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Geometry of D1–D5–P bound states

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Abstract

Supersymmetric solutions of 6D supergravity (with two translation symmetries) can be written as a hyper-Kähler base times a 2D fiber. The subset of these solutions which correspond to true bound states of D1–D5–P charges give microstates of the 3-charge extremal black hole. To understand the characteristics shared by the bound states we decompose known bound state geometries into base–fiber form. The axial symmetry of the solutions make the base Gibbons–Hawking. We find the base to be actually 'pseudo-hyper-Kähler': The signature changes from $(4, 0)$ to $(0, 4)$ across a hypersurface. 2-charge D1–D5 geometries are characterized by a 'central curve' S^1; the analogue for 3-charge appears to be a hypersurface that for our metrics is an orbifold of $S^1 \times S^3$.

1. Introduction

The black hole information puzzle [1] suggests that we lack a key factor in our understanding of large dense collections of matter. Some computations in string theory suggest that the traditional picture of a black hole as 'empty space with a central singularity' might be incorrect, and the degrees of freedom accounting for the black hole entropy are distributed throughout the hole. The simplest object with entropy is the 2-charge extremal system, which can be realized as a bound state of D1 and D5 branes. In this case it was found that the microstates are not point-like but have a certain typical 'size'. Different microstates have different geometries; none has a horizon or singularity but if we draw a surface bounding the typical state then the area A of this surface satisfies $\frac{A}{4G} \sim S$, where $S = 2\sqrt{2}\pi\sqrt{n_1 n_5}$ is the microscopic entropy of the 2-charge system [2,3].

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We can wonder if a similar 'swelling up' would happen for the 3-charge extremal D1–D5–P states. The microscopic entropy of the system is $S_{\text{micro}} = 2\pi\sqrt{n_1 n_5 n_p}$. This time the naive geometry is a black hole whose horizon area satisfies $S_{\text{Bek}} \equiv \frac{A}{4G} = S_{\text{micro}}$ [4]. If the microstates 'swelled up' to fill this horizon then there would be no information puzzle—the black hole microstates would just be large 'fuzzballs' that radiated like any other ball of matter. No individual state would have a horizon; the horizon would only be an effective construct arising upon coarse graining over microstates.

In [5–7] the geometries for some 3-charge microstates were constructed, and it was found that these geometries were regular, with no horizon. The 'throat' of the 3-charge solution, instead of ending in a horizon, ended in a 'cap', just like the 2-charge geometries. But these geometries were a small subset of all the states of the 3-charge system, and were also not very generic states; in particular, they carried a significant amount of rotation.

The generic state is not expected to be well-described by a classical geometry. Even in the 2-charge case, where the geometries were generated by a vibrating string profile, the string profile could be taken to be classical only when we take a high occupation number for each harmonic, so that eigenstates can be replaced by coherent states and a classical configuration achieved. But in the 2-charge case the classical geometries helped deduce the size of the generic microstate, and we expect to get significant insight from classical geometries in the 3-charge case as well. The extremal solutions we seek are BPS and thus the geometries preserve supersymmetry. The D1–D5–P system is obtained by compactifying IIB string theory to $M_{4,1} \times S^1 \times T^4$. Dimensionally reducing on T^4, we get supersymmetric solutions in 6D. In the classical limit of large charges we expect translation invariance along the time directions t and the S^1 direction y [2].

In [8] the general class of supersymmetric 6D supergravity geometries (with these translation symmetries) was described. All such solutions can be written as a 4-dimensional hyper-Kähler base with a 2-dimensional (t, y) fiber over this base. In an interesting set of recent papers, such formulations have been used to construct large families of 3-charge BPS solutions [9–12].

But generic solutions constructed this way include regular ones as well as ones with pathologies (horizons, singularities, closed timelike curves). They include true bound states of D1–D5–P charges as well as superpositions of such bound states. Unlike the situation with D1–D5, we do not have as yet a way to isolate the solutions that describe the true bound states of D1–D5–P, which are the ones that describe the microstates of the 3-charge black hole.

To get some insight into the characteristics of bound states, in this paper we return to the geometries constructed in [5,6]. These are known to describe true bound states, since they were constructed by starting with 2-charge D1–D5 bound states and doing spectral flow to add the P charge; a further class was obtained by applying S, T dualities to these geometries so that we again get bound states. We cast these solutions in the form of [8], identifying the base and the fiber. From the result we can immediately make some observations. A hyper-Kähler metric is usually Euclidean, with signature $(4, 0)$. We find that the base for our geometries is actually 'pseudo-hyper-Kähler': the signature on the outer region is $(4, 0)$ but inside a certain boundary it changes to $(0, 4)$.[1]

The place where the signature changes is given by the intersection of a surface $f = 0$ in 6D with the 4D base. In the 2-charge D1–D5 case the $f = 0$ condition defined the 'central curve' of the KK-monopole tube which characterized the geometry: different shapes of this curve gave different bound states. This motivates us to study the $f = 0$ surface in the 3-charge case as well.

[1] The full 6D metric, however, retains signature $(5, 1)$ everywhere.

We find that this surface (at $t = \text{const}$) is an orbifold of $S^3 \times S^1$ by an orbifold group Z_k; the group acts without fixed points so the $f = 0$ surface is smooth. When the P charge vanishes the S^3 collapses to zero and we get the S^1 of the 2-charge D1–D5 system.

The geometries of [5,6] are regular everywhere except for possible ALE type singularities arising from fixed points of an orbifold action. In the above base–fiber split the metric of the base turns out to have certain orbifold singularities, and we investigate how the fiber at these locations behaves so as to yield the singularities of the full 6D metric. We end with some conjectures on the role of the $f = 0$ surface: since different shapes of the $f = 0$ curve described all different bound states of the 2-charge D1–D5 system, it might be that different shapes of the $f = 0$ surface in the 3-charge case characterize all different D1–D5–P bound states.

2. Writing the metrics as base × fiber

Consider Type IIB string theory on $M_{5,1} \times S^1 \times T^4$. Let the length of S^1 be $2\pi R$ and the volume of T^4 be V. Wrap n_5 D5 branes on $S^1 \times T^4$ and n_1 D1 branes on S^1. In addition, let there be n_p units of momentum along S^1. We are interested in the geometries created by the BPS bound states of these charges.

Dimensionally reducing on T^4 we get supersymmetric solutions in 6D. In our solutions we can choose moduli such that the solution can be represented as a solution of minimal supergravity in 6D (in particular, the dilaton becomes a constant). Further, we expect that in the classical limit which we consider the solutions will be translationally invariant in the time direction t and the S^1 direction y [2].

2.1. General form of the 6D metrics

It was shown in [8] that the most general supersymmetric solution to minimal supergravity in 6 dimensions with translation symmetry along

$$u = t + y, \qquad v = t - y \tag{2.1}$$

can be written as

$$ds^2 = -H^{-1}\big(dv + \sqrt{2}\beta\big)\left(du + \sqrt{2}\omega + \frac{F}{2}\big(dv + \sqrt{2}\beta\big)\right) + H h_{mn}\, dx^m\, dx^n. \tag{2.2}$$

Here x^m ($m = 1, \dots, 4$) are coordinates in the noncompact spatial directions. H and F are functions and β and ω are 1-forms on this 4-dimensional space. $h_{mn}\, dx^m\, dx^n$ gives a hyper-Kähler metric on this 4-dimensional space; we call this the 'base metric'.

To write the field equations satisfied by these variables, define the self-dual 2-form on the base

$$\mathcal{G}^+ = H^{-1}\left(\frac{d\omega + \star d\omega}{2} + \frac{F}{2}\, d\beta\right), \tag{2.3}$$

where the $\star$ operation is defined with respect to the hyper-Kähler base metric. Then the equations of motion are equivalent to the following nonlinear system of equations for H, F, β and ω:

$$d\beta = \star d\beta, \tag{2.4}$$

$$d\mathcal{G}^+ = 0, \tag{2.5}$$

$$d \star dH + d\beta\, \mathcal{G}^+ = 0, \tag{2.6}$$

$$d \star dF + \big(\mathcal{G}^+\big)^2 = 0. \tag{2.7}$$

2.2. Writing 3-charge solutions in the general form

In [5,6] certain solutions of IIB supergravity were constructed carrying 3 charges, Q_1, Q_5, Q_p, and 2 angular momenta (parametrized by γ_1, γ_2). When the Q_1 and Q_5 charges are set equal ($Q_1 = Q_5 = Q$), and the moduli at infinity chosen appropriately, the dilaton vanishes and the 3-charge solution reduces to a solution of minimal supergravity. The resulting 6D metric is

$$\begin{aligned} ds^2 = {} & -\frac{1}{h}\big(dt^2 - dy^2\big) + \frac{Q_p}{hf}(dt-dy)^2 + hf\left(\frac{dr^2}{r^2+(\gamma_1+\gamma_2)^2\eta} + d\theta^2\right) \\ & + h\left(r^2 + \gamma_1(\gamma_1+\gamma_2)\eta - \frac{Q^2(\gamma_1^2-\gamma_2^2)\eta\cos^2\theta}{h^2f^2}\right)\cos^2\theta\, d\psi^2 \\ & + h\left(r^2 + \gamma_2(\gamma_1+\gamma_2)\eta + \frac{Q^2(\gamma_1^2-\gamma_2^2)\eta\sin^2\theta}{h^2f^2}\right)\sin^2\theta\, d\phi^2 \\ & + \frac{Q_p(\gamma_1+\gamma_2)^2\eta^2}{hf}\big(\cos^2\theta\, d\psi + \sin^2\theta\, d\phi\big)^2 \\ & - \frac{2Q}{hf}\big(\gamma_1\cos^2\theta\, d\psi + \gamma_2\sin^2\theta\, d\phi\big)(dt-dy) \\ & - \frac{2Q(\gamma_1+\gamma_2)\eta}{hf}\big(\cos^2\theta\, d\psi + \sin^2\theta\, d\phi\big)\, dy, \end{aligned} \tag{2.8}$$

with

$$\begin{aligned} & Q_p = -\gamma_1\gamma_2, \qquad \eta = \frac{Q}{Q+2Q_p}, \\ & f = r^2 + (\gamma_1+\gamma_2)\eta\big(\gamma_1\sin^2\theta + \gamma_2\cos^2\theta\big), \\ & h = 1 + \frac{Q}{f}. \end{aligned} \tag{2.9}$$

These geometries are dual to specific microstates of the D1–D5–P system. Thus the angular momenta γ_1, γ_2 take specific values; for these values the geometry (2.8) has no horizon, no closed time-like curves and the only singularities are orbifold singularities (which can be understood as degenerations of smooth geometries). The values of γ_1, γ_2 fall into two discrete series. Let

$$a = \frac{Q}{R} \tag{2.10}$$

(R is the radius of the y circle). The first series is

$$\gamma_1 = -an, \qquad \gamma_2 = a\left(n + \frac{1}{k}\right), \quad n, k \in \mathbb{Z}. \tag{2.11}$$

The second series corresponds to geometries obtained from the first by S, T dualities which interchange the D1 and P charges. These geometries have

$$\gamma_1 = -a\frac{k}{n_5(kn+1)}, \qquad \gamma_2 = a\frac{1}{n_5 n}, \quad n, k \in \mathbb{Z}, \quad n_5 \in \mathbb{N}. \tag{2.12}$$

The metric (2.8) can be written in the form (2.2), with the following values for H, F, β, ω and h_{mn}:

$$H = h, \qquad \frac{F}{2} = -\frac{Q_p}{f},$$

$$\sqrt{2}\beta = \frac{Q}{f}(\gamma_1+\gamma_2)\eta\left(\cos^2\theta\, d\psi + \sin^2\theta\, d\phi\right),$$
$$\sqrt{2}\omega = \frac{Q}{f}\left[\left(2\gamma_1 - (\gamma_1+\gamma_2)\eta\left(1-2\frac{Q_p}{f}\right)\right)\cos^2\theta\, d\psi + \left(2\gamma_2 - (\gamma_1+\gamma_2)\eta\left(1-2\frac{Q_p}{f}\right)\right)\sin^2\theta\, d\phi\right], \tag{2.13}$$

$$\begin{aligned} h_{mn}\, dx^m\, dx^n = {} & f\left(\frac{dr^2}{r^2+(\gamma_1+\gamma_2)^2\eta} + d\theta^2\right) \\ & + \frac{1}{f}\Big[\big[r^4 + r^2(\gamma_1+\gamma_2)\eta\big(2\gamma_1 - (\gamma_1-\gamma_2)\cos^2\theta\big) \\ & + (\gamma_1+\gamma_2)^2\gamma_1^2\eta^2\sin^2\theta\big]\cos^2\theta\, d\psi^2 \\ & + \big[r^4 + r^2(\gamma_1+\gamma_2)\eta\big(2\gamma_2 + (\gamma_1-\gamma_2)\sin^2\theta\big) \\ & + (\gamma_1+\gamma_2)^2\gamma_2^2\eta^2\cos^2\theta\big]\sin^2\theta\, d\phi^2 \\ & - 2\gamma_1\gamma_2(\gamma_1+\gamma_2)^2\eta^2\sin^2\theta\cos^2\theta\, d\psi\, d\phi\Big]. \end{aligned} \tag{2.14}$$

2.3. *The base metric in Gibbons–Hawking form*

Since the geometry (2.8) satisfies the equations of motion, the base metric $h_{mn}\, dx^m\, dx^n$ (Eq. (2.14)) should turn out to be hyper-Kähler. In fact this base metric is more special—it has two commuting isometries corresponding to translations along ϕ and ψ. Then, according to a theorem of [13], at least one linear combination of the two isometries is tri-holomorphic (i.e., commutes with the 3 complex structures of the hyper-Kähler space) and $h_{mn}\, dx^m\, dx^n$ should be writable in Gibbons–Hawking form

$$h_{mn}\, dx^m\, dx^n = H_2^{-1}(d\tau + \chi\, d\tilde{\phi})^2 + H_2\, ds_3^2. \tag{2.15}$$

Here τ and $\tilde{\phi}$ are linear combinations of ϕ and ψ, ds_3^2 is the flat metric on $\mathbb{R}^3$. H_2 is a function on $\mathbb{R}^3$ harmonic in the metric ds_3^2 and $\chi\, d\tilde{\phi}$ is a 1-form on $\mathbb{R}^3$ that satisfies

$$dH_2 = \star_3 d(\chi\, d\tilde{\phi}) \tag{2.16}$$

($\star_3$ is the Hodge dual with respect to the metric ds_3^2).

We can cast (2.14) in the form (2.15) if we choose

$$\tau = \psi - \phi, \qquad \tilde{\phi} = \psi + \phi. \tag{2.17}$$

Then one finds, starting from (2.14),

$$H_2 = \frac{4f}{(r^2+(\gamma_1+\gamma_2)^2\eta\cos^2\theta)(r^2+(\gamma_1+\gamma_2)^2\eta\sin^2\theta)}, \tag{2.18}$$

$$\chi = \frac{\gamma_1}{\gamma_1+\gamma_2}\frac{r^2\cos 2\theta + (\gamma_1+\gamma_2)^2\eta\cos^2\theta}{r^2+(\gamma_1+\gamma_2)^2\eta\cos^2\theta} + \frac{\gamma_2}{\gamma_1+\gamma_2}\frac{r^2\cos 2\theta - (\gamma_1+\gamma_2)^2\eta\sin^2\theta}{r^2+(\gamma_1+\gamma_2)^2\eta\sin^2\theta}, \tag{2.19}$$

$$ds_3^2 = \frac{(r^2+(\gamma_1+\gamma_2)^2\eta\cos^2\theta)(r^2+(\gamma_1+\gamma_2)^2\eta\sin^2\theta)}{4}\left(\frac{dr^2}{r^2+(\gamma_1+\gamma_2)^2\eta}+d\theta^2\right) + \frac{r^2(r^2+(\gamma_1+\gamma_2)^2\eta)}{4}\sin^2\theta\cos^2\theta\, d\tilde{\phi}^2. \tag{2.20}$$

The metric ds_3^2 can be brought to the manifestly flat form

$$ds_3^2 = d\tilde{r}^2 + \tilde{r}^2\, d\tilde{\theta}^2 + \tilde{r}^2\sin^2\tilde{\theta}\, d\tilde{\phi}^2 \tag{2.21}$$

by the change of coordinates

$$\tilde{r} = \frac{r^2+(\gamma_1+\gamma_2)^2\eta\sin^2\theta}{4}, \tag{2.22}$$

$$\tilde{r}\cos^2\frac{\tilde{\theta}}{2} = \frac{r^2}{4}\cos^2\theta. \tag{2.23}$$

It is helpful to note several algebraic relations following from this coordinate change. We have

$$\tilde{r}\sin^2\frac{\tilde{\theta}}{2} = \frac{r^2+(\gamma_1+\gamma_2)^2\eta}{4}\sin^2\theta. \tag{2.24}$$

Defining the vector

$$\vec{c} \equiv (0,0,c), \qquad c \equiv \left(\frac{\gamma_1+\gamma_2}{2}\right)^2\eta, \tag{2.25}$$

we get

$$\tilde{r}_c \equiv |\tilde{x}+\vec{c}| = \frac{r^2+(\gamma_1+\gamma_2)^2\eta\cos^2\theta}{4}. \tag{2.26}$$

Then

$$\frac{(\tilde{r}+\tilde{r}_c-c)(\tilde{r}+\tilde{r}_c+c)}{2\tilde{r}} = \tilde{r}+\tilde{r}_c+c\cos\tilde{\theta},$$
$$\cos\tilde{\theta}(\tilde{r}+\tilde{r}_c)+c = \frac{\tilde{r}_c-\tilde{r}}{c}(\tilde{r}+\tilde{r}_c+c\cos\tilde{\theta}), \tag{2.27}$$

$$f = 4\frac{\tilde{r}\gamma_1+\tilde{r}_c\gamma_2}{\gamma_1+\gamma_2}, \qquad \cos 2\theta = \frac{\tilde{r}_c-\tilde{r}}{c}. \tag{2.28}$$

Using the coordinates $\tilde{r}$ and $\tilde{\theta}$ the Gibbons–Hawking potential H_2 and the magnetic potential χ become

$$H_2 = \frac{1}{\gamma_1+\gamma_2}\left(\frac{\gamma_2}{\tilde{r}}+\frac{\gamma_1}{\tilde{r}_c}\right),$$
$$\chi = \frac{\gamma_2}{\gamma_1+\gamma_2}\cos\tilde{\theta} + \frac{\gamma_1}{\gamma_1+\gamma_2}\frac{\tilde{r}\cos\tilde{\theta}+c}{\tilde{r}_c}. \tag{2.29}$$

In this form H_2 is explicitly harmonic.

It was shown in [8] that the 6D metric over a Gibbons–Hawking base can be expressed through a set of harmonic functions H_i ($i = 1,\ldots,6$) on ds_3^2. In Appendix A we find these harmonic functions H_i for our geometries.

3. The surface $f = 0$

3.1. The 'pseudo-hyper-Kähler' geometry of the base

The field equations for supersymmetric solutions require the base to be a 4D hyper-Kähler manifold. A conventional hyper-Kähler manifold has signature $(4, 0)$; i.e., $(++++)$. We can get a similar geometry with signature $(0, 4)$ by simply reversing the sign of the 4D metric. But consider Eq. (2.29) for H_2. From (2.11), (2.12) we see that γ_1 and γ_2 have opposite signs. This means that H_2 has opposite signs near $\tilde{r}, \tilde{r}_c$. For $|\gamma_2| > |\gamma_1|$ we find that the change of sign happens over a 2D surface in the flat space ds_3^2 in (2.15); this surface is topologically a S^2 which surrounds the point $\vec{c} = (0, 0, c)$.[2] For $|\gamma_1| > |\gamma_2|$ the S^2 surrounds the origin $(0, 0, 0)$. From (2.15) we see that the base metric $h_{mn}\, dx^m\, dx^n$ has signature $(4, 0)$ if we look at the region outside this S^2 and signature $(0, 4)$ inside. Since this is not the nature of a traditional hyper-Kähler manifold we call the base 'pseudo-hyper-Kähler'.

From the form (2.28) of f we see that

$$H_2 = \frac{f}{4\tilde{r}\tilde{r}_c}, \tag{3.1}$$

so that H_2 changes sign when f passes through zero. Looking at the complete metric (2.2) and recalling that

$$H = h = 1 + \frac{Q}{f}, \tag{3.2}$$

we see that the 6D metric does not change signature; H changes sign at the same place that the base metric changes sign.[3] This is of course expected since the 6D geometry has signature $(5, 1)$ everywhere.

The condition $f = 0$ will generically give a 5D surface in 6D. The place where the base metric changes sign is the intersection of the surface $f = 0$ with the base. In the following we will often describe the $f = 0$ surface by giving the 4D section obtained by restricting $f = 0$ to constant t.

In the 2-charge D1–D5 geometries the surface $f = 0$ degenerates to a closed curve S^1. This curve can be regarded as the center of a 'KK-monopole tube' (i.e., a KK monopole $\times S^1$) [14], and different shapes of the S^1 give the set of different 2-charge geometries [2]. It thus appears interesting to investigate the $f = 0$ surface in more detail for the 3-charge case, which we do below.

3.2. Topology of the surface $f = 0$

We have

$$f = r^2 + (\gamma_1 + \gamma_2)\eta\big(\gamma_1 \sin^2\theta + \gamma_2 \cos^2\theta\big). \tag{3.3}$$

For generic values of γ_1 and γ_2, the equation $f = 0$ defines a hypersurface with no boundary. We are interested in the possible significance of this surface in terms of quantities in the dual CFT, so we take the limit $R \gg \sqrt{Q}$ which gives a large AdS type region and hence allows us to extract

[2] This is easy to see from the electrostatic analogy where we have charges γ_2, γ_1 placed at $\tilde{r}, \tilde{r}_c$.

[3] It was shown in [5,6] that $Q + f > 0$ everywhere.

the dual of the CFT in the near horizon limit $r \ll \sqrt{Q}$ [15].[4] We see that in this limit

$$Q_p \ll Q, \quad \eta \to 1. \tag{3.4}$$

3.2.1. The geometries (2.11)

Consider first the family (2.11). In this near horizon limit of (2.8) the change of variables

$$\begin{aligned}
&\rho = k\frac{r}{a}, \qquad \hat{t} = \frac{1}{k}\frac{t}{R}, \qquad \hat{y} = \frac{1}{k}\frac{y}{R},\\
&\hat{\psi} = \psi - \frac{\gamma_1}{a}\frac{t}{R} - \frac{\gamma_2}{a}\frac{y}{R} = \psi + n\frac{t}{R} - \left(n + \frac{1}{k}\right)\frac{y}{R},\\
&\hat{\phi} = \psi - \frac{\gamma_2}{a}\frac{t}{R} - \frac{\gamma_1}{a}\frac{y}{R} = \phi - \left(n + \frac{1}{k}\right)\frac{t}{R} + n\frac{y}{R},
\end{aligned} \tag{3.5}$$

gives the metric

$$\frac{ds^2_{\text{n.h.}}}{Q} = -\left(\rho^2 + 1\right)d\hat{t}^2 + \frac{d\rho^2}{\rho^2 + 1} + \rho^2\, d\hat{y}^2 + d\theta^2 + \cos^2\theta\, d\hat{\psi}^2 + \sin^2\theta\, d\hat{\phi}^2, \tag{3.6}$$

which is locally but not globally equivalent to $AdS_3 \times S^3$. Indeed, the change of variables (3.5) induces the following identifications on $\hat{y}$, $\hat{\psi}$ and $\hat{\phi}$:

$$\left(\frac{\hat{y}}{R}, \hat{\psi}, \hat{\phi}\right) \sim \left(\frac{\hat{y}}{R}, \hat{\psi}, \hat{\phi}\right) + 2\pi m\left(\frac{1}{k}, -\frac{1}{k}, 0\right), \qquad m \in \mathbb{Z}, \tag{3.7}$$

and for $k > 1$ the space (3.6) is a $\mathbb{Z}_k$ orbifold of $AdS_3 \times S^3$.

Consider now the hypersurface $f = 0$ in this space. The topology of the surface is different in the two cases: (i) $n > 0$, which gives $\gamma_2 > 0$, $\gamma_1 < 0$, (ii) $n < 0$, which gives $\gamma_2 < 0$ and $\gamma_1 > 0$.

(i) The case $n > 0$. Solving the equation $f = 0$ for r in terms of θ, one sees that in the (r, θ) plane the surface is simply an interval

$$\theta \in I_{n>0} = [\bar{\theta}, \pi/2], \tag{3.8}$$

where

$$\bar{\theta} = \tan^{-1}\left(-\frac{\gamma_2}{\gamma_1}\right)^{1/2} = \tan^{-1}\left(\frac{kn+1}{kn}\right)^{1/2} \tag{3.9}$$

corresponds to $r = 0$. Over this interval one has the three orthogonal directions $\hat{y}$, $\hat{\psi}$ and $\hat{\phi}$. The length of the circle $\hat{\phi}$ never vanishes over the interval (3.8) and thus it just gives an overall S^1 factor. The cycle $\hat{y}$ shrinks at one end of the interval ($r = 0$) and $\hat{\psi}$ shrinks at the other end ($\theta = \pi/2$), so that they form, together with the interval itself, a sphere S^3.[5] The identifications (3.7) act on this sphere without fixed points since the $\hat{y}$ and $\hat{\psi}$ circles do not shrink at the same time. Thus the resulting orbifold space is the (smooth) Lens space $S^3/\mathbb{Z}_k$. We conclude that in the case $n > 0$ the hypersurface $f = 0$ is topologically $S^1 \times (S^3/\mathbb{Z}_k)$.

[4] We are interested only in the topology of this surface, and we do not expect this topology to be altered by continuous changes of the metric; in particular, if we change the asymptotically AdS geometry to an asymptotically flat geometry then there will be deformations of this surface but not a change of topology.

[5] This follows by comparison with the metric $ds^2 = d\theta^2 + \cos^2\theta\, d\psi^2 + \sin^2\theta\, d\phi^2$ which gives S^3 for $0 < \theta < \frac{\pi}{2}$, $0 < \psi < 2\pi$, $0 < \phi < 2\pi$. We easily check that the surface is smooth at the ends of the interval (3.8) by inspection of the defining equation $f = 0$.

(ii) The case $n<0$. The interval in the (r,θ) plane is

$$\theta \in I_{n<0} = [0, \bar\theta]. \tag{3.10}$$

This time the $\hat\psi$ circle remains of nonzero size everywhere and provides an S^1 factor. At $\theta = 0$ it is the $\hat\phi$ circle that shrinks, and at $\theta = \bar\theta$ the $\hat y$ circle shrinks. We thus get $S^1 \times S^3$, with the S^1 spanned by $\hat\psi$ and the S^3 made up by $\hat y$, $\hat\phi$ and the interval $I_{n<0}$. This surface is to be divided by the orbifold action (3.7) which this time acts on both the S^1 and the S^3. Noting that the $\hat\psi$ circle does not shrink anywhere we see that the orbifold action has no fixed points, and the resulting hypersurface is a smooth orbifold $(S^1 \times S^3)/\mathbb{Z}_k$.

3.2.2. The geometries (2.12)

A similar analysis can be done for the $f=0$ surface for the family of geometries with γ_1, γ_2 given by (2.12). In the near horizon limit these geometries reduce again to the $AdS_3 \times S^3$ form (3.6) after the coordinate transformation

$$\rho = n_5 n(kn+1)\frac{r}{a}, \qquad \hat t = \frac{1}{n_5 n(kn+1)}\frac{t}{R}, \qquad \hat y = \frac{1}{n_5 n(kn+1)}\frac{y}{R},$$
$$\hat\psi = \psi + \frac{k}{n_5(kn+1)}\frac{t}{R} - \frac{1}{n_5 n}\frac{y}{R}, \qquad \hat\phi = \phi - \frac{1}{n_5 n}\frac{t}{R} + \frac{k}{n_5(kn+1)}\frac{y}{R}. \tag{3.11}$$

We see from (3.11) that these coordinates are subject to the identifications

$$\left(\frac{\hat y}{R}, \hat\psi, \hat\phi\right) \sim \left(\frac{\hat y}{R}, \hat\psi, \hat\phi\right) + 2\pi m\left(\frac{1}{n_5 n(kn+1)}, -\frac{1}{n_5 n}, \frac{k}{n_5(kn+1)}\right), \quad m \in \mathbb{Z}. \tag{3.12}$$

We thus get a $\mathbb{Z}_{n_5 n(kn+1)}$ orbifold group; let ω be the generator of this group. We examine the cases $n>0$ and $n<0$ separately.

(i) The case $n>0$. The intersection of the surface $f=0$ with the (r,θ) plane is again the interval $I_{n>0}$ defined in (3.8). As in the previous case, the circles $\hat y$ and $\hat\psi$ fibered over $I_{n>0}$ form an S^3 and $\hat\phi$ gives a finite size S^1. However, the orbifold group $\mathbb{Z}_{n_5 n(kn+1)}$ now acts simultaneously on the S^3 and the S^1, according to (3.12), and the $f=0$ surface is an orbifold $(S^3 \times S^1)/\mathbb{Z}_{n_5 n(kn+1)}$ with orbifold action given by (3.12).

This orbifold action again has no fixed points. To see this, consider first the place where the $\hat y$ circle shrinks to zero. To get a fixed point we would need to get a trivial action on the two nonshrinking circles $\hat\psi$, $\hat\phi$. This means that we must take an element of the orbifold group ω^s, where $\frac{s}{n_5 n}$ and $\frac{ks}{n_5(kn+1)}$ are both integers. A little thought then shows that $\frac{s}{n_5 n(kn+1)}$ must be integral as well, so that we get only the trivial element of the orbifold group. A similar analysis shows that there is no fixed point when the $\hat\psi$ circle shrinks. Thus there is no fixed point of the orbifold group and the surface $f=0$ is smooth.

(ii) The case $n<0$. As in the previous case, the $f=0$ surface is an orbifold $(S^3 \times S^1)/\mathbb{Z}_{n_5 n(kn+1)}$, but this time the S^3 is made up by the circles $\hat y$ and $\hat\phi$ fibered over the interval $I_{n<0}$ of Eq. (3.10) and the S^1 is generated by $\hat\psi$. The orbifold action is given by (3.12). An analysis similar to the one for case (i) shows that there are no fixed points of the orbifold action and the surface $f=0$ is smooth.

4. Regularity of the 3-charge solution

It was found in [5,6] that the metric (2.8) is regular everywhere apart from possible orbifold singularities. We wish to examine this singularity structure from the point of view of the base–

fiber decomposition (2.2) of the metric; this might help us to understand how to find more general bound states using a formalism like [8].

4.1. *The geometries* (2.11)

We first look at the geometries with γ_1, γ_2 given by (2.11). We know from [6] that these are completely smooth spaces for $k = 1$ while for $k > 1$ they have an orbifold singularity along an S^1. (Transverse to the S^1 this is an ALE type singularity which arise from the collision of k KK monopoles [14].)

Let us first start by looking at the base metric. As can be seen from the form of the Gibbons–Hawking potential H_2 in (2.29), potential singularities of the base can occur at: (i) $\tilde{r} = 0$ (i.e., $r = 0$ and $\theta = 0$); (ii) $\tilde{r}_c = 0$ (i.e., $r = 0$ and $\theta = \pi/2$); (iii) $\gamma_2\tilde{r}_c + \gamma_1\tilde{r} = 0$ (i.e., $f = 0$). We will examine cases (i) and (ii) here and return to (iii) in Section 4.3 below.

4.1.1. *The* $\tilde{r} \to 0$ *limit*

We find that the base space metric has in general an orbifold singularity at $\tilde{r} = 0$. This is seen as follows. Around the point $\tilde{r} = 0$ define the new coordinates r', θ', ψ' and ϕ' as follows[6]:

$$r'^2 = 4\frac{\gamma_2}{\gamma_1+\gamma_2}\tilde{r} = 4(kn+1)\tilde{r}, \qquad \theta' = \frac{\tilde{\theta}}{2},$$
$$\psi' = \frac{\gamma_1+\gamma_2}{\gamma_2}\psi = \frac{1}{kn+1}\psi, \qquad \phi' = -\frac{\gamma_1}{\gamma_2}\psi + \phi = \frac{kn}{kn+1}\psi + \phi. \tag{4.1}$$

In these coordinates the base metric is

$$h_{mn}\,dx^m\,dx^n \approx dr'^2 + r'^2\,d\theta'^2 + r'^2\cos^2\theta'\,d\psi'^2 + r'^2\sin^2\theta'\,d\phi'^2, \tag{4.2}$$

which is the form of the metric for flat $\mathbb{R}^4$. However, the new angular coordinates ψ' and ϕ' are subject to the identifications

$$(\psi',\phi') \sim (\psi',\phi') + 2\pi n_1\left(\frac{1}{kn+1}, 1 - \frac{1}{kn+1}\right) + 2\pi n_2(0,1), \qquad n_1, n_2 \in \mathbb{Z}, \tag{4.3}$$

so that the base metric around $\tilde{r} = 0$ is equivalent to an orbifold $\mathbb{R}^4/\mathbb{Z}_{kn+1}$, where the group $\mathbb{Z}_{kn+1}$ acts on both the ψ' and ϕ' cycles according to the identifications (4.3). Since at $\tilde{r} = 0$ both these cycles shrink to a point, $\tilde{r} = 0$ is a fixed point for the orbifold action and the space has an ALE type singularity.

In the total 6D space, however, there is no singularity at $\tilde{r} = 0$. To see this, consider the behavior of H, F, β and ω around $\tilde{r} = 0$:

$$H \approx 1 + \frac{Q}{\gamma_2(\gamma_1+\gamma_2)\eta}, \qquad \frac{F}{2} \approx \frac{\gamma_1}{(\gamma_1+\gamma_2)\eta},$$
$$\sqrt{2}\beta \approx -\sqrt{2}\omega \approx \frac{Q}{\gamma_2}\,d\psi \equiv \sqrt{2}\,\beta_\psi\,d\psi. \tag{4.4}$$

[6] For definiteness we are assuming here that $n > 0$ and thus $\gamma_2 > 0$ and $\gamma_1 < 0$. In the case $n < 0$ the appropriate definition of r'^2 has a negative sign compared to the definition in (4.1) and the base space has signature $(0,4)$ around $\tilde{r} = 0$.

The 6D metric thus has the form

$$ds^2 \approx \left(1+\frac{k^2 Q}{a^2\eta(kn+1)}\right)^{-1}\left[\left(-dt^2+dy'^2\right)+\frac{kn}{\eta}(dt-dy')^2\right] + \left(1+\frac{k^2 Q}{a^2\eta(kn+1)}\right)\left[dr'^2+r'^2\,d\theta'^2+r'^2\cos^2\theta'\,d\psi'^2+r'^2\sin^2\theta'\,d\phi'^2\right], \tag{4.5}$$

where we have defined

$$\frac{y'}{R}=\frac{y}{R}-\frac{\sqrt{2}\beta_\psi}{R}\psi=\frac{y}{R}-\frac{k}{kn+1}\psi. \tag{4.6}$$

Note that in these variables the t, y' subspace is orthogonal to the other four coordinate directions. These variables are subject to the identifications

$$\left(\psi',\phi',\frac{y'}{R}\right)\sim\left(\psi',\phi',\frac{y'}{R}\right)+2\pi n_1\left(\frac{1}{kn+1},1-\frac{1}{kn+1},-\frac{k}{kn+1}\right). \tag{4.7}$$

The y' circle does not shrink to a point at $\tilde r=0$. Since k and $kn+1$ cannot share a common factor, we see that there will be a nonzero shift in the y' direction when we move from one orbifold image of (ψ',ϕ') to another. Thus $\mathbb{Z}_{kn+1}$ does not have fixed points on the total space. We conclude that the 6-dimensional metric (2.8) is regular around $\tilde r=0$.

4.1.2. The $\tilde r_c\to 0$ limit

We similarly find an orbifold singularity in the base space metric at $\tilde r_c=0$, but in this case we will also encounter an orbifold singularity in the total 6D space if $k>1$. This latter singularity is just the orbifold singularity found for the total space in [6].

This analysis around $\tilde r_c=0$ is connected to the analysis around $\tilde r=0$ by exchanging γ_1 with γ_2, ψ with ϕ, θ with $\pi/2-\theta$. At the point $\tilde r_c=0$ the base metric is reduced to locally flat form—but in this case with signature $(0,4)$—by the change of coordinates

$$r''^2=-4\frac{\gamma_1}{\gamma_1+\gamma_2}\tilde r_c=4kn\tilde r,\qquad \theta''=\frac{\tilde\theta_c}{2},$$
$$\psi''=\psi-\frac{\gamma_2}{\gamma_1}\phi=\psi+\frac{kn+1}{kn}\phi,\qquad \phi''=\frac{\gamma_1+\gamma_2}{\gamma_1}\phi=-\frac{1}{kn}\phi, \tag{4.8}$$

where $\tilde\theta_c$ is the polar coordinate around the point $\tilde r_c=0$:

$$\cos\tilde\theta_c=\frac{\tilde r\cos\tilde\theta+c}{\tilde r_c}. \tag{4.9}$$

The periodic identifications on the ψ'' and ϕ'' coordinates

$$(\psi'',\phi'')\sim(\psi'',\phi'')+2\pi n_1(1,0)+2\pi n_2\left(1+\frac{1}{kn},-\frac{1}{kn}\right),\qquad n_1,n_2\in\mathbb{Z}, \tag{4.10}$$

give the orbifold group $\mathbb{Z}_{kn}$. The group action has a fixed point at $\tilde r_c=0$ and the base space has an ALE singularity at this point.

Let us now look at the total 6D space. Around $\tilde r_c=0$ we have

$$H\approx 1+\frac{Q}{\gamma_1(\gamma_1+\gamma_2)\eta},\qquad \frac{F}{2}\approx\frac{\gamma_2}{(\gamma_1+\gamma_2)\eta},$$

$$\sqrt{2}\beta \approx -\sqrt{2}\omega \approx \frac{Q}{\gamma_1}\,d\phi \equiv \sqrt{2}\beta_\phi\,d\phi, \tag{4.11}$$

and thus the metric of the total space is[7]

$$ds^2 \approx \left(\frac{kQ}{a^2\eta n}-1\right)^{-1}\left[\left(dt^2-dy''^2\right)+\frac{kn+1}{\eta}(dt-dy'')^2\right] + \left(\frac{kQ}{a^2\eta n}-1\right)\left[dr''^2+r''^2\,d\theta''^2+r''^2\cos^2\theta''\,d\psi''^2+r''^2\sin^2\theta''\,d\phi''^2\right], \tag{4.12}$$

where

$$\frac{y''}{R}=\frac{y}{R}-\frac{\sqrt{2}\beta_\phi}{R}\phi=\frac{y}{R}+\frac{1}{n}\phi. \tag{4.13}$$

Thus, the 6-dimensional space defined by the metric (4.12) is the quotient of a smooth space by the $\mathbb{Z}_{kn}$ group that identifies the points

$$\left(\psi'',\phi'',\frac{y''}{R}\right)\sim\left(\psi'',\phi'',\frac{y''}{R}\right)+2\pi n_2\left(1+\frac{1}{kn},-\frac{1}{kn},\frac{1}{n}\right). \tag{4.14}$$

Let ω be the generator of this orbifold group. The element ω^n acts trivially on y''. The fixed point group is thus $\mathbb{Z}_k$, and we have an orbifold singularity in the 6D space of order k if $k>1$.

4.2. *The geometries* (2.12)

We can do a similar analysis for the geometries with γ_1,γ_2 given by (2.12). We will again find that there are orbifold singularities in the base metric at both $\tilde{r}=0$ and $\tilde{r}_c=0$ (in fact, the orbifold groups are the same as those for the geometries (2.11)). But the full 6D metric also has orbifold singularities at each of these locations, while the class (2.11) had an orbifold singularities only at $\tilde{r}_c=0$. These singularities of the full 6D metric are of course just the same ones found directly in [6], but here we are interested in seeing how they arise from the base $\times$ fiber decomposition of the metric.

Consider first the location $\tilde{r}=0$. We insert the values of γ_1,γ_2 given in (2.12) into the expressions (4.1), (4.4), and find the following form of the 6D metric near $\tilde{r}=0$:

$$ds^2 \approx \left(1+\frac{n^2n_5^2(kn+1)Q}{a^2\eta}\right)^{-1}\left[\left(-dt^2+dy'^2\right)+\frac{kn}{\eta}(dt-dy')^2\right] + \left(1+\frac{n^2n_5^2(kn+1)Q}{a^2\eta}\right)\left[dr'^2+r'^2\,d\theta'^2+r'^2\cos^2\theta'\,d\psi'^2+r'^2\sin^2\theta'\,d\phi'^2\right]. \tag{4.15}$$

The variables here are now defined as

$$r'^2=4(kn+1)\tilde{r},\qquad \theta'=\frac{\tilde{\theta}}{2},$$
$$\psi'=\frac{1}{kn+1}\psi,\qquad \phi'=\frac{kn}{kn+1}\psi+\phi,\qquad \frac{y'}{R}=\frac{y}{R}-n_5 n\psi, \tag{4.16}$$

[7] Note that $\frac{kQ}{a^2\eta n}-1=\frac{1}{na^2}[kQ+na^2(2kn+1)]$, which is positive since we are working with $n>0$. The signature of the total space is thus everywhere $(5,1)$.

and are thus subject to the identifications

$$\left(\psi', \phi', \frac{y'}{R}\right) \sim \left(\psi', \phi', \frac{y'}{R}\right) + 2\pi n_1 \left(\frac{1}{kn+1}, 1 - \frac{1}{kn+1}, -n_5 n\right). \tag{4.17}$$

If we look at just the coordinates ψ', ϕ' which lie in the base metric, then since both these circles shrink to zero at $\tilde{r} = 0$, we find an orbifold singularity $\mathbb{Z}_{kn+1}$. This is the same singularity that we found for the base when looking at the family (2.11).[8] Since y' shifts by an integer multiple of 2π under this orbifold action, the full 6D space also has an orbifold singularity $\mathbb{Z}_{kn+1}$ at $\tilde{r} = 0$.

Let us now examine the singularities at $\tilde{r}_c = 0$. From (4.8), (4.11), we find that the metrics (2.12) behave as

$$\begin{aligned} ds^2 \approx & \left(\frac{n n_5^2 (kn+1)^2 Q}{a^2 k \eta} - 1\right)^{-1} \left[\left(dt^2 - dy''^2\right) + \frac{kn+1}{\eta}(dt - dy'')^2\right] \\ & + \left(\frac{n n_5^2 (kn+1)^2 Q}{a^2 k \eta} - 1\right) \\ & \times \left[dr''^2 + r''^2 \, d\theta''^2 + r''^2 \cos^2\theta'' \, d\psi''^2 + r''^2 \sin^2\theta'' \, d\phi''^2\right], \end{aligned} \tag{4.18}$$

where

$$\begin{aligned} & r''^2 = 4kn\tilde{r}, \qquad \theta'' = \frac{\tilde{\theta}_c}{2}, \\ & \psi'' = \psi + \frac{kn+1}{kn}\phi, \qquad \phi'' = -\frac{1}{kn}\phi, \qquad \frac{y''}{R} = \frac{y}{R} + \frac{n_5(kn+1)}{k}\phi. \end{aligned} \tag{4.19}$$

These coordinates are subject to the identifications

$$\left(\psi'', \phi'', \frac{y''}{R}\right) \sim \left(\psi'', \phi'', \frac{y''}{R}\right) + 2\pi n_2 \left(1 + \frac{1}{kn}, -\frac{1}{kn}, \frac{n_5(kn+1)}{k}\right). \tag{4.20}$$

The base metric has an orbifold singularity $\mathbb{Z}_{kn}$ since the ψ'', ϕ'' circles shrink at $\tilde{r}_c = 0$. The y'' coordinate shifts under the orbifold action however, and the y'' circle does not shrink here. Let ω be the generator of the orbifold action (4.20). If n_5, k share no common factor then points of the full 6D metric are left invariant by ω^k, and we find an orbifold singularity $\mathbb{Z}_n$. (There is, of course, no singularity if $n = 1$.) If n_5 and k share a common factor m then y'' is left invariant under $\omega^{\frac{k}{m}}$, and we have an orbifold singularity $\mathbb{Z}_{nm}$.

4.3. *The $f \to 0$ limit*

We have observed that the Gibbons–Hawking base space is such that at the intersection with the hypersurface $f = 0$ the Gibbons–Hawking potential H_2 vanishes and the base metric has a severe singularity: its signature changes from $(4, 0)$ to $(0, 4)$. From the analysis in [5,6] we know that the 6D metric (2.8) has no singularity at $f = 0$, so singularities of the base must cancel contributions from the fiber in some way to make the total space smooth. We study the cancellation of the leading order terms in some detail, since we expect that there will be an

[8] The definitions ψ', ϕ' in (4.1) involve the *ratios* of the γ_i, and these ratios are the same for the families (2.11) and (2.12); thus the singularities of the base will turn out to be the same in the two cases.

analogue of the surface $f=0$ for more general 3-charge geometries, and the fiber and base should to be such that the singularities cancel.

Note that H, F and β have simple poles at $f=0$, ω has a double pole proportional to Q_p, while H_2 vanishes linearly:

$$\begin{aligned}
&H \to \frac{Q}{f}, \qquad \frac{F}{2} = -\frac{Q_p}{f},\\
&\sqrt{2}\beta = \frac{Q}{f}(\gamma_1+\gamma_2)\eta\big(\cos^2\theta\, d\psi + \sin^2\theta\, d\phi\big),\\
&\sqrt{2}\omega \to \frac{2QQ_p}{f^2}(\gamma_1+\gamma_2)\eta\big(\cos^2\theta\, d\psi + \sin^2\theta\, d\phi\big),\\
&H_2 \to -\frac{\gamma_2}{4\tilde{r}^2\gamma_1} f.
\end{aligned} \tag{4.21}$$

We observe that in this limit the 1-forms β and ω are parallel, being proportional to $\cos^2\theta\, d\psi + \sin^2\theta\, d\phi$. The part of the base metric which diverges is also parallel to this form, a fact which can be seen as follows. As $f \to 0$,

$$\frac{\tilde{r}}{\tilde{r}_c} \to -\frac{\gamma_2}{\gamma_1}, \qquad \chi \to -\frac{(\gamma_1+\gamma_2)\gamma_2\eta}{4\tilde{r}}, \qquad \cos 2\theta \to -\frac{4\tilde{r}}{(\gamma_1+\gamma_2)\gamma_2\eta}, \tag{4.22}$$

and thus

$$\cos^2\theta\, d\psi + \sin^2\theta\, d\phi = \frac{d\tilde{\phi} + \cos 2\theta\, d\tau}{2} \;\to\; -\frac{2\tilde{r}}{(\gamma_1+\gamma_2)\gamma_2\eta}(d\tau + \chi\, d\tilde{\phi}), \tag{4.23}$$

and we note that the part $(d\tau + \chi\, d\tilde{\phi})^2$ of the base metric is the one with the divergent coefficient.

We rewrite (4.21) as

$$\begin{aligned}
\sqrt{2}\beta &\to -\frac{2Q\tilde{r}}{\gamma_2 f}(d\tau + \chi\, d\tilde{\phi}) \equiv \sqrt{2}\beta_0(d\tau + \chi\, d\tilde{\phi}),\\
\sqrt{2}\omega &\to -\frac{4QQ_p\tilde{r}}{\gamma_2 f^2}(d\tau + \chi\, d\tilde{\phi}) \equiv \sqrt{2}\omega_0(d\tau + \chi\, d\tilde{\phi}).
\end{aligned} \tag{4.24}$$

We can now substitute the above expressions into (2.2) and collect the coefficients of the leading divergence as $f \to 0$. We get

$$ds^2_{6\mathrm{D}} \to \left[-H^{-1}\sqrt{2}\beta_0\left(\sqrt{2}\omega_0 + \frac{F}{2}\sqrt{2}\beta_0\right) + HH_2^{-1}\right](d\tau + \chi\, d\tilde{\phi})^2. \tag{4.25}$$

While each term in the above expression diverges as $\sim 1/f^2$, we find using the above limiting forms of the coefficient functions that the coefficient of $1/f^2$ cancels. Note that this cancellation occurs for generic values of γ_1 and γ_2.

Solutions with Gibbons–Hawking base can be written in terms of harmonic functions H_i on ds_3^2 [8] (the H_i for our bound states are given in Appendix A). If we try to make new geometries by superposing the harmonic functions found for bound states then we do not in general get a cancellation of this $1/f^2$ singularity. Thus, we cannot easily make new bound state geometries by exploiting the linearity of the space of harmonic functions.

5. Discussion

It is interesting that supersymmetric solutions in 6D (with translation symmetry along t, y) can be written as a hyper-Kähler base times a (t, y) fiber. Further, the equations of motion allow the coefficient functions in this decomposition to be expressed in terms of a set of harmonic functions [8,9].

Large classes of classical 3-charge extremal solutions may be written down using such formalisms [9–11], but to address the questions relevant to black holes we need to find the solutions that correspond to actual *bound* states of the D1, D5, P charges. A generic choice of harmonic functions would not give a bound state; we can already see this from the 2-charge D1–D5 case where we can superpose harmonic functions to make nonbound states from bound states. In particular (as noted in Appendix A) superposing harmonic functions to create new 3-charge geometries gives solutions that have pathologies and are not likely to be bound states.

To gain some insight into the nature of the bound state geometries we have in this paper expressed the geometries of [5,6] in the 'base × fiber' form. These geometries are known to represent bound states since they were obtained by applying exact symmetries (spectral flow and S, T dualities) to 2-charge bound state geometries. (These 2-charge geometries in turn were known to be bound states because they were generated by S, T dualities applied to a single fundamental string carrying vibrations [2].)

From this analysis we observed that the base is not hyper-Kähler but 'pseudo-hyperkahler'; the signature changes from $(4, 0)$ to $(0, 4)$ across a hypersurface in the base. This hypersurface is the intersection of the base with the surface $f = 0$ defined in the full 6D space. In the 2-charge case the surface $f = 0$ (when restricted to $t = \text{const}$) degenerates to a simple closed curve S^1, and different shapes of this curve map out the different microstates [2]. It is thus interesting to investigate the $f = 0$ surface in the 3-charge case as well. We found that in the 3-charge case the $f = 0$ surface (again restricted to $t = \text{const}$) was $S^1 \times S^3$ divided by an orbifold group. This group acted without fixed points, so the $f = 0$ surface was smooth. Roughly speaking, the S^1 is the same S^1 as in the 2-charge case, and the S^3 collapses to a point when the momentum charge vanishes. It is possible that different shapes of the $f = 0$ surface give the different 3-charge microstates, just as different shapes of the $f = 0$ curve in the 2-charge case described the microstates.

It is interesting to compute the area of the $f = 0$ hypersurface (at fixed t). For the family (2.11) we find

$$A = 4\pi^2(2\pi R)\frac{\sqrt{kn+1}-1}{2kn+1}\sqrt{Q_1 Q_5 Q_p}$$

(we have replaced Q^2 by $Q_1 Q_5$ to achieve a form more symmetric between the charges). Dividing by the 6D Newton constant we get

$$\frac{A}{G^{(6)}} \sim \frac{\sqrt{Q_1 Q_5 Q_p}}{G^{(5)}}, \tag{5.1}$$

where $G^{(5)} = G^{(6)}/(2\pi R)$ is the 5D Newton constant obtained upon reduction along y. The above relation is reminiscent of the entropy relation for the black hole in 5D, though we do not have any horizon in the geometry. It will be interesting to see if a surface analogous to $f = 0$ with such an area exists for generic 3-charge microstates, and to relate this surface to the supertube description of bound states [16–20].

To gain further insight into the base × fiber form, we analyzed the singularities occurring in the base metric. The base is in our case a Gibbons–Hawking geometry, with the harmonic function having two centers. The full 6D geometry is known to have orbifold singularities except in special cases. In the base $\times$ fiber decomposition we find that there are orbifold singularities in the base at the two centers of the harmonic function. We saw explicitly how putting together the base with the fiber generates the appropriate orbifold singularities for the full 6D metric. If one could understand the general singularity structure of the base and the regularity conditions needed on the full 6D metric then one might be able to extract the D1–D5–P bound state geometries out of the general family of supersymmetric solutions.

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Appendix A. Harmonic functions for the 3-charge metric

In [8] an interesting observation was made: when the base metric is of Gibbons–Hawking type, as turns out to be the case for our 3-charge metric (2.8), one can rewrite the quantities H, F, β, ω appearing in (2.13) as some combinations of harmonic functions on flat $\mathbb{R}^3$. Conversely, any choice of these harmonic functions generates a solution that will satisfy the equations of motion (2.4)–(2.7).

In this appendix, we will derive the harmonic functions corresponding to the 3-charge metric (2.13).

Let us introduce a convenient basis of 1-forms

$$\sigma \equiv d\tau + \chi\, d\tilde\phi, \qquad d\tilde\alpha^i \equiv \{d\tilde r, d\tilde\theta, d\tilde\phi\} \tag{A.1}$$

in terms of which the $\star$ operation reads (only the components which are used in the following computation are written down)

$$\begin{aligned} &\star(d\tilde r\,\sigma) = -H_2\tilde r^2 \sin\tilde\theta\, d\tilde\theta\, d\tilde\phi, && \star(d\tilde\theta\,\sigma) = H_2 \sin\tilde\theta\, d\tilde r\, d\tilde\phi,\\ &\star(d\tilde r\, d\tilde\phi) = H_2^{-1}\sin^{-1}\tilde\theta\, d\tilde\theta\,\sigma, && \star(d\tilde\theta\, d\tilde\phi) = -H_2^{-1} r^{-2}\sin^{-1}\tilde\theta\, d\tilde r\,\sigma. \end{aligned} \tag{A.2}$$

Let us expand the 1-forms β and ω and the self-dual 2-form $\mathcal{G}^+$ in this basis[9]:

$$\beta = \beta_0\sigma + \beta_i\, d\tilde\alpha^i, \qquad \omega = \omega_0\sigma + \omega_i\, d\tilde\alpha^i, \tag{A.3}$$

$$\mathcal{G}^+ = C_i\, d\tilde\alpha^i\,\sigma - \frac{\epsilon_{ij}^{\ \ k}}{2} C_k H_2\, d\tilde\alpha^i\, d\tilde\alpha^j, \tag{A.4}$$

where we used self-duality of $\mathcal{G}^+$.

As shown in [8], the equation of motion (2.4) implies

$$\beta_0 = H_2^{-1} H_3, \qquad \star_3 d\big(\beta_i\, d\tilde\alpha^i\big) = -dH_3 \tag{A.5}$$

with H_3 harmonic on $\mathbb{R}^3$. For the 3-charge metric (2.8) we have

$$\sqrt{2}\beta_0 = \frac{Q(\gamma_1+\gamma_2)\eta}{2}\frac{\cos 2\theta}{f} = \frac{Q}{2}\frac{\tilde r_c - \tilde r}{\gamma_1\tilde r + \gamma_2\tilde r_c}, \tag{A.6}$$

[9] The 3-dimensional indices i, j, k are raised and lowered with the 3-dimensional flat metric g_3 defined in (2.21), and $\epsilon_{\tilde r\tilde\theta\tilde\phi} = \sqrt{g_3} = \tilde r^2 \sin\tilde\theta$.

from which we find

$$\sqrt{2}H_3 = \frac{Q}{2(\gamma_1+\gamma_2)}\left(\frac{1}{\tilde r} - \frac{1}{\tilde r_c}\right). \tag{A.7}$$

The closure of $\mathcal{G}^+$, Eq. (2.5), implies

$$C_i = \partial_i\left(H_2^{-1}H_4\right), \tag{A.8}$$

with H_4 harmonic on $\mathbb{R}^3$. The value of $\mathcal{G}^+$ for the metric (2.8) can be computed from the definition (2.3) and from (2.13). The result for C_i, derived with the help of MATHEMATICA, is

$$\begin{aligned}
\sqrt{2}C_{\tilde r} &= \frac{(\gamma_1+\gamma_2)^3\gamma_1\gamma_2\eta}{16}\frac{4\tilde r\cos\tilde\theta + (\gamma_1+\gamma_2)^2\eta}{\tilde r_c(\gamma_1\tilde r+\gamma_2\tilde r_c)^2},\\
\sqrt{2}C_{\tilde\theta} &= \frac{(\gamma_1+\gamma_2)^3\gamma_1\gamma_2\eta}{4}\frac{\tilde r^2\sin\tilde\theta}{\tilde r_c(\gamma_1\tilde r+\gamma_2\tilde r_c)^2},\\
\sqrt{2}C_{\tilde\phi} &= 0.
\end{aligned} \tag{A.9}$$

From this we see that

$$\sqrt{2}C_i = \partial_i\left[\frac{\gamma_1+\gamma_2}{2}\frac{\gamma_1\tilde r - \gamma_2\tilde r_c}{\gamma_1\tilde r+\gamma_2\tilde r_c}\right], \tag{A.10}$$

and thus

$$\sqrt{2}H_4 = \frac{1}{2}\left(\frac{\gamma_1}{\tilde r_c} - \frac{\gamma_2}{\tilde r}\right) + \text{const}\cdot H_2. \tag{A.11}$$

The value of the constant in the equation above can be chosen in such a way that H_4 vanishes in the two charge limit ($\gamma_1\gamma_2 = 0$):

$$\text{const} = \frac{\gamma_2-\gamma_1}{2}. \tag{A.12}$$

With this choice we get

$$\sqrt{2}H_4 = \frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2}\left(\frac{1}{\tilde r_c} - \frac{1}{\tilde r}\right). \tag{A.13}$$

The remaining equations of motion (2.6) and (2.7) imply

$$H = H_1 + H_2^{-1}H_3H_4, \tag{A.14}$$

$$F = -H_5 - H_2^{-1}H_4^2, \tag{A.15}$$

where again H_1 and H_5 are harmonic on $\mathbb{R}^3$. The values for the our 3-charge metrics are

$$H = 1 + \frac{Q}{4}\frac{\gamma_1+\gamma_2}{\gamma_1\tilde r+\gamma_2\tilde r_c} \implies H_1 = 1 + \frac{Q}{4(\gamma_1+\gamma_2)}\left(\frac{\gamma_1}{\tilde r} + \frac{\gamma_2}{\tilde r_c}\right), \tag{A.16}$$

$$\frac{F}{2} = \frac{\gamma_1\gamma_2}{4}\frac{\gamma_1+\gamma_2}{\gamma_1\tilde r+\gamma_2\tilde r_c} \implies H_5 = -\frac{\gamma_1\gamma_2}{2(\gamma_1+\gamma_2)}\left(\frac{\gamma_1}{\tilde r} + \frac{\gamma_2}{\tilde r_c}\right). \tag{A.17}$$

Finally, from the definition (2.3) of $\mathcal{G}^+$ we find that

$$\omega_0 = H_6 + H_2^{-2}H_3H_4^2 + H_2^{-1}H_1H_4 + \frac{1}{2}H_2^{-1}H_3H_5,$$

$$\star_3 d\big(\omega_i\, d\tilde{\alpha}^i\big) = H_2\, d\omega_0 - \omega_0\, dH_2 - 2(H_1 H_2 + H_3 H_4)\, d\left[\frac{H_4}{H_2}\right] - \big(H_4^2 + H_2 H_5\big)\, d\left[\frac{H_3}{H_2}\right], \tag{A.18}$$

where H_6 is yet another harmonic function. The metric (2.8) has

$$\sqrt{2}\omega_0 = \frac{Q}{8}\frac{\gamma_1^2 - \gamma_2^2 + \frac{4(\tilde{r}_c - \tilde{r})}{\eta} - 4(\tilde{r}_c - \tilde{r})\big(1 + \frac{\gamma_1\gamma_2(\gamma_1+\gamma_2)}{2(\gamma_1\tilde{r}+\gamma_2\tilde{r}_c)}\big)}{\gamma_1\tilde{r} + \gamma_2\tilde{r}_c}, \tag{A.19}$$

and thus

$$\sqrt{2}H_6 = \frac{Q}{8(\gamma_1+\gamma_2)}\left(\frac{\gamma_1^2}{\tilde{r}} - \frac{\gamma_2^2}{\tilde{r}_c}\right). \tag{A.20}$$

We thus explicitly identify the six harmonic functions H_i ($i = 1, \ldots, 6$) that describe 3-three charge solutions (2.8). Any choice of harmonic functions gives a solution to the equations of motion (2.4)–(2.7), but generic solutions constructed this way will not be true bound states of the 3-charge system. In particular, if we superpose the harmonic functions H_i corresponding to different values of the parameters γ_1, γ_2 to generate new geometries then we find pathologies at the $f = 0$ surface for instance, which we would not expect for an actual 3-charge bound state.

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