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Quasidiagonal Solutions of the Yang–Baxter Equation, Quantum Groups and Quantum Super Groups

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Abstract. This paper answers a few questions about algebraic aspects of bialgebras, associated with the family of solutions of the quantum Yang–Baxter equation in *Acta Appl. Math.* **41** (1995), pp. 57–98. We describe the relations of the bialgebras associated with these solutions and the standard deformations of GL_n and of the supergroup $\mathrm{GL}(m \mid n)$. We also show how the existence of zero divisors in some of these algebras are related to the combinatorics of their related matrix, providing a necessary and sufficient condition for the bialgebras to be a domain. We consider their Poincaré series, and we provide a Hopf algebra structure to quotients of these bialgebras in an explicit way. We discuss the problems involved with the lift of the Hopf algebra structure, working only by localization.

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Introduction

In [34], Reshetikhin *et al*. described a construction (also discovered by others and called FRT construction according to most literature) which associates a bialgebra to every matrix with coefficients in a given field K. If the matrix in question is a solution of the quantum Yang–Baxter equation, one gets what is called a 'dual quasitriangular bialgebra' or 'coquasitriangular bialgebra'. This is 'dual quasitriangular', as a bialgebra is equivalent to the fact that the category of its corepresentations is braided (see [16, 20, 21], or [10] for a weaker result).

In [11], one finds a class of solutions of the quantum Yang–Baxter equation, which we call 'quasidiagonal' because the matrices belonging to this family are such that $R^{ab}_{cd} = 0$ unless $\{a, b\} = \{c, d\}$. Our purpose is to study the structure of the dual quasitriangular bialgebras associated with these solutions. We will mainly be interested in Poincaré series, in the existence of zero divisors, in the possibility of providing Hopf algebra structures as explicitly as possible, and in the description of those bialgebras in terms of well-known objects of quantum group theory. We will show how, even though we can always reduce them to standard deformations, some peculiar phenomena still arise.

We will first focus on bialgebras associated with a particular case of quasidiagonal solution, namely those for which either $R^{ab}_{ba} = 0$ or $R^{ba}_{ab} = 0$ for every $a, b \in \{1, \ldots, n\}$. We call them 'type II' according to Hazewinkel's terminology. We describe these bialgebras in terms of (twists) of standard deformations of the general linear group or supergroup. In the case where the bialgebra has no nilpotent elements, this is just the multiparameter deformation on the algebra of functions of a matrix semigroup. This is already pointed out in [11], and this bialgebra has been studied by dozens of people. In particular, in [1] one finds how it can be embedded in a Hopf algebra, and the fact that it is a twist of the one parameter standard deformation of the function algebra on the semigroup of $n \times n$ matrices with coefficients in the field K.

If a bialgebra of type II has nilpotents, we show that its Poincaré series is the same as that of the algebra of functions on a supermanifold of matrices, but it is clear that it cannot be a deformation of that object, since it is a bialgebra and not a super bialgebra. However, we show that it is a twist of a sub-bialgebra of the 'bosonization' of Manin's deformation of the algebra of functions on $\mathrm{Mat}(p, n - p)$. Bosonization is a process that associates to a Hopf superalgebra (resp. a super bialgebra) a genuine Hopf algebra (resp. bialgebra). This process has been described by D. Fischman in [7], for the universal enveloping algebra of the Lie superalgebra of endomorphisms of a super vector space and it is a type of Radford biproduct. This process has been introduced in the more general context of braided categories by S. Majid in [25]. A survey of some of the results in the area can be found in [32].

The author discovered after the completion of this work that a very similar relation between deformations of $\mathrm{Mat}(p, n - p)$ and bialgebras of type II were also found by S. Majid and M. J. Rodriguez-Plaza in [27] and [28] (which also contain the results in [27]) using the superization process, which is essentially the inverse of bosonization.

The identification of a type II bialgebra with a sub-bialgebra of the bosonization of a Hopf superalgebra (see [12] for an explicit description of the antipode construction), allows us to find easily a minimal Hopf algebra containing the type II bialgebra by localization. The importance of this computation does not lie so much in the description of the antipode, which is basically that in [12], but in the rigorous proof that the localization we perform makes sense according to Ore's noncommutative localization theory. This allows us to say more about algebraic properties of the localized algebra: for instance, that it does not collapse and that the bialgebra we started with is really embedded in its localization. The case $p = 1$, $n = 2$ was already handled in [14], and [8], but in that case it was not clear at all that what one should invert is some sort of quantum determinant.

At this point it becomes very easy to see that if we factor a type II bialgebra by the ideal generated by its nilpotent elements, we again obtain a bialgebra which is isomorphic to the twist of a tensor product of two standard deformations of the algebras of functions on two matrix semigroups, which recalls the obvious classical properties of $\mathrm{GL}(p \mid n-p)$.

Once the bialgebras of type II are neatly described, we can go through the whole family of solutions in [11]. In this case, the associated bialgebra behaves poorly. It will have zero divisors which are not nilpotent, and/or it will have a too fast growing Poincaré series.

A key result in this paper is a necessary and sufficient condition for the matrix R in order to have an associated bialgebra that is a domain. Moreover, once one limits the study to those bialgebras whose Poincaré series does not grow too fast, it turns out that there is a straightforward way to factor out zero divisors so that the quotient is a twist of a tensor product of bialgebras of type II. In particular, those bialgebras can be factored and their quotient is a domain which can easily be described in terms of standard deformations of matrix semigroups and can be embedded in a Hopf algebra. A standard way to lift the Hopf algebra structure explicitly is still an open question, even though, by theorems in [29] and [18] we know that formal solutions are always possible, and that a weak antipode always exists. Besides, our factorization is too natural not to be compatible with their construction. We show what kind of difficulties can arise if one wants to extend the antipode to an extension of $A(R)$ by localization.

We also provide a different quotient which is an amusing domain, although it is no longer a bialgebra, but only a comodule algebra: we also show what its relation is with quantum planes.

The first two sections are merely introductory: in Section 1 we give a short description of Hazewinkel's ‘quasidiagonal’ solutions, while in Section 2 we recall the FRT construction.

In Section 3 we describe in detail the bialgebras associated with a quasidiagonal solution of the quantum YB equation such that for every $a \neq b$ either $R_{ba}^{ab} \neq 0$ or $R_{ab}^{ba} \neq 0$, i.e. those of type II. We describe their relation with Manin's deformations of the general linear supergroup and we provide their Hopf envelope by finding a proper Ore set.

In Section 4, we show the existence of zero divisors in a general bialgebra associated with any quasidiagonal solution of the quantum YB equation, and we describe explicitly the relations between this algebraic property and the combinatorics of the matrix R.

The presence of zero divisors cannot be avoided in most cases, but we build different kinds of factor algebra: the first quotient turns out to be a twist of a tensor product of bialgebras of the type described in Section 3. This factor algebra is easily seen to be a bialgebra, and we are able to embed it into a Hopf algebra in a way that is consistent with the embedding of bialgebras of type II into a Hopf algebra.

In the Appendix, we provide a description of the construction of twisting a comodule algebra by means of a 2-cocycle. This is actually very standard material but we decided to include it in this paper as an appendix, since notations differ widely in the literature. We show the relation with the three different definitions in the survey [32], in [1] and in [6] and [33]. For further reading on the subject, one can consult [18].

The field K over which we will work shall be always be of characteristic different from 2.

1. The 'Quasidiagonal Solutions' of the Yang–Baxter Equation

Let $R = (r^{ab}_{cd})$ be an $n^2 \times n^2$ invertible matrix over a field K that we fix from now on.

R is said to be 'quasidiagonal' if $r^{ab}_{cd} = 0$ unless $\{a, b\} = \{c, d\}$. We can view R as an operator on $V \otimes V$ for some vector space V with basis $\{e^1, \dots, e^n\}$ as follows

$$R(e^i \otimes e^j) = \sum_{k,l} r^{ij}_{kl} e^k \otimes e^l.$$

In this section, we describe the conditions presented in [11] that R needs to fulfill in order to be both quasidiagonal and a solution of the quantum Yang–Baxter equation (q-YB equation) in the form:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{1.1}$$

In the above formula, if R represents the operator acting on $V \otimes V$, R_{ij} stands for the operator acting, on $V \otimes V \otimes V$, as R on the ith and jth component, and as the identity elsewhere. We will stick to this notation from now on.

In order to find these conditions, Hazewinkel assumed that R is a quasidiagonal solution of the q-YB equation, and looked for the relations that these assumptions imply for the matrix entries. Then, he showed that these relations are also sufficient for a quasidiagonal matrix to be a solution of the q-YB equation. We present his procedure here.

Let the R be a quasidiagonal solution of (1.1). Each of its entries has two upper and two lower indices all belonging to the index set $I := \{1, 2, \dots, n\}$. We define a relation on I in the following way:

$$a \leqslant b \Leftrightarrow r^{ab}_{ba} \neq 0. \tag{1.2}$$

With the given assumption on R, $\leqslant$ turns out to be a pre-order (not antisymmetric). Then, one can define a relation of 'connectedness', denoted by $\sim$, on I, namely:

$$a \sim b \Leftrightarrow a \leqslant b \quad \text{or} \quad b \leqslant a. \tag{1.3}$$

Provided that the assumption on R holds, $\sim$ turns out to be an equivalence relation. An equivalence class for '$\sim$' will be called a '*block*'.

Since $\leqslant$ is not antisymmetric, it may happen that inside a block, one has $a \leqslant b$ and $b \leqslant a$ with $a \neq b$: such a and b will be said to be '*strongly connected*' (notation: $a \simeq b$). Strong connectedness is an equivalence relation inside a block, and its equivalence classes will be called '*components*'.

Then, one finds the following condition on components and blocks, in order to have a solution of (1.1):

PROPOSITION 1.1 ([11]). *Let C be a component of a block B determined by a quasidiagonal solution of* (1.1). *Then, there is a $\lambda \neq 0$ in K, such that for all $a, b \in C$ $(a \neq b)$*:

$$r_{aa}^{aa} = r_{bb}^{bb} = r_{ba}^{ab} = r_{ab}^{ba} = \lambda, \qquad r_{ab}^{ab} = r_{ba}^{ba} = 0. \tag{1.4}$$

Since a block B consists of several components $C_1, C_2, \ldots, C_p$ and all its elements are connected, we can renumber the components in such a way that $C_1 < C_2 < \cdots < C_p$, where the ordering of the components is the one that agrees with the ordering of the indices belonging to the component. Namely, $C_k < C_j$ if and only if $a \leqslant b$, and $b \nleqslant a$ for every $a \in C_k$ and $b \in C_j$.

A description follows of the submatrix R_B whose entries are indexed only by elements in block B.

PROPOSITION 1.2 ([11]). *Let R be a quasidiagonal solution of* (1.1), *B be a block of I determined by R and $C_1 < C_2 < \cdots < C_p$ be the components of B. Let λ_j be the scalar corresponding to the component C_j according to Proposition* 1.1, *for all $1 \leqslant j \leqslant p$. Then, there are scalars $y \neq 0$ and $z \neq 0$ such that for all $a \in C_i$ and $b \in C_j$ with $i < j$*:

$$r_{ab}^{ba} = 0,\ r_{ba}^{ab} = y \quad \textit{and} \quad r_{ab}^{ab} r_{ba}^{ba} = z. \tag{1.5}$$

Moreover, all the λ_j's satisfy the same quadratic equation

$$\lambda_j^2 = y\lambda_j + z. \tag{1.6}$$

The following proposition tells us how blocks should match with each other:

PROPOSITION 1.3 ([11]). *Let $B_1, \ldots, B_m$ be the blocks of $\{1, 2, \ldots, n\}$, for the quasidiagonal solution R of* (1.1). *Then, there are nonzero scalars $z_{st} = z_{ts}$ for $s \neq t \in \{1, \ldots, m\}$, such that*

$$r_{ab}^{ab} r_{ba}^{ba} = z_{st} \quad \forall a \in B_s, b \in B_t, s \neq t. \tag{1.7}$$

In the following theorem, one essentially sees that the conditions in Propositions 1.1–1.3, are also sufficient, for a quasidiagonal matrix R to be a solution of the quantum Yang–Baxter equation.

THEOREM 1.4 ([11]). *Let K be a field. Let $I = \{1, \ldots, n\}$. Divide I into subsets and call them 'blocks'. Split the blocks into subsets and call those subsets 'components'. Choose scalars in K as follows*:

(i) *For each block B_s consisting of a single component C choose $\lambda_s \neq 0$*;

(ii) *For each block B_s with more components, choose $y_s \neq 0$ and $z_s \neq 0$ and for each component C_j^s in B_s choose λ_j^s satisfying $(\lambda_j^s)^2 = y_s\lambda_j^s + z_s$;*
(iii) *For each two distinct blocks B_s, B_t, choose $z_{st} = z_{ts} \neq 0$;*
(iv) *For each $a,\ b \in B_s$ with $a > b$, choose a scalar $x_{ab} \neq 0$;*
(v) *For each $a \in B_s$ and $b \in B_t$ with $s > t$, choose $x_{ab} \neq 0$.*

Now, define the r_{cd}^{ab} as follows:

(I) $r_{aa}^{aa} = r_{bb}^{bb} = r_{ba}^{ab} = r_{ab}^{ba} = \lambda_j^s$, *and* $r_{ab}^{ab} = r_{ba}^{ba} = 0$ *for* $a \neq b$, $a, b \in C_j^s \subset B_s$;
(II) $r_{ba}^{ab} = y_s$, $r_{ab}^{ba} = 0$, $r_{ab}^{ab} = z_s x_{ba}^{-1}$, $r_{ba}^{ba} = x_{ba}$ *for* $a, b \in B_s$, $a < b$;
(III) $r_{ab}^{ab} = x_{ab}$, $r_{ba}^{ba} = z_{st} x_{ab}^{-1}$, $r_{ba}^{ab} = r_{ab}^{ba} = 0$ *for* $a \in B_s$, $b \in B_t$ *and* $s < t$;
(IV) $r_{cd}^{ab} = 0$ *if* $\{a, b\} \neq \{c, d\}$.

Then, the matrix R thus specified constitutes a quasidiagonal solution of the quantum Yang–Baxter equation. Moreover, up to a permutation of $\{1, \ldots, n\}$*, every solution satisfying* (IV) *can be described this way.*

Remarks. If R is quasidiagonal, then all powers of R will be quasidiagonal as well, and therefore all elements in $K[R]$ will have the same property. If R is also invertible, all the elements in $K[R, R^{-1}]$ will be quasidiagonal. This is a consequence of Cayley–Hamilton's theorem, for instance.

After [11] was published, an article by Markl and Majid was published on the glueing of Yang–Baxter operators [26]. In their Theorem 2.7, they introduce a procedure that associates to two given solutions of the quantum Yang–Baxter equation, a third one. They mainly focus on a special case, namely when the operators are q-Hecke. This means in our notation that there is an invertible $q \in K$ such that $(PR)^2 = (q - q^{-1})PR + 1$ where P is the permutation matrix $P_{cd}^{ab} = \delta_d^a \delta_c^b$. It is possible to reduce the solutions in [11] to an iterated application of their results, starting with multiples of the permutation matrix, and scalar matrices. On the other hand, the matrices in [11] are not all obtained by glueing, since they are not necessarily Hecke. Indeed, the characteristic polynomial of PR is

$$\begin{aligned}\det(PR - \beta) \ &= \ \prod_{i=1}^{n}(r^{aa} - \beta) \prod_{a<b;\, a\sim b; a\simeq b} (\lambda_a - \beta)^2 \times \\ &\times \prod_{a<b\, a\sim b,\, a\not\simeq b} (\beta^2 - \beta y_a + \lambda_a \mu_a) \prod_{a<b\, a\not\sim b} (\beta^2 - z_{st}).\end{aligned}$$

Hence, its minimal polynomial might have a degree higher than 2. What is very useful in [11] is that we have a very explicit combinatorial description of how blocks must match with each other, which is very simple, and that it is a classification. On the other hand, the results in [26] refer to a less limited family. Majid and Markl also treat the bialgebras and Hopf algebras related to the q-Hecke operators arising by glueing. An interested reader could find in their results a categorical explanation for the choices and the phenomena in Section 4, although the algebras we describe are not necessarily associated with q-Hecke operators.

2. The FRT Construction

We start this section with a silly remark about notation. The 'R-matrices' that we will use from now on, are the transposed form of those appearing in most of the literature. This choice was made for convenience, since our main tool was the classification provided in [11], and we preferred to be faithful to the notation adopted there. This will not, however, change the essence of our results. Indeed, the bialgebras one meets in the literature are very close to ours (i.e. they have opposite algebra or coalgebra structure or they are isomorphic to ours), so they have essentially the same properties as far as being PBW algebras, integral domains or being a Hopf algebra is concerned.

We assume that the reader is acquainted with the definitions of bialgebra, Hopf algebra, braided category. Good references for these notions are the surveys [4, 15] or [18]. They contain a description of most of the standard results in the theory.

Let us start with some definitions, and terminology:

DEFINITION 2.1. A 'dual quasitriangular bialgebra' $(H, m, u, \Delta, \varepsilon, r)$ is a bialgebra H together with a linear form r on $H \otimes H$ such that: r is invertible under the (convolution) product defined in $(H \otimes H)^*$ (linear dual), and

$$m\sigma = r * m * r^{-1}, \tag{2.1}$$

$$r(m \otimes \mathrm{id}) = r_{13} * r_{23}, \tag{2.2a}$$

$$r(\mathrm{id} \otimes m) = r_{13} * r_{12}. \tag{2.2b}$$

Here, r_{12}, r_{23}, and r_{13} are linear forms on $H^{\otimes 3}$ defined by

$$r_{12} = r \otimes \varepsilon, \quad r_{23} = \varepsilon \otimes r, \quad r_{13} = (\varepsilon \otimes r)(\sigma_{H,H} \otimes \mathrm{id}).$$

The form r is called the 'universal R-form' of H. A Hopf algebra is dual quasitriangular if the underlying bialgebra is.

The property of a bialgebra being dual quasitriangular is important when we deal with its representations. Indeed, if H is dual quasitriangular then, given two H-comodules V and W, there is a standard comodule isomorphism $c: V \otimes W \to W \otimes V$ defined by means of r. This is built in such a way that the category of finite dimensional corepresentations is a 'braided category' (see [10, Prop. 1.1, 16, 21] or the survey in [15]).

The FRT construction is a standard procedure to get a bialgebra starting from any invertible solution of the q-YB equation. This was discovered by many people independently. In order to unify notations, we give the statements of the existence theorem, and describe the construction, following the survey [15].

THEOREM 2.2. *Let V be a finite-dimensional vector space and R an endomorphism of $V \otimes V$. There exists a bialgebra $A(R)$ together with a linear map $\Delta_V: V \to A(R) \otimes V$ such that*

(i) *the map Δ_V equips V with the structure of a comodule over $A(R)$;*

(ii) *the map RP becomes a comodule map with respect to this structure* (P *is the flip operator associated to the permutation matrix*);

(iii) *if A' is another bialgebra coacting on V with linear map Δ'_V such that* (ii) *is satisfied, there exists a unique bialgebra morphism $f\colon A(R) \to A'$ such that $\Delta'_V = (f \otimes \mathrm{id}_V) \circ \Delta_V$.*

The bialgebra $A(R)$ is unique up to isomorphism.

Proof. A complete proof of these statements is to be found, for instance, in [34]. We give only a short description of the construction of $A(R)$. Let $\{e^i\}_{1\leqslant i\leqslant n}$ be a basis of V, so that R is represented by the matrix r^{ab}_{cd} such that

$$R(e^i \otimes e^j) = \sum_{k,l} r^{ij}_{kl} e^k \otimes e^l.$$

We pick n^2 indeterminates t^i_j, $1 \leqslant i, j \leqslant n$. Then, the bialgebra $A(R)$ is defined as the quotient of the free algebra generated by the t^i_j's by the two-sided ideal $I(R)$ generated by all elements

$$Sp^{ab}_{cd} = \sum_{ij} r^{ab}_{i,j} t^i_c t^j_d - \sum_{k,l} r^{kl}_{cd} t^b_l t^a_k \tag{$*$}$$

for every $a, b, c, d \in \{1, \dots, n\}$. Those relations can also be expressed in terms of matrix products as follows. Let I_n denote the $n \times n$ identity matrix. Define the matrices T, T_1 and T_2 as $T_{ij} := (t^i_j)$, $T_1 := T \otimes I_n$ and $T_2 := I_n \otimes T$, so that $(T_1)^{ij}_{kl} = (t^i_k \delta^j_l)$ and $(T_2)^{ij}_{kl} = (\delta^i_k t^j_l)$. Then the relations for $A(R)$ can be expressed by means of

$$RT_1T_2 = T_2T_1R. \tag{$**$}$$

One can check that there is a unique bialgebra structure on $A(R)$ such that

$$\Delta(t^i_j) = \sum_k t^i_k \otimes t^k_j, \qquad \varepsilon(t^i_j) = \delta^i_j. \tag{$***$}$$

The coaction on V is given by $\Delta_V(e^i) = \sum_j t^i_j \otimes e^j$. □

To clear the air a little, we point out that one does not need a solution of the q-YB equation in order to construct a bialgebra. The observation that if R is a solution of q-YB equation one gets a dual quasitriangular bialgebra is due independently to Majid in [20] and to Larson and Towber (see [16]). This is now a standard fact that can be found in [18] and [15].

THEOREM 2.3. *If R is as in Theorem* 2.1 *and satisfies the Yang–Baxter equation* (1.1), *there is a unique linear form r on $A(R)$ turning $A(R)$ into a dual quasi-triangular bialgebra such that the isomorphism RP is the standard isomorphism $V^{\otimes 2} \to V^{\otimes 2}$ defined by means of r. We have $r(t^i_j \otimes t^k_l) = r^{ik}_{jl}$.*

Remark. We recall that if R is invertible, then r is invertible under convolution and $r^{-1}(t^i_j \otimes t^l_k) = (R^{-1})^{il}_{jk}$.

We end this section describing two well-known bialgebras that can be obtained by means of this construction.

EXAMPLE 2.1. Take an $n^2 \times n^2$ quasidiagonal solution R of (1.1) such that one has one single block B consisting of one single component. We will call this type of block (and their associated matrix R_B), '*type I*'. In this case, the relations in $A(R)$ are all trivial. Indeed

$$\begin{aligned}
Sp^{aa}_{cc} &= \textstyle\sum_{ij} r^{aa}_{i,j} t^i_c t^j_c - \sum_{k,l} r^{kl}_{cc} t^a_l t^a_k = \lambda t^a_c t^a_c - \lambda t^a_c t^a_c \equiv 0 \qquad (a = b, c = d),\\
Sp^{aa}_{cd} &= \textstyle\sum_{ij} r^{aa}_{i,j} t^i_c t^j_d - \sum_{k,l} r^{kl}_{cd} t^a_l t^a_k = \lambda t^a_c t^a_d - \lambda t^a_c t^a_d \equiv 0 \qquad (a = b, c \neq d),\\
Sp^{ab}_{cc} &= \textstyle\sum_{ij} r^{ab}_{i,j} t^i_c t^j_c - \sum_{k,l} r^{kl}_{cc} t^b_l t^a_k = \lambda t^b_c t^a_c - \lambda t^b_c t^a_c \equiv 0 \qquad (a \neq b, c = d),\\
Sp^{ab}_{cd} &= \textstyle\sum_{ij} r^{ab}_{i,j} t^i_c t^j_d - \sum_{k,l} r^{kl}_{cd} t^b_l t^a_k = \lambda t^b_c t^a_d - \lambda t^b_c t^a_d \equiv 0 \qquad (a \neq b, c \neq d).
\end{aligned}$$

Therefore $A(R)$ is isomorphic, as an algebra, to the free algebra on n^2 generators. Once we give to each t^i_j degree 1, and call $A(R)_r$ the homogeneous component of degree r, we see that the Poincaré series of $A(R)$ is

$$P(A(R), t) = \sum_{r \geqslant 0} \dim(A(R)_r) t^r = \sum_{r \geqslant 0} n^{2r} t^r = (1 - n^2 t)^{-1}.$$

EXAMPLE 2.2. Let R be a quasidiagonal solution of (1.1) such that I is a single block B with all components C_i of size 1, i.e. such that $|C_i| = 1$. We will call this sort of block (and their corresponding matrices R_B) '*type II*'. Suppose also that $\lambda_i = \lambda$ is constant for all components in the block. Then, $A(R)$ is the standard multiparameter deformation of the algebra of functions on $M_n(K)$, the semigroup of $n \times n$ matrices with entries in K. One can always multiply the matrix by a constant so that the second solution μ of Equation (1.6) is equal to -1 (so $z = \lambda$), as in [1], or to $-\lambda^{-1}$ (so, $z = 1$), as in most of the literature.

The parameters p_{ji} for $j > i$ and λ in [1], correspond, respectively, to $r^{ij}_{ij}\lambda^{-1}$ and $\lambda/(\lambda - y)$. The relations for $A(R)$ are

$$\begin{aligned}
t^i_j t^l_k &= (r^{il}_{il})^{-1} r^{jk}_{jk} t^l_k t^i_j + y (r^{il}_{il})^{-1} t^l_j t^i_k \quad (i > l, j > k),\\
t^i_j t^i_k &= (r^{kj}_{kj})^{-1} \lambda t^i_k t^i_j \quad (j > k),\\
t^i_j t^l_k &= (r^{il}_{il})^{-1} r^{jk}_{jk} t^l_k t^i_j \quad (i > l, j \leqslant k).
\end{aligned}$$

It is well known that $A(R)$ is Noetherian as an algebra, it is a domain and that its Poincaré series is the same as that of the ring of polynomials in n^2 commuting indeterminates, for $\lambda \neq \mu$. Namely,

$$P(A(R), t) = \sum_{r \geqslant 0} \dim(A(R)_r) t^r = \sum_{r \geqslant 0} \binom{n^2 + r - 1}{r} t^r = (1 - t)^{-n^2}.$$

$A(R)$ coincides with $K[t^i_j;\ 1 \leqslant i, j \leqslant n]$ if $\lambda = r^{ij}_{ij} = 1$ for every $i, j \in I$, and $\mu = -1$, i.e. it is a genuine deformation of the algebra of functions on $M_n(K)$.

3. Bialgebras of Type II

The purpose is of this section is to study the bialgebras associated with a quasidiagonal solution of the q-YB equation, whose set of indices I consists of a single block with all components of size one. Following [11], we call them 'bialgebras of type II'. We will show their relation with the quantum matrix supergroups (very similar results for a bigger class of R-matrices were also obtained in [28]) and we embed $A(R)$ into a Hopf algebra $H(R)$, by means of an Ore localization. We will show in the next section that for other matrices R this is not always possible, because we have zero divisors. We think this is an important issue, although it is often neglected in the literature. Although we discovered recently that Proposition 3.11 is not a new result, we still present it here because it is useful for the definition of the antipode for $H(R)$, while using only the isomorphism in [28] would cost a little more work.

One has a partition of $I = E \cup O$, such that, if λ and μ are the two solutions of Equation (1.6), $r^{aa}_{aa} = \lambda$ if $a \in E$ and $r^{aa}_{aa} = \mu$ if $a \in O$. If $E = I$ or $O = I$, then we fall back on Example 2.2. In this case, it is well known that the bialgebra $A(R)$ can be embedded in a Hopf algebra which is the Ore localization of $A(R)$ at the quantum determinant, and that $A(R)$ is a twist of the standard 1-parameter deformation of $M_n(K)$.

Suppose now that the partition of I is nontrivial. Then, it defines an equivalence class for its elements, which we denote by $\equiv'$. We have the following lemma:

LEMMA 3.1. *Let R be any quasidiagonal solution of Equation* (1.1). *If $r^{aa}_{aa} \neq r^{bb}_{bb}$, then $(t^a_b)^2 = 0$ in $A(R)$. In particular, if $A(R)$ is of type* II, *and the partition of I is nontrivial, $(t^a_b)^2 = 0$ whenever $a \not\equiv' b$.*

Proof. The relation $Sp^{aa}_{bb} = 0$ gives

$$r^{aa}_{aa} t^a_b t^a_b = r^{bb}_{bb} t^a_b t^a_b, \text{ i.e. } (r^{aa}_{aa} - r^{bb}_{bb}) t^a_b t^a_b = 0. \qquad \square$$

We introduce an ordering on the monomials in the t^i_j's with $i, j \in I$ as follows. We associate to each t^i_j degree 1, and we order the monomials first by their degree (monomial of lower degree < monomial of higher degree), and then if two monomials have the same degree, we order them in lexicographic order, considering the upper index before the lower one ($t^a_b > t^c_d$ if $a > c$ or if $a = c$ and $b > d$, and then again lexicographically for monomials of higher degree).

We rewrite now the relations for $A(R)$ in such a way that we have a single monomial on the left-hand side, and a linear combination of monomials which are strictly smaller than the above-mentioned monomial on the right-hand side. From now on we write r^{ab} instead of r_{ab}^{ab} for an entry of R belonging to the diagonal, and we write y to denote $\lambda + \mu$. Then the relations for $A(R)$ are

$$\begin{aligned} t_j^i t_j^i &= 0 \quad (i \not\equiv' j), \\ t_j^i t_k^l &= (r^{il})^{-1} r^{jk} t_k^l t_j^i + y (r^{il})^{-1} t_j^l t_k^i \quad (i > l,\, j > k), \\ t_j^i t_k^i &= (r^{kj})^{-1} r^{ii} t_k^i t_j^i \quad (j > k), \\ t_j^i t_k^l &= (r^{il})^{-1} r^{jk} t_k^l t_j^i \quad (i > l,\, j \leqslant k). \end{aligned}$$

We can check that this is a set of rewriting rules whose overlap ambiguities are confluent. Moreover, the monomials in the t_j^i's such that for every two subwords $t < t'$ in the monomial, t is on the left of t', and no t_j^i such that $i \not\equiv' j$ occurs twice in the monomial, form a basis of $A(R)$. This type of monomial will be called 'normally ordered'.

In particular, the Poincaré series of $A(R)$ is then the same as that of the function algebra of the supervariety $\mathrm{Mat}(|E|, |O|)$, namely

$$\begin{aligned} P(A(R), t) &= \sum_{r \geqslant 0} \dim(A(R)_r) t^r \\ &= \sum_{r \geqslant 0}^{\infty} \left[\sum_{k=0}^{r} \binom{|E|^2 + |O|^2 + r - k - 1}{r - k} \times \right. \\ &\quad \left. \times \binom{2|E||O|}{k} \right] t^r = (1 - t)^{-|E|^2 - |O|^2} (1 + t)^{2|E||O|}. \end{aligned}$$

The case of $A(R)$ with $n = 2$, and $|E| = |O| = 1$ has been studied by Jing, who has found a concrete Hopf algebra in which $A(R)$ can be embedded (see [14]). The problem of finding a concrete Hopf algebra structure for this type of $A(R)$ for any n can be solved by generalizing his work and finding an appropriate localization, as we will show.

To simplify computations, we shall assume from now on that all the indices in E precede all the indices in O. This can always be made possible by reordering the indices, making sure one rewrites the relations in a suitable way.

We start with the following, that resembles a classical property for the algebras of functions on the supervariety $\mathrm{Mat}(|E| \mid |O|)$.

PROPOSITION 3.2. *Let $A(R)$ be a bialgebra of type* II, *let $|E| = p$ and let N be the ideal generated by the t_j^i's such that $i \not\equiv' j$. Then, N is a bialgebra ideal, and $A(R)_{\mathrm{I}} := A(R)/N$ is a domain. If the field K contains $\pm\beta$, the square roots of $-\lambda\mu$, then $A(R)_{\mathrm{I}}$ is isomorphic to a twist of the tensor product of standard deformations of the algebras of functions on $M_p(K)$ and $M_{n-p}(K)$, with deformation parameter $q = \lambda/\beta$ and $q' = \mu/\beta$, respectively.*

Proof. For the first statement, we have to prove that $\Delta(N) \subset N \otimes A(R) + A(R) \otimes N$ and that $\varepsilon(N) = 0$. If $i \not\equiv' j$, then

$$\Delta(t_j^i) = \sum_k t_k^i \otimes t_j^k = \sum_{k \equiv' i} t_k^i \otimes t_j^k + \sum_{k \equiv' j} t_k^i \otimes t_j^k \in A(R) \otimes N + N \otimes A(R).$$

Since Δ is an algebra homomorphism, $\Delta(N) \subset N \otimes A(R) + A(R) \otimes N$. Again, if $i \not\equiv' j$, clearly $i \neq j$ hence $\varepsilon(t^i_j) = 0$. Hence, $\varepsilon(N) = 0$ and $A(R)_{\mathrm{I}}$ inherits a bialgebra structure from $A(R)$.

It is easy to see that $A(R)_{\mathrm{I}}$ is isomorphic to the algebra A generated by the t^i_j's with $i \equiv' j$, with relations:

$$\begin{aligned} t^i_j t^l_k &= (r^{il})^{-1} r^{jk} t^l_k t^i_j + y (r^{il})^{-1} t^l_j t^i_k \quad (i > l, j > k, i \equiv' l), \\ t^i_j t^i_k &= (r^{kj})^{-1} r^{ii} t^i_k t^i_j \quad (j > k), \\ t^i_j t^l_k &= (r^{il})^{-1} r^{jk} t^l_k t^i_j \quad (i > l, j \leqslant k \text{ or } i > l, i \not\equiv' k), \end{aligned}$$

because

$$\begin{aligned} \phi\colon A(R) &\to A, \\ t^i_j &\mapsto \begin{cases} t^i_j & \text{if } i \equiv' j, \\ 0 & \text{if } i \not\equiv' j \end{cases} \end{aligned} \tag{3.1}$$

is a surjective bialgebra homomorphism whose kernel is exactly N. This follows by looking at the bases of both spaces.

$A(R)_{\mathrm{I}}$ is a domain because the graded ring associated with the filtration given by the degree firstly, and the lexicographic order secondly on the monomials is a domain.

As far as the twist is concerned: this is again a standard fact. We show a proof because this technique will be used frequently in this paper. Let us denote the deformed algebras of $M_p(K)$ and $M_{n-p}(K)$ by $M_{q,p}$ and $M_{q',n-p}$. Let their generators be, respectively, u_{kl} for $1 \leqslant k, l \leqslant p$ and u_{rs} for $p+1 \leqslant r, s \leqslant n$. We want to twist the bialgebra $M_{q,p} \otimes M_{q',n-p}$ (with the usual tensor product comultiplication) by a cocycle σ_1 on the left and σ_1^{-1} on the right. $\sigma_1 \in ((M_{q,p} \otimes M_{q',n-p})^{\otimes 2})^*$ is given as follows:

$$\sigma_1(u_{vj} \otimes u_{lk}) = \begin{cases} \delta_{vj}\delta_{lk} r^{lv} \beta^{-1} & \text{if } v \leqslant p \text{ and } l > p, \\ \delta_{vj}\delta_{lk} \beta^{-1} r^{lv} & \text{if } v < l \text{ and } v \equiv' l, \\ \delta_{vj}\delta_{lk} & \text{if } s = t \text{ and } v \geqslant l \text{ or if } t > s \end{cases}$$

on the generators of $((M_{q,p} \otimes M_{q',n-p})^{\otimes 2}$, and it is extended multiplicatively on the other elements of $(M_{q,p} \otimes M_{q',n-p})^{\otimes 2}$, i.e. for the monomials $u := (u_{i_1 j_1})^{e_1} \cdots (u_{i_t j_t})^{e_t}$ and $v := (u_{k_1 l_1})^{f_1} \cdots (u_{k_g l_g})^{f_g}$:

$$\sigma_1(u \otimes v) = \prod_{c=1}^{t} \delta_{i_c j_c} \prod_{v=1}^{g} \delta_{k_v l_v} \prod_{i_c < k_v} (\beta^{-1} r^{k_v i_c})^{e_c f_v}$$

and $\sigma_1^{-1}(u_{ij} \otimes u_{lk}) = \delta_{ij}\delta_{lk}(\sigma_1(u_{ii} \otimes u_{kk}))^{-1}$. Then the map

$$\begin{aligned} \phi\colon {}_{\sigma_1}(M_{q,p} \otimes M_{q',n-p})_{\sigma_1^{-1}} &\to A(R)_{\mathrm{I}} \\ u_{lk} &\mapsto t^l_k \end{aligned}$$

with $q = \lambda/\beta$, $q' = \mu/\beta$ is a well defined bialgebra isomorphism, as it can be checked by direct computation. □

Remark. By the previous proposition it follows that the product of two elements of $A(R)$ whose expressions in terms of the basis does not contain any t^i_j with $i \not\equiv' j$ is zero if and only if at least one of the two elements are zero, since this is what happens in $A(R)_{\mathrm{I}}$.

Besides, if a $t^i_j \in N$, and there is an $a \in A(R)$ such that $at^i_j = 0$, then t^i_j must appear as a factor in the 'highest' (in the ordering of the monomials of the basis) normally ordered monomial in the expression of a, unless $a = 0$. Indeed, let $a = c_A t^A +$ lower order terms, where $0 \neq c_A \in K$, t^A follows the usual multi-index notation, and A is an n^2 matrix with entries in $\mathbb{Z}_{\geqslant 0}$ and such that $a_{kl} \in \{0, 1\}$ if $k \not\equiv' l$. Then $0 = at^i_j = c'c_A t^{A+E_{ij}} +$ lower order terms, where E_{ij} is the n^2 matrix with all 0 entries except for that indexed by i, j, which is 1. Hence, $t^{A+E_{ij}} = 0$ which implies that $a_{ij} = 1$ otherwise the monomial would belong to the basis.

COROLLARY 3.3. *The Poincaré series of $A(R)_{\mathrm{I}}$ is the same as that of the algebra on $p^2 + (n-p)^2$ commuting variables with p, and n as before. Moreover, $A(R)_{\mathrm{I}}$ is a twist of the tensor product of the standard multiparameter deformations of the algebras of functions on $M_p(K)$ and $M_{n-p}(K)$.*

Proof. The first statement is obvious. The second statement follows from the fact that the multiparameter deformation of $M_n(K)$ is nothing but a twist of the one parameter deformation. The needed cocycle $\sigma_2 \in ((M'_{q,p} \otimes M'_{q_r,n_r})^{\otimes 2})^*$ will be given by (same notation as above)

$$\sigma_2(u_{vj} \otimes u_{lk}) = \begin{cases} \delta_{vj}\delta_{lk} r^{lv}(r^{jj})^{-1} & \text{if } v < l,\, v \not\equiv' l, \\ \delta_{vj}\delta_{lk} & \text{otherwise.} \end{cases}$$

□

PROPOSITION 3.4. *$A(R)_{\mathrm{I}}$ is again a dual quasitriangular bialgebra with universal R-form $u(t^i_j \otimes t^l_k) = r^{il}_{jk}$ whenever $i \equiv' j$, and $u(t^i_j \otimes t^l_k) = 0$ if $i \not\equiv' j$.*

Proof. One checks (2.1), (2.2a) and (2.2b), knowing that they hold for $A(R)$ and that the projection is a bialgebra homomorphism. For instance, one sees that (2.1) in $A(R)$ corresponds with $r^{ab}_{ij} t^i_c t^j_d = r^{uv}_{cd} t^b_v t^a_u$. Projecting onto $A(R)_{\mathrm{I}}$ one gets (2.1) for $A(R)_{\mathrm{I}}$, by observing that if a product of some t^i_j's appears with coefficient r^{ab}_{cd}, with $a \not\equiv' c$, then it belongs to the kernel of the projection, or it is involved in a trivial relation. The other formulae are checked in a similar way, knowing that $\varepsilon_{A(R)} = \varepsilon_{A(R)_{\mathrm{I}}}\phi$. □

PROPOSITION 3.5. *Let $A(R)$ be a type II bialgebra. Then, there are two left comodule algebras* Sym(R) *and* $\wedge(R)$ *for $A(R)$ such that $A(R)$ can be also defined as the universal bialgebra coacting on* Sym(R) *and* $\wedge(R)$, *i.e. any other bialgebra coacting on* Sym(R) *and* $\wedge(R)$ *is a homomorphic image of $A(R)$.*

Proof. If the partition of I is trivial, the assertion is well known, being $\mathrm{Sym}(R)$ and $\wedge(R)$ the usual quantum symmetric and antisymmetric algebra (see for instance [30], or [4]). The same construction can be used in this case. Let S be the matrix of the operator defined by the operator RP where P is the flip operator on $V \otimes V$ (V as in Section 1) associated with the permutation matrix. We recall that the entries of S are given by $S^{ab}_{cd} = R^{ba}_{cd}$. We can prove that S is diagonalizable, and its minimal polynomial is $(X-\lambda)(X-\mu)$. Then, $\mathrm{Sym}(R)$ is the associative algebra obtained by the quotient of the free algebra on generators $x_1, \ldots, x_n$ by the ideal generated by the entries of $(S-\lambda)(X \odot X)$ where $X \odot X$ (notation as in [4]) is the n^2 column matrix with entries $(X \odot X)^{ij}_{11} = x_i x_j$. $\mathrm{Sym}(R)$ is also called the 'quantum symmetric algebra' defined by the matrix R. By $\wedge(R)$ we denote instead the associative algebra obtained as the quotient of the free algebra on generators $\xi_1, \ldots, \xi_n$ by the ideal generated by the entries of $(S-\mu)(\Xi \odot \Xi)$, where $\Xi \odot \Xi$ (notation as in [4]) is the n^2 column matrix with entries $(\Xi \odot \Xi)^{ij}_{11} = \xi_i \xi_j$. $\wedge(R)$ is also called the 'quantum antisymmetric algebra' defined by the matrix R. The relations for $\mathrm{Sym}(R)$ are then

$$x_i x_j = \begin{cases} -\mu^{-1} r^{ji} x_j x_i & \text{if } i > j, \\ 0 & \text{if } i = j \in O. \end{cases}$$

The relations for $\wedge(R)$ are

$$\xi_i \xi_j = \begin{cases} -\lambda^{-1} r^{ji} \xi_j \xi_i & \text{if } i > j, \\ 0 & \text{if } i = j \in E. \end{cases}$$

It is then a standard fact that the relations on $A(R)$ are equivalent to the fact that $\mathrm{Sym}(R)$ and $\wedge(R)$ are comodule algebras for $A(R)$, with coaction $\delta(x_i) = \sum t^i_j \otimes x_j$ and $\delta'(\xi_i) = \sum t^i_j \otimes \xi_j$. □

Our purpose is now to find a suitable Hopf algebra $H(R)$ in which $A(R)$ could be embedded. This can be done by means of localizing at a certain Ore set. We know already that a twist of $A(R)_{\mathrm{I}}$ can be embedded in a Hopf algebra, because we know that $M_{qp}(K) \otimes M_{q',n-p}(K)$ can be embedded in the Hopf algebra $\mathbf{GL}_{qp} \otimes \mathbf{GL}_{q',n-p}$, by localizing at the quantum determinants. Moreover, twisting the tensor product of the quantum exterior algebras of $M_{qp}(K)$ and $M_{q',n-p}(K)$ on the left by σ_1 of Proposition 3.3, we can again find a left comodule algebra for $A(R)_{\mathrm{I}}$ which is graded, and whose degree n component L_n is one-dimensional. As in the standard case, one can define a sort of quantum determinant by requiring it to be the element d of $A(R)_{\mathrm{I}}$ such that if $L_n = Kv$, $\delta(v) = d \otimes v$. This can be computed, and it turns out that $d = d_E d_0 = d_0 d_E$ where d_E (respectively d_O) is the usual quantum determinant of $M_{qp}(K)$ (respectively $M_{q',n-p}(K)$). It is possible to show that the multiplicatively closed set $\{d^k \mid k \in \mathbb{Z}_{\geqslant 0}\}$ satisfies Ore condition. In particular, we can localize $A(R)_{\mathrm{I}}$ at d and the localized ring can be provided of a Hopf algebra structure. We do not show this here, since it is follows by straightforward but tedious computations. Anyway, we use this as a motivation. We would like

$H(R)$ to respect this result, so that we try and see if localizing $A(R)$ at d_E and d_O viewed as elements of $A(R)$, we obtain a decent algebra, that can be given a Hopf algebra structure.

We recall that

$$d_E = \sum_{\pi \in S_p} \prod_{\substack{1 \leqslant j<k \leqslant p \\ \pi(j)>\pi(k)}} (-\lambda^{-1} r^{\pi(k)\pi(j)}) t^1_{\pi(1)} \cdots t^p_{\pi(p)} \tag{3.1E}$$

and that

$$d_O = \sum_{\pi \in S_{n-p}} \prod_{\substack{p+1 \leqslant j<k \leqslant n \\ \pi(j)>\pi(k)}} (-\mu^{-1} r^{\pi(k)\pi(j)}) t^1_{\pi(1)} \cdots t^p_{\pi(p)}, \tag{3.2O}$$

where S_{n-p} denotes the subgroup of S_n which permutes the set $\{p+1, \dots n\}$.

Ore conditions, for a multiplicatively closed set M, are conditions in order to localize a noncommutative ring R at the set M, i.e. they tell in which cases we are allowed to 'add the inverses of the elements in M to the ring R'. Those relations make sure that the ring does not collapse and that nothing bad happens. By this theory of localization we can say exactly what the kernel of the map $R \to R_M$ is, where R_M is the localized ring and we can state algebraic properties about R_M as: R_M is Noetherian if R is, there is a sort of reciprocity for nilpotent and prime ideals, composition of localizations give isomorphic rings, and more (see [35] and references therein).

The conditions read as follows:

(a) For any $s \in M$ and $r \in R$ there exist $r' \in R$ and $s' \in M$ such that $s'r = r's$;
(b) If $rs = 0$ for some $r \in R$ and $s \in M$, then there is an $s' \in M$ such hat $s'r = 0$.

In particular, condition (b) is always satisfied if the elements in M are not zero divisors. Besides, the kernel of the obvious map $R \to R_M$, the localized ring at M, is $\{r \in R \mid sr = 0 \text{ for some } s \in M\}$. Hence, if the set M contains only regular elements, the map mentioned above is an injection.

Let us define $M = \{d_E^{e_1} d_0^{o_1} \cdots d_E^{e_r} d_O^{o_r} \mid e_i, o_i \in \mathbb{Z}_{\geqslant 0}\}$. We will show now that the multiplicative set M generated by d_O and d_E is an Ore set (i.e. it satisfies ore conditions).

LEMMA 3.6. *In the context above described, we have the following relations:*

(i) $d_S t^i_j = C^S_{ij} t^i_j d_S$, *where* $S = E, O$, $i \not\equiv' j$ *and* C^S_{ij} *is a nonzero constant depending on* i, j *and* S.
(ii) $d_S t^i_j = C^S_{ij} t^i_j d_S$ *for* $S = E, O$, $i, j \in S$ *and* C^S_{ij} *is a nonzero constant depending on* S, i *and* j.
(iii) $d_S t^i_j = C^S_{ij} t^i_j d_S + \nu$ *for* $S = E, O$, $i, j \notin S$ C^S_{ij} *is a nonzero constant and* $\nu \in N^2$.

Proof. (i) We prove the statement for $S = E$, $i \in E$ and $j \in O$.

The case $S = E$, $i \in O$ and $j \in E$ is proven in a similar way using the alternative formula for d_E given by

$$d_E = \sum_{\pi \in S_p} \prod_{\substack{j<k \\ \pi(j)>\pi(k)}} (-\lambda^{-1} r^{\pi(j)\pi(k)}) t_1^{\pi(1)} \cdots t_p^{\pi(p)}.$$

The other two statements in (i) are proven in the same way.

From now on, for any pair of indices $s < k$, and any $\pi \in S_p$, $T^{\pi}_{s\cdots k}$ will denote the monomial $t^s_{\pi(s)} t^{s+1}_{\pi(s+1)} \cdots t^k_{\pi(k)}$.

$$\begin{aligned} d_E t^i_j &= \sum_{\pi \in S_p} \prod_{\substack{1 \leqslant j<k \leqslant p \\ \pi(j)>\pi(k)}} (-\lambda^{-1} r^{\pi(k)\pi(j)}) t^1_{\pi(1)} \cdots t^p_{\pi(p)} t^i_j \\ &= \sum_{\pi \in S_p} \prod_{\substack{1 \leqslant j<k \leqslant p \\ \pi(j)>\pi(k)}} (-\lambda^{-1} r^{\pi(k)\pi(j)}) \left(\prod_{u=i}^{p} (r^{ui})^{-1} r^{\pi(u)j} \right) T^{\pi}_{1\cdots i-1} t^i_j T^{\pi}_{i\cdots p}. \end{aligned}$$

We now put $C_{ip} = (\prod_{u=i}^{p} (r^{ui})^{-1} r^{\pi(u)j})$, and we study the product $C_{ip} T^{\pi}_{1\cdots i-1} t^i_j T^{\pi}_{i\cdots p}$. We have

$$C_{ip} T^{\pi}_{1\cdots i-1} t^i_j T^{\pi}_{i\cdots p} = C_{ip} \prod_{u=1}^{i-1} (r^{ui})^{-1} r^{\pi(u)j} t^i_j T^{\pi}_{1\cdots p} - (\lambda + \mu) D_{\pi},$$

where $D_{\pi} = \sum_{t=1}^{i-1} D^t_{\pi}$ and

$$D^t_{\pi} = \left(\prod_{k=i-t+1}^{p} (r^{ki})^{-1} r^{k,i-t} \right) (r^{i-t,i})^{-1} T^{\pi}_{1\cdots i-t-1} t^i_{\pi(i-t)} T^{\pi}_{i-t+1\cdots p} t^{i-t}_j.$$

Our purpose is to show that

$$\sum_{\pi \in S_p} \prod_{\substack{1 \leqslant j<k \leqslant p \\ \pi(j)>\pi(k)}} (-\lambda^{-1} r^{\pi(k)\pi(j)}) D_{\pi} = 0.$$

We shall prove this by showing by induction that $\forall t$, $1 \leqslant t \leqslant i-1$,

$$\sum_{\pi \in S_p} \prod_{\substack{1 \leqslant j<k \leqslant p \\ \pi(j)>\pi(k)}} (-\lambda^{-1} r^{\pi(k)\pi(j)}) D^t_{\pi} = 0.$$

Let $t = 1$ and let τ denote the transposition $(i-1\ i) \in S_p$. Then,

$$\begin{aligned} &\sum_{\pi \in S_p} \prod_{\substack{1 \leqslant j<k \leqslant p \\ \pi(j)>\pi(k)}} (-\lambda^{-1} r^{\pi(k)\pi(j)}) D^1_{\pi} \\ &\quad = \sum_{\pi \in S_p} \prod_{\substack{1 \leqslant j<k \leqslant p \\ \pi(j)>\pi(k)}} (-\lambda^{-1} r^{\pi(k)\pi(j)}) \left(\prod_{k=i}^{p} (r^{ki})^{-1} r^{k,i-1} \right) (r^{i-1,i})^{-1} \times \end{aligned}$$

$$
\begin{aligned}
&\times T^{\pi}_{1\cdots i-2} t^{i}_{\pi(i-1)} t^{i}_{\pi(i)} T^{\pi}_{i+1\cdots p} t^{i-1}_{j} \\
&= \left(\prod_{k=i}^{p} (r^{ki})^{-1} r^{k,i-1}\right) (r^{i-1,i})^{-1} \sum_{\substack{\pi\in S_p \\ \pi(i-1)<\pi(i)}} \prod_{\substack{1\leqslant j<k\leqslant p \\ \pi(j)>\pi(k)}} (-\lambda^{-1} r^{\pi(k)\pi(j)}) \times \\
&\quad \times T^{\pi}_{1\cdots i-2} \,[t^{i}_{\pi(i-1)} t^{i}_{\pi(i)} - \lambda^{-1} r^{\pi(i-1)\pi(i)} t^{i}_{\pi\tau(i-1)} t^{i}_{\pi\tau(i)}]\, T^{\pi}_{i+1\cdots p} t^{i-1}_{j} \\
&= 0
\end{aligned}
$$

since the difference in square brackets is $t^{i}_{\pi(i-1)} t^{i}_{\pi(i)} - \lambda^{-1} r^{\pi(i-1)\pi(i)} t^{i}_{\pi(i)} t^{i}_{\pi(i-1)} = 0$.

Now let $w \geqslant 2$ be the smallest positive integer such that the expression $\mathrm{Diff}_w = \sum_{\pi\in S_p} \prod_{1\leqslant j<k\leqslant p, \pi(j)>\pi(k)} (-\lambda^{-1} r^{\pi(k)\pi(j)}) D^{w}_{\pi} \neq 0$. Then let σ denote the transposition $(i-w\ i-w+1) \in S_p$. Then,

$$
\begin{aligned}
\mathrm{Diff}_w \ &= \sum_{\pi\in S_p} \prod_{\substack{1\leqslant j<k\leqslant p \\ \pi(j)>\pi(k)}} (-\lambda^{-1} r^{\pi(k)\pi(j)}) \left(\prod_{k=i-w+1}^{p} (r^{ki})^{-1} r^{k,i-w} \right) (r^{i-w,i})^{-1} \times \\
&\quad \times T^{\pi}_{1\cdots i-w-1}\ t^{i}_{\pi(i-w)} t^{i-w+1}_{\pi(i-w+1)}\ T^{\pi}_{i-w+2\cdots p} t^{i-w}_{j} \\
&= L_{wi} \sum_{\substack{\pi\in S_p \\ \pi(i-w)<\pi(i-w+1)}} \prod_{\substack{1\leqslant j<k\leqslant p \\ \pi(j)>\pi(k)}} (-\lambda^{-1} r^{\pi(k)\pi(j)}) T^{\pi}_{1\cdots i-w-1} \times \\
&\quad \times \Big[(r^{i,i-w+1})^{-1} r^{\pi(i-w)\pi(i-w+1)} t^{i-w+1}_{\pi(i-w+1)} t^{i}_{\pi(i-w)} + \\
&\quad - \lambda^{-1} r^{\pi\sigma(i-w+1)\pi\sigma(i-w)} t^{i}_{\pi\sigma(i-w)} t^{i-w+1}_{\pi\sigma(i-w+1)} \Big]\ T^{\pi}_{i-w+2\cdots p} t^{i-w}_{j} \\
&= L_{wi} \sum_{\substack{\pi\in S_p \\ \pi(i-w)<\pi(i-w+1)}} \prod_{\substack{1\leqslant j<k\leqslant p \\ \pi(j)>\pi(k)}} (-\lambda^{-1} r^{\pi(k)\pi(j)}) T^{\pi}_{1\cdots i-w-1} \times \\
&\quad \times \Big[(r^{i,i-w+1})^{-1} r^{\pi(i-w)\pi(i-w+1)} t^{i-w+1}_{\pi(i-w+1)} t^{i}_{\pi(i-w)} + \\
&\quad - \lambda^{-1} (\mu+\lambda) r^{\pi(i-w)\pi(i-w+1)} (r^{i,i-w+1})^{-1} t^{i-w+1}_{\pi(i-w+1)} t^{i}_{\pi(i-w)} + \\
&\quad + \mu (r^{i,i-w+1})^{-1} t^{i-w+1}_{\pi(i-w)} t^{i}_{\pi(i-w+1)} \Big] T^{\pi}_{i-w+2\cdots p} t^{i-w}_{j} \\
&= L_{wi} \sum_{\substack{\pi\in S_p \\ \pi(i-w)<\pi(i-w+1)}} \prod_{\substack{1\leqslant j<k\leqslant p \\ \pi(j)>\pi(k)}} (-\lambda^{-1} r^{\pi(k)\pi(j)}) \mu (r^{i,i-w+1})^{-1} \times \\
&\quad \times T^{\pi}_{1\cdots i-w-1} \Big[-\lambda^{-1} r^{\pi(i-w)\pi(i-w+1)} t^{i-w+1}_{\pi(i-w+1)} t^{i}_{\pi(i-w)} + \\
&\quad + t^{i-w+1}_{\pi(i-w)} t^{i}_{\pi(i-w+1)} \Big] T^{\pi}_{i-w+2\cdots p} t^{i-w}_{j}
\end{aligned}
$$

where L_{wi} is clearly nonzero. On the other hand,

$$
\begin{aligned}
0 \ &= \ \mathrm{Diff}_{w-1} := \sum_{\pi\in S_p} \prod_{\substack{1\leqslant j<k\leqslant p \\ \pi(k)<\pi(j)}} (-\lambda^{-1} r^{\pi(k)\pi(j)}) D^{w-1}_{\pi} \\
&= \sum_{\pi\in S_p} \prod_{\substack{1\leqslant j<k\leqslant p \\ \pi(j)>\pi(k)}} (-\lambda^{-1} r^{\pi(k)\pi(j)}) \prod_{k=i-w+2}^{p} (r^{ki})^{-1} r^{k,i-w+1} \times
\end{aligned}
$$

$$\times\ (r^{i-w+1,i})^{-1}\ T^{\pi}_{1\cdots i-w} t^{i}_{\pi(i-w+1)} T^{\pi}_{i-w+2\cdots p} t^{i-w+1}_{j}$$

$$= \sum_{\substack{\pi\in S_p \\ \pi(i-w)<\pi(i-w+1)}} \prod_{\substack{1\leqslant j<k\leqslant p \\ \pi(j)>\pi(k)}} (-\lambda^{-1} r^{\pi(k)\pi(j)}) \times$$

$$\times \left(\prod_{k=i-w+2}^{p} (r^{ki})^{-1} r^{k,i-w+1} \right) (r^{i-w+1,i})^{-1} T^{\pi}_{1\cdots i-w-1} \times$$

$$\times \big[t^{i-w}_{\pi(i-w)} t^{i}_{\pi(i-w+1)} +$$

$$- \lambda^{-1} r^{\pi(i-w)\pi(i-w+1)} t^{i-w}_{\pi(i-w+1)} t^{i}_{\pi(i-w)} \big] T^{\pi}_{i-w+2\cdots p} t^{i-w+1}_{j}.$$

Hence

$$\sum_{\substack{\pi\in S_p \\ \pi(i-w)<\pi(i-w+1)}} \prod_{\substack{1\leqslant j<k\leqslant p \\ \pi(j)>\pi(k)}} (-\lambda^{-1} r^{\pi(k)\pi(j)}) T^{\pi}_{1\cdots i-w-1} \times$$

$$\times \big[t^{i-w}_{\pi(i-w)} t^{i}_{\pi(i-w+1)} - \lambda^{-1} r^{\pi(i-w)\pi(i-w+1)} t^{i-w}_{\pi(i-w+1)} t^{i}_{\pi(i-w)} \big] \times$$

$$\times\ T^{\pi}_{i-w+2\cdots p} t^{i-w+1}_{j} = 0.$$

Now we compare this last result with the expression we obtained for $L^{-1}_{wi}\,\mathrm{Diff}_w$. The last result is zero if and only if

$$A_{w-1} := \sum_{\substack{\pi\in S_p \\ \pi(i-w)<\pi(i-w+1)}} \prod_{\substack{1\leqslant j<k\leqslant p \\ \pi(j)>\pi(k)}} (-\lambda^{-1} r^{\pi(k)\pi(j)}) T^{\pi}_{1\cdots i-w-1} \times$$

$$\times \big[t^{i-w}_{\pi(i-w)} t^{i}_{\pi(i-w+1)} - \lambda^{-1} r^{\pi(i-w)\pi(i-w+1)} t^{i-w}_{\pi(i-w+1)} t^{i}_{\pi(i-w)} \big] T^{\pi}_{i-w+2\cdots p}$$

$$= 0$$

by the remark after Proposition 3.2, since no t^{i-w+1}_{j} appears in the expression of this term. By definition of w, we have

$$A_{w} := \sum_{\substack{\pi\in S_p \\ \pi(i-w)<\pi(i-w+1)}} \prod_{\substack{1\leqslant j<k\leqslant p \\ \pi(j)>\pi(k)}} (-\lambda^{-1} r^{\pi(k)\pi(j)}) T^{\pi}_{1\cdots i-w-1} \times$$

$$\times \big[-\lambda^{-1} r^{\pi(i-w)\pi(i-w+1)} t^{i-w+1}_{\pi(i-w+1)} t^{i}_{\pi(i-w)} + t^{i-w+1}_{\pi(i-w)} t^{i}_{\pi(i-w+1)} \big] T^{\pi}_{i-w+2\cdots p}$$

$$\neq 0$$

since $L^{-1}_{wi}\,\mathrm{Diff}_w \neq 0$. But the above expression is the same as A_{w-1} except for the fact that the $(i-w)$th row of t^{k}_{l}'s has been substituted by the $(i-w+1)$th.

Now, if we rewrite A_{w-1} and A_w as linear combinations of elements of the basis, which means just ‘pushing the $t^{i}_{\pi(i-w)}$’s and the $t^{i}_{\pi(i-w+1)}$’s, respectively, forward in the monomials’, we will have again the same expression for both, with the t^{i-w}_{k}’s replaced by the t^{i-w+1}_{k}’s, since on the left of the $t^{i}_{\pi(i-w)}$ and $t^{i}_{\pi(i-w+1)}$ the two expressions coincide. Hence, since the elements of the basis are linearly independent, it follows that all coefficients for A_{w-1} must be zero, hence they are zero also for A_w, since they are the same. Therefore, also $\mathrm{Diff}_w = 0$ despite our assumption. So, for every t,

$$\sum_{\pi \in S_p} \prod_{\substack{1 \leqslant j < k \leqslant p \\ \pi(j) > \pi(k)}} (-\lambda^{-1} r^{\pi(k)\pi(j)}) D_\pi^t = 0,$$

hence

$$\sum_{\pi \in S_p} \prod_{\substack{1 \leqslant j < k \leqslant p \\ \pi(j) > \pi(k)}} (-\lambda^{-1} r^{\pi(k)\pi(j)}) D_\pi = 0.$$

This implies that

$$\begin{aligned} d_E t_j^i \; = \; & \prod_{u=1}^{p} (r^{ui})^{-1} r^{uj} \sum_{\pi \in S_p} \prod_{\substack{1 \leqslant j < k \leqslant p \\ \pi(j) > \pi(k)}} (-\lambda^{-1} r^{\pi(k)\pi(j)}) t_j^i T_{1\cdots p}^\pi + \\ & - (\lambda + \mu) \sum_{\pi \in S_p} \prod_{\substack{1 \leqslant j < k \leqslant p \\ \pi(j) > \pi(k)}} (-\lambda^{-1} r^{\pi(k)\pi(j)}) D_\pi \\ = \; & \prod_{u=1}^{p} (r^{ui})^{-1} r^{uj} t_j^i d_E + 0. \end{aligned}$$

(ii) follows by the fact that this relation is true inside $A(R_S)$ where R_S is the submatrix of R whose entries are those that have index in S, and $A(R_S)$ is its associated bialgebra.

(iii) follows by straightforward computation. Indeed, for $S = E$, i, $j \in O$, and for any $\pi \in S_p$

$$\begin{aligned} t_j^i T_{1\cdots p}^\pi \; = \; & \prod_{u=1}^{p} (r^{ui})^{-1} r^{ju} T_{1\cdots p}^\pi t_j^i + \\ & + y \sum_{l=1}^{p} (r^{il})^{-1} \prod_{v=1}^{l-1} (r^{iv})^{-1} r^{j\pi(v)} T_{1\cdots l-1}^\pi t_j^l t_{\pi(l)}^i T_{l+1\cdots p}^\pi, \end{aligned}$$

hence

$$\begin{aligned} t_j^i d_E \; = \; & \sum_{\pi \in S_p} \prod_{t<k, \pi(t) > \pi(k)} (-\lambda^{-1} r^{\pi(k)\pi(t)}) t_j^i T_{1\cdots p}^\pi \\ = \; & \prod_{u=1}^{p} (r^{ui})^{-1} r^{ju} \sum_{\pi \in S_p} \prod_{t<k, \pi(t) > \pi(k)} (-\lambda^{-1} r^{\pi(k)\pi(t)}) T_{1\cdots p}^\pi t_j^i + \\ & + y \sum_{\pi \in S_p} \sum_{l=1}^{p} (r^{j\pi(l)})^{-1} \prod_{t<k \;\; \pi(t) > \pi(k)} (-\lambda^{-1} r^{\pi(k)\pi(t)}) \times \\ & \times \prod_{v=1}^{l-1} (r^{iv})^{-1} r^{j\pi(v)} [T_{1\cdots l-1}^\pi t_{\pi(l)}^i] \cdot [t_j^l T_{l+1\cdots p}^\pi]. \end{aligned}$$

The second term in the sum is a linear combination of products of two elements belonging to N, hence it belongs to N^2. □

LEMMA 3.7. *N is a nilpotent ideal of $A(R)$.*

Proof. In order to prove this statement, we introduce the concept of 'level of degeneracy' of an element of $A(R)$. For a monomial, not necessarily ordered, this is the number of generators of the form t^i_j with $i \not\equiv' j$ occurring in the monomial. So far, the level of degeneracy need not be well defined for an element of $A(R)$, since not every relation preserves it. Indeed, if $i, j \in E$ and $k, l \in O$, $t^l_k t^i_j = (r^{li})^{-1} r^{kj} t^i_j t^l_k + y r^{kj} t^i_k t^l_j$. On the other hand, the level of degeneracy of any basis element is well defined. Hence, we will say that the level of degeneracy $ld(a)$ of an element $a \in A(R)$, is the minimum level of degeneracy belonging to the basis elements that occur in the expression of a. We observe that $ld(a) = k$ if and only if $a \in N^k$. Indeed, if we take any monomial in $A(R)$, we obtain a sum in which at least one term has the same level of degeneracy of the monomial we started with, and a sum of elements whose level of degeneracy is greater or equal to that one. This can be checked considering case by case all the relations in $A(R)$ with all the possible partitions of indices occurring in the relation. Hence, we can read it as a sum of products of $\geqslant ld(a)$ elements in N. Hence, $a \in N^{ld(a)}$. It is clear then that any element (thus also any monomial in the t^i_j's) is zero if its level of degeneracy is bigger than $2p(n-p) = |\{t^i_j \mid i \not\equiv' j\}|$. Therefore $N^k = 0$ for $k > 2p(n-p)$. □

There follows:

THEOREM 3.8. *The multiplicatively closed set M satisfies Ore conditions.*

Proof. By Lemma 3.6 it follows that $\forall m \in M$ and $\forall r \in N^k$, there exist an $r' \in N^k$ such that $sr' - rs \in N^{k+2}$.

Since the ideal N is nilpotent, to prove condition (a) it would be enough to show that, once we have an $r' \in N^k$ and an $s' \in M$ such that $sr' - rs' \in N^{k+2}$ for any given $r \in N^k$ and $s \in M$, we can construct an $s'' \in M$ and an $r'' \in N^k$ such that $sr'' - rs'' \in N^{k+4}$. Let r, s, r', s', be as above. Then, $sr' - rs' - n_{k+2} = 0$, for some $n_k \in N^{k+2}$.

Let ν_{k+2} be the element of N^{k+2} such that $n_{k+2}s - s\nu_{k+2} \in N^{k+4}$, whose existence follows by Lemma 3.6. Then, one has: $-s\nu_{k+2} + n_{k+2}s + sr's - rs's - n_{k+2}s = s(r's - \nu_{k+2}) - r(s's) \in N^{k+4}$. Hence, by induction, for every $r \in A(R)$ and every $s \in M$, there exist a $\rho \in A(R)$ and a $\sigma \in M$ such that $s\rho = r\sigma$.

We need to show that condition (b) holds as well. This follows by regularity of d_E and d_O. This can be checked again in the associated graded ring (filtration as before), and follows by the fact that no element in the expression of d_E or d_O has a positive level of degeneracy. □

As a consequence of the theorem, we have an embedding of $A(R)$ in its localization $H(R)$ at M. We remark then that the ideal $N' \subset H(R)$ generated by the t^i_j's with $i \not\equiv' j$ is again nilpotent, since M is an Ore set (see [35]). Moreover, one can check that a basis for $H(R)$ is given by the elements of the form $d_E^{-k} d_O^{-l} m$ where m is a normally ordered monomial in the t^i_j's such that $\min(k, e_{ij}, i, j \in E) = \min(l, e_{ij}, i, j \in O) = 0$ where e_{ij} is the exponent of t^i_j in m.

THEOREM 3.9. *$H(R)$ defined above is a Hopf algebra.*

Proof. The bialgebra structure can be extended to $H(R)$ by putting for $S = E, O$: $\varepsilon(d_S^{-1}) = \varepsilon(d_S)^{-1} = 1$ and $\Delta(d_S^{-1}) = \Delta(d_S)^{-1} \in H(R) \otimes H(R)$. The inverse of $\Delta(d_S)$ exists in $H(R) \otimes H(R)$ since $\Delta(d_S) \in d_S \otimes d_S + N \otimes A(R) + A(R) \otimes N$, i.e. it is an invertible element modulo a nilpotent one. $(d_S^{-1} \otimes d_S^{-1}) N \otimes N \subset N' \otimes N'$, hence it consists of nilpotent elements, therefore we have completed the proof. The extension of the comultiplication clearly respects the algebra structure. □

The problem now is the construction of an antipode. We define here the value of the candidate-antipode ι on the generators of $A(R)$, in such a way that the necessary property is satisfied.

Since for every k, l one should have $\sum_{j=1}^n t^k_j \iota(t^j_l) = \sum_{j=1}^n \iota(t^k_j) t^j_l = \delta_{jl}$, we must actually look for a multiplicative inverse of the matrix T.

Let us divide T into four submatrices:

$$T = \begin{pmatrix} T_{EE} & T_{EO} \\ T_{OE} & T_{OO} \end{pmatrix},$$

where T_{SP} is the submatrix with upper index in S and lower index in P, where S and P can be E or O. We look for a matrix U with entries in $H(R)$ such that $TU = UT = \mathrm{Id}$. If such a U exists, then we define $\iota(t^i_j) = U^i_j$. We write then, in the same fashion,

$$U = \begin{pmatrix} U_{EE} & U_{EO} \\ U_{OE} & U_{OO} \end{pmatrix}.$$

We know that the inverse matrix of T_{EE} (respectively T_{OO}) exists, since this is indeed the case for $H(R_E)$ (respectively $H(R_O)$), the Hopf algebra associated with the submatrix of R with indices in E (respectively O), which is the one described in [1]. $H(R_E)$ (respectively $H(R_O)$) is isomorphic, as an algebra, to the subalgebra of $H(R)$ generated by the t^i_j's with $i, j \in E$ (respectively in O) and d_E^{-1} (respectively d_O^{-1}). Hence, it makes sense to write T_{EE}^{-1} and T_{OO}^{-1}. We can then multiply the relation $TU = \mathrm{Id}$ on the left by $\begin{pmatrix} T_{EE}^{-1} & 0 \\ 0 & T_{OO}^{-1} \end{pmatrix}$. We obtain a system i n the U_{SP} whose solution is

$$U_{EE} = (T_{EE} - T_{EO} T_{OO}^{-1} T_{OE})^{-1},$$

$$U_{EO} = -T_{EE}^{-1} T_{EO} (T_{OO} - T_{OE} T_{EE}^{-1} T_{EO})^{-1};$$

$$U_{OE} = -T_{OO}^{-1}T_{OE}(T_{EE} - T_{EO}T_{OO}^{-1}T_{OE})$$

and

$$U_{OO} = (T_{OO} - T_{OE}T_{EE}^{-1}T_{EO})^{-1}.$$

The inverses in these formulae exist in $\mathrm{Mat}(n, H(R))$. Indeed, the quantum determinant of the matrices that we want to invert, belongs to M modulo N'. Therefore, these quantum determinants are themselves invertible, since M is an Ore set and N' is nilpotent. It follows that, using the standard procedure together with the fact that M is an Ore set and that N' is nilpotent, we can invert the matrices as well.

We can check that the matrix U is then also a right inverse for T. It remains to define $\iota(d_E^{-1})$, $\iota(d_O^{-1})$, and to check that ι can be extended to an algebra antihomomorphism. We will postpone this discussion to the end of this section.

We describe now $A(R)$ in terms of objects that are better known. For this purpose, we introduce E_q, Manin's deformation of the function algebra on the supermanifold $\mathrm{Mat}(a \mid b)$, and its bosonization $B(E_q)$, which is a genuine bialgebra.

We will show that for a particular choice of the parameters of R, $A(R)$ is a sub bialgebra of the bosonized object, and that one can then define an antipode for this particular $A(R)$ using the fact that a Hopf super algebra H in which E_q is contained has been computed by Manin and by Ho Hai, so that its bosonization has an antipode as well.

Finally, we will show that all the other bialgebras of type II can be twisted into one of the $A(R)$'s that can be embedded in $B(H)$, and that the only essential datum is the partition of I.

Manin's Quantum General Linear Supergroups. In order to define commutation relations for E_q, we have to fix a 'format', i.e. an n-tuple $\{a_1, \dots, a_n\}$ with elements in $\mathbb{Z}_2$, and a family of $\binom{n}{2}$ nonzero elements in the field K: $\{q_{ij}\}_{1\leqslant i<j\leqslant n}$ such that $q_{ij} = \varepsilon_{ij}q$ where $\varepsilon_{ij} = \pm 1$, q is a given parameter. Then E_q is the algebra generated by the z_i^k's with $1 \leqslant k, i \leqslant n$, subject to the following relations:

$$\begin{aligned}
(z_i^k)^2 &= 0 \quad \text{for } a_i + a_k \equiv 1,\\
z_i^l z_i^k &= (-1)^{(a_k+1)(a_l+1)} q_{kl}^{-1} z_i^k z_i^k \quad \text{for } a_i \equiv 1 \text{ and } k < l,\\
z_i^l z_i^k &= (-1)^{a_k a_l} q_{kl} z_i^k z_i^l \quad \text{for } a_i \equiv 0 \text{ and } k < l,\\
z_j^k z_i^k &= (-1)^{a_i a_j} q_{ij} z_i^k z_j^k \quad \text{for } a_k \equiv 0 \text{ and } j > k,\\
z_j^k z_i^k &= (-1)^{(a_i+1)(a_j+1)} q_{ij}^{-1} z_i^k z_j^k \quad \text{for } a_k \equiv 1 \text{ and } j > i,\\
z_j^k z_i^l &= (-1)^{(a_j+a_k)(a_i+a_l)} \varepsilon_{ij}\varepsilon_{kl} z_j^k z_i^l \quad \text{for } l > k \text{ and } j > i,\\
z_j^l z_i^k &= \varepsilon_{ij}\varepsilon_{kl}(-1)^{(a_i+a_k)(a_j+a_l)} z_i^k z_j^l + \varepsilon_{ij}(-1)^{(a_i a_j + a_i a_l + a_j a_l)}(q - q^{-1}) z_i^l z_j^k\\
&\quad \text{for } l > k \text{ and } j > i.
\end{aligned}$$

We say that z_j^i is 'even' if $a_i + a_j \equiv 0$ and that it is otherwise 'odd', so that E_q becomes a superalgebra. It is actually a super bialgebra with comultiplication defined on the generators as $\Delta(z_i^j) = \sum_k z_i^k \otimes z_k^j$, and extended to the whole of E_q in the unique way that provides a superalgebra homomorphism $E_q \to E_q \otimes E_q$. The counit is given by $\varepsilon(z_k^i) = \delta_k^i$.

We define an ordering on the z_j^i's by $z_i^j < z_k^l$ if either $i > k$ or $i = k$ and $j > l$. We call 'normally ordered monomials' those monomials in the z_j^i's such that smaller subwords occur on the left of the bigger ones, and no z_j^i with $a_i + a_j \equiv 1$ occurs twice. By Theorem 3.12 in [31], normally ordered monomials form a basis E_q. Therefore, E_q is a deformation of the algebra of functions on the supermanifold $\mathrm{Mat}(a \mid b)$ where a and b denote, respectively, the amount of even and odd terms in the format.*

Manin then defines the deformation of the general linear supergroup as the Hopf envelope of E_q. This is a Hopf superalgebra H_q together with a super bialgebra map $\gamma\colon E_q \to H_q$ having universal properties with respect to all superbialgebra maps from E_q to a Hopf superalgebra H'. It is built formally in [31], where the existence of a quantum Berezinian is also shown. In [12], one can find an explicit computation of the Hopf envelope for a multiparameter deformation of $M(a \mid b)$, using the existence of the quantum Berezinian, so that we have an explicit description of a Hopf superalgebra $H(E_q)$ containing E_q.

We show the link between E_q and $A(R)$. This is given by bosonization:

Bosonization of a Hopf Superalgebra. Given a super bialgebra, or a Hopf superalgebra, one can construct an ordinary bialgebra or Hopf algebra in such a way that the super object and the ordinary one, have 'equivalent' representation theories (see [25] for further details). We describe the process for a Hopf superalgebra pointing out that in case we are dealing with a super bialgebra, we can simply forget about the existence of the antipode, holding all the results involving only the bialgebra structure.

Let's recall briefly what a Hopf super algebra is. Let $H = H_0 \oplus H_1$ be a $\mathbb{Z}_2$-graded algebra, having also a $\mathbb{Z}_2$-graded coalgebra structure, i.e. $\Delta(H_i) \subset \sum_{k+l\equiv i} H_k \otimes H_l$ and $\varepsilon(H_1) = 0$, such that comultiplication and counit are algebra maps where the multiplication on $H \otimes H$ is defined as $(m \otimes m) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id})$ where τ is the graded flip operator. Such an H is called a super bialgebra. If H also possesses an antipode, which is a $\mathbb{Z}_2$-graded map, H is called a Hopf superalgebra. Sometimes, we will write $|a| = i$ if $a \in H_i$.

PROPOSITION 3.10 ([25, Corollary 4.3]). *Let H be a Hopf superalgebra over a field K. Then, H is a subalgebra of the Hopf algebra $B(H)$, defined as follows. As an algebra, $B(H)$ is the extension of H by adjoining an element j such that $j^2 = 1$, $jb = (-1)^i bj$ for $b \in H_i$. The comultiplication, counit and antipode are*

* By Theorem 3.12 in [31] we also see that our omission of the η_{ij}'s in the defining relations does not really limit the generality.

given by j being grouplike, $\Delta_{B(H)}(b) = \sum b_{i1} j^{|b_{i2}|} \otimes b_{i2}$ *if* $\Delta_H(b) = \sum b_{i1} \otimes b_{i2}$; $\varepsilon_H(b) = \varepsilon_H(b)$ *and* $S_{B(H)}(b) = j^{|b|} S_H(b)$ *for all homogeneous* $b \in H$. $B(H)$ *is called the 'bosonization' of* H.

Now let now $B(E_q)$ be the bosonization of E_q, associated with a given format $\{a_1, \dots, a_n\}$ and a given choice of ε_{ij} for every $1 \leqslant i < j \leqslant n$. Let us take an $n^2 \times n^2$ quasidiagonal solution of the q-YB equation R such that R is of type II, and such that $E = \{l \in I | a_l \equiv 0\}$. In particular, $l \equiv' k \Leftrightarrow a_l \equiv a_k$. If the field of definition K contains the square root of $z = -\lambda\mu$, (λ and μ are as usual, the roots of Equation (1.6)), then one can always make sure that the relations of $A(R)$ can be defined by a matrix R' with $\mu' = -(\lambda')^{-1}$, by multiplying R by the scalar matrix $\beta^{-1} Id$ with $\beta^2 = z$.

We will assume from now on that the above condition on K is always fulfilled so that it will not be restrictive to assume that for R, $\lambda\mu = -1$.

Let us now take $r^{eb} = (-1)^{a_e a_b} \varepsilon_{eb}$ for $e < b$, and $\lambda = q$ of Manin's E_q. This gives a well defined bialgebra $A(R)$.

We define the linear map

$$\begin{aligned} \psi\colon A(R) &\to B(E_q) \\ t_k^i &\mapsto z_i^k j^{a_k}. \end{aligned}$$

PROPOSITION 3.11. *The map ψ above defined is a bialgebra homomorphism.*

Proof. ψ is a coalgebra map since $\varepsilon\psi(t_k^l) = \varepsilon(z_l^k)\varepsilon(j^{a_k}) = \delta_{kl}$ and

$$\begin{aligned} (\psi \otimes \psi)\Delta(t_k^l) &= (\psi \otimes \psi)\Big(\sum_r t_r^l \otimes t_k^r\Big) \\ &= \sum_r z_l^r j^{a_r} \otimes z_r^k j^{a_k} = \Big(\sum_r z_l^r j^{(a_r + a_k)} \otimes z_r^k\Big)(j^{a_k} \otimes j^{a_k}) \\ &= \Delta(z_l^k)\Delta(j^{a_k}) = \Delta(z_l^k j^{a_k}) = \Delta\psi(t_k^l). \end{aligned}$$

We look at the image of the relations in $A(R)$: if $k \not\equiv' l$

$$\psi(t_l^k t_l^k) = z_l^k j^{a_k} z_l^k j^{a_k} = (-1)^{(a_k + a_l)a_k} z_l^k z_l^k = 0$$

by the first relation of E_q.

Now we look at the second relation of $A(R)$. If $e > b$ and $c > d$,

$$\begin{aligned} &\psi(t_c^e t_d^b - (r_{eb}^{eb})^{-1} r_{cd}^{cd} t_d^b t_c^e - y(r_{eb}^{eb})^{-1} t_c^b t_d^e) j^{(a_d + a_e)} \\ &\quad = \big[(-1)^{a_c(a_b + a_d)} z_e^c z_b^d - \varepsilon_{be}\varepsilon_{dc}(-1)^{a_e(a_b + a_d)} z_b^d z_e^c - \\ &\qquad - (q - q^{-1})\varepsilon_{be}(-1)^{(a_e a_b + a_e a_c + a_d a_c)} z_b^c z_e^d\big] \end{aligned}$$

which is zero by the last relation for E_q.

As far as the third relation for $A(R)$ is concerned, we have, for $c > d$,

$$\begin{aligned} &\psi(t_c^i t_d^i - (r_{dc}^{dc})^{-1} r_{ii}^{ii} t_d^i t_c^i) \\ &\quad = \big[z_i^c z_i^d (-1)^{(a_i + a_d)a_c} - \varepsilon_{dc}(-1)^{a_i a_d} r_{ii}^{ii} z_i^d z_i^c\big] j^{a_c + a_d} = 0 \end{aligned}$$

by the second and third relation for E_q, depending on the parity of a_i.

Finally, the image under ψ of the last generating relation for $A(R)$ is given by the following: for $i > l$ and $c < k$:

$$\begin{aligned}&\psi(t^i_c t^l_k - (r^{il}_{il})^{-1} r^{ck}_{ck} t^l_k t^i_c)\\&\quad = \left[(-1)^{a_c(a_k+a_l)} z^c_i z^k_l - \varepsilon_{li}\varepsilon_{ck}(-1)^{(a_k+a_l)a_i} z^k_l z^c_i\right] j^{a_c+a_k} = 0\end{aligned}$$

by the second last relation for E_q.

If, at last, $i > l$ and $c = k$, the relation becomes instead

$$\begin{aligned}&\psi(t^i_c t^l_c - (r^{il}_{il})^{-1} r^{cc}_{cc} t^l_c t^i_c)\\&\quad = (-1)^{a_c(a_c+a_l)} z^c_i z^c_l - \varepsilon_{li} r^{cc}_{cc}(-1)^{(a_c+a_l)a_i+a_c} z^c_l z^c_i = 0\end{aligned}$$

by the fourth and fifth relation for E_q, depending on the parity of a_c. □

PROPOSITION 3.12. *The map ψ above described is injective, hence $A(R)$ can be identified with a sub bialgebra of $B(E_q)$.*

Proof. By construction, $B(E_q) \cong E_q \otimes K\mathbb{Z}_2$ as a vector space, hence a basis for $B(E_q)$ is given by elements of the form mj^p where m is a normally ordered monomial in E_q, and $p \in \mathbb{Z}_2$. Therefore, those elements are linearly independent. Now, any element of the basis of $A(R)$ goes over to a different element of this basis for $B(E_q)$, up to a sign. Hence, ψ is injective. □

Remark. One can easily check that $\psi(A(R))$ is the subalgebra of $B(E_q)$ whose basis is given by the elements of the form mj^p where m is a normally ordered monomial in E_q and p is the sum of all a_k's with the right multiplicity such that k occurs as an upper index in the monomial m.

If we have a Hopf superalgebra H_q, which is the Hopf envelope of E_q, its bosonization $B(H_q)$ will also have an antipode, and $\psi(A(R)) \subset B(H_q)$. We may wonder whether $H(R)$ can also be embedded in $B(H_q)$, and whether the antipodes of the two objects correspond to each other. The answer is positive, as we will show. In order to do this, we need to recall the construction of H_q in [12].

Let A, B, C, D denote, respectively, the submatrices of $Z = (z^i_k)$ as follows. A contains only the entries such that both indices are in E; B has only entries such that the upper index is in O and the lower index is in E; C has entries such that the upper index is in E and the lower index is in O, and D has only the entries with both indices in O. Given a square matrix M of elements in E_q, we can use the usual formula for the quantum determinant in order to define $\det_q(M)$ (see [12, 30, 31]). If the determinant turns out to be invertible in some extension of E_q, then we can write the inverse of the matrix M using the usual formulae (see, for instance, [1]). Then we have the following theorem:

THEOREM 3.13 ([12]). *The quantum linear supergroup H_q can be derived from E_q by localizing the elements* $\det_q A$ *and* $\det_q(D - CA^{-1}B)$. *In* $\mathbf{GL}_q(a|b)$ *the* *elements* $\det_q(D)$ *and* $\det_q(A - BD^{-1}C)$ *are also invertible, and the quantum Berezinian is* $\det_q(A)\det_q(D - CA^{-1}B)^{-1} = \det_q(D)^{-1}\det_q(A - BD^{-1}C)$.

In particular, one can obtain H_q by localizing E_q at $\det_q(D)$ and $\det_q(A)$. Therefore, $B(H_q)$ can be obtained by localizing $B(E_q)$ at $\det_q(D)$ and $\det_q(A)$. A basis for H_q is given by the elements of the form $\det_q(A)^{-k}\det_q(D)^{-l}m$ where m is a normally ordered monomial in the z_q^p's such that $\min(k, e_{pq}; a_p \equiv a_q \equiv 0) = \min(l, e_{pq}; a_p \equiv a_q \equiv 1) = 0$ where e_{pq} is the exponent of z_q^p in m. Therefore, a basis for $B(H_q)$ is given by the elements of the form aj^e where a is an element of the basis of H_q and $e \in \{0, 1\}$.

PROPOSITION 3.14. *Let R be a type* II *matrix with $r^{eb} = \varepsilon_{eb}(-1)^{a_e a_b}$, $\lambda = q$ and $\mu = -q^{-1}$. Then, one can extend the map ψ to an injection of $H(R)$ in $B(H_q)$ and $H(R)$ can be identified with a Hopf subalgebra of $B(H_q)$.*

Proof. Since $\psi(d_E) = \det_q(A)$ and $\psi(d_O) = \det_q(D)j^{n-p}$, we can extend ψ to d_E^{-1} and d_O^{-1} by $\psi(d_E^{-1}) = \det_q(A)^{-1}$ and $\psi(d_O^{-1}) = \det_q(D)^{-1}j^{n-p}$, and this clearly extends to a bialgebra homomorphism. This map is again injective because it sends different elements of the basis of $H(R)$ to different elements of the basis of $B(H_q)$ up to a sign.

Claim. On the t_j^i's, $\psi \circ \iota = S_{B(H_q)} \circ \psi$.

As a consequence of the claim, ι can be extended to an algebra antihomomorphism $H(R) \to H(R)$, since ψ is an algebra embedding and S can be extended to an algebra antihomomorphism. Indeed, ι can be extended to an algebra antihomomorphism $A(R) \to H(R)$. Hence, $\iota(d_E)$ and $\iota(d_O)$ are well defined, and they are invertible because their image under ψ is so. Therefore, we can define $\iota(d_E^{-1}) = \iota(d_E)^{-1}$ and $\iota(d_O^{-1}) = \iota(d_O)^{-1}$, respecting the relation $\psi \circ \iota = S_{B(H_q)} \circ \psi$.

We still have to prove the claim. We do it for T_{EE} and T_{EO}:

$$\begin{aligned}
&\psi(\iota(T_{EE}))\\
&\quad = \psi(T_{EE} - T_{EO}T_{OO}^{-1}T_{OE})^{-1} = (A - BjD^{-1}j^{(n-p)+(n-p-1)}C)^{-1}\\
&\quad = (A - BD^{-1}C)^{-1} = j^0 S_{H_q}(A) = S_{B(H_q)}(\psi(T_{EE})),
\end{aligned}$$

$$\begin{aligned}
&\psi(\iota(T_{EO}))\\
&\quad = -\psi(-T_{EE}^{-1}T_{EO}(T_{OO} - T_{OE}T_{EE}^{-1}T_{EO})^{-1}) = -A^{-1}Bj(Dj - CA^{-1}Bj)^{-1}\\
&\quad = -A^{-1}Bj^2(D - CA^{-1}B)^{-1} = S_{H_q}(B) = jS_{H_q}(Bj) = S_{B(H_q)}(\psi(B)).
\end{aligned}$$

For T_{OE} one has to use another form of writing U_{OE}, namely $U_{OE} = -U_{OO}T_{OE}T_{EE}^{-1}$. □

So far we have shown that for any element H_q of Manin's family of deformation of the general linear supergroup, there is a particular R matrix of type II such that

$H(R)$ is a Hopf subalgebra of $B(H_q)$. Using again the cocycle twists, one sees that each $A(R)$ has essentially the property stated in Propositions 3.11, 3.12.

PROPOSITION 3.15. *Every algebra $A(R)$ of type* II *can be twisted into one of the particular types described above, provided that K contains the square roots of $-\lambda\mu$.*

Proof. We have shown that, once K satisfies the given condition, we can always make sure that $\lambda\mu = -1$, so that $r^{ij}r^{ji} = 1$ for every $i \neq j \in I$. For any choice of $\varepsilon_{ij} \in \{-1, 1\}$, we can provide a 2-cocycle $\sigma \in (A(R) \otimes A(R))^*$ such that ${}_\sigma A(R)_{\sigma^{-1}}$ is isomorphic to the bialgebra $A(P)$ associated with the matrix P of type II with entries $p^{ij} = \varepsilon_{ij}(-1)^{a_i a_j}$, $p^{ii} \in \{\lambda, -\lambda^{-1}\}$ depending on the parity of a_i, $p^{ij}_{ji} = \lambda - \lambda^{-1}$ if $i < j$. The cocycle is defined on the generators of $A(R) \otimes A(R)$ as follows, and extended in the usual way:

$$\sigma(t^i_j \otimes t^l_k) = \begin{cases} \delta_{ij}\delta_{lk} & \text{if } i \leqslant l, \\ \delta_{ij}\delta_{lk} r^{il}\varepsilon_{il}(-1)^{a_i a_j} & \text{if } i > l. \end{cases}$$

□

Remark. Let $A(R)$ as in Proposition 3.11, and let $A(R) \triangledown \mathbb{Z}_2$ be the bialgebra obtained by extending $A(R)$ by the element g such that $g^2 = 1$, g is grouplike and $gt^i_j = (-1)^{a_i + a_j} t^i_j g$. Then it is immediate to prove that the map $\phi A(R) \triangledown \mathbb{Z}_2 \to B(E_q)$ sending g to j extends the map ψ to a bialgebra isomorphism. This is Theorem IV.3 in [28].

4. More Blocks and Zero Divisors

We study now the behaviour of $A(R)$ when R is a quasidiagonal solution of the quantum Yang–Baxter equation, not necessarily of type II. We start showing the existence of not nilpotent zero divisors in $A(R)$ in most of the cases.

PROPOSITION 4.1. *Let R be a quasidiagonal solution of* (1.1) *having at least two blocks. If at least one block has more than one component, then $A(R)$ has zero divisors which are not nilpotent.*

Proof. We use the same notation as in Section 1. Let B_s and B_t be two distinct blocks, and let B_t have more than one component.

We divide the proof into two cases:

Case 1. $(r^{aa})^2$ is not constant for all $a \in B_s \cup B_t$.
Then let $a \in B_t \cup B_s$, $c \in B_t$ and $d \in B_s$. By relations $Sp^{aa}_{cd} = 0$ and $Sp^{aa}_{dc} = 0$ we get

$$r^{aa}t^a_c t^a_d = r^{cd}t^a_d t^a_c = (r^{aa})^{-1}r^{cd}r^{dc}t^a_c t^a_d.$$

Hence, whenever $r^{cd}r^{dc} = z_{st} \neq (r^{aa})^2$, we have a family of zero divisors. An a for which $z_{st} \neq (r^{aa})^2$ exists always because z_{st} depends only on the pair of blocks.

In particular, taking either $c = a$ or $d = a$ (depending on whether a belongs to B_s or B_t) we have either that $t_a^a t_d^a = 0$ or $t_c^a t_a^a = 0$, neither of the two elements can be zero because relations have a degree of at least 2. Moreover, t_a^a cannot be nilpotent because $\varepsilon(t_a^a) = 1$.

Case 2. $(r^{aa})^2$ is constant for all $a \in B_s \cup B_t$.

In particular, the above condition, together with the hypothesis on B_t imply that r^{aa} is constant on B_t. Indeed, it follows that for any $a, b \in B_t$, $r^{aa} = \pm r^{bb}$ but the fact that B_t has more than one component implies that the minus sign has to be excluded. Otherwise, for some c and d, and for $r^{aa} = -r^{bb}$ there would follow that $0 \neq r_{dc}^{cd} = y_t = \lambda_t + \mu_t = r^{aa} + r^{bb} = 0$.

Hence, we have $r^{aa} = \lambda_t$ for every $a \in B_t$ and $r^{aa} = \pm\lambda_t$ for every a in B_s. Now, if $z_{st} \neq (r^{aa})^2$ we get the same zero divisors as in Case 1. So we might as well restrict to the case that $z_{st} = \lambda_t^2$. Then we consider the product $t_b^a(\lambda_t (r^{bd})^{-1} t_b^b t_d^d - t_d^b t_b^d)$ for $a \in B_s$, for $b, d \in B_t$, $b < d$.

By the relation $Sp_{bb}^{ba} = 0$ we have $t_b^a t_b^b = \lambda_t^{-1} r^{ba} t_b^b t_b^a$, by relations $Sp_{db}^{ba} = 0$ and $Sp_{bd}^{ab} = 0$ we get $t_b^a t_d^b = \lambda_t^{-1} r^{bd} t_d^a t_b^b$; by the relations $Sp_{bd}^{ad} = Sp_{db}^{ad} = Sp_{bd}^{da} = 0$ we get $t_d^a t_b^d = \lambda_t (r^{bd})^{-1} t_b^a t_d^d$, finally, by the relation $Sp_{bd}^{ba} = 0$ we get $t_b^b t_d^a = (r^{ba})^{-1} r^{bd} t_d^a t_b^b$.

Hence, for the above product we have

$$\begin{aligned} &t_b^a(\lambda_t (r^{bd})^{-1} t_b^b t_d^d - t_d^b t_b^d) \\ &\quad = (r^{bd})^{-1} r^{ba} t_b^b t_b^a t_d^d - \lambda_t^{-1} r^{bd} t_d^a t_b^b t_b^d \\ &\quad = \lambda_t^{-1} r^{ba} t_b^b t_d^a t_b^d - \lambda_t^{-1} r^{bd} r^{ba} (r^{bd})^{-1} t_b^b t_d^a t_b^d = 0. \end{aligned}$$

t_b^a is nonzero because it has degree 1, and $(\lambda_t (r^{bd})^{-1} t_b^b t_d^d - t_d^b t_b^d)$ is not nilpotent and nonzero since $\varepsilon(\lambda_t (r^{bd})^{-1} t_b^b t_d^d - t_d^b t_b^d) = \lambda_t (r^{bd})^{-1} \neq 0$. □

In particular, if all blocks have only components of size one, the proposition below states that $A(R)$ behaves in a 'strange' way unless R is diagonal.

We are now able to make a classification of the quasidiagonal solutions of the quantum Yang–Baxter equation R for which the bialgebra $A(R)$ is a domain.

THEOREM 4.2. *Let K contain the square roots of z_{st}, and r^{aa} for every s, t and a. In the setting above, and if the square roots of z_{st} exist in K; $A(R)$ is a domain if and only if either R is a scalar multiple of the permutation matrix P, or there is a λ such that $r^{aa} = \lambda$ for every $a \in I$, and R is q-Hecke up to a scalar factor.*

Proof. The case of R multiple of the permutation matrix (i.e. R corresponds to one single component) is clear, since $A(R)$ is the free algebra in this case. Let's assume now that $R \neq \alpha P$ for any scalar α.

We recall that rescaling R gives the same set of algebra relations, hence the fact that R is q-Hecke up to a scalar factor is equivalent to the fact that PR is diagonalizable and that it has two eigenvalues if K is big enough.

(⇒). If R is a domain, it follows from Lemma 3.1 that r^{aa} must be constant on I, so necessity of the first condition is clear. The characteristic polynomial of $S = PR$ is

$$\begin{aligned}\det(S-\beta) &= \prod_{i=1}^{n}(r^{aa}-\beta)\prod_{a<b;a\sim b;a\simeq b}(\lambda_a-\beta)^2\times\\ &\quad\times\prod_{a<ba\sim b,a\not\simeq b}(\beta^2-\beta y_a+\lambda_a\mu_a)\prod_{a<ba\not\sim b}(\beta^2-z_{st}).\end{aligned}$$

Since R is a domain, we know from Proposition 4.1 that either there is only one block, or each block has only one component. Hence, the characteristic polynomial is either of the form

$$\begin{aligned}\det(S-\beta) &= \prod_{i=1}^{n}(r^{aa}-\beta)\prod_{a<b;a\sim b;a\simeq b}(\lambda_a-\beta)^2\times\\ &\quad\times\prod_{a<ba\sim b,a\not\simeq b}(\beta^2-\beta y_a+\lambda_a\mu_a)\end{aligned}$$

or of the form

$$\det(S-\beta)=\prod_{i=1}^{n}(r^{aa}-\beta)\prod_{a<b;a\sim b;a\simeq b}(\lambda_a-\beta)^2\prod_{a<ba\not\sim b}(\beta^2-z_{st}).$$

Hence, the eigenvalues are only λ and μ in the first case, and λ and $\pm\sqrt{z_{st}}$ in the second case. Moreover, by the proof of Proposition 4.2 we see that if $A(R)$ is a domain, $z_{st} = \lambda^2$ for every s and t. Hence, also in the second case there are only two distinct eigenvalues. It is also easy to see that in both settings PR is diagonalizable, hence R is q-Hecke.

(⇐). Suppose that R is Hecke and that $r^{aa} = \lambda$. By the analysis of the characteristic polynomial it follows that there are only two cases possible. Either there is only one block associated to R, or there are more blocks associated to R, each block consists of only one component, and $z_{st} = \lambda^2$ for every block B_s and B_t. One works out those two cases and sees that $A(R)$ is a domain. Indeed, the relations for the one-block case are

$$t_c^a t_d^b = \begin{cases}(r^{dc})^{-1}\lambda t_d^a t_c^b & \text{if } a\simeq b \text{ and } c>d,\\ \lambda(r^{ab})^{-1}t_a^b t_d^c & \text{if } a>b \text{ and } c\simeq d,\\ (r^{ab})^{-1}r^{cd}t_d^b t_c^a & \text{if } a>b \text{ and } c\leqslant d,\\ (r^{ab})^{-1}r^{cd}t_d^b t_c^a+(\lambda+\mu)(r^{ab})^{-1}t_c^b t_d^a & \text{if } a>b \text{ and } c>d\end{cases}$$

and the relations for the more blocks case are

$$t_c^a t_d^b = \begin{cases}r^{cd}\lambda^{-1}t_d^a t_c^b & \text{if } a\sim b \text{ and } c>d,\\ (r^{ab})^{-1}\lambda t_c^b t_d^a & \text{if } a>b \text{ and } c\sim d,\\ r^{cd}(r^{ab})^{-1}t_d^b t_c^a & \text{if } a>b \text{ and } c\not\sim d,\end{cases}$$

where by $a > b$ we mean in both formulae $a \not\geq b$ and $a > b$. In both cases one checks the confluence of the rewriting rules and by the diamond lemma (see [3]) one concludes that in both cases a basis for $A(R)$ is given by monomials of the form $A_1^1 A_2^1 \cdots A_r^1 A_1^2 \cdots A_{r-1}^r A_r^r$ where A_j^i is *any* monomial in the t_v^u's with $u \in C_i$ and $v \in C_j$ and $C_1, C_2, \ldots, C_r$ are the components related to R. In both cases we can build a filtration on $A(R)$ given by the degree first, then by the lexicographic order on the A_j^i's viewed as an undecomposable entity, and then by lexicographic order on the single monomials. One checks that the associated graded ring is a domain, hence $A(R)$ is a domain. □

Remark. The theorem says that $A(R)$ is a domain if and only if PR is diagonalizable and it has at most two eigenvalues. One can compute the Poincaré series of such an $A(R)$. It is given by

$$P(A(R), t) = \sum_{d \geqslant 0} \left[\sum_{\sum d_{ij} = d} \left(\prod_{i,j=1}^{r} (c_i c_j)^{d_{ij}} \right) \right] t^d$$

where r is the number of components, c_j is the size of the component C_j and the indices i and j in d_{ij} run from 1 to r.

In particular, an easy consequence of the theorem is the analysis of the case that R is diagonal.

COROLLARY–PROPOSITION 4.3. *Let R be a diagonal solution of the q-YB equation. Then, $A(R)$ is a domain if and only if $r^{ii} = \lambda =$ constant, and $r^{ij} r^{ji} = \lambda^2$ for all $i \neq j$. In this case $A(R)$ is isomorphic as an algebra to a twist of $K[u_{ij}; 1 \leqslant i, j \leqslant n]$, taken with the usual comultiplication: $\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}$. The Poincaré series of the above-mentioned bialgebras are therefore the same.*

Proof. The first statement is a consequence of Theorem 4.2. The second fact is standard and rather easy. The cocycle $\sigma_d \in (K[u_{ij}; 1 \leqslant i, j \leqslant n] \otimes K[u_{ij}; 1 \leqslant i, j \leqslant n])^*$ that does the job is

$$\sigma_d(u_{ij} \otimes u_{lk}) = \begin{cases} \delta_{ij}\delta_{lk} r_{li}^{li} \lambda^{-1} & \text{if } i < l, \\ \delta_{ij}\delta_{lk} & \text{if } i \geqslant l. \end{cases}$$

□

Now the question becomes whether factoring out a choice of the zero divisors provides a domain which is still a bialgebra. We would like the dual quasitriangular structure to descend to the quotient, and this quotient to be embedded in a Hopf algebra.

From [18] we know that if the dual quasitriangular structure of $A(R)$ descends to a quotient, then, in order to have a Hopf algebra we need R to be bi-invertible. This means that R and R^{t_2} (t_2 stands for transposing R only with respect to the second factor in the tensor product, i.e. $(R^{t_2})_{cd}^{ab} = R_{cb}^{ad}$) should both be invertible. The condition on R^{t_2} is equivalent to the fact that *all components associated with*

R are of size one, as it is easy to check. Therefore, we will reduce from now on to this case.

The first attempt is factoring out the ideal $I_\sim$ generated by the elements of type t^i_j with $i \not\sim j$. This seems the right starting point because of the following proposition:

PROPOSITION 4.4. *$I_\sim$ is also a coideal, hence $A(R)/I_\sim$ is a bialgebra.*

Proof. As in Proposition 3.2. We point out that this property holds independently of the size of the components. □

We are going to describe this quotient in terms of the objects studied in the previous section, namely in terms of bialgebras of type II. We see that their description is analogous to that in Proposition 3.2.

At first it is straightforward to check that $A(R)/I_\sim$ is isomorphic as an algebra to the subalgebra of $A(R)$ generated only by the t^i_j's with $i \sim j$.

Inside a given block, we again use the equivalence relation $\equiv'$, defined as in Section 3. If we then write $i \not\equiv' j$, this implies also that $i \sim j$. If B_t is one of the blocks in which the index set I is parted, we denote by λ_t and μ_t the solutions of Equation (1.6) corresponding to B_t.

The relations for $A(R)/I_\sim$ are then

$$t^i_j t^l_k = \begin{cases} (r^{il})^{-1} r^{jk} t^l_k t^i_j & \text{if } i \not\sim l \text{ or if } i > l \text{ and } j <\leqslant k \\ 0 & \text{if } i = l,\ j = k \text{ and } i \not\equiv' l \\ (r^{il})^{-1} r^{jk} t^l_k t^i_j + y (r^{il})^{-1} t^l_j t^i_k & \text{if } i \sim l,\ i > l \text{ and } j > k \\ r^{ii} (r^{kj})^{-1} t^i_k t^i_j & \text{if } i = l \text{ and } j < k. \end{cases}$$

Then we have:

PROPOSITION 4.5. *Let R be a quasidiagonal solution of the q-YB equation, such that each block in which the set of indices is partitioned, has only components of size one. Then let $B_1, \dots, B_p$ be the blocks of I, and let $A(R)_i$ be the bialgebra corresponding to the submatrix R_i of R that has as entries only those which are indexed by elements in B_i. Then, $A(R)/I_\sim$ is isomorphic to a twist of the tensor product of the $A(R)_i$'s. The Poincaré series of $A(R)/I_\sim$ is then the same as that of the function algebra on* $\mathrm{Mat}(a \mid b)$ *with a being the amount of indices j in I such that $r^{jj}_{jj} = \lambda_j$ and b the amount of indices j in I such that $r^{jj}_{jj} = \mu_j$.*

Proof. We have to find the suitable 2-cocycle σ for $(A(R)_1 \otimes \cdots \otimes A(R)_p)$ provided by the obvious coalgebra structure. Let us denote the generators of $A(R)_k$ by u_{ij} for $\sum_{t=1}^{k-1} |B_t| < i, j \leqslant \sum_{t=1}^{k} |B_t|$. Then σ is defined on the generators $u_{ij} \otimes u_{rs}$ of $(A(R)_1 \otimes \cdots \otimes A(R)_p)^{\otimes 2}$, by

$$\sigma(u_{ij} \otimes u_{kl}) = \begin{cases} \delta_{ij}\delta_{kl}(r^{ik})^{-1} & \text{if } i > j \text{ and } i \not\sim j, \\ \delta_{ij}\delta_{kl} & \text{otherwise.} \end{cases}$$

Now it is easy to show that ${}_\sigma(A(R)_1 \otimes \cdots \otimes A(R)_p)_{\sigma^{-1}} \simeq A(R)$ as a bialgebra. The statement about the Poincaré series then follows easily. □

Hence, for R as in the proposition above, there always exists a quotient bialgebra of $A(R)$ which is a domain. The next question is whether this is again dual quasitriangular. The answer is affirmative.

PROPOSITION 4.6. *Let R be as above. Then, $A(R)/I_{\sim}$ is a dual quasitriangular bialgebra with universal R-form U given by $U(t^i_j \otimes t^l_k) = R^{il}_{jk}$ on the generators.*

Proof. Straightforward from the fact that $A(R)/I_{\sim}$ is isomorphic as an algebra to a subalgebra of $A(R)$, and from Proposition 4.4. □

We know by Proposition 4.5 and Section 3 that, up to a twist, $A(R)/I_{\sim}$ can be embedded in a Hopf algebra. We try to get rid of the 'up to a twist' now.

PROPOSITION 4.7. *Let R, $A(R)$, $A(R)_i$, $A(R)/I_{\sim}$ be as above. Let $E_j = \{k \in B_j \mid r^{kk} = \lambda_j\}$ and $O_j = \{k \in B_j \mid \mu_j = r^{kk}\}$, for a block B_j of I. Let d_{E_j} (resp. d_{O_j}) denote the element of formula* (3.1E) *(resp.* 3.2O*) relative to $A(R)_j$ and let $d_j = d_{E_j} d_{O_j}$. Then the corresponding element D_j of d_j in $A(R)/I_{\sim}$ after the twist by σ in Proposition 4.5 has exactly the same expression as d_j, so that we can identify them. The D_j's commute with each other as elements of $A(R)/I_{\sim}$, their product D is grouplike up to nilpotent elements, and the multiplicatively closed set generated by the trivial lift of D in $A(R)$ is an Ore set.*

Proof. The fact that the expression is the same is clear because the twist by σ does not affect products of elements with indices in the same block. The fact that the D's commute with each other follows by direct computation. Indeed, for summands $T_{j\pi}$ and $T_{k\tau}$ in the expressions of D_j and D_k respectively, with $j > k$, we have:

$$T_{j\pi} := t^{N_j+1}_{\pi(N_j+1)} t^{N_j+2}_{\pi(N_j+2)} \cdots t^{N_j+|B_j|}_{\pi(N_j+|B_j|)}$$

and

$$T_{k\tau} := t^{N_k+1}_{\pi(N_k+1)} t^{N_k+2}_{\pi(N_k+2)} \cdots t^{N_k+|B_k|}_{\pi(N_k+|B_k|)},$$

where $N_j = \sum_{t=1}^{j} |B_t|$, $N_k = \sum_{t=1}^{j} |B_t|$, π is a permutation in $S_{|E_j|} \times S_{|O_j|}$ and τ is a permutation in $S_{|E_k|} \times S_{|O_k|}$. Then,

$$\sigma(T_{j\pi} \otimes T_{k\tau}) = \begin{cases} 0 & \text{if } \pi \neq \text{id and /or } \tau \neq \text{id}, \\ \prod_{i \in B_j; s \in B_k} (r^{is})^{-1} r^{\pi(i)\tau(s)} = 1 & \text{if } \pi = \text{id and } \tau = \text{id}. \end{cases}$$

The fact that D is grouplike modulo a nilpotent element follows from the fact that it is so in $A(R)_1 \otimes \cdots \otimes A(R)_p$. The fact that $\{D^k \mid K \geqslant 0\}$ is an Ore set follows from the fact that each D_j satisfies Ore conditions with respect to the elements containing only indices belonging to B_j. Indeed, twisting by σ does not affect products of elements with indices all belonging to the same block. Each D_j commutes then up to a constant factor with the generators with indices in all the other blocks. Indeed, for t^i_j with $i, j \in B_t$

$$D_k \bar{t}^i_j = \sum_{s,l \in B_t} \sigma(D_k \otimes t^i_s) \sigma^{-1}(D_k \otimes t^l_j) \overline{dt^s_l} + 0 = \prod_{t \in B_k} (r^{ti})^{-1} r^{\pi(t)k} \overline{dt^i_j},$$

where overlining stands for the identification between $A(R)/I_\sim$ and the tensor product of the $A(R)_i$'s. □

We have just seen that it makes sense to localize $A(R)/I_\sim$ at D, and that the expression of D and d are the same. One can now extend the 2-cocycle σ to $(A(R)_1 \otimes \cdots \otimes A(R)_p)_d$, the localization of $(A(R)_1 \otimes \cdots \otimes A(R)_p)$ at d. Indeed, if we pose $\sigma(d^{-1} \otimes a) = \sigma^{-1}(d \otimes a)\varepsilon(a)$ for a monomial $a \in (A(R)_1 \otimes \cdots \otimes A(R)_p)$, we see that σ is again a 2-cocycle and that

$$_\sigma[(A(R)_1 \otimes \cdots \otimes A(R)_p)_d]\sigma^{-1} \simeq (A(R)/I_\sim)_D,$$

the localization of $A(R)/I_\sim$ at D. Hence, by the fact that the twist of a Hopf algebra is again a Hopf algebra, we know that $A(R)/I_\sim$ is a Hopf algebra. Besides, since the sub-bialgebras generated by elements with indices in a single block of I, say B_t, together with D_t^{-1} have always an antipode, from what we saw in Section 3, we also have a definition of the antipode on the generators, since the antipode is always unique. Hence we have:

PROPOSITION 4.8. *$(A(R)/I_\sim)_D$ is a Hopf algebra.*

Remark. We might wonder whether the construction of an antipode for $(A(R)/I_\sim)_D$ could be lifted to an antipode for some extension of $A(R)$ itself. In particular, we would like to describe the Hopf envelope H of $A(R)$ in this way. By the universal property of H, it follows that there must be a unique Hopf algebra map $\chi: H \to (A(R)/I_\sim)_D$ such that $\chi \circ \iota = \pi$, where π is the map $A(R) \to A(R)/I_\sim \to (A(R)/I_\sim)_D$. In particular, this implies that for any lift $\tilde{D}$ of D, $\iota(\tilde{D}) + \ker(\chi)$ must be invertible modulo $\ker(\chi)$. Since $\iota(\ker(\pi)) \subset \ker(\chi)$, it is not absurd to wonder whether also H could be constructed by localization at some set in $\iota(A(R))$. Of course this is not necessary, but it is reasonable to assume that the invertibility of $\iota(\tilde{D}) + \ker(\chi)$ modulo $\ker(\chi)$ implies genuine invertibility of some element in $\iota(\tilde{D}) + \ker(\chi) \cap \iota(A(R)) = \iota(\tilde{D}) + \iota(I_\sim)$. Then a straightforward assumption would be to assume that this element is exactly $\iota(\tilde{D})$, but this is too strong an assumption. Indeed, localizing $A(R)$ itself at $\tilde{D}$ leads in general no further than to localizing $A(R)/I_\sim$ at D, as we see from the following proposition.

PROPOSITION 4.9. *Let $\tilde{D}_j$ be the obvious lift of D_j in $A(R)$, i.e. and let $\tilde{D} = \prod_{j=1}^{p}(\tilde{D}_j)$. Then, if every E_j and O_j have never size* 1, *$I_\sim$ is equal to* $\mathrm{Ann}(\tilde{D})$, *the annihilator of $\tilde{D}$.*

Proof. Let $t_i^j \in I_\sim$, with $i \in B_r$. Then, $t_i^j \tilde{D} = t_i^j(\tilde{D}_{E_j}\tilde{D}_{O_j}) \times \text{others} = C t_i^j(\tilde{D}_{O_j}\tilde{D}_{E_j}) \times \text{others}$ for some constant C.

If $r^{jk}r^{kj} \neq (r^{ii})^2$ for every $k \in B_r$, then, as in the proof of Proposition 4.1, $t_i^j t_i^k = 0$. Since t_i^j commutes with the elements of type $t_{\pi(a)}^a$ up to a constant, it follows that the above product is zero.

Hence we might assume that $z_{jr} = (r^{ii})^2$. We look then at the product $t_i^j \tilde{D}_{P_r}$ for $P_r = E_r$ if $i \in E_r$, and $P_r = O_r$ if $I \in O_r$. Then, if $|P_r| \geqslant 2$:

$$t_i^j \left(\sum_{\pi} \left(\prod_{a<b;\ \pi(a)>\pi(b)} (-(r^{ii})^{-1} r^{\pi(b)\pi(a)}) \right) t_{\pi(s_1)}^{s_1} \cdots t_{\pi(s_p)}^{s_p} \right)$$
$$= \sum_{\pi(s_1)<\pi(s_2)} q \text{ length}(\pi) t_i^j \left(t_{\pi(s_1)}^{s_1} t_{\pi(s_2)}^{s_2} - (r^{ii})^{-1} r^{\pi(s_1)\pi(s_2)} t_{\pi(s_2)}^{s_1} t_{\pi(s_1)}^{s_2} \right) t_{\pi(s_3)}^{s_3} \cdots t_{\pi(s_p)}^{s_p},$$

where $q \text{ length}(\pi) = \left(\prod_{a<b;\ \pi(a)>\pi(b)} (-(r^{ii})^{-1} r^{\pi(b)\pi(a)})\right)$.

We look at the product: $t_i^j (t_{\pi(s_1)}^{s_1} t_{\pi(s_2)}^{s_2} - (r^{ii})^{-1} r^{\pi(s_1)\pi(s_2)} t_{\pi(s_2)}^{s_1} t_{\pi(s_1)}^{s_2})$: one can check as in the proof of Proposition 4.1 that this is always zero. Since the elements of the form t_i^j with $i \not\sim j$ quasicommutes with all the elements except the t_l^k's with $k \sim j$ and $i \sim l$, $I_\sim \tilde{D} = 0$. □

From the proof of this proposition we see that the annihilator of $\tilde{D}$ is *at least* the ideal generated by all the t_i^j's with i not belonging to an E_r or an O_r of size one, and it might be bigger, for instance if the $z_{st} \neq (r^{ii})^2$ for every s, t and i.

Hence if we localized $A(R)$ at $\tilde{D}$ (supposing localization were possible) the kernel of the map $A(R) \to (A(R))_{\tilde{D}}$ would be, under the hypothesis of Proposition 4.9, the whole $I_\sim$ since the elements of the form t_i^j with $i \not\sim j$ quasicommutes with all the elements except the t_l^k's with $k \sim j$ and $i \sim l$.

We might now look for another lift of D in $A(R)$ and wonder if its annihilator can be zero. We show with an example what can go wrong.

EXAMPLE. Let R be the following 9×9 solution of the quantum Yang–Baxter equation: the blocks are $B_1 = \{1\}$ and $B_2 = \{2, 3\}$, every component has size one and $0 \neq r^{11} = \lambda_1 \neq \pm\lambda_2 = r^{22} = r^{33} \neq 0$, $r_{32}^{23} = \lambda_2 + \mu \neq 0$, $r^{13}r^{31} = r^{12}r^{21} = \lambda_2^2$, $r^{32}r^{23} = -\lambda_2\mu \neq \lambda_1^2$. Then we know from Proposition 4.1 that $A(R)$ is not a domain. $I_\sim$ is generated by $t_2^1, t_3^1, t_1^3, t_1^2$ and $\tilde{D}_1 = t_1^1$, $\tilde{D}_2 = t_2^2 t_3^3 - \lambda_2^{-1} r^{23} t_3^2 t_2^3$, and $t_1^2 \tilde{D}_2 = t_1^3 \tilde{D}_2 = 0$ by Proposition 4.9. Since $r^{1j} r^{j1} = (\lambda_2)^2 \neq (\lambda_1)^2$ for $j \in B_2$, it follows that $t_1^1 t_j^1 = 0 = t_1^1 t_1^j$, hence $I_\sim = \text{Ann}(\tilde{D})$. We see by the relations that $t_2^k t_j^1 = (r^{k1})^{-1} r^{2j} t_j^1 t_2^k$ and that $t_j^1 t_3^k = (r^{1k})^{-1} r^{j3} t_3^k t_j^1$ for $j, k \in B_2$. Moreover, $t_1^j t_k^2 = (r^{j2})^{-1} r^{1k} t_k^2 t_1^j$ and $t_k^3 t_1^j = (r^{3j})^{-1} r^{k1} t_k^2 t_1^j$, hence $I_\sim$ quasicommutes with elements with indices only in B_2 or in B_1. We can easily see that elements in $I_\sim$ also quasicommute with each other. Hence, a lift of $D \in A(R)/I_\sim$ would be of the form $F = t_1^1(t_2^2 t_3^3 - \lambda_2^{-1} r^{23} t_3^2 t_2^3) + a_{12} t_2^1 + a_{21} t_1^2 + a_{31} t_1^3 + a_{13} t_3^1$, with the $a_{ij} \in A(R)$ not necessarily unique.

But then, taking for instance t_1^j, with $j \in B_2$ we would have $F t_1^j = a_{12} t_2^1 t_1^3 + a_{13} t_3^1 t_1^3$ because $(t_1^j)^2 = 0$, and $t_1^2 t_1^3 = 0 = t_1^3 t_1^2$ by taking then t_k^1 with $k \in B_2$ we have finally that $F t_1^j t_k^1 = 0$. Hence, all the elements of the form $t_1^j t_k^1$ for $j, k \in \{2, 3\}$ are in the annihilator of such an F, so that localizing at F (for a suitable choice of the a_{ij}'s that makes it an Ore set) would still not be an embedding, being those elements in the kernel. In this case we also observe that $I_{\sim}^3 = 0$ since all its generators are nilpotent and quasicommute with all elements in $A(R)$. Still, $A(R)/I_{\sim}$ is a domain and it can be embedded in the Hopf algebra localized at $\pi(t_t^1(t_2^2 t_3^3 - \lambda_2^{-1} r^{23} t_3^2 t_2^3)$.

We have seen so far what can go wrong in lifting an antipode from a quotient of $A(R)$. We would like to recall that the question of the antipode has been treated by Manin, with a formal construction of the Hopf envelope, and Majid, who presents different approaches to the problem, for instance by the definition of a weak antipode, that always exists, by localization, and by formal extension (see [29] and [18] for further reading).

We want to devote the last part of this section to another choice for a quotient of $A(R)$. This is not again a bialgebra, but has amusing properties. It can be made more general, but we only want to give an idea of what can happen.

Let R from now on be such that all components are of size one, $r^{aa} = \lambda$ for every $a \in I$, and such that $z_{st} = \lambda^2$ for every s and t, and suppose that R is not diagonal. If there is more than one block, $A(R)$ is not a domain. One can then factor out $A(R)$ by the ideal $I_{\times}$ generated by all elements of the form $t_j^i t_k^l - \lambda (r^{kj})^{-1} t_k^i t_j^l$ with $i < l$ and $j > k$. Some of those elements are zero already in $A(R)$. We want to give a presentation of the new algebra $A(R)_{\mathrm{II}} = A(R)/I_{\times}$ by generators and relations. One can check that the relations of $A(R)$, combined with the new ones, provide a set of rewriting rules for $A(R)_{\mathrm{II}}$ with confluent overlap ambiguities. They are, as follows:

$$t_j^i t_k^l = (r_{kj}^{kj})^{-1} \lambda t_k^i t_j^l \quad (i \geqslant l,\ j > k), \tag{4.a}$$

$$t_j^i t_k^l = (r_{il}^{il})^{-1} r_{jk}^{jk} t_k^l t_j^i \quad (i > l,\ j \leqslant k), \tag{4.b}$$

$$t_j^i t_k^l = (r_{kj}^{kj})^{-1} \lambda t_k^i t_j^l \quad (i > l,\ j < k). \tag{4.c}$$

A basis for this algebra is given by all the monomials such that both upper and lower indices are in nondecreasing order: $t_{j_1}^{i_1} \cdots t_{j_k}^{i_k}$ with $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k$ and $j_1 \leqslant j_2 \leqslant \cdots \leqslant j_k$.

Again we have

PROPOSITION 4.10. *The algebra $A(R)_{\mathrm{II}}$ is a domain.*

Proof. Standard reasoning as before. As usual, the degree and the lexicographic ordering* provide an ordering of the monomials of the basis. Then, any two elements a and b in $A(R)_{\mathrm{II}}$ can be written as $a = c_A t^A +$ lower order terms, and $b = c_B t^B +$ lower order terms. Here A and B are $n \times n$ matrices with nonnegative

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* We say that $\alpha < \beta$ if degree$(\beta) >$ degree(α) or if degree$(\beta) =$ degree(α) and β precedes α in lexicographic order.

integer entries satisfying the conditions $\forall i, j \in \{1, \dots, n\}$, $a_{ij} \neq 0 \Rightarrow a_{kl} = 0$ unless $k \leqslant i$ and $j \leqslant l$ or $i \leqslant k$ and $l \leqslant j$. Then, the leading term of ab is

$$c_A c_B \prod_{i>l, j>k} [\lambda^2 (r_{il}^{il} r_{jk}^{jk})^{-1}]^{a_{ij} b_{lk}} \cdot \prod_{i \geqslant l, j \leqslant k} [\lambda (r_{il}^{il})^{-1}]^{a_{ij} b_{lk}} \cdot \prod_{i<l, j>k} [\lambda (r_{kj}^{kj})^{-1}]^{a_{ij} b_{lk}} t^C,$$

where C is the (unique) n^2 matrix with coefficients in $\mathbb{Z}_{\geqslant 0}$ satisfying the conditions

$$\forall i, j \in \{1, \dots n\} : c_{ij} \neq 0 \Rightarrow c_{kl} = 0 \quad \text{unless } k \leqslant i \text{ and } j \leqslant l \text{ or } i \leqslant k \text{and } l \leqslant j$$

and

$$\forall i, j \in \{1, \dots n\} \sum_{j=1}^{n} c_{ij} = \sum_{j=1}^{n} (a_{ij} + b_{ij}) \quad \text{and} \quad \sum_{j=1}^{n} c_{ji} = \sum_{j=1}^{n} (a_{ji} + b_{ji}).$$

Therefore, $ab \neq 0$. □

We now have a ‘negative’ result:

PROPOSITION 4.11. *The ideal $I_\times$ defined in Section* 4 *is not a bialgebra ideal. However, $\Delta(I_\times) \subset A(R) \otimes I_\times$, i.e. $I_\times$ is a left coideal.*

Proof. If $i < j$, $\varepsilon(t_j^i t_i^j - \lambda(r_{ij}^{ij})^{-1} t_i^i t_j^j) \neq 0$. The second statement follows by direct computation. □

So, even if one does not have a bialgebra structure on $A(R)_{\mathrm{II}}$, one does still have a comultiplication coming from that of $A(R)$, which is an algebra homomorphism from $A(R)_{\mathrm{II}}$ to $A(R) \otimes A(R)_{\mathrm{II}}$, i.e. $A(R)_{\mathrm{II}}$ is a comodule algebra. One can say more about the algebra structure of $A(R)_{\mathrm{II}}$. It turns out to be isomorphic to a subalgebra of a tensor product of two different skew polynomial algebras (quantum planes).

Let X (resp. Y) be the algebra of polynomials with coefficients in K in the commuting variables $x_1, \dots, x_n$ (resp. $y_1, \dots, y_n$). They are left comodules for $F(M_n)$, the bialgebra of polynomial functions on $M_n(K)$ with the usual comultiplication. Therefore, one can twist X (resp. Y) on the left by means of a 2 cocycle σ_X (resp. σ_Y) in $(F(M_n) \otimes F(M_n))^*$. Let σ_X (resp. σ_Y) be

$$\sigma_X(u_{ij} \otimes u_{lk}) = \begin{cases} \delta_{ij} \delta_{lk} \lambda (r_{il}^{il})^{-1} & \text{if } i > l, \\ \delta_{ij} \delta_{lk} & \text{if } i \leqslant l. \end{cases}$$

$$\sigma_Y(u_{ij} \otimes u_{lk}) = \begin{cases} \delta_{ij} \delta_{lk} \lambda (r_{kj}^{kj})^{-1} & \text{if } j > k, \\ \delta_{ij} \delta_{lk} & \text{if } j \leqslant k. \end{cases}$$

Then ${}_{\sigma_X}X$ is the skew polynomial algebra with generators $x_1, \dots, x_n$ and relations $x_i x_j = \lambda (r_{ij}^{ij})^{-1} x_j x_i$ for $i > j$, and ${}_{\sigma_Y}Y$ is the skew polynomial algebra with generators $y_1, \dots, y_n$ and relations $y_i y_j = \lambda (r_{ji}^{ji})^{-1} y_j y_i$ for $i > j$.

Their tensor product ${}_{\sigma_X}X \otimes_{\sigma_Y} Y$ is $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$-graded, once we have given the usual grading to the two components ${}_{\sigma_X}X$ and ${}_{\sigma_Y}Y$. We consider then the subalgebra G generated by all elements whose degree belongs to $\{(k, k) \mid k \in \mathbb{Z}_{\geqslant 0}\}$, that is $G = \sum_{k \in \mathbb{Z}_{\geqslant 0}} ({}_{\sigma_X}X)_k \otimes ({}_{\sigma_Y}Y)_k$. Clearly, G is $\mathbb{Z}_{\geqslant 0}$-graded. We have the following description of $A(R)_{\mathrm{II}}$.

PROPOSITION 4.12. *The algebra $A(R)_{\mathrm{II}}$ is isomorphic to G.*

Proof. The relations for $A(R)_{\mathrm{II}}$ can be rewritten as follows:

$$t^i_j t^l_k = \begin{cases} \lambda^2 (r^{il}_{il} r^{kj}_{kj})^{-1} t^l_k t^i_j & \text{if } i > l \text{ and } j > k, \\ \lambda (r^{il}_{il})^{-1} t^l_j t^i_k & \text{if } i \geqslant l \text{ and } j \leqslant k, \\ \lambda (r^{kj}_{kj})^{-1} t^i_k t^l_j & \text{if } i < l \text{ and } j > k. \end{cases}$$

Then, it is easy to see that the map sending t^i_j to $x_i \otimes y_j$ is an algebra isomorphism. □

This gives an easy way to compute the Poincaré series of $A(R)_{\mathrm{II}}$.

COROLLARY 4.13. *The Poincaré series of $A(R)_{\mathrm{II}}$ is given by $\sum_{r \geqslant 1} \binom{r+n-1}{r}^2 t^r$.*

We end this section with a remark that will not be surprising to an attentive reader. We work under the same assumption on R as above. Let $A(B_s)$ be the subalgebra of $A(R)$ corresponding to the block B_s. This is isomorphic, as an algebra, to the standard multiparameter deformation of the algebra of functions on $M_{n_s}(K)$. Again we denote by d_s the quantum determinant of this subalgebra. Then we have the following proposition:

PROPOSITION 4.14. *With the assumption above, d_s belongs to the kernel of the projection* proj: $A(R) \to A(R)_{\mathrm{II}}$, *i.e. to the ideal $I_\times$, for each block B_s with at least two elements.*

Proof. One easily computes

$$\begin{aligned} \mathrm{proj}(d_s) &= \sum_{\pi \in S_{n'_s}} \Bigg(\prod_{i<j, \pi(i)>\pi(j)} (-\lambda)^{-1} r^{\pi(j)\pi(i)}_{\pi(j)\pi(i)} \Bigg) \times \\ &\quad \times \Bigg(\prod_{i<j, \pi(i)>\pi(j)} \lambda (r^{\pi(j)\pi(i)}_{\pi(j)\pi(i)})^{-1} \Bigg) t^{1'}_{1'} \cdots t^{n'_s}_{n'_s} \\ &= \Bigg(\sum_{\pi \in S_{n'_s}} \mathrm{sign}(\pi) \Bigg) t^{1'}_{1'} \cdots t^{n'_s}_{n'_s} = 0. \end{aligned}$$

□

Appendix

In this appendix we want to describe in detail the construction of twists of bialgebras. The notion of twist that we used in this paper is the same that appears in

[1, 32]. It is a standard concept in mathematics, due to many people before them. It is, as we show here, a special case of the dual of the notion of twist defined in [4, 6, 33] (Drinfel'd gives an even more general definition, though). We add these computations for completeness, sake and readability, although we do not claim any originality for them. Another treatment of these results is, for instance, to be found in [18]. A deeper discussion on specializations would also be needed, in order to be clear about what we mean by tensor products, and duality, but we would walk 'too far from our path'. Our purpose is just to show how to compute twists, and what they represent. We start with Drinfel'd's type of twist.

DEFINITION–PROPOSITION. *Let* $(A, m, i, \Delta, \varepsilon, S)$ *be a Hopf algebra over a commutative ring. Let* F *be an invertible element of* $A \otimes A$ *such that*

(a) $F_{12}(\Delta \otimes \mathrm{id})(F) = F_{23}(\mathrm{id} \otimes \Delta)(F)$,
(b) $(\varepsilon \otimes \mathrm{id})(F) = 1 = (\mathrm{id} \otimes \varepsilon)(F)$.

Then $v = m(\mathrm{id} \otimes S)(F)$ *is an invertible element of* A *with* $v^{-1} = m(S \otimes \mathrm{id})(F^{-1})$. *Moreover, if we define*

$$\Delta^F : A \to A \otimes A \quad \textit{and}$$
$$S^F : A \to A$$

by

(c) $\Delta^F(a) = F\Delta(a)F^{-1}$ (*product in* $A \otimes A$),
(d) $S^F(a) = vS(a)v^{-1} \quad \forall a \in A$,

then $(A, m, i, \Delta^F, \varepsilon, S^F)$ *is again a Hopf algebra, denoted by* A^F *and called the 'co-twist' of* A *by* F.

This proposition is to be found in the survey [4], Chapter 4, where the 'co-twist' is called 'twist'(see there for further references, and, of course, [6], where the case of quasi Hopf algebras is treated). Actually, one can apply the same construction even if A is just a bialgebra, obtaining a 'cotwisted' bialgebra.

Suppose now that we have a non degenerate pairing of Hopf algebras (or bialgebras), A and B, and suppose we perform the 'co-twist' to A. We look for some sort of 'twist' for B in order to get again a pairing. Of course, since 'co-twisting' doesn't affect the multiplication, the new notion of twist should leave the comultiplication μ in B unchanged. We make use of the following construction, as described in the survey [32].

DEFINITION–PROPOSITION. *Let* H *be a Hopf algebra over the field* K, *and* B *be a left* (*resp. right*) H*-comodule algebra* (*i.e.* B *is an algebra, and an* H*-comodule such that the comodule map* ρ *is also an algebra homomorphism*). *Let* σ *be a linear map* $\sigma\colon H \otimes H \to K$ *satisfying*

$$(\mathrm{e}_{\mathrm{left}}) \sum \sigma(k_1 \otimes m_1)\sigma(h \otimes k_2 m_2) = \sum \sigma(h_1 \otimes k_1)\sigma(h_2 k_2 \otimes m),$$

(*respectively* e_{right})

$$\sum \sigma(k_2 \otimes m_2)\sigma(h \otimes k_1 m_1) = \sum \sigma(h_2 \otimes k_2)\sigma(h_1 k_1 \otimes m))$$

and

$$\text{(f) } \sigma(h \otimes 1) = \sigma(1 \otimes h) = \varepsilon(h),\ \forall\, h, k, m \in H$$

with $\mu(h) = \sum h_1 \otimes h_2$, *etc.*

Then, the σ*-left* (*resp.* σ*-right*) *twisted comodule algebra* ${}_\sigma B$ (*resp.* B_σ) *is an algebra with the same underlying vector space as* B, *and product given by*

$$(\text{g}_{\text{left}})\ \bar{a} \cdot \bar{b} = \sum \sigma(h_a \otimes h_b)\overline{c_a c_b} \text{ if } \rho(a) = \sum h_a \otimes c_a,\ a, b \in B$$

(*respectively* g_{right})

$$\bar{a} \cdot \bar{b} = \sum \overline{c_a c_b}\sigma(h_a \otimes h_b) \text{ if } \rho(a) = \sum c_a \otimes h_a,\ a, b \in B),$$

where one denotes by $a \mapsto \bar{a}$ *the identification of vector spaces.*

A linear map $\sigma\colon H \otimes H \to K$ satisfying (e_{left}), (resp. (e_{right})) and f, is called a 2-left (resp. right) cocycle.

If B is a Hopf algebra, given a 2-cocycle, one can perform such a twist to B, viewed as a left (resp. right) B-comodule algebra. Then, one sees that $\mu_\sigma\colon {}_\sigma B \to B \otimes_\sigma B$ is an algebra homomorphism, but in general this does not hold for $\mu_\sigma\colon {}_\sigma B \to {}_\sigma B \otimes_\sigma B$, so that ${}_\sigma B$ is not a bialgebra in general. However, one can see that, if σ is a left-2-cocycle and is invertible as an element of $(B \otimes B)^*$ (which is always an algebra)*, then σ^{-1} is a right-2-cocycle. The double twist ${}_\sigma B_{\sigma^{-1}}$ is again a bialgebra. In fact, one does not even find it necessary that B has an antipode. Everything can be done for any bialgebra, with an invertible 2-cocycle.

Now let A and B be bialgebras, so that (A, B) is a pairing. We 'cotwist' A by an F satisfying (a) and (b). $F \in (A \otimes A) \subset B^* \otimes B^* \subset (B \otimes B)^*$, and it is easily seen that F satisfies (a) and (b) $\Leftrightarrow$ F satisfies (f) and (e_l). Therefore, we can twist the multiplication in B by F on the left and F^{-1} on the right, obtaining ${}_F B_{F^{-1}}$ which is a bialgebra. Moreover, we can define a pairing $A^F \times {}_F B_{F^{-1}} \to K$ using the identifications $A \to A^F$ and $B \to {}_F B_{F^{-1}}$: $\langle \bar{f}, a \rangle_F := \langle f, a \rangle\ \forall \bar{a} \in {}_F B_{F^{-1}}$ and $\bar{f} \in A^F$. This form is clearly bilinear and nondegenerate, since the underlying vector spaces are the same as before. One can easily check that it defines a bialgebra pairing. Indeed $\forall f, g \in A, a, b \in B$

$$\langle \bar{f} \otimes \bar{g}, \mu(\bar{a}) \rangle_F = \langle f \otimes g, \mu(a) \rangle = \langle fg, a \rangle = \langle \overline{fg}, \bar{a} \rangle_F,$$
$$\varepsilon_{A^F}(\bar{f}) = \langle \bar{f}, 1 \rangle_F = \langle f, 1 \rangle = \varepsilon_A(f),$$
$$\varepsilon_{{}_F B_{F^{-1}}}(\bar{a}) = \varepsilon_B(a) = \langle 1_A, a \rangle = \langle 1, \bar{a} \rangle_F$$

* Namely, there is a τ in $(B \otimes B)^*$ such that $\forall a$ and b in B $m(\tau \otimes \sigma)\Delta_{B \otimes B}(a \otimes b) = m(\sigma \otimes \tau)\Delta_{B \otimes B}(a \otimes b) = \varepsilon(a)\varepsilon(b)$.

and

$$\begin{aligned}
&\langle \bar{f}, m(\bar{a}\otimes\bar{b})\rangle_F\\
&\langle \bar{f}, (F\otimes m_B\otimes F^{-1})\sigma_{2354}(\mu\otimes\mathrm{id}\otimes\mu\otimes\mathrm{id})(\mu\otimes\mu)(\bar{a}\otimes\bar{b})\rangle_F\\
&\quad=\sum\langle F, a_1\otimes b_1\rangle\langle\Delta(f), a_2\otimes b_2\rangle\langle F^{-1}, a_3\otimes b_3\rangle\\
&\quad=\langle\Delta^F(\bar{f}), \bar{a}\otimes\bar{b}\rangle_F.
\end{aligned}$$

Here, σ_{ijkl} denotes the action of the cycle $(ijkl)$ by permuting factors in the tensor product. If A is an Hopf algebra with antipode S, then B is also an Hopf algebra, and the antipode also satisfies a pairing condition. In this case, we can also provide ${}_FB_{F^{-1}}$ with the antipode $(S^F)^* = k^F := (v\otimes k\otimes v^{-1})(\mu\otimes\mathrm{id})\mu$. With this antipode, ${}_FB_{F^{-1}}$ is a Hopf algebra in pairing with A^F. The fact that k^F is indeed an antipode follows from nondegeneracy of $\langle,\rangle_F$. Indeed, for every $\bar{f}\in A^F$ and every $\bar{a}\in {}_FB_{F^{-1}}$, one can check that

$$\begin{aligned}
&\langle \bar{f}, m(\mathrm{id}\otimes k^F)\mu(\bar{a})\rangle_F\\
&\quad=\langle\varepsilon(\bar{f})1_{A^F}, \bar{a}\rangle_F\\
&\quad=\varepsilon_A(f)\varepsilon_B(a)=\langle\bar{f}, \varepsilon(\bar{a})1_{{}_FB_{F^{-1}}}\rangle_F.
\end{aligned}$$

Because of the duality between these twisting processes, one could also start from the twist of a given algebra B, and try to cotwist the dual algebra A, but in this case it is not at all obvious that a 2-cocycle $\sigma\in(B\otimes B)^*$ is an element of $A\otimes A$. The way of escape is to observe that $A\otimes A\subset(B\otimes B)^*$ and that the latter is always an algebra. For this reason, the product $\sigma\Delta(f)\sigma^{-1}$ still makes sense in $(B\otimes B)^*$, where one can check if the necessary conditions are satisfied. However, we cannot be sure that Δ^σ lands in $A\otimes A$, therefore, we may need some sort of completed tensor product or restrict to the algebra A^σ defined as $\{a\in A\mid\Delta^\sigma(a)\in A\otimes A\}$. Δ^σ lands in the tensor product of this idea is that all those things work once one finds a suitable context in which the computations make sense, which may differ case by case.

The link between the twist in [1] and the one we described here is given as follows. In [1] the authors just look for a 2-cocycle c in $\mathrm{Hom}(G, K^*)$ for some Abelian group G by means of which the algebra is graded. In their paper this is given by the free Abelian group with generators t_i, $1\leqslant i\leqslant n$. This fits into the picture we gave as follows. There is a natural projection π from their quantum $\mathbf{GL}_n$ onto the group algebra KG given by $u_{ij}\mapsto\delta_{ij}t_i$, i.e., KG is the quantum subgroup corresponding to the torus, and one can check that if c is a 2-cocycle as defined in [1], then, $c\circ\pi$ is a cocycle as it is defined in this section.

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