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Heat semigroup and functions of bounded variation on Riemannian manifolds

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Abstract. Let M be a connected Riemannian manifold without boundary with Ricci curvature bounded from below and such that the volume of the geodesic balls of centre x and fixed radius $r > 0$ have a volume bounded away from 0 uniformly with respect to x, and let $\big(T(t)\big)_{t \geqq 0}$ be the heat semigroup on M. We show that the total variation of the gradient of a function $u \in L^1(M)$ equals the limit of the L^1-norm of $\nabla T(t)u$ as $t \to 0$. In particular, this limit is finite if and only if u is a function of bounded variation.

Introduction

Functions of bounded variation in $\mathbb{R}^n$ are by now deeply studied, and the spaces BV are a well-established tool for studying variational problems, often with some geometric flavour. This is due to the possibility of having discontinuities along $(n-1)$-dimensional surfaces, which is not the case for functions in Sobolev spaces. In fact, a very important particular case of BV functions are characteristic functions of sets with finite perimeter in the sense of Caccioppoli-De Giorgi, the most natural class of sets where the isoperimetric problem can be formulated and solved. After various attempts to generalise to $\mathbb{R}^n$ the classical notion of BV functions of only one real variable, the new idea that opened the way for the modern theory was the definition proposed by E. De Giorgi in [11]. It is based upon a regularisation with a Gaussian convolution kernel and can be rephrased using the heat semigroup $\big(T(t)\big)_{t \geqq 0}$ in $\mathbb{R}^n$ as follows. Given a function $u \in L^1(\mathbb{R}^n)$, define its *variation* by

$$(1) \qquad |Du|(\mathbb{R}^n) = \lim_{t \to 0} \|\nabla T(t)u\|_{L^1(\mathbb{R}^n)}$$

and say that u has bounded variation, $u \in BV(\mathbb{R}^n)$, if $|Du|(\mathbb{R}^n)$ is finite. It has been shown in [11] that u has finite variation if and only if its distributional gradient is an $\mathbb{R}^n$-valued measure with finite total variation (in the sense of measures) given by $|Du|(\mathbb{R}^n)$. As a consequence, we have the equality

$$(2) \qquad |Du|(\mathbb{R}^n) = \sup\left\{ \int_{\mathbb{R}^n} u \operatorname{div} g \, dx : g \in [\mathscr{C}_c^1(\mathbb{R}^n)]^n, \|g\|_\infty \leqq 1 \right\}.$$

This last formula, first used in [21], defines directly BV functions, and can be easily generalised to Riemannian manifolds (see (1.4) below). By the way, let us just mention that further characterisations of BV functions are available, which can be used in even more general contexts, such as metric spaces endowed with a doubling measure (see e.g. [22], [2]). Moreover, notice that further connections between isoperimetric inequalities and the heat semigroup have been pointed out, even in non-euclidean contexts, such as Gaussian spaces (see e.g. [2] and also [23]).

In this paper, we address the question whether equality (1) holds on a connected Riemannian manifold M, where the left-hand side is defined as in (2), see (1.4), and in the right-hand side the heat semigroup on M is used. The answer is affirmative, at least under some geometric hypotheses, see (H1), (H2) below. We require that the Ricci curvature of M is bounded below, and that the volume of the (geodesic) balls of a given radius is bounded below by a constant independent of the centre. These assumptions bound the geometry of M in two opposite directions, see Remarks 1.2 and 1.3, and in fact both hold trivially for compact manifolds, and more generally for manifolds of bounded geometry. Moreover, (H1) and (H2) imply the uniqueness of the bounded solution of the Cauchy problem for heat equation. We deeply use hypotheses (H1) and (H2) and some of their relevant consequences, such as the Sobolev embedding and the induced regularity of the heat semigroup (see e.g. Theorem 2.1 and Theorem 2.6). We point out that we have used two different approaches for the proof of Theorem 2.6; in the compact case we get a direct proof as in [5], whereas in the non-compact case we have used Gaussian estimates, that hold true under assumptions (H1) and (H2) (see Subsection 1.1).

It seems to be worth mentioning that the equality

$$\lim_{t\to 0} \int_M |\nabla T(t)u(x)|\, d\mu(x) = |Du|(M)$$

yields an approximation in variation of BV functions by regular functions. An approximation result of this kind can be shown on general manifolds using a partition of unity argument and the corresponding result on $\mathbb{R}^n$ (see Proposition 1.4 below), and in fact we also use this as an intermediate result, but we point out that the approximation given by the heat semigroup is global and intrinsic.

The main steps in our proof are two. First, we show that the limit in (1), with M in place of $\mathbb{R}^n$, exists (this is trivial in $\mathbb{R}^n$ by monotonicity with respect to time), and then, we show that the value of the limit is the variation of u defined in (1.4).

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1. Notation and preliminaries

In this section we recall some basic facts concerning Riemannian manifolds and Sobolev and bounded variation functions.

1.1. Riemannian manifolds and geometric hypotheses. The framework of this paper is given by an n-dimensional manifold M (not necessarily compact) without boundary, i.e., $\partial M = \emptyset$, with $n \geqq 2$, verifying the geometric hypotheses (H1), (H2) stated below in this subsection. We start by fixing some notation. We denote by TM and T^*M the tangent and co-tangent bundles of M, respectively. Given a vector bundle E, we denote by $\Gamma(E)$ the smooth sections of E; in particular, $\mathscr{T}_q^p(M) = \Gamma\big((T^*M)^p \times (TM)^q\big)$ is the space of tensors of type (p, q) on M, $\Gamma(TM)$ is the space of vector fields on M and $\Gamma(T^*M)$ is the space of 1-forms on M.

The Riemannian structure on M is defined by a symmetric metric tensor $g \in \mathscr{T}_0^2(M)$. The metric g defines an isomorphism $i : \Gamma(TM) \to \Gamma(T^*M)$ in such a way that for any $X \in \Gamma(TM)$,

$$i(X)(Y) = g(X, Y), \quad \forall Y \in \Gamma(TM).$$

Using the map i, it is possible to define the tensor $\bar{g} \in \mathscr{T}_2^0(M)$ through

$$\bar{g}(\omega^1, \omega^2) = g(i^{-1}\omega^1, i^{-1}\omega^2),$$

where $i^{-1} : \Gamma(T^*M) \to \Gamma(TM)$ is the inverse of i. This tensor defines an inner product on $\Gamma(T^*M)$ such that the map i becomes an isometry.

For $X \in T_xM$ and $\omega \in T_x^*M$, we set

$$|X| = \sqrt{g(X, X)}, \quad |\omega| = \sqrt{\bar{g}(\omega, \omega)}.$$

We denote by $\langle \cdot, \cdot \rangle$ and by $|\cdot|$ the inner product and the norm induced by the metric g on any tensor T of type (p, q) in order to satisfy

$$|T| = \langle T, T \rangle^{1/2} = \sup\{T(X_1, \dots, X_p, \omega^1, \dots, \omega^q) : |X_1|, \dots, |X_p|, |\omega^1|, \dots, |\omega^q| \leqq 1\}.$$

A vector bundle E such that the fibres are endowed with an inner product is called a Riemannian vector bundle. If M is not compact, given a general Riemannian vector bundle E, we denote by $\Gamma_c(E)$ the smooth sections of E with compact support, and by $\Gamma_0(E)$ the closure of $\Gamma_c(E)$ with respect to the norm

$$\|T\|_\infty = \sup\{|T(x)| : x \in M\}, \quad T \in \Gamma(E).$$

Of course, if M is compact then $\Gamma_0(E) = \Gamma_c(E) = \Gamma(E)$.

The Riemannian metric g induces a geodesic distance d on M; we always assume that the metric space (M, d) is complete. Moreover, we denote by $B_r(x)$ the geodesic open ball centred at $x \in M$ and with radius $r > 0$. In this setting, there is a natural way of defining a measure μ on M, also without assuming M orientable; the measure μ is given in local coordinates by

$$d\mu = \sqrt{\det g}\, dx.$$

By $\nabla : \mathscr{T}_q^p(M) \to \mathscr{T}_q^{p+1}(M)$ we denote the Levi-Civita connection on M, that is the unique connection compatible with the metric g, in the sense that

$$X\langle T,S\rangle = \langle \nabla_X T,S\rangle + \langle T,\nabla_X S\rangle, \quad \forall T,S \in \mathscr{T}_q^p(M).$$

Here the notation $\nabla_X T \in \mathscr{T}_q^p(M)$ means that

$$(\nabla_X T)(X_1,\dots,X_p,\omega^1,\dots,\omega^q) := (\nabla T)(X,X_1,\dots,X_p,\omega^1,\dots,\omega^q).$$

The Levi-Civita connection is a torsion-free connection which induces the Riemann curvature tensor $\mathrm{R} \in \mathscr{T}_0^4(M)$ defined by

$$\mathrm{R}(X,Y,W,Z) = \langle \nabla_X\nabla_Y W - \nabla_Y\nabla_X W - \nabla_{[X,Y]}W, Z\rangle, \quad \forall X,Y,W,Z \in \Gamma(TM).$$

We denote by Ric the Ricci tensor of type $(2,0)$ defined pointwise by

$$\mathrm{Ric}(X,Y) = \sum_{i=1}^n \mathrm{R}(X,e_i,Y,e_i), \quad \forall X,Y \in T_xM,$$

where $\{e_i\}$ is an orthonormal basis of T_xM. The connection ∇ on a function u simply defines the covariant derivative of u, i.e.,

$$\nabla u = i(\mathrm{grad}\, u),$$

where grad is the standard gradient defined using local coordinates.

We denote by div : $\mathscr{T}_q^p(M) \to \mathscr{T}_q^{p-1}(M)$ the operator defined using the formula

$$\int_M \langle \nabla T,S\rangle \, d\mu = -\int_M \langle T, \mathrm{div}\, S\rangle \, d\mu, \quad \forall T \in \mathscr{T}_q^{p-1}(M), \forall S \in \mathscr{T}_q^p(M).$$

In this way, given a function $u \in \mathscr{C}^\infty(M)$, the Laplace-Beltrami operator applied to u is defined by

$$\Delta u = \mathrm{div}\, \nabla u;$$

notice that the operator defined above is negative definite. The Hessian of a function $u \in \mathscr{C}^\infty(M)$ is given by the tensor $\mathrm{Hess}\, u = \nabla^2 u \in \mathscr{T}_0^2(M)$. Finally, we recall the Bochner-Lichnerowitz-Weitzenböck formula; for any $u \in \mathscr{C}^\infty(M)$,

$$\frac{1}{2}\Delta|\nabla u|^2 = |\mathrm{Hess}\, u|^2 + \langle \nabla\Delta u, \nabla u\rangle + \mathrm{Ric}(\nabla u, \nabla u). \tag{1.1}$$

In the whole paper, we consider Riemannian manifolds satisfying the following two hypotheses:

(H1) There exists $K \geqq 0$ such that $\mathrm{Ric} \geqq -K$, i.e.,

$$\mathrm{Ric}(X,X) \geqq -K|X|^2, \quad \forall X \in \Gamma(TM).$$

(H2) There exists $v > 0$ such that

$$\inf_{x\in M} \mu(B_1(x)) \geqq v.$$

Hypothesis (H1), in particular, implies that the metric measure space (M, d, μ) is *locally* doubling. Indeed, Cheeger, Gromov and Taylor [7] (see also Hajłasz and Koskela [17], Section 10.1) proved that for every $x \in M$ and for every $R > 0$

$$\mu\big(B_{2R}(x)\big) \leqq 2^n \exp\{2R\sqrt{(n-1)K}\}\mu\big(B_R(x)\big). \tag{1.2}$$

The local doubling condition implies the following growth bound for the measure of balls.

Lemma 1.1. *Assuming* (H1), *for every* $x \in M$ *and* $0 < r < R$ *the inequality*

$$\frac{\mu\big(B_R(x)\big)}{\mu\big(B_r(x)\big)} \leqq 2^n \left(\frac{R}{r}\right)^n \left(\frac{e^{4R}}{e^{2r}}\right)^{\sqrt{(n-1)K}}$$

holds.

Proof. Choose $j \in \mathbb{N}$ such that

$$2^{j-1} r < R \leqq 2^j r;$$

as an immediate consequence, we have that j satisfies the inequality

$$2^j < \frac{2R}{r}.$$

Iterating the local-doubling condition (1.2), since $B_R(x) \subset B_{2^j r}(x)$ we obtain that

$$\begin{aligned}
\mu\big(B_R(x)\big) &\leqq \mu\big(B_{2^j r}(x)\big) \leqq 2^n \exp\{2^j r\sqrt{(n-1)K}\}\mu\big(B_{2^{j-1}r}(x)\big) \\
&\leqq 2^{jn} \exp\left\{ r\sum_{k=1}^{j} 2^k \sqrt{(n-1)K} \right\} \mu\big(B_r(x)\big) \\
&\leqq 2^n \left(\frac{R}{r}\right)^n \exp\{2r\sqrt{(n-1)K}(2^j - 1)\}\mu\big(B_r(x)\big) \\
&\leqq 2^n \left(\frac{R}{r}\right)^n \exp\{2r\sqrt{(n-1)K}(2R/r - 1)\}\mu\big(B_r(x)\big)
\end{aligned}$$

and the desired estimate follows. □

Remark 1.2. Condition (H2) together with Lemma 1.1 gives the following uniform lower bound for balls of a given radius $0 < \varrho < 1$:

$$\inf_{x \in M} \mu\big(B_\varrho(x)\big) \geqq v\left(\frac{\varrho}{2}\right)^n \exp\{(2\varrho - 4)\sqrt{(n-1)K}\},$$

where v is the constant in (H2). Moreover, by Lemma 1.1 a manifold M satisfying (H2) is compact if and only if $\mu(M)$ is finite.

Moreover, for a suitable constant $c > 0$, the volume growth estimate

$$\mu(B_\varrho(x)) \leqq c\varrho^n e^{c\varrho} \tag{1.3}$$

holds for all $x \in M$, $\varrho > 0$.

Remark 1.3. A counterpart of the previous remark directly follows from hypothesis (H1); in fact, see e.g. [12], Theorem 3.98, for every $x \in M$

$$\mu(B_\varrho(x)) = \omega_n \varrho^n \left(1 - \frac{s(x)}{6(n+1)}\varrho^2 + o(\varrho^2)\right),$$

where $s(x)$ is the scalar curvature at $x \in M$, i.e., the trace of the Ricci curvature. This, together with (H1), implies that there are $R_0 > 0$, $c > 0$ such that for every $x \in M$ and every $0 < \varrho < R_0$

$$\mu(B_\varrho(x)) \leqq c\varrho^n.$$

1.2. Sobolev spaces and functions of bounded variation. In this subsection we recall the definition and the basic properties of Sobolev spaces and functions of bounded variation on a manifold. We refer to [18] for more information on Sobolev spaces on manifolds, and to [1] for a discussion of BV in the Euclidean setting.

For $1 \leqq p < \infty$ and $k \in \mathbb{N}$, we denote by $H^{k,p}(M)$ the completion of the space

$$\mathscr{C}^p_k(M) := \left\{ u \in \mathscr{C}^\infty(M) : \|u\|_{k,p} := \|u\|_p + \sum_{j=1}^{k} \left(\int_M |\nabla^j u|^p \, d\mu \right)^{1/p} < +\infty \right\}$$

with respect to the norm $\|\cdot\|_{k,p}$.

Given a function $u \in L^1(M)$, define the *variation* of u by

$$|Du|(M) = \sup\left\{ \int_M u \operatorname{div} \omega \, d\mu : \omega \in \Gamma_c(T^*M), |\omega| \leqq 1 \right\}. \tag{1.4}$$

A function $u \in L^1(M)$ has bounded variation, $u \in BV(M)$, if $|Du|(M) < +\infty$. Notice that $H^{1,1}(M) \subset BV(M)$. A function $u \in BV(M)$ defines an element $Du \in (\Gamma_0(T^*M))'$, the dual space of $\Gamma_0(T^*M)$; in fact, the map

$$u \mapsto (Du, \omega) := -\int_M u \operatorname{div} \omega \, d\mu, \quad \forall \omega \in \Gamma_c(T^*M),$$

is well defined and, thanks to condition $|Du|(M) < +\infty$, can be extended by continuity to the whole space $\Gamma_0(T^*M)$. This dual space, unlike the Euclidean space, cannot be naturally identified with a vector valued measure space; what is possible to say, is that a BV function u defines, as in the Euclidean case, a finite measure $|Du|$ and a $|Du|$-measurable section $\sigma_u : M \to T^*M$ with $|\sigma_u| = 1$ almost everywhere and such that the distributional derivative Du of u is given by

$$(Du, \omega) = \int_M \langle \sigma_u, \omega \rangle d|Du|, \quad \forall \omega \in \Gamma_0(T^*M).$$

A measurable set $E \subset M$ has finite perimeter if $|D\chi_E|$ is finite, where we denote by χ_E its characteristic function. We denote the perimeter of E in a Borel set A by

$$P(E, A) = |D\chi_E|(A). \tag{1.5}$$

Assumptions (H1) and (H2) are crucial in the context of $H^{k,p}$ and BV spaces. In fact, under assumption (H1), condition (H2) is equivalent to the Sobolev embedding $H^{1,p}(M) \subset L^q(M)$, $1/q - 1/p = 1/n$, $BV(M) \subset L^{n/(n-1)}(M)$, see [25] and also [18], Theorem 3.3. As a consequence of the last embedding, the following isoperimetric inequality holds:

$$\min\{\mu(E), \mu(M \backslash E)\} \leqq c_I P(E, M)^{\frac{n}{n-1}} \tag{1.6}$$

for every $E \subset M$ and for some constant $c_I > 0$ depending only upon n, K, v. If we replace (H1) by $\mathrm{Ric} \geqq 0$, the situation is simpler. In particular, by [7], μ is doubling and then by [6], the isoperimetric inequality and the Sobolev embedding hold and (H2) follows (see also [17], Section 10).

We recall that, since for every $\omega \in \Gamma_c(T^*M)$ the map

$$u \mapsto \int_M u \operatorname{div} \omega \, d\mu$$

is continuous with respect to the $L^1(M)$ topology, then the map

$$u \mapsto |Du|(M)$$

is L^1-lower semi-continuous. Obviously, it is impossible to approximate BV function *in norm* by smooth functions: what can be done is to get an approximation *in variation*. This is well-known in the Euclidean setting (see e.g. [1], Theorem 3.9) and can be adapted to manifolds via a partition of unity argument. Notice that the following statement is true also without hypotheses (H1) and (H2).

Proposition 1.4. *For every $u \in BV(M)$ there exists a sequence $(f_j)_j \subset \mathscr{C}_c^\infty(M)$ such that $f_j \to u$ in $L^1(M)$ and*

$$|Du|(M) = \lim_{j\to\infty} \int_M |\nabla f_j| \, d\mu. \tag{1.7}$$

Proof. We fix some notation; given an open set V, for $\tau > 0$ we define the following sets:

$$V^\tau = \{x \in M : \operatorname{dist}(x, V) < \tau\}, \quad V_\tau = \{x \in M : \operatorname{dist}(x, V^c) > \tau\}.$$

If M is compact, we may take $\mathscr{C}^\infty$ instead of $\mathscr{C}_c^\infty$ and we don't need the following statement. If M is not compact, let us prove the following

Claim. For every $u \in BV(M)$ and for every $\varepsilon > 0$ there exists a function $u_\varepsilon \in BV(M)$ with compact support in M such that

$$\|u - u_\varepsilon\|_{L^1(M)} < \varepsilon, \quad |Du_\varepsilon|(M) < |Du|(M) + \varepsilon.$$

In fact, if we fix $\varepsilon > 0$, there exists a relatively compact open set B such that

$$\int_{M\setminus B} |u|\,d\mu < \varepsilon, \quad \int_{B^\varepsilon\setminus B} |u|\,d\mu < \frac{\varepsilon^2}{2}, \quad |Du|(M\setminus B) < \varepsilon;$$

moreover, we can consider $\zeta \in \mathscr{C}^\infty(M)$ with $0 \leqq \zeta \leqq 1$, $\zeta = 1$ on B, $\operatorname{supp}\zeta \subset B^\varepsilon$ and $|\nabla\zeta| \leqq 2/\varepsilon$, and take $u_\varepsilon = \zeta u$. For such a function we have

$$\int_M |u - u_\varepsilon|\,d\mu \leqq \int_{M\setminus B} |u|\,d\mu < \varepsilon,$$

and, if $\omega \in \Gamma_c(T^*M)$ with $|\omega| \leqq 1$, then

$$\int_M u_\varepsilon \operatorname{div}\omega\,d\mu = \int_M u \operatorname{div}(\zeta\omega)\,d\mu - \int_{B^\varepsilon\setminus B} u\langle\omega, \nabla\zeta\rangle\,d\mu.$$

Since $\zeta\omega \in \Gamma_c(T^*M)$ and $|\zeta\omega| \leqq 1$, we obtain that

$$\int_M u_\varepsilon \operatorname{div}\omega\,d\mu \leqq |Du|(M) + \int_{B^\varepsilon\setminus B} |u|\,|\langle\omega, \nabla\zeta\rangle|\,d\mu \leqq |Du|(M) + \varepsilon.$$

This in particular implies that $u_\varepsilon \in BV(M)$ and

$$|Du_\varepsilon|(M) \leqq |Du|(M) + \varepsilon$$

and the claim follows.

Given $u \in BV(M)$ with compact support, we consider a finite family of open bounded sets $(U_i)_{i=1,\dots,N}$ with the following properties:

(1) $U_i \cap U_j = \emptyset$ for $i \neq j$.

(2) $\exists\tau$ such that for every $0 < \eta < \tau$ the family U_i^η covers the support of u and U_i^τ is contained in a coordinate chart (V_i, ψ_i).

(3) $|Du|(\partial U_i) = 0$ for all i.

(4) $T^*(U_i^\eta) \simeq U_i^\eta \times \mathbb{R}^n$ and $d\psi_i$ is an isometry between $T_x^*(M)$ and $\mathbb{R}^n$ for every $x \in U_i^\eta$.

For every fixed $\varepsilon > 0$, we take $0 < \eta < \tau$ in such a way that

$$|Du|(M\setminus U_\eta) < \varepsilon, \quad \text{where } U_\eta = \bigcup_{i=1}^N U_{i,\eta}.$$

Let us show that for any $i = 1, \dots, N$ we can find a function $f_i \in \mathscr{C}^\infty(M)$ satisfying

$$(1.8) \qquad \int_{U_i^\eta} |u - f_i|\,d\mu < \frac{\eta\varepsilon}{2N}, \quad \int_{U_i^\eta} |\nabla f_i|\,d\mu < |Du|(U_i^\eta) + \frac{\varepsilon}{N}.$$

In fact,

$$|Du|(U_i^\eta) = \sup\left\{ \int_{U_i^\eta} u \operatorname{div}\omega \, d\mu, |\omega| \leqq 1, \operatorname{spt}\omega \subset\subset U_i^\eta \right\}.$$

Setting $w_j = g(\omega, dx^j)$ and $V_i^\eta = \psi_i(U_i^\eta)$, from

$$\operatorname{div}\omega = \frac{1}{\sqrt{\det g}} \operatorname{div}_{\mathbb{R}^n}\left(\sqrt{\det g} w\right)$$

we get

$$|Dv|(U_i^\eta) = \left\{ \int_{V_i^\eta} (v \circ \psi_i) \operatorname{div}(\sqrt{\det g} w) \, dx, |w| \leqq 1, w \in C_c^\infty(V_i^\eta) \right\} =: |D(v \circ \psi)|_\lambda$$

for every $v \in BV(U_i^\eta)$, where $\lambda = \sqrt{\det g(\psi_i)}$ and we denote by $|Dv|_\lambda$ the total variation of v in the weighted space BV_λ. The classical approximation result (see e.g. [1], Theorem 3.9) can be extended to weighted spaces, as shown in [4], Theorem 3.4, and the existence of the f_i as in (1.8) follows.

If φ_i is a partition of unity of $\operatorname{supp} u$ subordinated to the open covering U_i^η with $|\nabla\varphi_i| \leqq C/\eta$, we define

$$f = \sum_{i=1}^N \varphi_i f_i;$$

the function f is a smooth function with

$$f \equiv 0 \quad \text{on } M \setminus \bigcup_{i=1}^N U_i^\eta,$$

that is $f \in \mathscr{C}_c^\infty(M)$. Clearly

$$\int_M |u - f| \, d\mu = \int_{M \setminus \operatorname{supp} u} |f| \, d\mu + \sum_{i=1}^N \int_{\operatorname{supp} u} \varphi_i |u - f_i| \, d\mu \leqq 2 \sum_{i=1}^N \int_{U_i^\eta} |u - f_i| \, d\mu < \eta\varepsilon.$$

Moreover, for $x \in A_{i,\eta} = U_i^\eta \setminus U_{i,\eta}$, we denote by

$$I(i) = \{ j \in \{1, \dots, N\} : j \neq i, U_i^\eta \cap U_j^\eta \neq \emptyset \},$$

$$I'(x) = \{ j \in \{1, \dots, N\} : x \in U_j^\eta \},$$

$$I_i(x) = I'(x) \setminus \{i\} \subset I(i).$$

Using this notation, we have that

$$\nabla\varphi_i(x) = -\sum_{j \in I_i(x)} \nabla\varphi_j(x),$$

whence

$$\nabla f(x) = \sum_{j \in I'(x)} \varphi_j(x) \nabla f_j(x) + \sum_{j \in I_i(x)} \big(f_j(x) - f_i(x)\big) \nabla \varphi_j(x)$$

and then, since $\operatorname{supp} f \setminus U_\eta = \bigcup_{i=1}^{N} A_{i,\eta}$,

$$\begin{aligned}
\int_M |\nabla f|\, d\mu &\leqq \int_{U_\eta} |\nabla f|\, d\mu + \sum_{i=1}^{N} \int_{A_{i,\eta}} |\nabla f|\, d\mu \\
&\leqq \sum_{i=1}^{N} \int_{U_{i,\eta}} |\nabla f_i|\, d\mu + \sum_{i=1}^{N} \sum_{j \in I(i)} \int_{A_{i,\eta}} \varphi_j |\nabla f_j|\, d\mu + \sum_{i=1}^{N} \sum_{j \in I(i)} \int_{A_{i,\eta}} |f_j - f_i|\, |\nabla \varphi_j|\, d\mu \\
&\leqq |Du|(U_\eta) + \varepsilon + \sum_{i=1}^{N} \sum_{j \in I(i)} \int_{A_{i,\eta} \cap U_j^\eta} |\nabla f_j|\, d\mu + 2\frac{C}{\eta} \sum_{i=1}^{N} \sum_{j \in I(i)} \int_{A_{i,\eta} \cap A_{j,\eta}} |f_j - u|\, d\mu \\
&\leqq |Du|(U_\eta) + \varepsilon + N \sum_{i=1}^{N} \left(|Du|(A_{i,\eta}) + \frac{\varepsilon}{N} \right) + \frac{2CN}{\eta} \sum_{i=1}^{N} \int_{A_{i,\eta}} |u - f_i|\, d\mu \\
&\leqq |Du|(U_\eta) + 2\varepsilon + N^2 |Du|(M \setminus U_\eta) + 2CN\varepsilon. \quad \square
\end{aligned}$$

2. The heat semigroup on M

In this section, after sketching the construction of the heat semigroup on the manifolds we are considering and recalling some of its properties, we derive some pointwise bounds on the heat kernel and its derivatives. We refer to [9] for the missing proofs, unless otherwise stated. Consider the Dirichlet form

$$\mathscr{D}(u, v) = \int_M \langle \nabla u, \nabla v \rangle\, d\mu$$

with domain $H^{1,2}(M)$. It defines the operator Δ_2, that on smooth functions coincides with the Laplace-Beltrami operator Δ, defined as follows:

$$-\Delta_2 u = f \quad \Leftrightarrow \quad \mathscr{D}(u, v) = \int_M f v\, d\mu, \quad \forall v \in L^2(M).$$

Since Δ_2 is self-adjoint and nonpositive on $L^2(M)$, it generates a strongly continuous, positive, contractive and analytic semigroup $\big(T_2(t)\big)_{t \geqq 0}$ in $L^2(M)$, which extrapolates to a positive strongly continuous contractive semigroup $\big(T_p(t)\big)_{t \geqq 0}$ on $L^p(M)$, $1 \leqq p < +\infty$, which is even analytic for $p > 1$. The generator of $\big(T_p(t)\big)_{t \geqq 0}$ is denoted by Δ_p, and the semigroup gives the solution $u(t, x) = T(t) f(x)$ of the Cauchy problem

$$\begin{cases} \partial_t u = \Delta_p u, & t > 0,\ x \in M, \\ u(0, x) = f(x), & x \in M. \end{cases}$$

The case $p = 1$ is more delicate, and requires (H1). Analyticity was first proved by N. Varopoulos [25] without calculating the angle of sectoriality. With stronger hypotheses on the curvature (see Theorem 2.1 below) it was E. B. Davies [10] who obtained the angle $\pi/2$. We state the following theorem.

Theorem 2.1. *If M is a complete Riemannian manifold satisfying* (H1), *then the heat semigroup $(T(t))_{t\geqq 0}$ is analytic on $L^1(M)$. In addition, if the curvature of M is positive outside some compact subset of M, then it even holds that $(T(t))_{t\geqq 0}$ is bounded analytic on $L^1(M)$ with angle $\pi/2$.*

It is also important to point out that the heat semigroup has the following integral representation; for any $u \in L^1(M)$,

$$T(t)u(x) = \int_M p(x, y, t)u(y)\, d\mu(y),$$

where the heat kernel $0 < p \in \mathscr{C}^\infty(M \times M \times (0, +\infty))$ verifies always the condition

$$\int_M p(x, y, t)\, d\mu(y) \leqq 1,$$

for every $x \in M$, $t > 0$, but in our setting the equality $\int_M p = 1$ holds. This last condition is known as *stochastic completeness* and, in particular, is equivalent to the uniqueness of bounded solutions of the Cauchy problem for the heat equation (see e.g. [15]).

We also point out that some results on the heat equation for differential forms can be found in [19], [8].

2.1. Gaussian estimates for the heat kernel in the non-compact case. In this section we discuss some further properties of the heat kernel in the case where M is a non-compact manifold. We deduce inclusion (2.9) in Theorem 2.6 from the integral estimate (2.7), which is derived in [14], even though not explicitly stated as a consequence of the on-diagonal estimate (2.6). We are indebted to the referee for this observation, that allowed us to simplify the original treatment.

We start with the following definition (see Grigor'yan, [13], Definition 1.1).

Definition 2.2. A Λ-isoperimetric inequality is valid for a region $\Omega \subset M$ if for every subregion $D \subset \Omega$ the inequality

$$\lambda_1(D) \geqq \Lambda(\mu(D)) \tag{2.1}$$

holds, where $\lambda_1(D)$ is the first Dirichlet eigenvalue of D and Λ is a positive decreasing function.

The validity of isoperimetric inequalities on Riemannian manifolds is closely related to the validity of Gaussian estimates for the heat kernel. More precisely, Grigor'yan proves in [13], Theorem 5.2 the following estimate.

Theorem 2.3. *Suppose that in any ball $B_R(x)$ of a fixed radius $R > 0$ the Λ-isoperimetric inequality holds with the function $\Lambda = \Lambda_{x,R}$ defined as follows*:

$$\Lambda_{x,R}(v) = a(x, R)v^{-\nu},$$

where $a(x, R) > 0$, $\nu > 0$; then for all $x, y \in M$, for every $t > t_0 > 0$,

$$p(x,y,t) \leqq \frac{c_\nu \exp\{\lambda(t_0 - t)\}}{\min\{t_0, R^2\}^{1/\nu}\big(a(x,R)a(y,R)\big)^{1/2\nu}} \cdot \left(1 + \frac{d(x,y)^2}{t}\right)^{1+1/\nu} \exp\left\{-\frac{d(x,y)^2}{4t}\right\}, \tag{2.2}$$

where $\lambda = \lambda_1(M)$ is the spectral radius of the manifold M, i.e.,

$$\lambda_1(M) := \inf_{\Omega \subset\subset M} \lambda_1(\Omega).$$

In the same paper, Grigor'yan also proves that an isoperimetric inequality implies the Λ-isoperimetric inequality. More precisely, in [13], Proposition 2.4 it is proved that if for any $D \subset \Omega$ the inequality

$$\mathrm{Area}(\partial D) \geqq g\big(\mu(D)\big) \tag{2.3}$$

holds, for a function $g : (0, +\infty) \to (0, +\infty)$ such that $g(\xi)/\xi$ is decreasing, then the Λ-isoperimetric inequality holds for the region Ω with

$$\Lambda(\xi) = \frac{1}{4}\left(\frac{g(\xi)}{\xi}\right)^2.$$

Starting from these considerations and since the isoperimetric inequality (1.6) holds, we can state the following proposition.

Proposition 2.4. *Let M be a non-compact Riemannian manifold satisfying* (H1) *and* (H2); *then for every $0 < t \leqq 1$ and $x, y \in M$, the following Gaussian estimate holds*:

$$p(x,y,t) \leqq \frac{c_G}{t^{n/2}}\left(1 + \frac{d(x,y)^2}{t}\right)^{1+n/2} \exp\left\{-\frac{d(x,y)^2}{4t}\right\}, \tag{2.4}$$

where $c_G = c(n, K, c_I)$ is a constant depending only on the dimension n of the manifold M, the bound K of the Ricci curvature and the isoperimetric constant c_I.

Proof. Since, by Remark 1.2, $\mu(M) = +\infty$, we can apply (2.2) with an arbitrary $R > 1$ and then the isoperimetric inequality (1.6) reduces, for bounded sets $E \subset M$, to

$$\mu(E) \leqq c_I P(E, M)^{n/n-1}.$$

This last inequality is exactly (2.3) with

$$g(\xi) = \left(\frac{\xi}{c_I}\right)^{1-1/n}.$$

Then $B_R(x)$ admits a Λ-isoperimetric inequality with

$$\Lambda(\xi) = \frac{\xi^{-2/n}}{4c_I^{2-2/n}};$$

hence, (2.4) follows from (2.2) with $t \leqq 1$, $t_0 = t/2$, $v = 2/n$, $a(x, R) = 1/(4c_I^{2-2/n}) =: a$ and

$$c_G = \frac{2^{n/2} c_n \exp\{\lambda/2\}}{a^{n/2}}. \quad \square$$

We are now in a position to deduce an integral estimate from Proposition 2.4.

Proposition 2.5. *Let M be a non-compact Riemannian manifold satisfying* (H1) *and* (H2); *then for every $0 < t \leqq 1$ there exists $c > 0$ such that for every $y \in M$*

$$\int_M |\nabla p(x, y, t)| \, d\mu(x) \leqq \frac{c}{\sqrt{t}}. \tag{2.5}$$

Proof. Notice that for $x = y$ inequality (2.4) reads

$$p(x, x, t) \leqq \frac{c_G}{t^{n/2}} \tag{2.6}$$

from which (see [14], Section 3) the integral estimate

$$\int_M |\nabla_x p(x, y, t)|^2 \exp\left\{\frac{d^2(x, y)}{Dt}\right\} d\mu(x) \leqq \frac{c}{t^{n/2+1}} \tag{2.7}$$

follows for all $y \in M$, $0 < t \leqq 1$, for suitable $c > 0$, $D > 2$. Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\left(\int_M |\nabla_x p(x, y, t)| \, d\mu(x)\right)^2 &\leqq \int_M |\nabla_x p(x, y, t)|^2 \exp\left\{\frac{d^2(x, y)}{Dt}\right\} d\mu(x) \\ &\quad \times \int_M \exp\left\{-\frac{d^2(x, y)}{Dt}\right\} d\mu(x) \\ &\leqq \frac{c}{t^{n/2+1}} \int_M \exp\left\{-\frac{d^2(x, y)}{Dt}\right\} d\mu(x).\end{aligned}$$

To conclude, let us now show that

$$\int_M \exp\left\{-\frac{d^2(x, y)}{Dt}\right\} d\mu(x) \leqq c t^{n/2}$$

follows from the volume growth estimate (1.3). The proof is based on the same argument as in [16], Section 5.3, and is presented for completeness. To our end, it suffices to prove that there exists $c > 0$ such that for every $y \in M$

$$\int_M \exp\{-d_t(x, y)^2\} \, d\mu_t(x) \leqq c, \quad \forall y \in M,$$

where we have introduced the metric tensor $g_t = (Dt)^{-1} g$, with associated distance $d_t = (Dt)^{-1/2} d$ and measure $d\mu_t = (Dt)^{-n/2} \, d\mu$. For the measure μ_t, since

$$B_R^t(y) = \{x : d_t(x, y) < R\} = B_{R\sqrt{Dt}}(y),$$

by Lemma 1.1 the inequality

$$\frac{\mu_t(B_R^t(x))}{\mu_t(B_r^t(x))} \leqq 2^n \left(\frac{R}{r}\right)^n \left(\frac{e^{4R\sqrt{Dt}}}{e^{2r\sqrt{Dt}}}\right)^{\sqrt{(n-1)K}} \tag{2.8}$$

holds for every $y \in M$ and $0 < r < R$. Then, setting $B = B_{R_0}^t(y)$, with R_0 the radius coming from Remark 1.3, we can write

$$M = B \cup \bigcup_{j=0}^{\infty} A_j$$

where

$$A_j = B_{2^{j+1}R_0}^t(y) \backslash B_{2^j R_0}^t(y).$$

For $x \in A_j$ we have that $2^j R_0 \leqq d_t(x, y) < 2^{j+1} R_0$, and then

$$\exp\{-d_t(x, y)^2\} \leqq \exp\{-2^{2j} R_0^2\};$$

as a consequence, using (2.8) we obtain that

$$\begin{aligned}
&\int_{A_j} \exp\{-d_t(x, y)^2\}\, d\mu_t(x) \\
&\quad \leqq \mu_t(A_j) \exp\{-2^{2j} R_0^2\} \\
&\quad \leqq \mu_t\big(B_{2^{j+1}R_0}^t(y)\big) \exp\{-2^{2j} R_0^2\} \\
&\quad \leqq \mu_t(B) \exp\{-2^{2j} R_0^2\} 2^{n(j+2)} \exp\{2^{j+3} R_0 \sqrt{(n-1)Kt}\} \\
&\quad = \mu_t(B) \exp\{-4^j R_0^2 + 2^{j+3} R_0 \sqrt{(n-1)Kt} + n(j+2) \ln 2\} \\
&\quad = \mu_t(B) a_j,
\end{aligned}$$

where $a_j = a_j(n, K)$ are numbers depending only on the indicated parameters. A direct calculation gives that

$$\sum_{j=0}^{\infty} a_j < \infty,$$

and then we have

$$\begin{aligned}
&\int_M \exp\{-d_t(x, y)^2\}\, d\mu_t(x) \\
&\quad = \int_B \exp\{-d_t(x, y)^2\}\, d\mu_t(x) + \sum_{j=0}^{\infty} \int_{A_j} \exp\{-d_t(x, y)^2\}\, d\mu(x) \\
&\quad \leqq \mu_t(B)\left(1 + \sum_{j=0}^{\infty} a_j\right),
\end{aligned}$$

and then the assertion follows since by Remark 1.3 it holds that

$$\mu_t(B) = t^{-n/2}\mu\big(B_{R_0\sqrt{t}}(y)\big) \leqq cR_0^n$$

for every $t \leqq 1$. □

2.2. Regularity of the heat semigroup. In this subsection we investigate important regularity properties of the heat semigroup.

Theorem 2.6. *Under hypotheses* (H1), (H2)

$$D(\Delta_1) \subset H^{1,1}(M), \tag{2.9}$$

$$\Delta T(t)u \in H^{1,1}(M), \quad \forall t > 0, u \in L^1(M). \tag{2.10}$$

Proof. If M is a compact manifold the proof is achieved either by duality relying on [24], Théorème 4.2, or by the same direct argument as in [5], which we sketch below. Assume $n \geqq 3$ at first, let $T_k(y) = (-k) \vee (y \wedge k)$ be the truncation function, and let us denote by $\{u < k\}$ the set $\{x \in M : u(x) < k\}$. Multiplying the equation $u - \Delta u = f$ by $T_k(u)$ and using the divergence theorem, for $f \in L^1(M)$ we obtain

$$\int_{\{|u|<k\}} |\nabla T_k(u)|^2 \, d\mu \leqq k\|f\|_{L^1(M)}. \tag{2.11}$$

By the Sobolev embedding we know that

$$\begin{aligned}\left(\int_M |T_k(u)|^{2^*} d\mu\right)^{2/2^*} &= \left(\int_{\{|u|<k\}} |u|^{2^*} d\mu + k^{2^*}\mu(\{|u| \geqq k\})\right)^{2/2^*} \\ &\leqq c\int_M |\nabla T_k(u)|^2 \, d\mu,\end{aligned}$$

(where $2^* = 2n/(n-2)$) whence, using (2.11), we deduce

$$\mu(\{|u| \geqq k\}) \leqq c\frac{\|f\|_1^{n/(n-2)}}{k^{n/(n-2)}}, \tag{2.12}$$

which is true for every k. Now we come to the gradient estimate. Using (2.12) and (2.11) we get

$$\begin{aligned}\mu(\{|\nabla u| \geqq \lambda\}) &\leqq \mu(\{|u| \geqq k\}) + \frac{1}{\lambda^2}\int_{\{|u|<k\}\cap\{|\nabla u| \geqq \lambda\}} |\nabla u|^2 \, d\mu \\ &\leqq c\frac{\|f\|_1^{n/(n-2)}}{k^{n/(n-2)}} + \frac{k}{\lambda^2}\|f\|_1,\end{aligned}$$

which is true for every $k > 0$, $\lambda > 0$. Minimising over k we find a constant C such that $\mu(\{|\nabla u| \geqq \lambda\}) \leqq C\lambda^{n/(n-1)}$ for every $\lambda > 0$, and therefore $|\nabla u| \in L^p(M)$ for all $p < n/(n-1)$. If $n = 2$, the same argument can be used, with an arbitrary exponent $1 < q < \infty$ in place of 2^*.

If M is non-compact, we show that there is $C > 0$ such that if $u - \Delta u = f \in L^1(M)$, then $\|\nabla u\|_1 \leqq C\|f\|_1$. From the representation of the resolvent operator

$$R(\lambda, \Delta_1)f = \int_0^\infty e^{-\lambda t} T(t) f \, dt$$

we see that it is sufficient to prove that

$$\int_0^\infty e^{-\lambda t} \|\nabla T(t) f\|_1 \, dt \leqq C \|f\|_1. \tag{2.13}$$

From (2.5) we deduce for $t \leqq 1$

$$\begin{aligned} \|\nabla T(t) f\|_1 &= \int_M \left| \int_M \nabla_x p(x, y, t) f(y) \, d\mu(y) \right| d\mu(x) \\ &\leqq \int_M \int_M |\nabla_x p(x, y, t)| \, |f(y)| \, d\mu(y) \, d\mu(x) \\ &= \int_M |f(y)| \int_M |\nabla_x p(x, y, t)| \, d\mu(x) \, d\mu(y) \leqq \frac{c}{\sqrt{t}} \|f\|_1 \end{aligned}$$

whereas for $t > 1$ we have

$$\|\nabla T(t) f\|_1 \leqq \|\nabla T(1) T(t-1) f\|_1 \leqq c \|T(t-1) f\|_1 \leqq c \|f\|_1$$

by contractivity. Summing up,

$$\left\| \int_0^\infty e^{-\lambda t} \nabla T(t) f \, dt \right\|_1 \leqq c \int_0^1 \frac{e^{-\lambda t}}{\sqrt{t}} \|f\|_1 \, dt + c \int_1^\infty e^{-\lambda t} \, dt \leqq C \|f\|_1,$$

where the last constant C depends only upon the constants in the preceding inequalities.

It remains to prove (2.10). Since, as a consequence of analyticity, we know that $T(t)u \in D(\Delta_1)$ for every $t > 0$, we may write

$$\Delta T(t) u = \Delta T(t/2) T(t/2) u = T(t/2) \Delta T(t/2) u.$$

As noticed before, $T(t/2) u \in D(\Delta_1)$, so that

$$\Delta T(t/2) u \in L^1(M) \quad \text{and} \quad T(t/2) \Delta T(t/2) u \in D(\Delta_1) \subset H^{1,1}(M). \qquad \square$$

Remark 2.7. It will be important in the sequel to notice that Proposition 1.4 can be stated by saying that BV functions can be approximated in variation by functions in the domain of the Laplace-Beltrami operator, i.e., *for every $u \in BV(M)$ there exists a sequence (f_j) contained in $D(\Delta_1)$ such that $f_j \to u$ in $L^1(M)$ and* (1.7) *holds.*

3. A characterisation of total variation via heat semigroup

In this section we prove that, given a function $u \in L^1(M)$, the limit

$$\lim_{t\to 0} \int_M |\nabla T(t)u| \, d\mu \tag{3.1}$$

exists and is equal to the total variation of u defined by (1.4). We need an elementary calculus lemma.

Lemma 3.1. *Let $f \in \mathscr{C}^1(0,+\infty)$ be a positive function satisfying the condition*

$$f'(t) \leqq Kf(t), \quad \forall t > 0$$

for a positive constant $K > 0$. Then the function $t \mapsto e^{-Kt}f(t)$ is non-increasing; in particular, the limit

$$\lim_{t\to 0} f(t)$$

exists, either finite or infinite.

Proof. From the condition

$$f'(t) \leqq Kf(t)$$

and the fact that $f(t) > 0$, we can deduce that

$$\frac{f'(t)}{f(t)} \leqq K.$$

Then, with $0 < s < t$ fixed, we integrate the previous condition from s to t to obtain

$$\ln \frac{f(t)}{f(s)} \leqq K(t-s),$$

and then

$$e^{-Kt}f(t) \leqq f(s)e^{-Ks},$$

whence the monotonicity, and the thesis follows. □

We are now in a position to prove the following result.

Theorem 3.2. *Let M be a Riemannian manifold satisfying* (H1), (H2). *Then, for every $u \in L^1(M)$, the function*

$$t \mapsto e^{-Kt} \int_M |\nabla T(t)u| \, d\mu, \tag{3.2}$$

where K is the constant in (H1), *is non-increasing; in particular, the limit*

$$\lim_{t\to 0} \int_M |\nabla T(t)u(x)| \, d\mu$$

exists, either finite or infinite.

Proof. Let $u \in L^1(M)$ be a given function; we are going to prove that the function $f : (0, +\infty) \to \mathbb{R}$ defined by

$$f(t) := \|\nabla T(t)u\|_{L^1}$$

is differentiable and satisfies $f' \leqq Kf$ for every $t > 0$. Observe that for every $t > 0$ the function $T(t)u$ is analytic, and then the equality

$$\partial_t |\nabla T(t)u| = \frac{1}{|\nabla T(t)u|} \langle \nabla \Delta T(t)u, \nabla T(t)u \rangle$$

holds μ-a.e. in M. Since

$$\left| \frac{1}{|\nabla T(t)u|} \langle \nabla \Delta T(t)u, \nabla T(t)u \rangle \right| \leqq |\nabla \Delta T(t)u|,$$

by (2.10), we can conclude that f is differentiable and we can differentiate under the integral sign. Using the Bochner-Lichnerowitz-Weitzenböck formula (1.1), we obtain

$$\begin{aligned}
f'(t) &= \frac{d}{dt} \int_M |\nabla T(t)u| \, d\mu \\
&= \frac{d}{dt} \int_M \langle \nabla T(t)u, \nabla T(t)u \rangle^{1/2} \, d\mu \\
&= \frac{1}{2} \int_M \frac{1}{|\nabla T(t)u|} \partial_t \langle \nabla T(t)u, \nabla T(t)u \rangle \, d\mu \\
&= \int_M \frac{1}{|\nabla T(t)u|} \langle \partial_t \nabla T(t)u, \nabla T(t)u \rangle \, d\mu \\
&= \int_M \frac{1}{|\nabla T(t)u|} \langle \nabla \Delta T(t)u, \nabla T(t)u \rangle \, d\mu \\
&= \frac{1}{2} \int_M \frac{\Delta |\nabla T(t)u|^2}{|\nabla T(t)u|} \, d\mu - \int_M \frac{|\mathrm{Hess}\, T(t)u|^2}{|\nabla T(t)u|} \, d\mu - \int_M \frac{\mathrm{Ric}\big(\nabla T(t)u, \nabla T(t)u\big)}{|\nabla T(t)u|} \, d\mu.
\end{aligned}$$

Consider the following equality:

$$\begin{aligned}
\int_M \frac{\Delta |\nabla T(t)u|^2}{|\nabla T(t)u|} \, d\mu &= \int_M \frac{1}{|\nabla T(t)u|} \Delta |\nabla T(t)u|^2 \, d\mu \\
&= -\int_M \left\langle \nabla \frac{1}{|\nabla T(t)u|}, \nabla |\nabla T(t)u|^2 \right\rangle d\mu \\
&= \frac{1}{2} \int_M \frac{1}{|\nabla T(t)u|^3} \langle \nabla |\nabla T(t)u|^2, \nabla |\nabla T(t)u|^2 \rangle \, d\mu \\
&= \frac{1}{2} \int_M \frac{1}{|\nabla T(t)u|^3} |\nabla |\nabla T(t)u|^2|^2 \, d\mu.
\end{aligned}$$

Taking into account that

$$|\nabla|\nabla T(t)u|^2| = \sup_{|X| \leqq 1} |\nabla_X |\nabla T(t)u|^2|$$
$$= 2 \sup_{|X| \leqq 1} \langle \nabla_X \nabla T(t)u, \nabla T(t)u \rangle$$
$$\leqq 2 \sup_{|X| \leqq 1} |\nabla_X \nabla T(t)u| \cdot |\nabla T(t)u|$$
$$\leqq 2|\mathrm{Hess}\, T(t)u| \cdot |\nabla T(t)u|,$$

we obtain

$$\int_M \frac{\Delta|\nabla T(t)u|^2}{|\nabla T(t)u|} \, d\mu = -\frac{1}{2} \int_M \frac{1}{|\nabla T(t)u|^3} \langle \nabla|\nabla T(t)u|^2, \nabla|\nabla T(t)u|^2 \rangle \, d\mu$$
$$= -\frac{1}{2} \int_M \frac{1}{|\nabla T(t)u|^3} |\nabla|\nabla T(t)u|^2|^2 \, d\mu$$
$$\leqq 2 \int_M \frac{1}{|\nabla T(t)u|} |\mathrm{Hess}\, T(t)u|^2 \, d\mu.$$

In conclusion, taking into account that $\mathrm{Ric}\big(\nabla T(t)u, \nabla T(t)u\big) \geqq -K|\nabla T(t)u|^2$, we have obtained that

$$f'(t) \leqq K \int_M |\nabla T(t)u| \, d\mu = Kf(t);$$

applying Lemma 3.1, the conclusion follows. □

We can now prove that the limit (3.1) coincides with the total variation of $u \in BV(M)$.

Theorem 3.3. *Let M be a Riemannian manifold satisfying* (H1), (H2). *Then, for every $u \in L^1(M)$, the following formula holds*:

$$\lim_{t \to 0} \int_M |\nabla T(t)u(x)| \, d\mu(x) = |Du|(M). \tag{3.3}$$

Proof. Notice that, since $T(t)u \to u$ in L^1 as $t \to 0$, by the lower semicontinuity we have that

$$|Du|(M) \leqq \liminf_{t \to 0} \int_M |\nabla T(t)u| \, d\mu.$$

This inequality immediatly implies (3.3) if $u \in L^1(M) \backslash BV(M)$, both sides being $+\infty$. For $u \in BV(M)$, we prove that

$$\limsup_{t \to 0} \int_M |\nabla T(t)u(x)| \, d\mu(x) \leqq |Du|(M).$$

By Remark 2.7, there exists a sequence $(f_j)_j \subset D(\Delta_1)$ converging to u in $L^1(M)$ such that

$$\lim_{j\to+\infty} \int_M |\nabla f_j|\, d\mu = |Du|(M).$$

Since $(f_j)_j \subset D(\Delta_1)$, we have $T(t)f_j \to f_j$ as $t \to 0$ in the graph norm, and then, by the inclusion $D(\Delta_1) \subset H^{1,1}(M)$, also in the $H^{1,1}(M)$ norm. Therefore, for every $\varepsilon > 0$ we can find a sequence $(t_j)_j$ converging to 0 such that $\|\nabla T(t_j)f_j - \nabla f_j\|_1 < \varepsilon$, whence

$$\int_M |\nabla T(t_j)f_j|\, d\mu < \int_M |\nabla f_j|\, d\mu + \varepsilon.$$

Moreover, for every $f \in D(\Delta_1)$ and $t > 0$, from Theorem 3.2 we get

$$e^{-Kt} \int_M |\nabla T(t)f|\, d\mu \leqq \int_M |\nabla f|\, d\mu; \tag{3.4}$$

then, defining $g_j = T(t_j)f_j$, the sequence $(g_j)_j$ converges to u in $L^1(M)$, and also $T(t)g_j \to T(t)u$ in $L^1(M)$ as $j \to +\infty$ for every $t > 0$. From the lower semicontinuity of the total variation and (3.4) it follows that

$$\begin{aligned}
\limsup_{t\to0} \int_M |\nabla T(t)u|\, d\mu &= \limsup_{t\to0}\, e^{-Kt} \int_M |\nabla T(t)u|\, d\mu \\
&\leqq \limsup_{t\to0} \left(\liminf_{j\to+\infty} e^{-Kt} \int_M |\nabla T(t)g_j|\, d\mu \right) \\
&\leqq \limsup_{j\to+\infty} \int_M |\nabla g_j|\, d\mu = \limsup_{j\to+\infty} \int_M |\nabla T(t_j)f_j|\, d\mu \\
&\leqq \limsup_{j\to+\infty} \int_M |\nabla f_j|\, d\mu + \varepsilon = |Du|(M) + \varepsilon.
\end{aligned}$$

Since ε was arbitrary, the proof is complete. □

Added in proof. After this work was completed, our main result has been proved in a simpler way, and without requiring hypothesis (H2), by A. Carbonaro and G. Mauceri, see *A note on bounded variation and heat semigroup on Riemannian manifolds*, to appear in Bull. Austr. Math. Soc.

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