_2}[p_1],\ \widehat{a_3}[p_2],\ \ldots, \widehat{a_n}[p_{n-1}],\ \widehat{t}[p_n]$$

where $\widehat{A_1} \equiv \widehat{a_1}$, hence we introduce a new dependent type t^{-1} and finally we state that

$$t^{-1}[x_1 \in A_1, \ldots, x_n \in a_n^{-1}] \quad \text{is} \quad B_\Sigma \xrightarrow{\widehat{t}[p_n]} A_{\Sigma n} \xrightarrow{\widehat{a_n}[p_{n-1}]} \cdots\cdots\cdots\ A_{\Sigma 1} \xrightarrow{\widehat{A_1}} 1$$

where the Σ subscript is used for the interpretation of the series of judgements of proper types introduced by an object of $Pgr(\mathcal{P})$.

Moreover, given a sequence of fibred functors $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta$ of $Pgf(\mathcal{P})$, for every morphism c of $Pgr(\mathcal{P})$

$$\begin{array}{ccccc} & & A_n & \xrightarrow{\ c\ } & B \\ & & & {\scriptstyle id}\searrow \quad \swarrow {\scriptstyle \beta(id)} & \\ 1 & \xleftarrow{!_{A_1}} & A_1 & \cdots\cdots & A_n \end{array}$$

we introduce a new term c and we state that

$$c \in \beta(id)[x_1 \in A_1, \ldots, x_n \in \alpha_n] \quad \text{is} \quad \begin{array}{ccc} A_n & \xrightarrow{\ \bar{c}(id)\ } & B \\ & {\scriptstyle id}\searrow \quad \swarrow {\scriptstyle \beta(id)} & \\ & A_n & \end{array}$$

where $\bar{c}(id) \equiv c$.

Finally, we add all the types and terms of the HP typed calculus (see the appendix for the inference rules). This calculus is valid and complete with respect to the class of H-pretoposes and is described as follows [2]. Given a H-pretopos the terminal type corresponds to the terminal object, the extensional equality types to the equalizers, the indexed sum types to pullbacks, the falsum type to the initial object, the disjoint sum types with the axiom of disjointness to disjoint coproducts, the natural numbers type to the natural numbers object and all these types are already presented in the extensional version of Martin-Löf's type theory [Mar84]. The key point in finding the typed calculus of H-pretoposes is to have noticed that a monomorphism turns out to be the interpretation of a type with at most one proof, also called proof-irrelevant in the literature, but here called *mono type*. Therefore, the novelty of this calculus lies in the presence of the forall type, that is the product type restricted to mono types, and also in the presence of the quotient types based only on mono equivalence relations such that the effectiveness holds. Here, we describe in details the forall type, the quotient type and the natural numbers type and we refer to [Mai97] and [Hof94] for details on the other types. Note that we define the fibred functors only on objects of the various slice categories $\mathcal{P}/A$, since on morphisms they turn out to be defined by

[2] Our definition of internal language of a category follows [LS86], for instance, and it is different from that in [Tay97].

the universal property of pullback. Moreover, they turn out to be fibred by stability or Beck-Chevalley conditions of the categorical property involved.

The *Forall type* corresponds to the right adjoint of pullback functor on subobjects:

$$\forall)\ \frac{\gamma(y)[\Gamma_n, y \in \beta] \quad d \in Eq(\gamma, w, z)[\Gamma_n, y \in \beta, w \in \gamma(y), z \in \gamma(y)]}{\forall_{y\in\beta}\gamma(y)\ [\Gamma_n]}$$

$$\text{is} \quad \frac{C \stackrel{\gamma}{\rightarrowtail} B \stackrel{\beta}{\longrightarrow} A_n \dots}{\forall_\beta C \stackrel{\forall_\beta\gamma}{\rightarrowtail} A_n \dots}$$

where $\forall_\beta\gamma : \mathcal{P}/A_n \to \mathcal{P}^{\to}$ is the functor defined in the following manner: for every $t : D \to A_n$, we put $\forall_\beta\gamma(t) \equiv \forall_{\beta(t)}\gamma(q(t, \beta(id)))$. Note that $\gamma(id)$ turns out to be a monomorphism, because the interpretation of $z \in \gamma[\Gamma_n, y \in \beta, w \in \gamma(y), z \in \gamma(y)]$ and $w \in \gamma[\Gamma_n, y \in \beta, w \in \gamma(y), z \in \gamma(y)]$, which are isomorphic with the same isomorphism respectively to the first and second projections of the product $\gamma(id) \times \gamma(id)$, are equal by hypothesis and by the validity of the extensional elimination rule for the equality type.

$$\text{I-}\forall)\ \frac{c \in \gamma(y)[\Gamma_n, y \in \beta]}{\lambda y.c \in \forall_{y\in\beta}\gamma(y)[\Gamma_n]} \quad \text{is} \quad \frac{B \stackrel{c}{\longrightarrow} C}{A_n \stackrel{(\lambda y.c)}{\longrightarrow} \forall_{y\in\beta}C}$$

$$\text{E-}\forall)\ \frac{b \in \beta[\Gamma_n] \quad f \in \forall_{y\in\beta}\gamma(y)[\Gamma_n, y \in \beta]}{Ap(f, b) \in \gamma(b)[\Gamma_n]} \quad \text{is} \quad \frac{A_n \stackrel{b}{\longrightarrow} B \qquad A_n \stackrel{f}{\longrightarrow} \forall_\beta C}{A_n \stackrel{Ap(f,b)}{\longrightarrow} A_n \times C}$$

with $(\lambda y.c)(id) \equiv \psi(c(id))$ and $Ap(f, b)(id) \equiv b(id)^*(\psi^{-1}(f(id)))$, where

$$\psi : Sub(B)(\beta(id)^*(id_{A_n}), \gamma(id)) \to Sub(A_n)(id_{A_n}, \forall_{\beta(id)}(\gamma(id)))$$

is the bijection of the adjunction $\beta(id)^* \dashv \forall_\beta$, by putting $\beta(id)^*(t) \equiv q(t, \beta(id))$ for every $t : B \to A_n$, that is we are considering the choice of pullback given by the split fibration. The conversion rules, that are the usual β and η conversion rules as in the extensional version of Martin-Löf's type theory [Mar84], are also valid.

The *Quotient type* corresponds to the effective quotients of equivalence relations (in the premisses we omit to add the generic context Γ_n):

$$\text{Q)}\ \frac{\begin{array}{l}\rho(x,y)\ type\ [x \in \alpha, y \in \alpha], d \in Eq(\rho, z, w)[x \in \alpha, y \in \alpha, z \in \rho, w \in \rho],\\ c_1 \in \rho(x,x)[x \in \alpha], \qquad c_2 \in \rho(y,x)[x \in \alpha, y \in \alpha, z \in \rho(x,y)],\\ c_3 \in \rho(x,z)[x \in \alpha, y \in \alpha, z \in \alpha, w \in \rho(x,y), w' \in \rho(y,z)]\end{array}}{\alpha/\rho\ [\Gamma_n]}$$

corresponds to

$$\frac{R \stackrel{\rho(id)}{\rightarrowtail} A \times A \stackrel{\alpha(id)\cdot\pi_1}{\longrightarrow} A_n \dots}{A/R \stackrel{Q(\alpha)(id)}{\longrightarrow} A_n \dots}$$

where $\pi_1 \equiv \alpha(\alpha(id))$, $\pi_2 \equiv q(\alpha(id), \alpha(id))$ and $Q(\alpha)(id)$ is defined as follows. In the case of the forall type, we have already noticed that a mono type corresponds to a monomorphism. Here, we can prove that $\rho(id)$ turns out to be also an equivalence relation in $\mathcal{P}/A_n$. Therefore, there exists the coequalizer $c : A \to A/R$ of $\pi_1 \cdot \rho(id)$ and $\pi_2 \cdot \rho(id)$. Moreover, as $\alpha(id) \cdot (\pi_1 \cdot \rho(id)) = \alpha(id) \cdot (\pi_2 \cdot \rho(id))$, we get $Q(\alpha(id))$ such that the following triangle diagram commutes

$$\begin{array}{ccccc} R & \overset{\pi_1 \cdot \rho(id)}{\underset{\pi_2 \cdot \rho(id)}{\rightrightarrows}} & A & \xrightarrow{\quad c \quad} & A/R \\ & & & \alpha(id) \searrow \quad \swarrow Q(\alpha(id)) & \\ & & & A_n & \end{array}$$

Therefore we define $Q(\alpha) : \mathcal{P}/A_n \to \mathcal{P}^{\to}$ in the following manner: for every $t : D \to A_n$ we put $Q(\alpha)(t) \equiv Q(\alpha(t))$, where $Q(\alpha(t))$ is the unique morphism such that $\alpha(t) = Q(\alpha(t)) \cdot c(t)$ and $c(t)$ is the coequalizer of the equivalence relation $\rho(t)$. The *introduction* rule for the quotient type is the next one:

$$\text{I-Q)}\ \frac{a \in \alpha\ [\Gamma_n]}{[a] \in \alpha/\rho\ [\Gamma_n]} \quad \text{is} \quad \frac{A_n \xrightarrow{a(id)} A}{A_n \xrightarrow{c \cdot (a(id))} A/R}$$

and the following *equality rule* is valid

$$\text{eq)}\ \frac{a \in \alpha[\Gamma_n] \quad b \in \alpha[\Gamma_n] \quad d \in \rho(a, b)[\Gamma_n]}{[a] = [b] \in \alpha/\rho\ [\Gamma_n]}$$

By using the indexed sum type, the *elimination* and *conversion* rules of the quotient type for dependent types (see the appendix) are equivalent to the following weaker elimination and conversion rules of the quotient type for types not depending on α or α/ρ, which are also derivable in $T(\mathcal{P})$

$$\text{E}_s\text{-Q)}\ \frac{m(x) \in \mu\ [x \in \alpha] \quad m(x) = m(y) \in \mu\ [x \in \alpha, y \in \alpha, d \in \rho(x, y)]}{Q_s(m, z) \in \mu\ [z \in \alpha/\rho]}$$

$$\text{C}_{1s}\text{-Q)}\ \frac{a \in \alpha \quad m(x) \in \mu\ [x \in \alpha] \quad m(x) = m(y) \in \mu\ [x \in \alpha, y \in \alpha, d \in \rho(x, y)]}{Q_s(m, [a]) = m(a) \in \mu}$$

$$\text{C}_{2s}\text{-Q)}\ \frac{t(z) \in \mu\ [z \in A/R]}{Q_s((x)t([x]), z) = t(z) \in \mu\ [z \in \alpha/\rho]}$$

where $M \xrightarrow{\mu(id)} A_n$ and $A \xrightarrow{\alpha(id)} A_n$. In the weaker elimination rule E_s-Q

$$Q_s(m, z) \in \mu\ [\Gamma_n, z \in \alpha/\rho] \quad \text{is} \quad A/R \xrightarrow{\langle id, u \rangle} A/R \times M$$

where u is the morphism in $\mathcal{P}/A_n$ such that $u \cdot c = q(\alpha(id), \mu(id)) \cdot m(id)$, which exists because by hypothesis $m(id) \cdot (\pi_1 \cdot \rho(id)) = m(id) \cdot (\pi_2 \cdot \rho(id))$, and

c is the coequalizer of $\pi_1 \cdot \rho(id)$ and $\pi_2 \cdot \rho(id)$. Moreover, since by hypothesis $\mu(id) \cdot (q(\alpha(id), \mu(id)) \cdot m(id)) = \alpha(id)$ we also have that by uniqueness $\mu(id) \cdot u = Q(\alpha(id))$. The C_{1s} and C_{2s} conversion rules are valid. Besides, the axiom of *Effectiveness* also holds:

$$\frac{a \in \alpha \quad b \in \alpha \quad [a] = [b] \in \alpha/\rho}{f(a,b) \in \rho(a,b)} \quad \text{is} \quad \frac{A_n \xrightarrow{a(id)} A \qquad A_n \xrightarrow{b(id)} A}{A_n \xrightarrow{(id,t)} R}$$

where $\langle id, t\rangle$ is defined as follows. Since by hypothesis $c \cdot a(id) = c \cdot b(id)$ and since the quotient is effective in $\mathcal{P}/A_n$, then there exists a morphism $t : A_n \to R$ such that $(\pi_1 \cdot \rho(id)) \cdot t = a(id)$ and $(\pi_2 \cdot \rho(id)) \cdot t = b(id)$.

The *Natural Numbers type* corresponds to the natural numbers object:

$$\text{nat)} \quad N[\,] \quad \text{is} \quad \mathcal{N}_\Sigma \xrightarrow{\widehat{\mathcal{N}}(id)} 1$$

where $\widehat{\mathcal{N}} : \mathcal{P}/1 \to \mathcal{P}^{\to}$ is the functor defined in the following manner: for every $!_D : D \to 1$ we put $\widehat{\mathcal{N}}(!_D) \equiv (!_D)^*(!_{\mathcal{N}})$ and $\mathcal{N}$ is a natural numbers object of $\mathcal{P}$.

Now, we show the *introduction* rules.

$$\text{I}_1\text{-nat)} \quad 0 \in N[\Gamma_n] \qquad \text{is} \qquad A_n \xrightarrow{\langle id, o \cdot !_{A_n}\rangle} A_n \times \mathcal{N}$$

where $o : 1 \to \mathcal{N}$ is the zero map in the H-pretopos $\mathcal{P}$. From now on, we call $\pi_1 \equiv \widehat{\mathcal{N}}(!_{A_n})$ and $\pi_2 \equiv q(!_{A_n}, \widehat{\mathcal{N}}(id))$.

$$\text{I}_2\text{-nat)} \quad s(n) \in N[\Gamma_n, n \in N] \qquad \text{is} \qquad A_n \times \mathcal{N} \xrightarrow{\langle id, \bar{s} \cdot \pi_2\rangle} A_n \times \mathcal{N} \times \mathcal{N}$$

where $s : \mathcal{N} \to \mathcal{N}$ is the successor map in the H-pretopos $\mathcal{P}$, $\bar{s} \equiv id_1^*(s)$ and $\langle id, \bar{s} \cdot \pi_2\rangle$ is the unique morphism towards the pullback of $!_{A_n \times \mathcal{N}}$ and $\widehat{\mathcal{N}}(!_1)$. By using the indexed sum type, the *elimination* and *conversion* rules of the natural numbers type for dependent types, as in the extensional version of Martin-Löf's type theory [NPS90], are equivalent to the following weaker elimination and conversion rules of the natural numbers type for types not depending on $\mathcal{N}$, which are also derivable in $T(\mathcal{P})$

$$\text{E}_s\text{-nat)} \quad \frac{a \in L \quad l(y) \in L\ [y \in L]}{Rec_s(a, l, n) \in L\ [n \in N]}$$

$$\text{C}_1\text{-nat)} \quad \frac{a \in L \quad l(y) \in L\ [y \in L]}{Rec_s(a, l, 0) = a \in L}$$

$$\text{C}_2\text{-nat)} \quad \frac{a \in L \quad l(y) \in L\ [y \in L]}{Rec_s(a, l, s(n)) = l(Rec_s(a, l, n)) \in L\ [n \in N]}$$

$$\text{C}_3\text{-nat)}\ \frac{a \in L \qquad l(y) \in L\ [y \in L] \qquad f(n) \in L\ [n \in N] \qquad f(0) = a \in L\ \ f(s(n)) = l(f(n)) \in L}{Rec_s(a,l,n) = f(n) \in L\ [n \in N]}$$

In the weaker elimination rule E_s-nat

$$Rec_s(a,l,n) \quad \text{is} \quad A_n \times \mathcal{N} \xrightarrow{\langle id,r\rangle} (A_n \times \mathcal{N}) \times L$$

where r is the unique morphism that makes the diagram below commute, by the property of natural numbers object in $\mathcal{P}/A_n$ with $\pi_2^L \equiv q(\xi(id),\xi(id))$

$$\begin{array}{ccccc} A_n & \xrightarrow{\langle id,0\rangle} & A_n \times \mathcal{N} & \xrightarrow{\langle \pi_1, \bar{s}\cdot\pi_2\rangle} & A_n \times \mathcal{N} \\ & \searrow^{a(id)} & \downarrow r & & \downarrow r \\ & & L & \xrightarrow[\pi_2^L \cdot l(id)]{} & L \end{array}$$

The conversion rules for the natural numbers type are also valid.

3 The relation between the type theories and the H-pretoposes

There is a sort of equivalence between the type theories described in the previous section and the category of H-pretoposes. So we can state that the type theory $T(\mathcal{P})$ is the internal language of the H-pretopos $\mathcal{P}$. First of all, we define the following categories:

1. *Lang* whose objects are the type theories of H-pretoposes and whose morphisms are translations: they send types to types so as to preserve the type and term constructors, closed terms to closed terms and variables to variables; we call *Lang** the category whose objects are those of *Lang*, but whose morphisms are translations preserving type and term constructors up to isomorphisms;
2. $HPretop_o$ whose objects are H-pretoposes with a fixed choice of H-pretopos structure and whose morphisms are strict logical functors, that is functors preserving the H-pretopos structure w.r.t. the fixed choices; we call $HPretop$ the category whose objects are those of $HPretop_o$, but whose morphisms are functors preserving the H-pretopos structure up to isomorphisms.

Now, we define a functor from H-pretoposes to type theories

$$T : HPretop_o \longrightarrow Lang$$

that associates to every H-pretopos $\mathcal{P}$ the internal type theory $T(\mathcal{P})$ described in the previous section. The functor T associates to every morphism $F : \mathcal{P} \rightarrow$

$\mathcal{D}$ of $HPretop_o$ the translation $T(F) : T(\mathcal{P}) \to T(\mathcal{D})$ defined as follows. Given a fibred functor $\sigma : \mathcal{P}/A \to \mathcal{P}^{\to}$, corresponding to a type judgement, and a natural transformation c, corresponding to a term judgement, we define $T(F)(\sigma)$ and $T(F)(c)$ by induction on the signature of $T(\mathcal{P})$. Indeed, if $\sigma = \widehat{b}$ for any $b : B \to A$ of $\mathcal{P}$, then we put $F(\sigma) = \widehat{F(b)}$, since the chosen pullbacks of $\mathcal{P}$ are sent into the chosen pullbacks of $\mathcal{D}$ by F. If σ is obtained by an inference rule, then we simply define $F(\sigma)$ such that $F(\sigma)(id) = F(\sigma(id))$, in order to make $T(F)$ be a translation. For example, we put $F(\Sigma_\beta(\gamma)) \equiv \Sigma_{F(\beta)}(F(\gamma))$. This definition of $T(F)$ is good, since the functor F preserves the H-pretopos structure w.r.t. the fixed choices used in the internal type theories of $\mathcal{P}$ and $\mathcal{D}$.

Moreover, we define a functor from type theories to H-pretoposes

$$P : Lang \longrightarrow HPretop_o$$

that associates to every type theory $\mathcal{T}$ the category $P(\mathcal{T})$, whose objects are closed types $A, B, C, \ldots$ and whose morphisms are the expressions $(x)b(x)$ corresponding to $b(x) \in B[x \in A]$, where the type B does not depend on A. We can prove that $P(\mathcal{T})$ is a H-pretopos by fixing a choice of its structure[3] (see [Mai97]). The functor P associates to every morphism of $Lang$ $L : \mathcal{T} \to \mathcal{T}'$ the functor $P(L) : P(\mathcal{T}) \to P(\mathcal{T}')$ defined as follows. For every closed type A, we put $P(L)(A) \equiv L(A)$, which is well defined since a translation sends closed types to closed types. For every morphism $b(x) \in B[x \in A]$ of $P(\mathcal{T})$ we put

$$P(L)(b(x) \in B[x \in A]) \equiv L(b(x)) \in L(B)[x \in L(A)]$$

Since L is a translation, then $P(L)$ is a functor preserving the H-pretopos structure. In order to describe the relation between type theories and H-pretoposes, we have to consider a type theory $\mathcal{T}$ as a category. We think of $\mathcal{T}$ as the category whose objects are the same as $Pgr(P(\mathcal{T}))$, but whose morphisms are sequences of morphisms by which we built a series of commutative squares. More precisely, the objects of $\mathcal{T}$ are the dependent types under a context $B(x_1, \ldots, x_n)[x_1 \in A_1, \ldots, x_n \in A_n]$. The morphisms of $\mathcal{T}$ exist only from $B[x_1 \in A_1, \ldots, x_n \in A_n]$ to $B'[x'_1 \in A'_1, \ldots, x'_n \in A'_n]$ and they are[4]

$$b' \in B'(a'_1, \ldots, a'_n)[x_1 \in A_1, \ldots, x_n \in A_n, y \in B(x_1, \ldots, x_n)]$$

such that $a_1 \in A'_1[x_1 \in A_1]$ and $a'_i \in A'_i(a'_1, \ldots, a'_{i-1})[x_1 \in A_1, \ldots, x_i \in A_i]$ for $i = 1, \ldots, n$. The composition is the substitution and the identity is $y \in B(x_1, \ldots, x_n)[x_1 \in A_1, \ldots, x_n \in A_n, y \in B]$. Therefore, we can consider equivalences of type theories. In the following we mean with ID the identity functor.

[3] For the choices of finite limits and right adjoints see [See84], for coproducts use disjoint sum types and for quotients use quotient types with indexed sum types.

[4] One could also consider the usual morphisms of contexts.

Proposition 5. *Let $T : HPretop_o \to Lang$ and $P : Lang \to HPretop_o$ be the functors defined above. There are two natural transformations: η from ID to $T \cdot P$, thought as functors from Lang to Lang*, and ϵ from $P \cdot T$ to ID, thought as functors from $HPretop_o$ to $HPretop$, such that for every type theory $\mathcal{T}$ and for every H-pretopos $\mathcal{P}$, $\eta_{\mathcal{T}} : \mathcal{T} \to T(P(\mathcal{T}))$ and $\epsilon_{\mathcal{P}} : P(T(\mathcal{P})) \to \mathcal{P}$ are equivalences.*

Proof. In order to obtain the natural transformation η, for every type theory $\mathcal{T}$ we define

$$\eta_{\mathcal{T}} : \mathcal{T} \to T(P(\mathcal{T}))$$

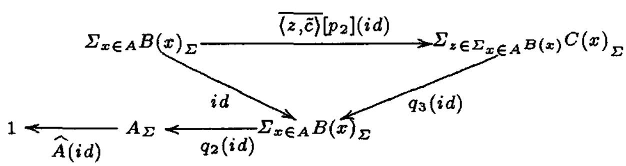
as follows. For any closed type $\eta_{\mathcal{T}}(A[\,]) \equiv \widehat{A}(id) : A_{\Sigma} \to 1$. For dependent type judgements, $\eta_{\mathcal{T}}(C(x,y)[x \in A, y \in B(x)])$ is the type judgement of $T(P(\mathcal{T}))$ corresponding to the sequence

$$\Sigma_{z \in \tilde{B}} C(x)_{\Sigma} \xrightarrow{q_3(id)} \Sigma_{x \in A} B(x)_{\Sigma} \xrightarrow{q_2(id)} A_{\Sigma} \xrightarrow{\widehat{A}(id)} 1$$

where $\tilde{B} \equiv \Sigma_{x \in A} B(x)$ and $q_i \equiv \widehat{\pi_1}[p_{i-1}]$ for $i = 2, 3$. This is the dependent type judgement arising from the following sequence

$$\Sigma_{z \in \Sigma_{x \in A} B(x)} C(x) \xrightarrow{\pi_1} \Sigma_{x \in A} B(x) \xrightarrow{\pi_1} A \xrightarrow{*} 1$$

in the internal type theory $T(P(\mathcal{T}))$, as it is described in the previous section. For term judgements, $\eta_{\mathcal{T}}(c \in C(x,y)[x \in A, y \in B(x)])$ is



where $\tilde{c} \equiv c[x/\pi_1(z), y/\pi_2(z)][z \in \Sigma_{x \in A} B(x)]$. This is the term judgement arising from $\langle z, \tilde{c} \rangle$ in the internal type theory $T(P(\mathcal{T}))$, as it is described in the previous section. We can obviously imagine how $\eta_{\mathcal{T}}$ is defined in the case of having a generic context of n types. We can see that η is a natural transformation, since translations preserve indexed sum types and projections. $\eta_{\mathcal{T}}$ is a translation up to isomorphisms and it is an equivalence of categories since the functor is faithfull, full and essentially surjective. Indeed, we can define a natural transformation η^{-1} such that, given a type theory $\mathcal{T}$, the component $\eta_{\mathcal{T}}^{-1} : T(P(\mathcal{T})) \to \mathcal{T}$ is defined as follows. Given a type judgement $B \xrightarrow{\beta(id)} A \xrightarrow{\alpha(id)} 1$ of $T(P(\mathcal{T}))$ we define

$$\eta_{\mathcal{T}}^{-1}(\alpha(id), \beta(id)) \equiv \beta(id)^{-1}(x)[x \in A]$$

where $\beta(id)^{-1}(x) \equiv \Sigma_{z\in B} Eq(A, \beta(id)(z), x)$, that is the fibers of $\beta(id)$. Given the term judgement $\begin{array}{ccc} A & \xrightarrow[id]{c(id)} & B \\ & \searrow \quad \swarrow_{\beta(id)} & \\ 1 \xleftarrow[\alpha(id)]{} A & & \end{array}$ of $T(P(\mathcal{T}))$, provided that $c(id)$ is $c(x) \in B[x \in A]$, $\eta_{\mathcal{T}}^{-1}$ associates to it the term judgement of $\mathcal{T}$

$$\langle c(x), eq\rangle \in \Sigma_{z\in B} Eq(A, \beta(id)(z), x)[x \in A]$$

We can see that η^{-1} is a natural transformation, since translations preserve indexed sum types, projections and equality types. We can prove that, for every type theory $\mathcal{T}$, $\eta_{\mathcal{T}}$ and $\eta_{\mathcal{T}}^{-1}$ give rise to an equivalence of categories (also see [See84]).

Moreover, we define a natural transformation ϵ such that for every H-pretopos $\mathcal{P}$ the component

$$\epsilon_{\mathcal{P}} : P(T(\mathcal{P})) \to \mathcal{P}$$

is defined as follows. $\epsilon_{\mathcal{P}}$ associates to every object $A \xrightarrow{\sigma(id)} 1$ of $P(T(\mathcal{P}))$ the object A and it associates to the morphism $\begin{array}{ccc} A & \xrightarrow[id]{b(id)} & A\times B \\ & \searrow \quad \swarrow_{\beta(!_A)} & \\ 1 \xleftarrow[\sigma(id)]{} A & & \end{array}$ the morphism $q(!_A, \beta(id))\cdot b(id) : A \to B$. We can easily prove that $\epsilon_{\mathcal{P}}$ is a functor preserving the H-pretopos structure up to isomorphisms[5]. We have that $\epsilon_{\mathcal{P}}$ gives rise to a natural transformation, since the functors preserve the H-pretopos structure w.r.t. the fixed choices. Moreover, $\epsilon_{\mathcal{P}}$ is an equivalence of categories, since it is faithfull by uniqueness of morphisms towards pullbacks, full because every section of a fibred functor has got a name in the language, and essentially surjective. Indeed, we can define a natural transformation ϵ^{-1} such that for every H-pretopos $\mathcal{P}$ the component $\epsilon_{\mathcal{P}}^{-1} : \mathcal{P} \to P(T(\mathcal{P}))$ is defined as follows. For every object A of $\mathcal{P}$, $\epsilon_{\mathcal{P}}^{-1}(A)$ is the closed type corresponding to $A_\Sigma \xrightarrow{\widehat{A}(id)} 1$. For every morphism $b : A \to B$ of $\mathcal{P}$, $\epsilon_{\mathcal{P}}^{-1}(b)$ is the term corresponding to $\begin{array}{ccc} A_\Sigma & \xrightarrow[id]{\overline{\langle id, b'\rangle}(id)} & A_\Sigma\times B_\Sigma \\ & \searrow \quad \swarrow_{\widehat{B}(!_{A_\Sigma})} & \\ 1 \xleftarrow[\widehat{A}(id)]{} A_\Sigma & & \end{array}$ where $b' = \pi_B^{-1} \cdot b \cdot \pi_A$ and where π_B and π_A are the second projections of the pullbacks of $!_A$ and $!_B$ along the identity. We conclude that for every H-pretopos $\mathcal{P}$, $\epsilon_{\mathcal{P}}$ and $\epsilon_{\mathcal{P}}^{-1}$ give rise to an equivalence of categories.

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[5] This is due to the choices of split fibration: see, for instance, the terminal object.

4 The free H-pretopos

The main idea is to generate a H-pretopos from a given category $\mathcal{C}$ by considering its objects as closed types and its morphisms as terms with a free variable. We can prove the universal property by the construction of the category of paths, which represents the dependent types in a categorical way.

Given a category $\mathcal{C}$, we consider the dependent type theory $T(\mathcal{C})$ generated by the inference rules as follows:

1. For every object $\boldsymbol{A}$ of $Ob\mathcal{C}$ we introduce a new type A and we state the closed type judgement $A\ [\,]$.
   Given $A \in Ob\mathcal{C}$ and $B \in Ob\mathcal{C}$ we state $A = B\ [\,]$, if they are the same object in $Ob\mathcal{C}$.
2. For every morphism $\boldsymbol{b : A \to B}$ in $\mathcal{C}$, we introduce a new term $b(x)$ and we state $b(x) \in B\ [x \in A]$, where A and B are closed types.
   Given $b : A \to B$ and $d : A \to B$ in $\mathcal{C}$, we state $b(x) = d(x) \in B\ [x \in A]$, provided that b and d are the same morphism in $\mathcal{C}$.
   Given $b : A \to B$ and $a : D \to A$ in $\mathcal{C}$, we state about composition $b(x)[x := a(y)] = (b \cdot a)(y) \in B\ [y \in D]$.
3. There are all the inference rules of the typed calculus for H-pretoposes as in the appendix.

Therefore $T(\mathcal{C})$ is a type theory of H-pretoposes.

Now, we can prove:

Proposition 6. *Let $P : Lang \longrightarrow HPretop_o$ be the functor described in section 3. The category $P(T(\mathcal{C}))$ is the free H-pretopos generated by the category $\mathcal{C}$.*

Proof. We know that $P(T(\mathcal{C}))$ is a H-pretopos from the definition of P. Given a functor $G : \mathcal{C} \to \mathcal{P}$, from the category $\mathcal{C}$ to the H-pretopos $\mathcal{P}$, we claim that there exists a unique functor $\tilde{G} : P(T(\mathcal{C})) \to \mathcal{P}$ in $HPretop_o$ such that the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\ I\ } & P(T(\mathcal{C})) \\ & {\scriptstyle G}\searrow \quad \swarrow{\scriptstyle \tilde{G}} & \\ & \mathcal{P} & \end{array}$$

commutes, where $I : \mathcal{C} \to P(T(\mathcal{C}))$ is the following functor: for every object $A \in Ob\mathcal{C}$ we put $I(A) \equiv A\ [\,]$ and for every morphism $b : A \to B$ we put $I(b) \equiv b(x) \in B[x \in A]$.

In order to define $\tilde{G}$ on $P(T(\mathcal{C}))$, we define an interpretation $\mathcal{J} : T(\mathcal{C}) \to Pgr(\mathcal{P})$, by passing to $Pgf(\mathcal{P})$, with the warning that we have to normalize the evaluation. This is done by adding the value of every fibred functor $\sigma \in Fib(\mathcal{P}/1, \mathcal{P}^{\to})$ on the empty, by induction on the signature, such that a type judgement will be interpreted by a sequence of $Pgr(\mathcal{P})$ like

$$\alpha_1(\emptyset), \alpha_2(id_{A_1}), ..., \alpha_n(id_{A_{n-1}})$$

The interpretation is the same as for the internal type theory, except for closed types and terms, which are interpreted in fibred functors evaluated on $\emptyset$. The reason is that we want to put $\tilde{G}(A[\,]) \equiv \mathrm{dom}\mathcal{J}(A[\,])$ and $\tilde{G}(b \in B[x \in A]) \equiv q(\mathcal{J}(A[\,]), \mathcal{J}(B[\,])) \cdot \mathcal{J}(b \in B[x \in A])$, but if we adopt for $\mathcal{J}$ the semantics defined in section 2, then $\tilde{G}$ would commute with G up to isomorphisms. So, for every object A of $Ob\mathcal{P}$, we extend the functor $\widehat{A}$ by adding $\widehat{A}(\emptyset) \equiv !_A$ and for every object B, $q(!_B, \widehat{A}(\emptyset))$ is the second projection of the pullback of $!_B$ and $\widehat{A}(\emptyset)$. For example, for the natural numbers $\mathcal{J}(N[\,]) \equiv \widehat{\mathcal{N}}(\emptyset) = !_{\mathcal{N}}$, instead of being interpreted as $!_{1\times\mathcal{N}}$ like in the semantics defined in section 2. Moreover, $\mathcal{J}(0 \in N[\,])$ is

$$\begin{array}{ccc} 1 & \xrightarrow{\widehat{o}(\emptyset)} & \mathcal{N} \\ & {\scriptstyle id_1}\searrow \quad \swarrow{\scriptstyle \widehat{\mathcal{N}}(\emptyset)} & \\ & 1 & \end{array}$$

where $\widehat{o}(\emptyset) \equiv o$ and $o : 1 \to \mathcal{N}$ is the zero map in $\mathcal{P}$. Finally, given a proper type arising from an object $A \in Ob\mathcal{C}$, we put $\mathcal{J}(A[\,]) \equiv \widehat{G(A)}(\emptyset)$ and given a proper term arising from a morphism $b : A \to B$ of $\mathcal{C}$, we put $\mathcal{J}(b \in B[x \in A]) \equiv \langle id_{G(A)}, G(b) \rangle$ section of $\widehat{G(B)}(\widehat{G(A)}(\emptyset)) : G(A) \times G(B) \to G(A)$. By definition $\tilde{G}$ preserves the H-pretopos structure and we get $\tilde{G} \cdot I = G$. Moreover, $\tilde{G}$ is obviously unique for fixed choices of the H-pretopos structure, which are required to interpret the type theory $T(\mathcal{C})$ into $Pgr(\mathcal{P})$.

The free structure gives rise to a monad. It would be interesting to investigate if the category $HPretop_o$ is monadic on *Cat* and *Graph*. Or at least, if we prove that $HPretop_o$ is essentially algebraic, as for the categorical models of *ITT* in[Obt89], we would get a representation theorem of $HPretop_o$ into a category of presheaves [AR94].

5 Some other free structures: the *Lex* and *ITT* categories

A similar correspondence to that one between type theories and H-pretoposes can be established for the category *Lex* and *ITT*. The category *Lex*, whose objects are the categories with finite limits and whose morphisms are functors strictly preserving finite limits, provides a valid and complete semantics for the typed calculus with terminal type, extensional equality types and indexed sum types. In the same way, the *ITT* category, whose objects are the locally cartesian closed categories with finite coproducts and a natural numbers object and whose morphisms are functors strictly preserving the *ITT* structure, provides a valid and complete semantics for the fragment of Martin-Löf's type theory with extensional equality and without universes and well-orders [Mar84]. These validity and completeness theorems can be proved in a similar way to that for H-pretoposes. We can easily notice that these dependent typed calculi allow us to build the free structure for *Lex* and *ITT* over *Cat*, in the same way we proved for the category $HPretop_o$. The free structures give a presentation of two monads, whose algebras correspond respectively to *Lex* and *ITT*, since *Lex* and *ITT* are monadic over *Graph* [Bur81] and admit an equational presentation.

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6 Appendix: the HP typed calculus

Terminal type

$$\text{Tr)}\ \top\ type \qquad \text{I-Tr)}\ \star \in \top \qquad \text{C-Tr)}\ \frac{t \in \top}{t = \star \in \top}$$

False type

$$\text{Fs)}\ \bot\ type \qquad \text{E-Fs)}\ \frac{a \in \bot \quad A\ type}{r_o(a) \in A}$$

Indexed Sum type

$$\Sigma)\ \frac{C(x)\ type\ \ [x \in B]}{\Sigma_{x \in B} C(x)\ type} \qquad \text{I-}\Sigma)\ \frac{b \in B \quad c \in C(b)}{\langle b, c \rangle \in \Sigma_{x \in B} C(x)}$$

$$\text{E-}\Sigma)\ \frac{d \in \Sigma_{x \in B} C(x) \quad m(x,y) \in M(\langle x, y \rangle)\ [x \in B, y \in C(x)]}{s(d, m) \in M(d)}$$

$$\text{C-}\Sigma)\ \frac{b \in B \quad c \in C(b) \quad m(x,y) \in M(\langle x, y \rangle)\ [x \in B, y \in C(x)]}{s(\langle b, c \rangle, m) = m(b, c) \in M(\langle b, c \rangle)}$$

Equality type

$$\text{Eq)}\ \frac{C\ type \quad c \in C \quad d \in C}{Eq(C, c, d)\ type} \qquad \text{I-Eq)}\ \frac{c = d \in C}{eq_C \in Eq(C, c, d)}$$

$$\text{E-Eq)}\ \frac{p \in Eq(C, c, d)}{c = d \in C} \qquad \text{C-Eq)}\ \frac{p \in Eq(C, c, d)}{p = eq_C \in Eq(C, c, d)}$$

Disjoint Sum type

$$\oplus)\ \frac{C\ type \quad D\ type}{C \oplus D\ type} \qquad \text{I}_1\text{-}\oplus)\ \frac{c \in C}{\epsilon_1(c) \in C \oplus D} \qquad \text{I}_2\text{-}\oplus)\ \frac{d \in D}{\epsilon_2(d) \in C \oplus D}$$

$$\text{E-}\oplus)\ \frac{w \in C \oplus D \quad a_C(x) \in A(\epsilon_1(x))[x \in C] \quad a_D(y) \in A(\epsilon_2(y))[y \in D]}{D(w, a_C, a_D) \in A(w)}$$

$$\text{C}_1\text{-}\oplus)\ \frac{c \in C \quad a_C(x) \in A(\epsilon_1(x))[x \in C] \quad a_D(y) \in A(\epsilon_2(y))[y \in D]}{D(\epsilon_1(c), a_C, a_D) = a_C(c) \in A(\epsilon_1(c))}$$

$$\text{C}_2\text{-}\oplus)\ \frac{d \in D \quad a_C(x) \in A(\epsilon_1(x))[x \in C] \quad a_D(y) \in A(\epsilon_2(y))[y \in D]}{D(\epsilon_2(d), a_C, a_D) = a_D(d) \in A(\epsilon_2(d))}$$

Disjointness

$$\frac{c \in C \quad d \in D \quad \epsilon_1(c) = \epsilon_2(d) \in C \oplus D}{m(c, d) \in \bot}$$

Forall type

$$\forall)\ \frac{C(x)\ type[x \in B] \quad d \in Eq(C(x), y, z)[x \in B, y \in C(x), z \in C(x)]}{\forall_{x \in B} C(x)\ type}$$

$$\text{I-}\forall)\ \frac{c \in C(x)[x \in B] \quad d \in Eq(C(x), y, z)[x \in B, y \in C(x), z \in C(x)]}{\lambda x^B.c \in \forall_{x \in B} C(x)}$$

$$\text{E-}\forall)\ \frac{b \in B \quad f \in \forall_{x \in B} C(x)}{Ap(f, b) \in C(b)}$$

$$\beta\text{C-}\forall)\ \frac{b \in B \quad c \in C(x)[x \in B] \quad d \in Eq(C(x), y, z)[x \in B, y \in C(x), z \in C(x)]}{Ap(\lambda x^B.c, b) = c(b) \in C(b)}$$

$$\eta\text{C-}\forall)\ \frac{f \in \forall_{x \in B} C(x)}{\lambda x^B.Ap(f, x) = f \in \forall_{x \in B} C(x)}$$

Quotient type

$$\text{Q})\ \frac{\begin{array}{l} R(x,y)\ type\ [x \in A, y \in A],\ d \in Eq(R(x,y), z, w)[x \in A, y \in A, z \in R(x,y), w \in R(x,y)] \\ c_1 \in R(x,x)[x \in A], \qquad c_2 \in R(y,x)[x \in A, y \in A, z \in R(x,y)] \\ c_3 \in R(x,z)[x \in A, y \in A, z \in A, w \in R(x,y), w' \in R(y,z)] \end{array}}{A/R\ type}$$

$$\text{I-Q})\ \frac{a \in A}{[a] \in A/R} \qquad \text{eq-}Q)\ \frac{a \in A \quad b \in A \quad d \in R(a,b)}{[a] = [b] \in A/R}$$

$$\text{E-Q})\ \frac{s \in A/R \quad l(x) \in L([x])[x \in A] \quad l(x) = l(y) \in L([x])[x \in A, y \in A, d \in R(x,y)]}{Q(l, s) \in L(s)}$$

$$\text{C-Q})\ \frac{a \in A \quad l(x) \in L([x])[x \in A] \quad l(x) = l(y) \in L([x])[x \in A, y \in A, d \in R(x,y)]}{Q(l, [a]) = l(a) \in L([a])}$$

Effectiveness

$$\frac{a \in A \quad b \in A \quad [a] = [b] \in A/R}{f(a,b) \in R(a,b)}$$

Natural Numbers type

$$\text{nat})\ N\ type \qquad \text{I}_1\text{-nat})\ 0 \in N \qquad \text{I}_2\text{-nat})\ \frac{n \in N}{s(n) \in N}$$

$$\text{E-nat})\ \frac{n \in N \quad a \in L(0) \quad l(x,y) \in L(s(x))[x \in N, y \in L(x)]}{Rec(a, l, n) \in L(n)}$$

$$\text{C}_1\text{-nat})\ \frac{a \in L(0) \quad l(x,y) \in L(s(x))[x \in N, y \in L(x)]}{Rec(a, l, 0) = a \in L(0)}$$

$$\text{C}_2\text{-nat})\ \frac{n \in N \quad a \in L(0) \quad l(x,y) \in L(s(x))[x \in N, y \in L(x)]}{Rec(a, l, s(n)) = l(n, Rec(a, l, n)) \in L(s(n))}$$

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