"Lost chains" in algebraic models

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How the complete classification of subalgebras can lead to new unexpected physics

L. Fortunato[1] and W.A. de Graaf[2]

[1] European Centre for Theoretical Studies in Nuclear Physics and Related Areas, Strada delle Tabarelle 286, I-38123 Villazzano (TN), Italy

[2] Dip. Matematica, Università di Trento, via Sommarive 24 ,I-38123 Povo (Trento), Italy

E-mail: fortunat@pd.infn.it

Abstract. The algebraic structure of some of the simplest algebraic models $u(2), u(3)$ and $u(4)$, widely used in several branches of physics either as toy models or as working instruments, are reanalyzed under a new perspective that releases the requirement that chains should terminate or pass through the angular momentum algebra. Unitary algebras are non-semisimple, therefore we first apply the Levi-Malcev decomposition. Then we use the theory of weighted Dynkin diagrams to identify conjugacy classes of $A_1 \sim su(2) \sim so(3)$ subalgebras: a complete classification of new angular momentum non conserving (AMNC) dynamical symmetries follows that we substantiate with examples.

1. Introduction

Unitary algebras arising in bosonic models of quantum mechanics play a fundamental role in several branches of physics such as nuclear, atomic, molecular and particle physics[1, 2, 3, 4, 5]. They are not semisimple, therefore the very first step is to use the well-known Levi-Malcev decomposition to extract the special unitary subgroup, that instead possesses this property [6]. To this algebra we apply the theory of weighted Dynkin diagrams [7, 8, 9] to classify the different conjugacy classes of A_1-type subalgebra (that is Cartan's notation for the isomorphic algebras $sl(2) \sim su(2) \sim so(3)$) in terms of weighted Dynkin diagrams (WDD). Within any Lie algebra one can identify $sl(2)$-triples, that is a set of three elements that formally satisfy the commutation relations of $sl(2)$. There are infinitely many such triples, but there is only a finite number of distinct nilpotent orbits. Nilpotent orbits are in one-to-one correspondence with certain distinguished semisimple orbits through the Jacobson-Morozov theorem, a theorem of Kostant and another of Mal'cev [7]. These orbits in turn can be labeled by their WDD. This yields a complete classification of A_1-type subalgebras that brings in the possibility of additional chains that do not conserve the (usual) angular momentum, but rather have some sort of angular momentum of their own. A somewhat unexpected discovery of spinor operators, occurring in the tensor analysis of the whole algebra with respect to this new angular momentum, has generated a useful discussion at the conference, that helped us in understanding it more deeply. All the algebras we deal with have also been implemented in the GAP4 programming language [10], that allows symbolic manipulations of algebraic objects with the help of a package [11] that has been especially designed to provide relevant computational algorithms. We have summarized in Fig.1 the total number of subalgebras and the number of distinct conjugacy classes that are found for each A_n with $n = 1, \cdots, 7$.

Type	# classes of subalgebras (A_n, B_n, C_n, D_n, etc.)	# nilpotent orbits # classes of A_1 subalg . # W.D.D .
A_1	1	1
A_2	2	2
A_3 - Vibron	8	4
A_4	14	6
A_5 - IBM	36	10
A_6	60	14
A_7	131	21

Figure 1. Number of classes of subalgebras and of nilpotent orbits, obtained using SLA (in GAP) from the recent classification of de Graaf [9]. Two examples that are used in algebraic models of molecular and nuclear physics are highlighted. See also next figure.

The dynamics of several physical system can be described in second quantization as being based on boson creation and annihilation operators $b_\alpha, b^\dagger_\alpha$ that are treated as elementary building blocks. The index specifies the number and any additional quantum number that can be useful to classify them. In particular they might carry an intrinsic angular momentum ℓ that has $2\ell+1$ components. They satisfy bosonic commutation rules of the form:

$$[b_\alpha, b^\dagger_{\alpha'}] = \delta_{\alpha,\alpha'} \tag{1}$$

$$[b_\alpha, b_{\alpha'}] = [b^\dagger_\alpha, b^\dagger_{\alpha'}] = 0 \tag{2}$$

The unitary algebra $u(n)$ is obtained by taking all the n^2 bilinear operators of the form $b^\dagger_\alpha b_{\alpha'}$.

Typical examples and very successful models are, for instance, the Interacting Boson Model for nuclei and the Vibron model for molecules, both of which have been investigated in great detail, mainly by F.Iachello and collaborators. A common feature of the majority of these models is that, once one has identified the relevant physical ingredients, that can be associated with boson creation and annihilation operators, one has a (larger) unitary algebra. Then one searches for all chains of subalgebras that end with the $so(3)$ angular momentum algebra, to ensure that the states, that are labeled by a suitable number of eigenvalues of Casimir operators of the chain, conserve angular momentum. This is certainly a very natural way to proceed, but the algebra is, mathematically speaking, richer than that. This richness can be appreciated in Fig. 2, where the A_3 and A_5 algebras (red points) and their subalgebras are plotted in a circular graph. The green points are A_1 subalgebras pertaining to different conjugacy classes. The number of chains of subalgebras that do not pass through the angular momentum algebra is much higher: although their existence has been known since the early days of algebraic models, they have not been much investigated in the past in connection with physical problems. The graph that corresponds to the IBM, albeit of huge importance in physics, is quite complicated, therefore we will restrict our discussion to some low rank cases, among which A_3 that corresponds to the vibron model. Our aim is to try and understand if and how the additional chains can be put in correspondence with the basic operators that define the algebra. In addition we will investigate their tensorial properties and suggest applications.

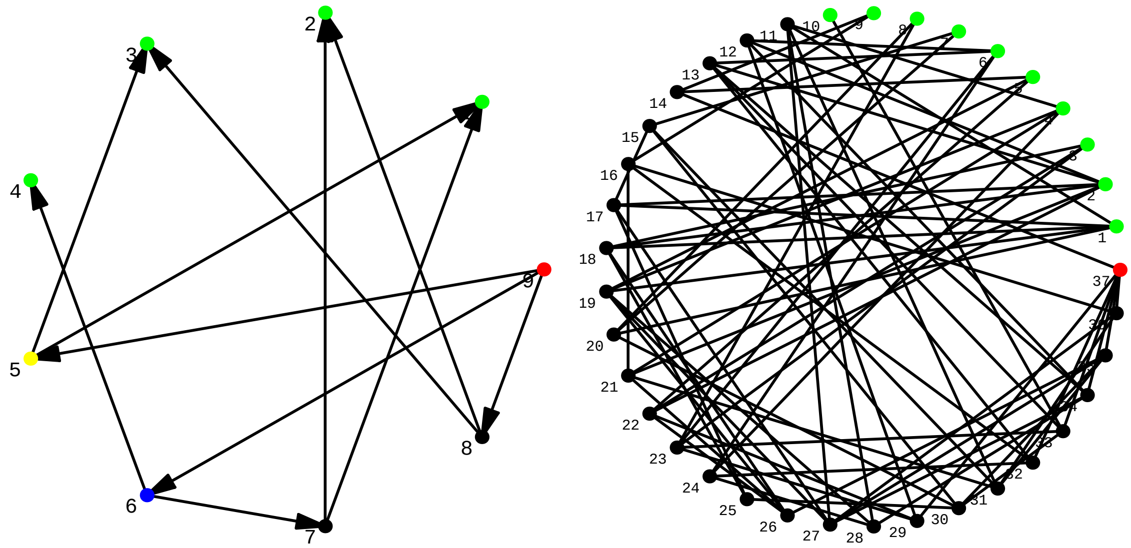

Figure 2. Circular graphs corresponding to the inclusions of subalgebras in A_3 (9 vertices, 11 edges) and A_5 (37 vertices, 79 edges) symbolized with red points. Green points are A_1 subalgebras. Black, yellow and blue in the left part indicate A_1A_1, A_2 and B_2 subalgebras respectively. Labels in the left part correspond to the ones in Fig. 5.

2. $u(2)$

The two dimensional harmonic oscillator problem arises in simple models of quantum mechanics [4, 12] and it is connected to the $u(2)$ Lie algebra. Two species of scalar bosons, called s and t can be invoked, that obey standard bosonic commutation rules of the form $[s, s^\dagger] = [t, t^\dagger] = 1$ and $[s, t] = [s, t^\dagger] = [s^\dagger, t] = [s^\dagger, t^\dagger] = 0$. The Lie algebra $u(2)$ is built with bilinear operators made up of these operators and it amounts to four elements:

$$\mathfrak{g}: \quad g_1 = s^\dagger s, \quad g_2 = s^\dagger t, \quad g_3 = t^\dagger s, \quad g_4 = t^\dagger t\ . \tag{3}$$

$\mathfrak{g}$ is not semisimple. Thanks to the Levi-Malcev decomposition it may be rewritten as the direct sum of a 1-dimensional subalgebra, the radical or maximal solvable ideal, and a 3-dimensional semisimple Levi subalgebra, as follows $u(2) \simeq \quad \mathfrak{r} \equiv u(1) \quad \oplus \quad \mathfrak{s} \equiv su(2)$, with $\mathfrak{r} = \langle g_1 + g_4 \rangle$ and $\mathfrak{s} = \langle g_2, g_3, g_1 - g_4 \rangle$. Notice that $\mathfrak{s}$ has a basis consisting of elements of the form given in Eq. (1.20) of Ref. [4], as the standard generators of $su(2)$ angular momentum algebra, namely $\hat{J}_x = (s^\dagger s - t^\dagger t)/2, \hat{J}_y = (t^\dagger s + s^\dagger t)/2, \hat{J}_z = i(t^\dagger s - s^\dagger t)/2$. Clearly the basis element of the radical is the total number of bosons operator, $\hat{N} = s^\dagger s + t^\dagger t$, that is known to commute with all the operators, thereby forming the center of the Lie algebra. Although in the present case it is almost trivial, the semisimple part is amenable to treatment with the theory of weighted Dynkin diagrams [7, 8]. It corresponds to A_1 in Cartan notation and admits only one nilpotent orbit, labeled by the weighted Dynkin diagram [2] (notice that WDD's are indicated in square brackets, not to be confused with references). Therefore there exists only one conjugacy class.

Two dynamical symmetries are present (see for example Ref. [2, 4, 12]), namely:

$$I)\quad \begin{array}{ccc} u(2) & \supset & u(1) \\ | & & | \\ [N] & & n_t \end{array} \qquad\qquad II)\quad \begin{array}{ccccc} u(2) & \supset & su(2) & \supset & so(2) \\ | & & | & & | \\ [N] & & j & & \mu \end{array}$$

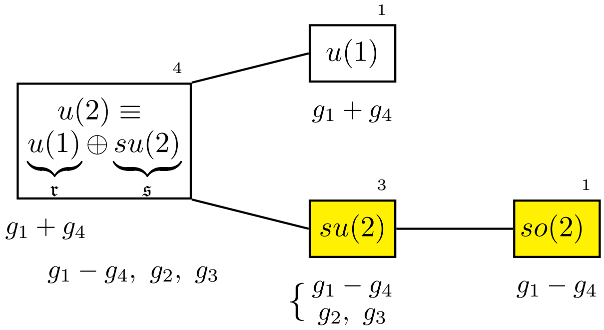

Figure 3. Classification of the $u(2)$ subalgebras. Semisimple algebras are indicated in yellow. Elements, according to the definitions in the text, are indicated below each frame and the order (dimension) is given as a small number in the upper-right corner.

Each one of them provides a basis that can be chosen for diagonalization. These two chains have a fundamental difference: one goes through the radical, the other through the semisimple Levi subalgebra. The eigenvalues of two invariants are needed to label the basis states and while they are fixed in the first chain $|\,[N]n_t\rangle$, there is more freedom in the second as they can be chosen either as $|\,[N]j\rangle$ or $|\,j\mu\rangle$. The second form exhibits in a natural way the splitting of eigenstates with different third component of angular momentum and it has prevailed in the specialized literature. The two subalgebras $u(1)$ and $so(2)$ are isomorphic, but when one is dealing with a complete classification of all subalgebras, some care must be taken. Although it is clear that the linear map that changes the sign of g_4 and leaves the rest unchanged, is actually swapping $u(1)$ and $so(2)$, *it does not send the whole reduction scheme of Fig. (3) into itself.* It can be easily proven that there is no Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{g}$ mapping $u(1)$ into $so(2)$.

3. $u(3)$

The algebra $u(3)$ is built upon bilinear combinations of three boson operators. These can be chosen either i) as three scalar bosons, that might be physically interpreted as appropriate combinations of the three Cartesian coordinates and momenta in a three dimensional harmonic oscillator, or ii) as the three components of a vector boson or finally iii) as a scalar boson plus two, so-called, circular bosons [3, 13]. Each construction has practical applications, especially to the study of molecular spectra. We study here the construction ii) made up in terms of a p ($\ell = 1$) boson, because it is necessary for the construction of the Vibron model [2, 4, 5]. The operators $p^\dagger_\mu$ and $\tilde{p}_\mu$ with $\mu = -1, 0, 1$ transform as the $\ell = 1$ representation of the rotation group. The nine bilinear operators built from tensor couplings close into the $u(3)$ Lie algebra. One can apply the same arguments as in the preceding section, namely the Levi-Malcev decomposition and the classification of A_1 subalgebras in terms of weighted Dynkin diagrams, to show that the lattice of subalgebras of $u(3)$ takes the form displayed in Fig. (4). The elements of the algebra can be written as:

$$\begin{array}{lllll} g_1 = [p^\dagger \times \tilde{p}]^{(0)}_0 & g_2 = [p^\dagger \times \tilde{p}]^{(1)}_{-1} & g_3 = [p^\dagger \times \tilde{p}]^{(1)}_0 & g_4 = [p^\dagger \times \tilde{p}]^{(1)}_1 & \\ g_5 = [p^\dagger \times \tilde{p}]^{(2)}_{-2} & g_6 = [p^\dagger \times \tilde{p}]^{(2)}_{-1} & g_7 = [p^\dagger \times \tilde{p}]^{(2)}_0 & g_8 = [p^\dagger \times \tilde{p}]^{(2)}_1 & g_9 = [p^\dagger \times \tilde{p}]^{(2)}_2 \end{array} \quad (4)$$

where one can identify a scalar $\hat{N} = \sqrt{3}g_1$, a vector $\hat{L}_\kappa = \sqrt{2}[p^\dagger \times \tilde{p}]^{(1)}_\kappa$ (3 components) and a quadrupole tensor $\hat{Q}_\kappa = [p^\dagger \times \tilde{p}]^{(2)}_\kappa$ (rank 2, 5 components) to be further discussed below [3]. The Levi-Malcev decomposition is clear from the division into white and yellow blocks in Fig. (4) and the scalar operator g_1 is actually always responsible for the radical part of the classification. From Figs. (3) and (4) one can see that the whole scheme can be divided in

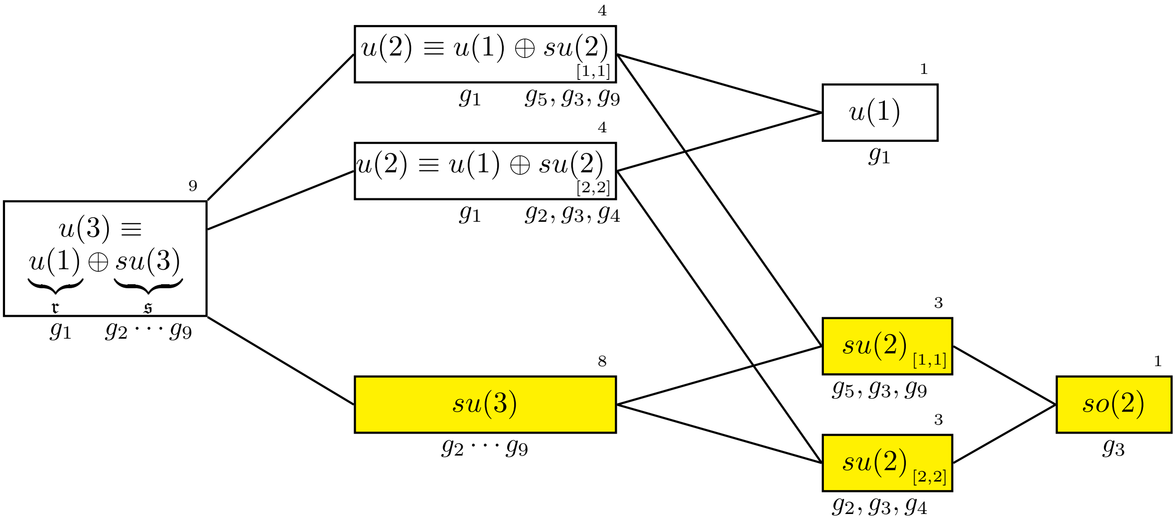

Figure 4. Classification of the $u(3)$ subalgebras. Semisimple algebras are indicated in yellow. Elements, according to the definitions in the text, are indicated below each frame and the order (dimension) is given as a small number above the upper-right corner. Weighted Dynkin diagrams are indicated in the lower-right corner: in the case of non-semisimple algebras they are referred only to the semisimple part. A total of 6 chains can be identified.

two parallel sheets, a *semisimple sheet*, containing all the semisimple Lie algebras, and a *non-semisimple sheet*, containing an exact copy of the structure of the lower one, where each algebra is multiplied by the radical. Of course $u(n)$ corresponds to $su(n)$ and so on, each corresponding pair is also connected by an inclusion relation from top to bottom with the exception of one dimensional subalgebras. The semisimple part starts at $su(3)$, called A_2 in Cartan's notation, that has two different types of A_1 subalgebras labeled by different weighted Dynkin diagrams, $[1, 1]$ and $[2, 2]$ respectively. Each of them is an entire conjugacy class of triplets of operators with the same WDD of which we choose just one representative (usually the simplest available or the one that has already been incorporated into an established model). The second one is the usual algebra of angular momentum, whose components are the components of the rank 1 tensor in Eq. (4). Indeed the three operators $\hat{L}_\kappa = \sqrt{2}[p^\dagger \times \tilde{p}]^{(1)}_\kappa$ satisfy the angular momentum algebra. This algebra and all the chains passing through it have been described by Iachello and the physical reasoning underlying this rightful choice is that the quantum mechanical description of our system and the basis states associated to the chain must conserve angular momentum.

It turns out, however, and this fact was unknown or mostly unnoticed till now, that the A_1 algebra labeled by $[1, 1]$ is made up of three objects that are not components of a vector, but that, nevertheless, have commutation relations that formally identify them as an angular momentum algebra

$$[\hat{W}_+, \hat{W}_-] = 2W_0 \qquad [\hat{W}_0, \hat{W}_\pm] = \pm\hat{W}_\pm \tag{5}$$

where the operators are defined as $\{\hat{W}_- = g_5, W_0 = g_3/\sqrt{2}, W_+ = g_9\}$. In usual terminology they are the $\hat{Q}_{\pm 2}$ components of the quadrupole tensor and the $\hat{L}_0$ component of the angular momentum. Historically only two other theoretical works have introduced something of this sort: Chen and Arima [14] discuss the origin of cylindrical bosons within the Interacting Boson Model, where they introduce the Δ spin that is built upon the highest and lowest components of the quadrupole tensor plus the zero component of the angular momentum, the difference being that their operators are made up of s and d bosons, while ours are made up of p bosons [1]; Elliott discusses, in the fundamental Ref. [15], a similar algebraic structure in the context of

[1] I'd like to thank P.van Isacker for bringing Ref.[14] to my attention.

the collective motion in the nuclear shell model.

Now the crucial point is that once we have an angular momentum algebra, say $\hat{J}$, (to be replaced either by $\hat{L}$ or by $\hat{W}$) we can define spherical tensors with respect to *that* algebra by means of the two formulas [4]:

$$[J_z, T_q^k] \;=\; q\,T_q^k \qquad [J_\pm, T_q^k] \;=\; \sqrt{(k \pm q + 1)(k \mp q)}\,T_{q\pm 1}^k\,. \tag{6}$$

We will use the name of J-tensors to refer to tensors with respect to a particular J-set. The whole $u(3)$ algebra is made up by an L-scalar, an L-vector and and L-tensor of rank 2, as outlined above after Eq. (4), with respect to the L-set. In this L-set, both $p^\dagger$ and $\tilde{p}$ transform as L-vectors. The analysis of the elements of $u(3)$ with respect to the W-set reveals instead that, together with the obvious W-vector given by $\hat{W}$ itself, we have two W-scalars and two W-spinors: this might come as a surprise since we are not expecting spin 1/2 objects to arise in a purely bosonic model. The proof is obtained through the following steps: g_1 and g_7 commute with all components of $\hat{W}$: they are therefore scalars; the objects $sp_1 = \{g_6, g_4\}$ and $sp_2 = \{-g_8, -g_2\}$, with components in the $\{-,+\}$ order, have the following commutation relation with $\hat{W}_0$:

$$[\hat{W}_0, sp_i^\pm] = \pm\frac{1}{2} sp_i^\pm \tag{7}$$

with $i = 1, 2$ and therefore they are the $-1/2$ and $+1/2$ components of a spinor. Finally the commutator with $\hat{W}_+$ and $\hat{W}_-$ terminate after two steps:

$$\begin{aligned} [\hat{W}_+, sp_i^-] &= sp_i^+ \qquad [\hat{W}_+, sp_i^+] = 0 \\ [\hat{W}_-, sp_i^+] &= sp_i^- \qquad [\hat{W}_-, sp_i^-] = 0 \end{aligned} \tag{8}$$

with $i = 1, 2$. The coefficients are compatible with $k = 1/2$ in Eq. (6), therefore we have found two spin$-1/2$ spinors. We propose the name of Chen-Arima spinors for spinors arising in bosonic models, because, to our knowledge, these authors have found the first case. Notice that one can always form other scalars from tensor coupling of Chen-Arima spinors obtaining:

$$[sp_1 \times sp_1]_0^{(W=0)} = -[sp_2 \times sp_2]_0^{(W=0)} = -\frac{\sqrt{3}}{2} g_7 \tag{9}$$

that is a W-scalar consistent with the observation made above.

It is interesting to note that the vectors $p^\dagger$ and $\tilde{p}$ don't satisfy anymore good tensorial properties with respect to the W-set (in other words they do not transform as vectors). Their role is replaced by other two objects $V = \{\sqrt{2}p_{-1}^\dagger p_{-1}^\dagger, 2p_{-1}^\dagger p_1^\dagger, \sqrt{2}p_1^\dagger p_1^\dagger\}$ and $U = \{\sqrt{2}\tilde{p}_{-1}\tilde{p}_{-1}, 2\tilde{p}_{-1}\tilde{p}_1, \sqrt{2}\tilde{p}_1\tilde{p}_1\}$ (with components ordered as $-1, 0, 1$) that are found to be good W-vectors by following the same argument outlined above. Their action creates or annihilates two bosons at the same time and this is consistent with the definitions of the components of the W-spin that are based on the highest and lowest ($\mu = \pm 2$) components of the quadrupole tensor.

4. $u(4)$

This is the case of the Vibron model [2, 3, 4, 5], that is built upon s and p bosons. Together with the nine generators in Eq. (4) one needs another seven generators containing s to close the algebra, namely:

$$\begin{aligned} &g_{10} = [s^\dagger \times \tilde{p}]_{-1}^{(1)} \quad g_{11} = [s^\dagger \times \tilde{p}]_0^{(1)} \quad g_{12} = [s^\dagger \times \tilde{p}]_1^{(1)} \\ &g_{13} = [p^\dagger \times \tilde{s}]_{-1}^{(1)} \quad g_{14} = [p^\dagger \times \tilde{s}]_0^{(1)} \quad g_{15} = [p^\dagger \times \tilde{s}]_1^{(1)} \quad g_{16} = [s^\dagger \times \tilde{s}]_0^{(0)} \end{aligned} \tag{10}$$

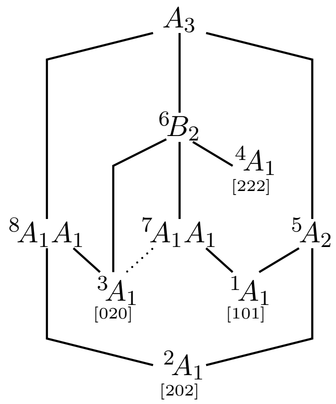

Figure 5. Classification of Lie subalgebras of A_3. Subalgebras carry an arbitrary upper left index. Three-dimensional subalgebras of type A_1 represent entire conjugacy classes. The corresponding WDD is given in square brackets. In theory all inclusions are possible, but in practice, in our particular realization, the dotted one is not possible simultaneously with all the others, although the direct inclusion $^6B_2 \supset {}^3A_1$ is still valid.

The radical of the $u(4)$ algebra is given by the total number of boson operator, $\hat{N} = \sqrt{3}g_1 + g_{16}$. The semisimple sheet consists (see Fig.5) of the seven chains that originate from A_3 and end up in one of the four possible A_1, the conjugacy classes of which are labeled by the WDD: [101], [202], [020] and [222]. In the Vibron model only two such chains, the ones passing through the standard angular momentum subalgebra [202], have been studied so far.

The standard well-known chains of the Vibron model are the most external paths of Fig. 5 connecting A_3 and $A_1[202]$ or, for the sake of clarity:

$$su(4) \sim A_3 \left\{ \begin{array}{ll} su(3) \sim & {}^5A_2 \\ & \\ so(4) \sim & {}^8A_1A_1 \end{array} \right\} so(3) \sim {}^2A_1 \tag{11}$$

that correspond to the nonrigid and rigid rotovibrator limits of the Vibron model. In the complete classification four classes of A_1 subalgebras are present and one has therefore four different "angular momenta" that can be used as J-sets to define spherical tensors. In Table 4 the usual tensor analysis of the algebra of the Vibron model with respect to the $^2A_1$ algebra with WDD [202] is given for reference. The W-angular momentum described above for $u(3)$ forms instead the $^1A_1$ algebra with WDD [101]. With respect to this W-set, the whole algebra $su(4)$ amounts to the W-vector $\hat{W}$, to four W-scalars and four W-spinors (cfr. Table 2). The scalars built with tensor couplings of $\hat{s}p_3$ with itself and $\hat{s}p_4$ with itself are identically zero, therefore one can define two new W-spinors $\hat{s}p'_3 = \{-g_2 - g_{10} - g_{13}, g_8 - g_{12} + g_{15}\}$ and $\hat{s}p'_4 = \{g_6 + g_{10} - g_{13}, g_4 + g_{12} + g_{15}\}$ as linear combinations. Their couplings give good W-scalars and they are also needed as elements of B_2.

We summarize in Table 2 the tensor analysis of A_3 with respect to $^1A_1[101]$ and we give in Tables 3 and 4 the tensor analysis of the whole algebra based on $^3A_1[020]$ and $^4A_1[222]$ respectively. A_3 is made up of 5 vectors with respect in the former case (among which the defining vector Y), while it is made up of a vector (the defining vector $T^{(1)}$), a quadrupole and an octupole tensor in the latter case.

One can write AMNC hamiltonians with the dynamical symmetry based for example on the

Table 1. Definitions and tensorial character of the operators forming the semisimple A_3 algebra with respect to the (standard) set forming the ${}^2A_1$ algebra with [202] WDD. The third column features alternative definitions for algebra elements that correspond to the second column with the following ordering $\mu = -k, .., 0, .., k$, where k is the rank.

operator	def^{rank}	alt.def.	components
$\hat{n}'$	$[s^\dagger \times \tilde{s}]^0_0 - \sqrt{\frac{1}{3}}[p^\dagger \times \tilde{p}]^0_0$	g_1	1
$\hat{L}_\mu/\sqrt{2}$	$[p^\dagger \times \tilde{p}]^1_\mu$	g_2, g_3, g_4	3
$\hat{Q}_\mu$	$[p^\dagger \times \tilde{p}]^2_\mu$	$g_5, \cdots, g_9$	5
$\hat{D}_\mu$	$i[p^\dagger \times \tilde{s} + s^\dagger \times \tilde{p}]^1_\mu$	g_{10}, g_{11}, g_{12}	3
$\hat{D}'_\mu$	$[p^\dagger \times \tilde{s} - s^\dagger \times \tilde{p}]^1_\mu$	g_{13}, g_{14}, g_{15}	3

Table 2. Definitions and tensorial character of the operators forming the A_3 algebra with respect to the non-standard set forming the ${}^1A_1$ algebra with [101] WDD (that can be obtained from the vector in the first line).

operator	definition	components
$\hat{s}_1, \cdots, \hat{s}_4$	g_1, g_7, g_{11}, g_{14}	1 (each)
$\hat{W}_\mu$	$-g_5/\sqrt{2}, g_3/\sqrt{2}, g_9/\sqrt{2}$	3
$\hat{sp}_1, \cdots, \hat{sp}_4$	$\{g_2, g_8\}, \{g_6, g_4\}, \{g_{10}, g_{12}\}, \{g_{13}, g_{15}\}$	2 (each)

Table 3. Definitions and tensorial character of the operators forming the A_3 algebra with respect to the non-standard set forming the ${}^3A_1$ algebra with [020] WDD (that can be obtained from the vector in the first line).

operator	definition	components
$\hat{Y}_\mu$	$i(g_2 + g_6)/\sqrt{2} + g_{10}, (g_1 + \sqrt{2}g_3 + \sqrt{2/3}g_7 + 2ig_{11})/2, i\sqrt{2}g_4 + g_{12} + g_{15}$	3
$\hat{B}_\mu$	$2\sqrt{2}g_9, \sqrt{2}i(g_4 - g_8) - 2g_{12}, 2i\sqrt{2}g_{11}$	3
$\hat{M}_\mu$	$-g_1 - \sqrt{2/3}g_7 + i(g_{14} - g_{11}), ig_6 + (g_{13} - g_{10})/\sqrt{2}, g_5$	3
$\hat{R}_\mu$	$-i(g_4 + g_8)/\sqrt{2} + g_{12}, -g_3/2 - \sqrt{3/8}g_7 - ig_{11}, i(g_2 + g_6)/\sqrt{2}$	3
$\hat{S}_\mu$	$-i(g_4/g_8)/\sqrt{2} + g_{15}, -g_1/\sqrt{2} - g_3/2 + \sqrt{1/12}g_7 - ig_{11}, -g_{10}$	3

chain $A_3 \supset B_2 \supset {}^7A_1A_1 \supset {}^1A_1$ or

$$\begin{array}{ccccccc} su(4) \sim so(6) & \supset & so(5) & \supset & so(4) & \supset & so(3) \\ | & & | & & | & & | \\ N & & t & & u & & w \end{array} \qquad (12)$$

where the labels in the last row are connected with the eigenvalues of the quadratic Casimir operators. The resulting energy formula for symmetric representations $E = \alpha N(N+4) + \beta t(t+3) + \gamma u(u+2) + \delta w(w+1)$ with $t = 0, \cdots, N$, $u = 0, \cdots, t$ and $w = 0, \cdots, u$, most probably is not a good choice for diatomic molecular spectra, but one can anyway use the basis states $\mid N, t, u, w\rangle$, that is actually a Gelfand-Tsetlin pattern for orthogonal algebras, to diagonalize hamiltonians based on the full spectrum generating algebra, with the *proviso* that, while N is the total boson number as in the Vibron model, the labels v, u and w *do not conserve angular momentum, but rather conserve* w. Following the established path, one would have not used

Table 4. Definitions and tensorial character of the operators forming the A_3 algebra with respect to the non-standard set forming the ${}^4A_1$ algebra with [222] WDD (that can be obtained from the vector in the first line).

oper.	definition	comp.
$\hat{T}^{(1)}_\mu$	$(g_2 - 7g_4 - 7g_6 - g_8 + 3g_{13})/2, -\frac{3}{2}g_1 - \sqrt{2}g_3 - 2g_5, g_2 + g_{10} + g_{12}$	3
$\hat{T}^{(2)}_\mu$	$-6\sqrt{2}(2g_3 + \sqrt{2}g_5 - \sqrt{2}g_9 + g_{14}), -3(g_2 + g_4 + g_6 - g_8 + g_{13}),$ $\sqrt{3/2}g_1 - \sqrt{3}g_3 - \sqrt{6}g_5 + g_7, (g_6 - g_2)/2 + g_{10} + g_{12}, (g_5 + \sqrt{2}g_{11})/4$	5
$\hat{T}^{(3)}_\mu$	$\frac{-72}{\sqrt{30}}(g_{13} - g_{15}), \frac{-12}{\sqrt{10}}(2g_2 + \sqrt{2}g_5 - \sqrt{2}g_9 - g_{14}), \frac{-6}{5\sqrt{2}}(3g_2 - g_4 - g_6 - 3g_8 - g_{13}),$ $(-\sqrt{6}g_1 + 2(\sqrt{3}g_3 + \sqrt{6}g_5 + 5g_7))/10, (g_2 + 5g_6 - 4g_{10} - 4g_{12})/(10\sqrt{2}),$ $(g_5/\sqrt{2} - g_{11})/\sqrt{40}, -g_{10}/\sqrt{120}$	7

this basis for the Vibron model, due to the difficulty in giving a precise physical meaning to the labels. One advantage is the ease of writing the branching rules due to the absence of missing labels. Several other AMNC dynamical symmetries, one for every possible path in Fig. 5 can be invoked with the same spirit: some of them would provide *at least* new and simpler diagonalization schemes and possibly applications to quantum many-body systems.

5. Conclusions

We have shown that the use of i) Levi-Malcev decomposition and ii) theory of weighted Dynkin diagrams allow a thorough classification of algebraic models arising in bosonic models of quantum mechanics providing i) a way of separating out semisimple from non-semisimple subalgebras and ii) a neat classification of all possible conjugacy classes of three dimensional subalgebras (A_1) respectively. Well-known algebraic models, either used as conventional or pedagogical toy models or actually applied to real systems, usually adopt subalgebra chains that end up in the standard angular momentum algebra. We have shown that one can "fill the gaps", i.e. write basis elements for the whole reduction scheme. In particular the elements of any additional A_1, having different WDD, can be used to define new angular momenta operators, with respect to which one can redefine tensors and give to the whole algebra elements a different tensorial character. With respect to one of these new angular momenta, it is found that other operators behave like spin-1/2 objects, a fact that was most surprising at first. This was hinted at by the old works of Elliott [15] and Chen-Arima [14] and our paper provides a more complete collocation for their findings. In particular Chen and Arima have found spinors arising within the Interacting Boson Model of the nucleus that is completely bosonic (built upon s and d bosons). We argue that they have found the $A_1[11011]$ subalgebra of $su(6)$, the analogous of our $A_1[101]$ subalgebra of $su(4)$. Another aspect worth mentioning again is that, although hamiltonian displaying dynamical symmetries based on AMNC chains might be unphysical, the basis states associated to them might provide an alternative (maybe easier) basis for diagonalization of complex hamiltonians.

Let us repeat once again that there is nothing wrong with the way algebraic models have been formulated: the chains that pass through the angular momentum algebra are certainly the most natural and the most useful. Since the mathematical structure is nevertheless larger, we simply wanted to understand if and how the other chains can be interpreted in terms of the basic operators that define the model, if this has some physical significance and if those alternative chains can be used either to model realistic hamiltonians (and this seems very improbable) or to furnish alternative diagonalization schemes (and this is clearly possible). We believe that there might be other insightful discoveries or advantages awaiting in the still little explored angular momentum non conserving chains of algebraic models.

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