About Effective Quotients in Constructive Type Theory*

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Abstract. We extend Martin-Löf's constructive set theory with effective quotient sets and the rule of uniqueness of propositional equality proofs. We prove that in the presence of at least two universes U_0 and U_1 the principle of excluded middle holds for small sets. The key point is the combination of uniqueness of propositional equality proofs with the effectiveness condition that allows us to recover information on the equivalence relation from the equality on the quotient set.

1 Introduction

Within the framework of Martin-Löf's Intuitionistic Type Theory [Mar84,NPS90], in order to generate some formal topologies [Sam87], the quotient sets are also desirable [NV97]. But some care is necessary in extending Martin-Löf's set theory with quotient sets if we want to keep constructivity.

Here, we consider the extension of intensional Martin-Löf's set theory (*MLTT*) with quotient sets as formulated in [Hof95] and we want to explore the possibility to make quotients effective. Intuitively, effectiveness for quotient sets means that if two elements of a set are in the same "equivalence class" as represented by an element of the quotient set, then the two elements satisfy the equivalence relation. A property with this name can be found in category theory as referred to an equivalence relation (see e.g. [MR77]). The usual constructions of quotients in classical set theory, in categorical universes like toposes and in the setoids made out of type theory enjoy this property.

In this paper we give an answer to the question of extending $MLTT$ with effective quotients, if we also add the rule of uniqueness of equality proofs [Hof95]. Indeed, even if the rule of uniqueness of equality proofs is not provable in the intensional version of Martin-Löf's set theory as proved by M. Hofmann and T. Streicher [HS95], however it is definable by pattern-matching [Coq92], which is a very useful tool for implementations of type theory.

To formulate effectiveness we need to pass to the extension of Martin-Löf's type theory, here called iTT, augmented with the true judgement A *true* (see [Mar84,Val95]). According to the paradigm in [Val95], the

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rules of iTT about true judgements, called admissible, are exactly those obtained from $MLTT$ such that we can prove at the level of the metalanguage the following preservation property: *the judgement "A true" is derivable in iTT iff there is a proof-term "a" such that "$a \in A$" is derivable in MLTT.*
We call iTT^Q the extension with true judgements corresponding to $MLTT^Q$, that is $MLTT$ augmented with intensional quotients and the uniqueness of equality proofs. Now, in iTT^Q we express the effectiveness condition in terms of true judgements as follows

$$\frac{a \in A \quad b \in A \quad \mathsf{Id}(A/R, [a], [b])\ true}{R(a, b)\ true}$$

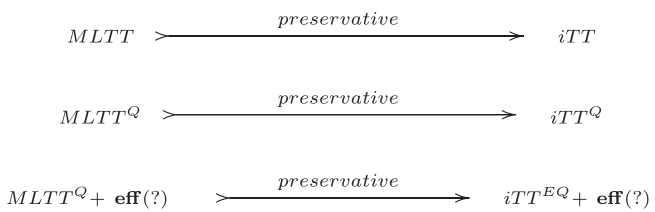
and we call iTT^{EQ} the extension of iTT^Q with this condition. We add effectiveness as a condition on true judgements, because we are not able to think of a constructive type theory with only the four kinds of judgements

$$A\ set \qquad A = B \qquad a \in A \qquad a = b \in A$$

that extends $MLTT^Q$ and whose extension with true judgements makes the effectiveness condition admissible. Indeed, in order to admit the effectiveness condition in the corresponding extension with true judgements, this claimed type theory should allow to derive the following rule

$$\textbf{eff} \quad \frac{a \in A \quad b \in A \quad p \in \mathsf{Id}(A/R, [a], [b])}{?(a, b, p) \in R(a, b)}$$

for some proof-term $?(a, b, p)$.



Actually, we will show here that we can not have such a theory where **eff** can be derived, since even in the extension iTT^{EQ} the principle of excluded middle holds for small sets. Indeed, in the presence of quotient sets with the effectiveness condition, the rule of uniqueness of propositional equality proofs and at least two universes U_0 and U_1, to which the codes of quotient sets are added, we can reproduce for small sets the proof of Diaconescu [Dia75] made within topos theory that the axiom of choice implies the principle of excluded middle. Therefore, to be clearer, if a constructed type theory including $MLTT^Q +$ **eff** existed, then its preservative extension with true judgements would admit the effectiveness condition. Hence, as shown here, we would be able to prove the principle of excluded middle for small sets at the level of true judgements and as a consequence of the preservation property in the pure type theory itself against its claimed constructivity.

In the framework of set theory the proof reproduced here shows the incompatibility, from the intuitionistic point of view, between the extensionality axiom and the axiom of choice. In the framework of topos theory it makes use of extensional powersets. Here, we will see that to reproduce extensionality in the context of intensional type theory, where the axiom of choice holds because of the presence of a strong existential quantifier, it is sufficient to have general effective quotients to be used on the first two universes and the rule of uniqueness of equality proofs at the propositional level. In fact, in the proof we mimic powersets by quotienting the first two universes under the relation of equiprovability. Then, we need the effectiveness condition to decode the extensional equality related to the quotients on the universes into the equiprovability relation. Finally, the rule of uniqueness of equality proofs seems crucial to identify the values of the choice function applied to two suitable extensionally equal subsets.

Of course, an analogous proof can be reproduced in the extensional version of Martin-Löf's set theory with the quotient sets as given in Nuprl [Con86] and only with the addition of the effectiveness condition.

We know that the effectiveness condition is surely derivable for decidable equivalence relations. But in general effectiveness is problematic, because it restores information that has been forgotten in the introduction rule for the equality of equivalence classes. This is confirmed by the proof given here.

The interest in the effectiveness condition arises from the mathematical practice of quotient sets. In order to keep effectiveness for quotient sets in the presence of uniqueness of equality proofs, an alternative strategy could be to let quotient sets based only on a proof-irrelevant equivalence relation, as it is in the type theory of Heyting pretoposes [Mai97].

2 The Idea of the Proof: Axiom of Choice versus Extensionality

We describe the idea behind the proof that in the extension of Martin-Löf set theory with effective quotient sets and the uniqueness of equality proofs the axiom of choice yields classical logic on small sets. We think that this proof can go through any other possible extension with analogous extensional constructors. The idea of the proof originally due to Diaconescu [Dia75] can be clearly understood in the framework of an intuitionistic set theory with basic axioms, as the empty axiom, the pair axiom and the comprehension axiom, also only for restricted formulas as in CZF [Acz78] (see e.g. [GM78,Bel97]). In this framework we can see how the axiom of choice is incompatible with the extensionality axiom from the constructive point of view, as we show in the following.

Let us consider a set A and the following subsets of the set $\{0, 1\}$, where $0 \equiv \emptyset$ and $1 \equiv \{\emptyset\}$:

$$V_0 \equiv \{x \in \{0,1\} :\ x = 0 \lor \exists y\, y \in A\} \qquad V_1 \equiv \{x \in \{0,1\} :\ x = 1 \lor \exists y\, y \in A\}$$

Now, if we apply the axiom of choice to the system of sets $\{V_0, V_1\}$ we get that the following proposition is true:

$$\forall z \in \{V_0, V_1\}\ \exists y \in \{0,1\}\ y \in z \longrightarrow$$
$$\exists f \in \{V_0, V_1\} \to \{0,1\}\ \ \forall z \in \{V_0, V_1\}\ \ f(z) \in z$$

Then we know that the premise of this implication is true by substituting y with 0 in the case of V_0 and with 1 in the case of V_1. Therefore we derive by modus ponens

$$\exists f \in \{V_0, V_1\} \to \{0,1\}\ \ \forall z \in \{V_0, V_1\}\ \ f(z) \in z$$

Then, applying the elimination of the existential quantifier, we can derive

$$(f(V_0) = 0\ \vee\ \exists y\ y \in A)\ \ \wedge\ \ (f(V_1) = 1\ \vee\ \exists y\ y \in A)$$

from which by distributivity we get

$$(f(V_0) = 0\ \wedge\ f(V_1) = 1)\ \vee \exists y\ y \in A$$

Now we are going to prove by $\vee$-elimination from the above proposition the principle of excluded middle for A. So, at first we assume $f(V_0) = 0 \wedge f(V_1) = 1$. Then note that if we also assume $\exists y\ y \in A$, from this by *extensionality* we get that $V_0 = V_1$, which combined with our first assumption yields $0 = 1$, which is falsum and lets us conclude $\neg\exists y\ y \in A$ and also $\exists y\ y \in A\ \vee\ \neg\exists y\ y \in A$. Since by assuming the second disjunct $\exists y\ y \in A$ we also get $\exists y\ y \in A\ \vee\ \neg\exists y\ y \in A$, by $\vee$-elimination applied on $(f(V_0) = 0\ \wedge\ f(V_1) = 1)\ \vee \exists y\ y \in A$ the principle of excluded middle for any set A

$$\exists y\ y \in A\ \ \vee\ \ \neg\exists y\ y \in A$$

is now derived. We can adapt the outline of this proof to the extension of Martin-Löf's set theory with effective quotient sets and the uniqueness of equality proofs, as we will show in the next sections. The uniqueness of equality proof seems crucial to reproduce the proof together with the extensionality captured by effective quotient sets.

3 Extension of *iTT* with Quotient Sets

In order to investigate the possibility of an extension with effective quotient sets, firstly we extend the intensional version of Martin-Löf's Intuitionistic Type Theory [NPS90], here called *MLTT*, with quotient sets and the rule of uniqueness of proofs for the intensional propositional equality as in [Hof95] (page 111) and we call this extension $MLTT^Q$. Then we consider its preservative extension iTT^Q with true judgements. Lastly we extend iTT^Q with the effectiveness condition and we call this extension iTT^{EQ}. As said in the introduction the meaning of a *true* judgement is the following: A *true* holds if and only if there exists a proof-element a such that $a \in A$ holds (for an account of this see [Mar84,Val95]). This is meaningful, since we identify *propositions* and *sets*. We call iTT the extension of *MLTT* with true judgements. The rules of iTT (iTT^Q) about true judgements are precisely those admissible by the rules of *MLTT* ($MLTT^Q$) according to the explained semantics, to which we add the following introduction rule

$$\text{(True Introduction)}\ \frac{a \in A}{A\ true}$$

such that iTT (iTT^Q) turns out to be a preservative extension of *MLTT* ($MLTT^Q$) in the sense stated in [Val95] and recalled in the introduction. For instance, among the admissible rules of iTT, we recall the case of the set of intensional propositional equality Id. The propositional equality is the internalization of the definitional equality between elements of a set at the level of propositions, considering two objects definitionally equal if they evaluate to the same normal form. Actually, there are two kinds of propositional equality characterizing intensional and extensional type theories: Id, which is intensional (see the rules below), and Eq, which is extensional (see [NPS90] and the section 5). Intensional propositional equality is entailed by definitional equality, that is two objects are propositionally equal if they are definitionally equal, but the other way around does not hold. On the contrary, extensional propositional equality is equivalent to definitional equality. The main difference is that in the presence of intensional propositional equality, definitional equality and type checking are decidable, but this is no longer true in the presence of extensional propositional equality.

The formation, introduction, elimination and conversion rules for the set Id are the following

Intensional equality set

$$\frac{A\ set \quad a \in A \quad b \in A}{\mathsf{Id}(A, a, b)\ set}$$

I-Id

$$\frac{a \in A}{\mathsf{id}(a) \in \mathsf{Id}(A, a, a)}$$

E- Id

$$\frac{d \in \mathsf{Id}(A, a, b) \quad c(x) \in C(x, x, \mathsf{id}(x)))\ [x : A]}{\mathsf{idpeel}(d, c) \in C(a, b, d)}$$

C-Id

$$\frac{a \in A \quad c(x) \in C(x, x, \mathsf{id}(x))\ [x : A]}{\mathsf{idpeel}(\mathsf{id}(a), c) = c(a) \in C(a, a, \mathsf{id}(a))}$$

In particular, the admissible rules corresponding to the elimination rule are the following:

$$\frac{d \in \mathsf{Id}(A, a, b) \quad \begin{matrix}[x : A] \\ | \\ C(x, x, \mathsf{id}(x))\ true\end{matrix}}{C(a, b, d)\ true} \qquad \frac{\mathsf{Id}(A, a, b)\ true \quad \begin{matrix}[x : A] \\ | \\ C(x, x)\ true\end{matrix}}{C(a, b)\ true}$$

Now, we extend iTT with quotient sets as formulated in [Hof95][1]:
Intensional Quotient set

$$\frac{\begin{array}{l} R(x,y)\ set\ [x \in A, y \in A] \\ c_1 \in R(x,x)[x \in A], \qquad c_2 \in R(y,x)[x \in A, y \in A, z \in R(x,y)] \\ c_3 \in R(x,z)[x \in A, y \in A, z \in A, w \in R(x,y), w' \in R(y,z)] \end{array}}{A/R\ set}$$

I-int.quotient

$$\frac{a \in A \quad A/R\ set}{[a] \in A/R}$$

eq-int.quotient

$$\frac{a \in A \quad b \in A \quad d \in R(a,b)}{\mathsf{Qax}(d) \in \mathsf{Id}(A/R, [a], [b])}$$

E-int.quotient

$$\frac{\begin{array}{l} s \in A/R \quad l(x) \in L([x])[x \in A] \\ h \in \mathsf{Id}(L([y]),\ \mathsf{sub}(\mathsf{Qax}(d), l(x)),\ l(y))\ [x \in A, y \in A, d \in R(x,y)] \end{array}}{\mathsf{Q}(l, h, s) \in L(s)}$$

where the term $\mathsf{sub}(c,d) \equiv \mathsf{idpeel}(c,(x)\lambda y.y)(d)$ for $c \in \mathsf{Id}(A,a,b)$ and $d \in L(a)$ (see also [NPS90] page 64) expresses substitution with equal elements;
C-int.quotient

$$\frac{\begin{array}{l} a \in A \quad l(x) \in L([x])[x \in A] \\ h \in \mathsf{Id}(L([y]),\ \mathsf{sub}(\mathsf{Qax}(d), l(x)),\ l(y))\ [x \in A, y \in A, d \in R(x,y)] \end{array}}{\mathsf{Q}(l, h, [a]) = l(a) \in L([a])}$$

We also want to make quotients effective and we require:
Effectiveness condition

$$\frac{a \in A \quad b \in A \quad \mathsf{Id}(A/R, [a], [b])\ true}{R(a,b)\ true}$$

Effectiveness expresses the fact that, as usual, every equivalence relation on a set A is the kernel of the function which maps an element of A into its equivalence class.

Note that effectiveness is expressed only as a condition in terms of true judgements, since we are not able to exhibit type-theoretical rules that make this effectiveness condition admissible, like for the rules of iTT^Q on true judgements, where by a type-theoretical rule we mean a rule expressed using judgements only of the following four kinds: $A\ set \quad A = B \quad a \in A \quad a = b \in A$. Indeed, in iTT^{EQ} we will prove a non-constructive principle, that is the principle of excluded middle on small sets, which lets us conclude that there are

[1] But we restrict the formation rule to quotient sets based on equivalence relations. In A/R we should record the proof terms c_1, c_2, c_3 and then the corresponding equality rule should say that varying c_1, c_2, c_3, the set A/R is the same.

no type-theoretical rules that make the effectiveness condition on quotient sets admissible and that in the same time follow the Heyting constructive semantics of connectives.

Finally, we add the rule of uniqueness of propositional equality proofs:
Id-Uni I

$$\frac{a \in A \quad p \in \mathsf{Id}(A, a, a)}{\mathsf{iduni}(a, p) \in \mathsf{Id}(\mathsf{Id}(A, a, a), p, \mathsf{id}(a))}$$

The corresponding conversion rule is the following:
Id-Uni conv

$$\frac{a \in A}{\mathsf{iduni}(a, \mathsf{id}(a)) = \mathsf{id}(\mathsf{id}(a)) \in \mathsf{Id}(\mathsf{Id}(A, a, a), \mathsf{id}(a), \mathsf{id}(a))}$$

By using Id-Uni and the elimination rule of the propositional equality on the proposition

$$\Pi_{w \in \mathsf{Id}(A,x,y)} \mathsf{Id}(\mathsf{Id}(A, x, y), w, z) \ [x \in A, y \in A, z \in \mathsf{Id}(A, x, y)]$$

Streicher proved that (see [Hof95] page 81) the set

$$\mathsf{Id}(\mathsf{Id}(A, x, y), w, z) \ \ [x \in A, y \in A, z \in \mathsf{Id}(A, x, y), w \in \mathsf{Id}(A, x, y)]$$

is inhabited by the proof-term

$$\mathsf{idpeel}(z, (x)\lambda w' \in \mathsf{Id}(A, x, x).\mathsf{iduni}(x, w'))(w)$$

Hence, the uniqueness of proofs of propositional equality set, called UIP, holds.

Remark 1. As we said in the introduction, the uniqueness of proofs of the propositional equality set is definable by pattern-matching [Coq92], but it is not derivable in the intensional version of Martin-Löf's set theory, as showed by M. Hofmann and T. Streicher (see [HS95]), who built a model where UIP is not valid.

Finally, we consider the first universe U_0, whose elements are called small sets [NPS90], and the second universe U_1, whose elements are called large sets and where U_0 is also coded (see [Mar84] and [Dyb97], but note that we do not give a new code to terms of the first universe into the second one to make formulas more readable in the following). We have also to add the following introduction rules for the codes of the quotient sets into the universes for $i = 0, 1$
UQ-I

$$\frac{\begin{array}{l} a \in U_i \qquad\qquad\qquad\qquad r(x,y) \in U_i \ [x \in T_i(a), y \in T_i(a)] \\ c_1 \in T_i(r(x,x)) \ [x \in T_i(a)],\, c_2 \in T_i(r(y,x)) \ [x \in T_i(a), y \in T_i(a), z \in T_i(r(x,y))] \\ c_3 \in T_i(r(x,z)) \ [x \in T_i(a), y \in T_i(a), z \in T_i(a), w \in T_i(r(x,y)), w' \in T_i(r(y,z))] \end{array}}{a \widehat{/} r \in U_i}$$

with the corresponding conversion rules

$$T_i(a \widehat{/} r) = T_i(a)/(x,y)T_i(r(x,y))$$

This extension of iTT, called iTT^{EQ}, is consistent because there is an interpretation of iTT^{EQ} into classical set theory (ZFC) with two strongly inaccessible cardinals. Indeed, we interpret the quotient sets in classical quotient sets and the first two universes respectively in the set of small sets and in the set of large sets, proved to be actual sets by the presence of the two strongly inaccessible cardinals.

4 Small Sets Are Classical

We are going to prove that for small sets in iTT^{EQ} the principle of excluded middle holds, i.e. for any element a of the first universe U_0, the judgement $T_0(a) \vee \neg T_0(a)$ *true* holds. This is a consequence of a particular application of the axiom of choice (AC). In topos theory the fact that AC implies the principle of excluded middle was first proved by Diaconescu [Dia75]. The same result is obtained in [MV99] within an extension of iTT with a powerset constructor by adapting the logical proof of [Bel88] about Diaconescu's theorem. Also Hofmann in [Hof95] claimed that the same result can be obtained in the Calculus of Constructions by adding proof-irrelevance at the level of propositions, equiprovability as equality between propositions and extensionality as equality between dependent propositions.

Here, we show that we can recover this proof in a predicative setting with effective quotient sets instead of an impredicative one like a topos. The key point is to simulate the powerset, by quotienting the first two universes under the relation of equiprovability among their elements.
Also in iTT^{EQ}, the so called *intuitionistic axiom of choice*

$$((\forall x \in A)(\exists y \in B)\ C(x, y)) \to ((\exists f \in A \to B)(\forall x \in A)\ C(x, f(x)))\ \mathit{true}$$

is proved by disjoint union sets, exactly as in [Mar84], concluding by true introduction.

We are going to use the axiom of choice on the quotients made out of the first two universes under the equivalence relation of equiprovability, i.e.

$$T_0(x) \leftrightarrow T_0(y)\ \mathit{set}\ [x \in U_0, y \in U_0] \qquad\qquad T_1(x) \leftrightarrow T_1(y)\ \mathit{set}\ [x \in U_1, y \in U_1]$$

Let us put the following abbreviations for $i = 0, 1$

$$\Omega_i \equiv U_i/\ (x, y)T_i(x) \leftrightarrow T_i(y)$$

Since there is a code for U_0 in U_1, i.e. $\widehat{U_0} \in U_1$, then there is inside U_1 the code $\widehat{\Omega_0}$ for Ω_o such that

$$T_1(\widehat{\Omega_0}) = \Omega_0$$

The reason to use the two universes is due to the possibility of deriving

$$\hat{\mathsf{Id}}(\hat{\Omega}_o, z, [\widehat{\top}]) \in U_1\ [z \in \Omega_0]$$

where $\top$ is the singleton set (see [NPS90]). We use the abbreviation $a =_A b$ for $\mathsf{Id}(A, a, b)$, when it is not coded in a universe.
Moreover, if A is a set, we will often write A to mean the judgement A *true*.

We also recall (see [NPS90]) that, in the presence of U_0, we can derive

$$\neg(\mathsf{true} =_{\mathsf{Bool}} \mathsf{false})$$

Now, we go on to show the claimed proof of the principle of excluded middle on small sets. As in [MV99], one of the key points is to internalize the truth of sets within the quotients on the universes, simulating the powersets. This is expressed by the following lemma, which is provable by the introduction equality rule on the quotient set in terms of true judgements and by the effectiveness condition.

Lemma 1. *For $i = 1, 2$ and any set $a \in U_i$, $[a] =_{\Omega_i} [\widehat{\top}]$ iff $T_i(a)$ true.*

Proof. From $[a] =_{\Omega_i} [\widehat{\top}]$ *true* by effectiveness of quotient sets we get $T_i(a) \leftrightarrow T_i(\widehat{\top})$ *true*, but $T_i(\widehat{\top}) = \top$ so $T_i(a)$ *true*. On the other hand, from $T_i(a)$ *true*, we get $T_i(a) \leftrightarrow T_i(\widehat{\top})$ and by the true version of the equality rule on the quotient set we conclude $[a] =_{\Omega_i} [\widehat{\top}]$.
■

Now, we consider the following abbreviations: for $z \in \Omega_0$

$$E(z) \equiv \mathsf{Id}(\Omega_0, z, [\widehat{\top}])$$

Hence, we prove:

Proposition 1. *In iTT^{EQ} the following proposition*

$$(\forall z \in \Sigma_{w \in \Omega_0 \times \Omega_0} [\widehat{E(\pi_1(w))} \hat{\vee}\ \widehat{E(\pi_2(w))}] =_{\Omega_1} [\widehat{\top}])$$
$$(\exists x \in \mathsf{Bool})\ (x =_{\mathsf{Bool}} \mathsf{true} \rightarrow E(\pi_1(\pi_1(z))))\ \wedge (x =_{\mathsf{Bool}} \mathsf{false} \rightarrow\ E(\pi_2(\pi_1(z))))$$

is true.

Proof. Suppose $z \in \Sigma_{w \in \Omega_0 \times \Omega_0} [\widehat{E(\pi_1(w))} \hat{\vee}\ \widehat{E(\pi_2(w))}] =_{\Omega_1} [\widehat{\top}]$. Then $\pi_1(z) \in \Omega_0 \times \Omega_0$ and $\pi_2(z)$ is a proof of $[\widehat{E(\pi_1(\pi_1(z)))} \hat{\vee}\ \widehat{E(\pi_2(\pi_1(z)))}] =_{\Omega_1} [\widehat{\top}]$. Thus, by lemma 1 and by the conversion rules for U_1, $E(\pi_1(\pi_1(z))) \vee E(\pi_2(\pi_1(z)))$. The result can now be proved by $\vee$-elimination, by putting for example $x = \mathsf{true}$ in the case $E(\pi_1(\pi_1(z)))$ *true*.
■

Thus, we can use the intuitionistic axiom of choice to obtain:

Proposition 2. *In iTT^{EQ} the following proposition*

$$(\exists f \in \Sigma_{w \in \Omega_0 \times \Omega_0} [\widehat{E(\pi_1(w))} \hat{\vee}\ \widehat{E(\pi_2(w))}] =_{\Omega_1} [\widehat{\top}] \rightarrow \mathsf{Bool})$$
$$(\forall z \in \Sigma_{w \in \Omega_0 \times \Omega_0} [\widehat{E(\pi_1(w))} \hat{\vee}\ \widehat{E(\pi_2(w))}] =_{\Omega_1} [\widehat{\top}])$$
$$(f(z) =_{\mathsf{Bool}} \mathsf{true} \rightarrow E(\pi_1(\pi_1(z))))\ \wedge\ (f(z) =_{\mathsf{Bool}} \mathsf{false} \rightarrow E(\pi_2(\pi_1(z))))$$

is true.

Suppose, now, that $a \in U_0$ is the code of a small set; then

$$\langle\langle[a], [\widehat{\top}]\rangle, \mathsf{Qax}(\langle\lambda y.\star, \lambda y'.\mathsf{inr}(\mathsf{id}([\widehat{\top}]))\rangle)\rangle$$

is an element of the set

$$\Sigma_{w\in\Omega_0\times\Omega_0}\ [E(\widehat{\pi_1(w)})\hat{\vee}\ E(\widehat{\pi_2(w)})] =_{\Omega_1}\ [\widehat{\top}]$$

where $\star \in \top$ is the only element of the singleton set. In fact, $\langle[a], [\widehat{\top}]\rangle \in \Omega_0 \times \Omega_0$ and

$$\langle\lambda y.\star, \lambda y'.\mathsf{inr}(\mathsf{id}([\widehat{\top}]))\rangle \in\ \mathsf{Id}(\Omega_0, [a], [\widehat{\top}]) \vee \mathsf{Id}(\Omega_0, [\widehat{\top}], [\widehat{\top}]) \leftrightarrow \top$$

from which, since

$$\mathsf{Id}(\Omega_0, [a], [\widehat{\top}]) \vee \mathsf{Id}(\Omega_0, [\widehat{\top}], [\widehat{\top}]) \leftrightarrow \top\ =\ T_1(\widehat{E([a])}\hat{\vee}\ \widehat{E([\widehat{\top}])}) \leftrightarrow T_1(\widehat{\top})$$

by the equality rule on the quotient set we get

$$\mathsf{Qax}(\langle\lambda y.\star, \lambda y'.\mathsf{inr}(\mathsf{id}([\widehat{\top}]))\rangle) \in\ [\widehat{E([a])}\hat{\vee}\ \widehat{E([\widehat{\top}])}] =_{\Omega_1}\ [\widehat{\top}]$$

Analogously,

$$\langle\langle[\widehat{\top}], [a]\rangle, \mathsf{Qax}(\langle\lambda y.\star, \lambda y'.\mathsf{inl}(\mathsf{id}([\widehat{\top}]))\rangle)\rangle$$

is an element of the set

$$\Sigma_{w\in\Omega_0\times\Omega_0}\ [E(\widehat{\pi_1(w)})\hat{\vee}\ E(\widehat{\pi_2(w)})] =_{\Omega_1}\ [\widehat{\top}]$$

Let us put for $w \in \Omega_0$

$$\mathsf{q}_1(w) \equiv \langle\langle w, [\widehat{\top}]\rangle, \mathsf{Qax}(\langle\lambda y.\star, \lambda y'.\mathsf{inr}(\mathsf{id}([\widehat{\top}]))\rangle)\rangle$$

and

$$\mathsf{q}_2(w) \equiv \langle\langle[\widehat{\top}], w\rangle, \mathsf{Qax}(\langle\lambda y.\star, \lambda y'.\mathsf{inl}(\mathsf{id}([\widehat{\top}]))\rangle)\rangle$$

Now, let f be the choice function obtained by $\exists$-elimination rule on the judgement in the proposition 2; then $f(\mathsf{q}_1([a])) =_{\mathsf{Bool}} \mathsf{true}\ \rightarrow\ E([a])$. But

$$(f(\mathsf{q}_1([a])) =_{\mathsf{Bool}} \mathsf{true})\ \vee\ (f(\mathsf{q}_1([a])) =_{\mathsf{Bool}} \mathsf{false})$$

since the set Bool is decidable (for a proof see [NPS90], page 177), and hence, by $\vee$-elimination, lemma 1 and a little intuitionistic logic, one gets that

$$(1)\quad T_0(a)\ \vee\ (f(\mathsf{q}_1([a])) =_{\mathsf{Bool}} \mathsf{false})$$

and in an analogous way

$$(2)\quad T_0(a)\ \vee\ (f(\mathsf{q}_2([a])) =_{\mathsf{Bool}} \mathsf{true})$$

Thus, by using distributivity on the conjunction of (1) and (2), one finally obtains

Proposition 3. *For any small set $a \in U_0$ in iTT^{EQ} the following proposition*

$$(\exists f \in \Sigma_{w \in \Omega_0 \times \Omega_0} \; [\widehat{E(\pi_1(w))} \hat{\vee} \; \widehat{E(\pi_2(w))}] =_{\Omega_1} \; [\widehat{\top}] \to \mathsf{Bool}) \\ T_0(a) \vee (f(\mathsf{q}_1([a])) =_{\mathsf{Bool}} \mathsf{false}) \;\wedge\; f(\mathsf{q}_2([a])) =_{\mathsf{Bool}} \mathsf{true})$$

is true.

Now, we proceed by $\exists$-elimination assuming for some proof-term f

$$T_0(a) \vee (f(\mathsf{q}_1([a])) =_{\mathsf{Bool}} \mathsf{false}) \;\wedge\; f(\mathsf{q}_2([a])) =_{\mathsf{Bool}} \mathsf{true})$$

on which we are going to apply $\vee$-elimination to prove the principle of excluded middle for $T_0(a)$.
But, first of all, note that if we assume $T_0(a)$ *true* then $[a] =_{\Omega_0} [\widehat{\top}]$ *true* by lemma 1 and hence

$$\mathsf{q}_1([a]) =_{\Sigma(\Omega_0 \times \Omega_0, \ldots)} \mathsf{q}_1([\widehat{\top}])$$

by the elimination rule of the intensional propositional equality with respect to the proposition

$$\mathsf{q}_1(x) =_{\Sigma(\Omega_0 \times \Omega_0, \ldots)} \mathsf{q}_1(y) \; [x \in \Omega_0, y \in \Omega_0]$$

Thus, $f(\mathsf{q}_1([a])) =_{\mathsf{Bool}} f(\mathsf{q}_1([\widehat{\top}]))$ and in a similar way from the same assumption we can also prove

$$f(\mathsf{q}_2([a])) =_{\mathsf{Bool}} f(\mathsf{q}_2([\widehat{\top}]))$$

Hence, since by the uniqueness of propositional equality proofs UIP we get a proof-term of

$$\pi_2(\mathsf{q}_1([\widehat{\top}])) =_{[\widehat{E([\widehat{\top}])} \hat{\vee} \; \widehat{E([\widehat{\top}])}] =_{\Omega_1} \; [\widehat{\top}]} \pi_2(\mathsf{q}_2([\widehat{\top}]))$$

as $\pi_1(\mathsf{q}_1([\widehat{\top}])) = \langle [\widehat{\top}], [\widehat{\top}] \rangle = \pi_1(\mathsf{q}_2([\widehat{\top}]))$, we conclude by the elimination rule of the propositional equality that

$$\mathsf{q}_1([\widehat{\top}])\rangle =_{\Sigma(\Omega_0 \times \Omega_0, \ldots)} \mathsf{q}_2([\widehat{\top}])$$

and therefore by transitivity

$$f(\mathsf{q}_1([a])) =_{\mathsf{Bool}} f(\mathsf{q}_2([a]))$$

Then if we also assume

$$(f(\mathsf{q}_1([a])) =_{\mathsf{Bool}} \mathsf{false}) \;\wedge\; (f(\mathsf{q}_2([a])) =_{\mathsf{Bool}} \mathsf{true}) \; true$$

we conclude $\mathsf{true} =_{\mathsf{Bool}} \mathsf{false}$ *true*. But we know that we can derive an element of $\neg(\mathsf{true} =_{\mathsf{Bool}} \mathsf{false})$. Hence, under the assumption

$$(f(\mathsf{q}_1([a])) =_{\mathsf{Bool}} \mathsf{false}) \;\wedge\; (f(\mathsf{q}_2([a])) =_{\mathsf{Bool}} \mathsf{true}),$$

the judgement $\neg T_0(a)$ *true* holds. So, from proposition 3, by $\exists$-elimination and by $\vee$-elimination applying $\vee$-introduction when the first disjunct is assumed and using the above argument when the latter disjunct is assumed, we can conclude $(T_0(a) \ \vee \ \neg T_0(a))$ *true* and

$$\Pi_{a \in U_0} \ T_0(a) \ \vee \ \neg T_0(a) \ true$$

To sum up the key points to reproduce the proof of the principle of excluded middle on small sets are the following:

- we use the axiom of choice, by quantifying on

$$\Sigma_{w \in \Omega_0 \times \Omega_0} \ [\widehat{E(\pi_1(w))} \hat{\vee} \ \widehat{E(\pi_2(w))}] =_{\Omega_1} \ [\widehat{\top}]$$

  instead of $\Sigma_{w \in \Omega_0 \times \Omega_0} \ E(\pi_1(w)) \vee \ E(\pi_2(w))$ in order to forget the proof-term of the disjunction and hence we need the second universe to encode

$$E(z) \equiv \mathsf{Id}(\hat{\Omega}_o, z, [\widehat{\top}]) \ [z \in \Omega_0]$$

  and to express at the propositional level when it is true;
- we exhibit a proof-term q_1 by means of the equality rule on the quotient set such that for $a \in U_0$

$$\mathsf{q}_1([a]) \ \in \ \Sigma_{\Omega_0 \times \Omega_0} \ [\widehat{E(\pi_1(w))} \hat{\vee} \ \widehat{E(\pi_2(w))}] =_{\Omega_1} \ [\widehat{\top}]$$

  in order to prove under the assumption $[a] =_{\Omega_0} [\widehat{\top}]$ *true*

$$\mathsf{q}_1([a]) =_{\Sigma_{w \in \Omega_0 \times \Omega_0} \ldots} \mathsf{q}_1([\widehat{\top}]) \ true \quad \text{and} \quad \mathsf{q}_2([a]) =_{\Sigma_{w \in \Omega_0 \times \Omega_0} \ldots} \mathsf{q}_2([\widehat{\top}]) \ true$$

- we use the uniqueness of propositional equality proofs in order to prove

$$\mathsf{q}_1([\widehat{\top}]) =_{\Sigma_{w \in \Omega_0 \times \Omega_0} \ldots} \mathsf{q}_2([\widehat{\top}])$$

In conclusion, if we had type-theoretical rules that make all the rules of iTT^{EQ} admissible such that we can prove that C *true* holds in iTT^{EQ} if and only if there exists a proof element for the proposition C, then we would have a proof element for the proposition $\Pi_{a \in U_0} T_0(a) \ \vee \ \neg T_0(a)$, which is expected to fail for small sets, according to an intuitionistic explanation of connectives.

5 Extensional Quotient Sets in Extensional Set Theory

The proof that effectiveness of quotient sets yields classical logic for small sets can also be done within the extensional version of Martin-Löf's Intuitionistic Type Theory with true judgements, called eTT, extended with the rules for quotient sets, as in Nuprl [Con86], to which we add the effectiveness condition and the introduction and conversion rules of the codes for quotient sets into the first two universes.

About the rules of true judgements, we only recall the case of the set of the extensional propositional equality Eq (see [NPS90]). The formation, introduction, elimination and conversion rules are the following:

Extensional Equality set

$$\text{Eq)}\ \frac{C\ set \quad c \in C \quad d \in C}{\mathsf{Eq}(C, c, d)\ set} \qquad \text{I-Eq)}\ \frac{c \in C}{\mathsf{eq_C}(c) \in \mathsf{Eq}(C, c, c)}$$

$$\text{E-Eq)}\ \frac{p \in \mathsf{Eq}(C, c, d)}{c = d \in C} \qquad \text{C-Eq)}\ \frac{p \in \mathsf{Eq}(C, c, d)}{p = \mathsf{eq_C}(\mathsf{c}) \in \mathsf{Eq}(C, c, d)}$$

In particular the elimination rule yields the admissibility of the following rule on true judgements:

$$\frac{\mathsf{Eq}(A, a, b)\ true}{a = b \in A}$$

We extend eTT with the rules of extensional quotient sets:

Quotient set

$$\frac{\begin{array}{l} R(x, y)\ set\ [x \in A, y \in A] \\ c_1 \in R(x, x)[x \in A], \qquad c_2 \in R(y, x)[x \in A, y \in A, z \in R(x, y)] \\ c_3 \in R(x, z)[x \in A, y \in A, z \in A, w \in R(x, y), w' \in R(y, z)] \end{array}}{A/R\ set}$$

I-quotient

$$\frac{a \in A \quad A/R\ set}{[a] \in A/R}$$

eq-quotient

$$\frac{a \in A \quad b \in A \quad d \in R(a, b)}{[a] = [b] \in A/R}$$

E-quotient

$$\frac{s \in A/R \quad l(x) \in L([x])\ [x \in A] \quad l(x) = l(y) \in L([x])\ [x \in A, y \in A, d \in R(x, y)]}{\mathsf{Q}(l, s) \in L(s)}$$

C-quotient

$$\frac{a \in A \quad l(x) \in L([x])\ [x \in A] \quad l(x) = l(y) \in L([x])\ [x \in A, y \in A, d \in R(x, y)]}{\mathsf{Q}(l, [a]) = l(a) \in L([a])}$$

Then we make extensional quotients effective through the following condition in terms of true judgements:

Effectiveness condition

$$\frac{a \in A \quad b \in A \quad [a] = [b] \in A/R}{R(a, b)\ true}$$

We also add the codes of quotient sets in the introduction rules of the first two universes and their corresponding conversion rules, as in section 3. Note that, like for the intensional propositional equality set, the introduction of equality on quotient sets yields the admissibility of the following rule:

$$\frac{a \in A \quad b \in A \quad R(a,b)\ true}{[a] = [b] \in A/R}$$

This extension of eTT, called eTT^{EQ}, is consistent, because there exists an interpretation in classical set theory (ZFC) with two strongly inaccessible cardinals. In the presence of the extensional propositional equality set, the rules for intensional quotient sets become equivalent to those of extensional quotient sets and the same holds with respect to the effectiveness condition. So, we can reproduce in eTT^{EQ} the proof of the previous section to derive

$$\Pi_{a \in U_0} T_0(a) \ \vee \ \neg T_0(a))\ true$$

which is expected to fail for small sets.

Note that this proof can not be recovered in the extensional version of set theory with effective quotient sets restricted to mono equivalence relations, that is equivalence relations inhabited by at most one proof. This kind of quotients is operating in the extensional type theory of Heyting pretoposes [Mai97] and also of toposes [Mai98], where even effectiveness can be type-theoretically expressed.

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