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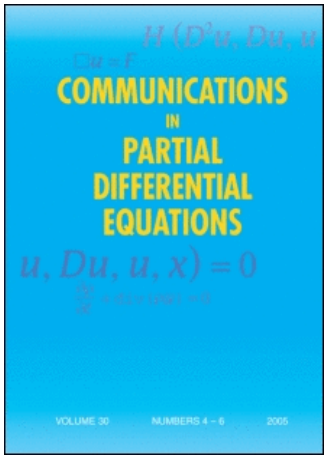

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VANISHING THEOREM FOR SHEAVES OF MICROFUNCTIONS AT THE BOUNDARY ON CR-MANIFOLDS

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Abstract. Let X be a complex analytic manifold. Consider $S \subset M \subset X$, real analytic submanifolds with $\operatorname{codim}_M^{\mathbf{R}} S = 1$, and let Ω be a connected component of $M \setminus S$. Let $p \in S \times_M T^*_M X$, where $T^*_M X$ denotes the conormal bundle to M in X, and denote by $\nu(p)$ the complex radial Euler field at p. Denote by $\mu_*(\mathcal{O}_X)$ (for $* = M, \Omega$) the microlocalization of the sheaf of holomorphic functions along $*$.

Under the assumption $\dim^{\mathbf{R}}(T_p T^*_M X \cap \nu(p)) = 1$, a theorem of vanishing for the cohomology groups $H^j \mu_M(\mathcal{O}_X)_p$ is proved in [**K-S 1**, Prop. 11.3.1], j being related to the number of positive and negative eigenvalues for the Levi form of M.

Under the hypothesis $\dim^{\mathbf{R}}(T_p T^*_S X \cap \nu(p)) = 1$, a similar result is proved here for the cohomology groups of the complex of microfunctions at the boundary $\mu_\Omega(\mathcal{O}_X)$. Stating this result in terms of regularity at the boundary for CR-hyperfuntions a local Bochner-type theorem is then obtained.

§1. NOTATIONS AND REVIEW

For the content of this section, we refer to [**S 1**] and [**K-S 2**].

1.1. Let X be a complex analytic manifold and $M \subset X$ a real analytic submanifold. One denotes by $\pi : T^*X \to X$ the cotangent bundle to X, by

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$\dot{\pi} : \dot{T}^*X \to X$ the cotangent bundle with the zero section removed, and by T^*_MX the conormal bundle to M in X. One denotes by α the canonical one-form on T^*X and sets $\sigma = -d\alpha$. A complex analytic submanifold $\Lambda \subset T^*X$ is called Lagrangian if it is so for the homogeneous symplectic structure induced by σ on T^*X. Let $\sigma^{\mathbf{R}}$, $\sigma^{\mathbf{I}}$, be twice the real and imaginary part of σ. These are symplectic forms on the real underlying manifold to T^*X. A real analytic submanifold Λ' is called $\mathbf{R}$-Lagrangian if it is so for the symplectic structure given by $\sigma^{\mathbf{R}}$. If Λ' is $\mathbf{R}$-Lagrangian, one says that Λ' is $\mathbf{I}$-symplectic if $\sigma^{\mathbf{I}}|_{\Lambda'}$ is non-degenerate.

1.2. For $p \in \dot{T}^*_MX$, we use the following notations:

$$\begin{aligned} E &= \text{the space } T_pT^*X \text{ endowed with the linear symplectic structure} \\ &\quad \text{given by the two-form } \sigma, \\ \nu(p) &= \text{the complex Euler radial field at } p, \\ \lambda_0(p) &= T_p(\pi^{-1}\pi(p)), \\ \lambda_M(p) &= T_pT^*_MX. \end{aligned}$$

We sometimes write ν, λ_0 and λ_M instead of $\nu(p)$, $\lambda_0(p)$ and $\lambda_M(p)$ respectively, for short.

If $\rho \subset E$ is an isotropic subspace of E, one denotes by E^ρ the space $\rho^\perp/\rho$ endowed with the symplectic structure induced by σ (here $\rho^\perp$ denotes the orthogonal to ρ with respect to σ). For $\lambda \subset E$ a real subspace, one sets $\lambda^\rho = ((\lambda \cap \rho^\perp) + \rho)/\rho \subset E^\rho$.

Let $\lambda \subset E$ be an $\mathbf{R}$-Lagrangian plane. For $\mu = \lambda \cap i\lambda$, one denotes by $L_{\lambda_0/\lambda}$ the Hermitian form on λ_0^μ defined for $(u,v) \in \lambda_0^\mu \times \lambda_0^\mu$ by $L_{\lambda_0/\lambda}(u,v) = \sigma^\rho(u,\overline{v})$, where $\overline{v}$ is the complex conjugate of v with respect to the isomorphism $\mathbf{C} \otimes_{\mathbf{R}} \lambda^\mu \cong E^\mu$. One can easily see that $L_{\lambda_0/\lambda}$ is non-degenerate on $\lambda_0^\mu/(\lambda^\mu \cap \lambda_0^\mu)^{\mathbf{C}}$.

The numbers $s^+(M,p)$, $s^-(M,p)$, of positive and negative eigenvalues for L_{λ_0/λ_M}, are given by the relations:

$$\begin{cases} s^+(M,p) + s^-(M,p) + \dim^{\mathbf{R}}(\lambda_M^\mu \cap \lambda_0^\mu) = n - \dim^{\mathbf{C}}\mu \\ s^+(M,p) - s^-(M,p) = \frac{1}{2}\tau(\lambda_M, i\lambda_M, \lambda_0), \end{cases}$$

where $\tau(\cdot,\cdot,\cdot)$ denotes the inertia index of three Lagrangian planes.

One also introduces

$$\gamma(M,p) = \dim^{\mathbf{C}}(\lambda_M \cap i\lambda_M \cap \lambda_0).$$

We sometimes write $s^\pm(M)$ and $\gamma(M)$ instead of $s^\pm(M,p)$ and $\gamma(M,p)$ respectively, for short.

One has the following result:

PROPOSITION 1.1. *(Cf.* [**D'A-Z**], [**S-T**]*)* $s^+(M)$, $s^-(M)$ *are the numbers of positive and negative eigenvalues of the Levi form of* M.

1.3. One denotes by $\mathrm{D}^b(X)$ the derived category of the category of bounded complexes of sheaves of $\mathbf{C}$-vector spaces and by $\mathrm{D}^b(X;p)$ the localization of $\mathrm{D}^b(X)$ at $p \in T^*X$ (cf. [**K-S 2**]).

For $A \subset X$ a locally closed subset, $\mathbf{C}_A$ denotes the sheaf which is 0 on $X \setminus A$ and the constant sheaf with fiber $\mathbf{C}$ on A. One sets, for short, $T^*_A X = \mathrm{SS}(\mathbf{C}_A) \subset T^*X$, where $\mathrm{SS}(\mathbf{C}_A)$ denotes the micro-support of $\mathbf{C}_A$.

Let $\mu_A(\mathcal{O}_X) = \mu\mathrm{hom}(\mathbf{C}_A, \mathcal{O}_X)$, where $\mathcal{O}_X$ is the sheaf of germs of holomorphic functions on X and $\mu\mathrm{hom}(\cdot,\cdot)$ is the bifunctor of microlocalization. Notice that the support of the complex $\mu_A(\mathcal{O}_X)$ is contained in $T^*_A X$.

§2. Statement of the result

2.1. Let $S \subset M$ be real analytic submanifolds of a complex analytic n-dimensional manifold X with $\mathrm{codim}^{\mathbf{R}}_M S = 1$ and $\mathrm{codim}^{\mathbf{R}}_X M = l$. Let Ω be a connected component of $M \setminus S$ in a neighborhood of $x_\circ \in S$ and take $p \in S \times_M \dot{T}^*_M X$ with $\pi(p) = x_\circ$.

In [**K-S 1**] the vanishing of $H^j \mu_M(\mathcal{O}_X)_p$ is related to the number of positive and negative eigenvalues for the Levi form of M as follows:

Theorem 2.1. *(Cf.* [**K-S 1**, Prop. 11.3.1, Prop. 11.3.5]*)*

(i) *Assume*

$$\dim^{\mathbf{R}}(\lambda_M(p) \cap \nu(p)) = 1. \tag{2.1}$$

Then $H^j \mu_M(\mathcal{O}_X)_p = 0$ *for* $j < l + s^-(M,p) - \gamma(M,p)$ *and for* $j > n - s^+(M,p) + \gamma(M,p)$.

(ii) *Assume (2.1) and moreover:*

$$s^-(M,p') - \gamma(M,p') \text{ is locally constant for } p' \in T^*_M X \text{ near } p. \tag{2.2}$$

Then $H^j \mu_M(\mathcal{O}_X)_p = 0$ *for* $j \neq l + s^-(M,p) - \gamma(M,p)$.

2.2. The aim of this paper is to prove analogous results for the complex of microfunctions at the boundary.

Theorem 2.2.

(i) *Assume*

$$\dim^{\mathbf{R}}(\lambda_S(p) \cap \nu(p)) = 1. \tag{2.3}$$

Then $H^j \mu_\Omega(\mathcal{O}_X)_p = 0$ *for* $j < l + s^-(M,p) - \gamma(M,p)$ *and for* $j > n - s^+(M,p) + \gamma(M,p)$.

(ii) *Assume (2.3) and moreover:*

$$\begin{cases} s^-(M,p') - \gamma(M,p') \text{ and } s^-(S,p') - \gamma(S,p') \text{ are} \\ \text{locally constant for } p' \in T^*_S X \text{ near } p. \end{cases} \tag{2.4}$$

Then $H^j \mu_\Omega(\mathcal{O}_X)_p = 0$ *for* $j \neq l + s^-(M,p) - \gamma(M,p)$.

REMARK 2.4. In the case of X being a complexification of M one recovers results of [**S 2**].

§3. PROOF OF THE THEOREM

3.1. We must first state some preliminary results of symplectic geometry. The following lemma is a slight generalization of a result of [**S 1**].

LEMMA 3.1. *(Cf.* [**S 1**, Prop. 1.9]*) Let λ_1 and λ_2 be two $\mathbf{R}$-Lagrangian planes of E, and assume:*

$$\operatorname{codim}^{\mathbf{R}}_{\lambda_1}(\lambda_1 \cap \lambda_2) = 1, \tag{3.1}$$

$$\dim^{\mathbf{R}}(\lambda_1 \cap \nu) = \dim^{\mathbf{R}}(\lambda_2 \cap \nu) = 1. \tag{3.2}$$

Then there exists a complex Lagrangian plane λ_0 such that:

$$\begin{cases} \dim^{\mathbf{R}}(\lambda_1 \cap \lambda_0) = \dim^{\mathbf{R}}(\lambda_2 \cap \lambda_0) = 1, \\ \text{the forms } L_{\lambda_0/\lambda_1},\ L_{\lambda_0/\lambda_2} \text{ are positive definite.} \end{cases} \tag{3.3}$$

For the reader's convenience, we give a proof here.

PROOF: Set $\rho = (\lambda_1 \cap i\lambda_1 \cap \lambda_2 \cap i\lambda_2) + \nu$.

Since $L_{\lambda_0/\lambda_1} = L^{\rho}_{\lambda_0^{\rho}/\lambda_1^{\rho}}$ and $L_{\lambda_0/\lambda_2} = L^{\rho}_{\lambda_0^{\rho}/\lambda_2^{\rho}}$, one reduces to work in the space E^{ρ}.

Setting $\mu = \lambda_1 \cap \lambda_2$, one may then assume from the beginning

$$\mu \cap i\mu = \{0\}, \tag{3.4}$$

and look for a complex Lagrangian plane λ_0 such that:

$$\begin{cases} \lambda_0 \text{ is transversal to } \lambda_1 \text{ and } \lambda_2, \\ \text{the forms } L_{\lambda_0/\lambda_1} \text{ and } L_{\lambda_0/\lambda_2} \text{ are positive definite.} \end{cases} \tag{3.3}'$$

Notice that if λ_i $(i = 1, 2)$ is degenerate (i.e. if $\lambda_i \cap i\lambda_i \neq \{0\}$), by (3.1), (3.4) we have $\lambda_i + i\lambda_i = \mu + i\mu$, (and hence $\lambda_i \cap i\lambda_i \supset \mu \cap i\mu$). Then either λ_1 or λ_2 is non-degenerate, for otherwise $\lambda_1 \cap i\lambda_1 = \lambda_2 \cap i\lambda_2 \neq 0$ which violates (3.4).

Assume that λ_1 is non-degenerate. Choose symplectic coordinates $(z, \zeta) = (x + iy; \xi + i\eta)$ in E so that $\lambda_1 = \{(z, \zeta); y = \xi = 0\}$ and μ is the hyperplane of λ_1 of equation $x_1 = 0$.

Consider the symplectic splitting $E = E_1 \oplus E'$ for $E_1 = \mathbf{C}_{z_1} \times \mathbf{C}_{\zeta_1}$, $E' = \mathbf{C}_{z'} \times \mathbf{C}_{\zeta'}$ and set $\lambda' = \{(z', \zeta'); y' = \xi' = 0\}$, $\rho_1 = \{(0, \zeta_1); \zeta_1 \in \mathbf{C}\}$ (so that $(\mu + i\mu)^{\perp} = \rho_1 \oplus \{0\}$).

One can see that $\lambda_1 = \tilde{\lambda}_1 \oplus \lambda'$, $\lambda_2 = \tilde{\lambda}_2 \oplus \lambda'$, where

$$\begin{cases} \tilde{\lambda}_1 = \{(z_1,\zeta_1); y_1 = \xi_1 = 0\}, \\ \tilde{\lambda}_2 = \begin{cases} \rho_1, & \text{if } \lambda_2 \text{ is degenerate,} \\ \{(z_1,\zeta_1); \xi_1 = \varepsilon x_1, y_1 = 0\}, & \text{if } \lambda_2 \text{ is non-degenerate } (\varepsilon > 0). \end{cases} \end{cases}$$

Choose $\lambda_0' \subset E'$ transversal to λ' and such that $L_{\lambda_0'/\lambda'}$ is positive definite.

Reasoning as in loc. cit. one may find $\tilde{\lambda}_0 \subset E_1$ transversal to both $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ such that $L_{\tilde{\lambda}_0/\tilde{\lambda}_1}$ is positive definite and $L_{\tilde{\lambda}_0/\tilde{\lambda}_2}$ is 0 (resp. positive definite) if $\tilde{\lambda}_2$ is degenerate (resp. non-degenerate).

One may then take $\lambda_0 = \tilde{\lambda}_0 \oplus \lambda_0'$. Q.E.D.

REMARK 3.2. From the preceeding proof it follows in particular that under the hypotheses of Lemma 3.1 either $\lambda_1 \cap i\lambda_1 \subset \lambda_2 \cap i\lambda_2$ or $\lambda_1 \cap i\lambda_1 \supset \lambda_2 \cap i\lambda_2$.

Noticing that (2.3) implies (2.1), one gets:

COROLLARY 3.3. *Let $S \subset M$ be real analytic submanifolds of a complex analytic manifold X with $\operatorname{codim}_M^{\mathbf{R}} S = 1$. Let $p \in S \times_M \dot{T}^*_M X$ and assume (2.3). Then there exists a germ of complex contact transformation χ near p which interchanges $(T^*X, T^*_M X, T^*_S X, p)$ and $(T^*Y, T^*_N Y, T^*_Z Y, q)$ where N and Z are hypersurfaces of Y satisfying:*

$$s^-(N,q) = s^-(Z,q) = 0.$$

LEMMA 3.4. *(Cf.* [**D'A-Z**, Prop. 2.1]) With the same notations of Corollary 3.3, one has the estimates:

$$s^{\pm}(S) - \gamma(S) \geq s^{\pm}(M) - \gamma(M) - 1.$$

PROOF: Let f be a real analytic function vanishing on M with $p = \mathrm{d}f(x_\circ)$. By Proposition 1.1 one has that $s^{\pm}(*)$ (for $* = M, S$) are the numbers of positive and negative eigenvalues for the Hermitian form on $T^{\mathbf{C}}_{x_\circ}(*)$ of matrix $(\partial_i \bar{\partial}_j f(x_\circ))_{i,j}$ (here one sets $T^{\mathbf{C}}_{x_\circ}(*) = T_{x_\circ}(*) \cap i\, T_{x_\circ}(*)$). One has

$$\dim^{\mathbf{C}} T^{\mathbf{C}}_{x_\circ} M = n - \operatorname{codim}^{\mathbf{R}}_X M + \gamma(M,p),$$

and hence $\operatorname{codim}^{\mathbf{C}}_{T^{\mathbf{C}}_{x_\circ} M} T^{\mathbf{C}}_{x_\circ} S = 1 + \gamma(M,p) - \gamma(S,p)$. The estimates follow. Q.E.D.

3.2. We give now a proof of our main theorem.

PROOF OF THEOREM 2.2: We use the notations of §2. Let Ω' be the other connected component of $M \setminus S$ near $x_\circ$ so that $M \setminus S = \Omega \cup \Omega'$.

From the exact sequence

$$0 \to \mathbf{C}_{\Omega} \oplus \mathbf{C}_{\Omega'} \to \mathbf{C}_M \to \mathbf{C}_S \to 0,$$

one gets the distinguished triangle

$$\mu_S(\mathcal{O}_X)_p \to \mu_M(\mathcal{O}_X)_p \to \mu_{\Omega}(\mathcal{O}_X)_p \oplus \mu_{\Omega'}(\mathcal{O}_X)_p \overset{+1}{\to} . \tag{3.5}$$

Assuming (2.3), by Theorem 2.1 (i), one has

(3.6)
$$\begin{cases} H^j\mu_M(\mathcal{O}_X)_p = 0 & \text{for } j \notin [l + s^-(M) - \gamma(M), n - s^+(M) + \gamma(M)], \\ H^j\mu_S(\mathcal{O}_X)_p = 0 & \text{for } j \notin [l + 1 + s^-(S) - \gamma(S), n - s^+(S) + \gamma(S)]. \end{cases}$$

We divide the proof of (i) in two steps:

a) We first show that

$$H^j\mu_{\Omega}(\mathcal{O}_X)_p = 0 \qquad \text{for } j > n - s^+(M) + \gamma(M). \tag{3.7}$$

By Lemma 3.4 one has:

$$n - s^+(M) + \gamma(M) \geq n - s^+(S) + \gamma(S) - 1$$

so that (3.7) follows from (3.5), (3.6).

b) Finally, we prove that

$$H^j\mu_{\Omega}(\mathcal{O}_X)_p = 0 \qquad \text{for } j < l + s^-(M) - \gamma(M). \tag{3.8}$$

By Lemma 3.4 one has:

$$l + s^-(M) - \gamma(M) \leq l + 1 + s^-(S) - \gamma(S) \tag{3.9}$$

If the inequality in (3.9) is strict, then (3.8) follows from (3.5), (3.6). Assume $l + s^-(M) - \gamma(M) = l + 1 + s^-(S) - \gamma(S)$. By (3.5), (3.6), it is enough to prove that the natural morphism:

$$H^{l+1+s^-(S)-\gamma(S)}\mu_S(\mathcal{O}_X)_p \longrightarrow H^{l+s^-(M)-\gamma(M)}\mu_M(\mathcal{O}_X)_p, \tag{3.10}$$

is injective.

Arguing as in the proof of Theorem 11.3.1 of [**K-S 1**], by Corollary 3.3 we may find a complex contact transformation χ in a neighborhood of p which interchanges $(T^*X, T^*_MX, T^*_SX, p)$ and $(T^*Y, T^*_NY, T^*_ZY, q)$ and such that N and Z are hypersurfaces of Y satisfying:

$$s^-(N) = s^-(Z) = 0.$$

(Of course $\gamma(N) = \gamma(Z) = 0$ since these are real hypersurfaces.) Moreover one can quantize this contact transformation by a kernel $K \in \mathrm{D}^b(X \times Y)$ and get isomorphisms in $\mathrm{D}^b(Y; q)$:

$$\begin{aligned} \Phi_K(\mathcal{O}_X) &\cong \mathcal{O}_Y, \\ \Phi_K(\mathbf{C}_M[-l - s^-(\dot{M}) + \gamma(M)]) &\cong \mathbf{C}_N[-1], \\ \Phi_K(\mathbf{C}_S[-l - 1 - s^-(S) + \gamma(S)]) &\cong \mathbf{C}_Z[-1] \end{aligned} \tag{3.11}$$

(cf. [**K-S 2**, ch. 11] as for quantized contact transformations). By (3.11) one gets isomorphisms:

$$\mu\mathrm{hom}(\mathbf{C}_M, \mathbf{C}_S)_p \cong \mu\mathrm{hom}(\mathbf{C}_N, \mathbf{C}_Z)_q, \tag{3.12}$$

$$\mu_M(\mathcal{O}_X)_p[l + s^-(M) - \gamma(M)] \cong \mu_N(\mathcal{O}_Y)_q[1], \tag{3.13}$$

$$\mu_S(\mathcal{O}_X)_p[l + 1 + s^-(S) - \gamma(S)] \cong \mu_Z(\mathcal{O}_Y)_q[1], \tag{3.14}$$

and (3.10) induces a morphism:

$$H^1\mu_Z(\mathcal{O}_X)_q \longrightarrow H^1\mu_N(\mathcal{O}_X)_q. \tag{3.15}$$

It then remains to prove that (3.15) is injective.

Denote by N^+ (resp. Z^+) the closed half spaces of Y with boundary N (resp. Z) such that $q \in T^*_{N^+}Y$ (resp. $q \in T^*_{Z^+}Y$).

By [**K-S 2**, Prop. 4.4.2], one has:

$$\begin{aligned} [\mathrm{R}\pi_!\mu\mathrm{hom}(\mathbf{C}_{N^+}, \mathbf{C}_{Z^+})]_{\pi(q)} &\cong [\mathrm{R}\mathcal{H}om(\mathbf{C}_{N^+}, \mathbf{C}_Y) \otimes \mathbf{C}_{Z^+}]_{\pi(q)} \\ &\cong (\mathbf{C}_{\mathrm{Int}\,(N^+)} \otimes \mathbf{C}_{Z^+})_{\pi(q)} \\ &= 0 \end{aligned}$$

since $\pi(q) \notin \mathrm{Int}\,(N^+)$. From the distinguished triangle:

$$\mathrm{R}\pi_!(\cdot) \to \mathrm{R}\pi_*(\cdot) \to \mathrm{R}\dot{\pi}_*(\cdot) \overset{+1}{\to},$$

applied to the complex $\mu\mathrm{hom}(\mathbf{C}_{N^+}, \mathbf{C}_{Z^+})$, we then get

$$\begin{aligned} \mu\mathrm{hom}(\mathbf{C}_{N^+}, \mathbf{C}_{Z^+})_q &\cong [\mathrm{R}\pi_*\mu\mathrm{hom}(\mathbf{C}_{N^+}, \mathbf{C}_{Z^+})]_{\pi(q)} \\ &\cong \mathrm{R}\mathcal{H}om(\mathbf{C}_{N^+}, \mathbf{C}_{Z^+})_{\pi(q)}. \end{aligned} \tag{3.16}$$

By (3.12), (3.16), one has:

$$\begin{aligned} \mu\mathrm{hom}(\mathbf{C}_M, \mathbf{C}_S)_p &\cong \mu\mathrm{hom}(\mathbf{C}_N, \mathbf{C}_Z)_q \\ &\cong \mu\mathrm{hom}(\mathbf{C}_{N^+}, \mathbf{C}_{Z^+})_q \\ &\cong \mathrm{R}\mathcal{H}om(\mathbf{C}_{N^+}, \mathbf{C}_{Z^+})_{\pi(q)}. \end{aligned} \tag{3.17}$$

Since $H^0(\mu\hom(\mathbf{C}_M, \mathbf{C}_S))_p \cong \mathbf{C}$, taking the zero$^{\text{th}}$ cohomology in (3.17) we get the isomorphism: $\Gamma_{N^+}(\mathbf{C}_{Z^+})_{\pi(q)} \cong \mathbf{C}$.

It follows that $N^+ \supset Z^+$ on a system of open neighborhoods V of $\pi(q)$ in Y, and that $\mathrm{Hom}_{\mathrm{D}^b(Y;q)}(\mathbf{C}_{N^+}, \mathbf{C}_{Z^+})$ is generated by the natural morphism $\mathbf{C}_{N^+ \cap V} \to \mathbf{C}_{Z^+ \cap V}$.

Then (3.15) is represented by the natural morphism:

$$\varinjlim_{V \ni \pi(q)} \frac{\mathcal{O}_Y(V \setminus Z^+)}{\mathcal{O}_Y(V)} \longrightarrow \varinjlim_{V \ni \pi(q)} \frac{\mathcal{O}_Y(V \setminus N^+)}{\mathcal{O}_Y(V)} \tag{3.18}$$

which is clearly injective. This complete the proof of (i).

As for (ii), one proceeds as above, noticing moreover that, due to hypotheses (2.3), one can apply [**K-S 1**, Prop. 11.3.5] and get:

$$\begin{cases} H^j \mu_M(\mathcal{O}_X)_p = 0 & \text{for } j \neq l + s^-(M) - \gamma(M), \\ H^j \mu_S(\mathcal{O}_X)_p = 0 & \text{for } j \neq l + 1 + s^-(S) - \gamma(S). \end{cases}$$

Q.E.D.

§4. An application

4.1. We first review here results of [**Z**] concerning the representation of sections of $\mu_\Omega(\mathcal{O}_X)$.

Let M be a real analytic submanifold of codimension l of a complex analytic manifold X of dimension n, let $\Omega \subset M$ be an open subset with real analytic boundary S and let $x_\circ \in S$. Let $\tau : TX \to X$ denote the tangent bundle and consider the projection $\sigma : M \times_X TX \to T_M X$. For A', A'' subsets of X one denotes by $C(A', A'') \subset TX$ the Whitney normal cone of A' along A''. For γ conic subset of TX one denotes by $\gamma^{\circ a} \subset T^*X$ its polar antipodal.

Proposition 4.1. *(Cf.* [**Z**, Theorem 2.1]*) Let γ be an open convex cone in $\overline{\Omega} \times_M T_M X$ such that $\tau(\gamma) \supset \overline{\Omega}$ in a neighborhood of $x_\circ$. Then:*

$$H^j(\mathrm{R}\Gamma_{\gamma^{\circ a}}(\mu_\Omega(\mathcal{O}_X)_{T^*_M X}))_{x_\circ} = \varinjlim_{B,U} H^{j-l}(B \cap U; \mathcal{O}_X),$$

where $B \subset X$ ranges through the family of open neighborhoods of $x_\circ$ and $U \subset X$ ranges through the family of open subsets such that

$$\sigma(M \times_X (TX \setminus C(X \setminus U, \overline{\Omega}))) \supset \gamma. \tag{4.1}$$

(This is a classical result for $\Omega = M$, cf. e.g. [**K-S 2**, Theorem 4.3.2].)

Definition 4.2. An open set $U \subset X$ satisfying (4.1) is called Ω-tuboid with profile γ.

REMARK 4.3. Let us discuss the meaning of (4.1).

(i) If X is a vector space, one has that $\theta \in (TX \setminus C(X \setminus U, \overline{\Omega}))_{x_\circ}$ iff there exist a neighborhood B of $x_\circ$ and an open cone $G \subset X$ containing θ such that $((\overline{\Omega} \cap B) + G) \cap B \subset U$.

(ii) Assume M be a hypersurface of X. In this case $T_M X \setminus M$ has two connected components and we denote by γ^+ the one such that $\operatorname{Int} \gamma^{\circ a} \ni p$. Choose complex analytic local coordinates $z = x + iy$ on X at $x_\circ$ with M defined by the equation $y_1 = f(z)$ (for a real analytic function f with $\mathrm{d}f(x_\circ) = 0$) and assume Ω given by $y_2 = 0$. Then the condition (4.1) is equivalent to the existence of a neighborhood B of $x_\circ$ such that

$$U \supset \begin{cases} B \cap \{z \in X;\ f(z) < y_1 < \varepsilon\, y_2,\ y_2 > 0\}, & \text{for } \gamma = \gamma^+, \\ B \cap \{z \in X;\ -\varepsilon\, y_2 < y_1 < \varepsilon\, y_2,\ y_2 > 0\}, & \text{for } \gamma = \overline{\Omega} \times_M T_M X. \end{cases}$$

4.2. Assume M is generic (i.e. $TM +_M iTM = M \times_X TX$) and denote by $M^{\mathbf{C}}$ a complexification of M. A complexification of X is given by $X \times \overline{X}$, where $\overline{X}$ denotes the complex conjugate of X. Let $\phi : M^{\mathbf{C}} \to X$ be the composite of the immersion $M^{\mathbf{C}} \hookrightarrow X \times \overline{X}$ (induced by the embedding $M \hookrightarrow X$) with the projection $X \times \overline{X} \to X$. By the hypothesis of genericity, ϕ is smooth. The coherent $\mathcal{D}_{M^{\mathbf{C}}}$-module $\overline{\partial}_b = \phi^*(\mathcal{D}_X)$ (here ϕ^* denotes the inverse image in the category of $\mathcal{D}$-modules) is called *induced Cauchy-Riemann system* on M. Let $or_{./.}$ denote the relative orientation sheaf and (for $* = M, \Omega$) set

$$\mathcal{C}_* = \mu_*(\mathcal{O}_{M^{\mathbf{C}}}) \otimes or_{M/M^{\mathbf{C}}}[n],$$
$$\mathcal{C}_{*/X} = \mu_*(\mathcal{O}_X) \otimes or_{M/X}[l].$$

(Notice that $\mathcal{C}_M$ is the sheaf of Sato's microfunctions and $\mathcal{C}_\Omega$ is the complex of microfunctions at the boundary of Schapira [**S 2**].) One has the isomorphisms

$$\mathcal{C}_{*/X} \cong \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\overline{\partial}_b, \mathcal{C}_*).$$

(For $* = M$ we refer to [**K-K**]. Results related to the case $* = \Omega$ are obtained in [**D'A-D'A-Z**].) This means that the complex $\mathcal{C}_{M/X}$ (resp. $\mathcal{C}_{\Omega/X}$) is isomorphic to the complex of CR-microfunctions on M (resp. to the complex of $\mathcal{C}_\Omega$-solutions to $\overline{\partial}_b$).

LEMMA 4.4. *The complex $(\mathcal{C}_{\Omega/X})_{T^*_M X}$ is concentrated in degree ≥ 0.*

PROOF: The result is an immediate consequence of Theorem 2.2 (i) recalling that, since M is generic, $\gamma(M, p) = 0$ for every $p \in \dot{T}^*_M X$. Q.E.D.

4.3. Let $p \in (\dot{T}^*_M X)_{x_\circ}$ and assume that

$$s^-(M, p) \geq 1. \tag{4.2}$$

Under this hypothesis, it is well known that the complex $(\mathcal{C}_{M/X})_p = 0$ is concentrated in degree > 0. Representing the sections of $\mathcal{C}_{M/X}$ as boundary values of holomorphic functions, one can rephrase this result as a criterion of holomorphic extension for functions defined in tuboids along M (cf. Definition 4.2 with $\Omega = M$ as for tuboids along M) with profile γ verifying $\gamma^{\circ a} \ni p$. In the case of M being a hypersurface, if one denotes by M^+ the connected component of $X \setminus M$ at $x_\circ$ which has p as exterior conormal, this simply means that any holomorphic function on M^+ holomorphically extends to a full neighborhood of $x_\circ$ (i.e. holomorphic functions cross the boundary of pseudo-concave domains).

Similarly, under the hypothesis (4.2) it follows from Theorem 2.2 (i) that the complex $(\mathcal{C}_{\Omega/X})_p$ is concentrated in degree > 0. By Proposition 4.1, the sections of $\mathcal{C}_{\Omega/X}$ may also be represented as boundary values of holomorphic functions and hence, once again, this result may be rephrased in terms of holomorphic extension. Proposition 4.5 below states this fact, and we refer to Remark 4.3 for a description of the geometry of the involved sets.

PROPOSITION 4.5. *Assume $s^-(M,p) \geq 1$. Then there exists an open neighborhood $\Lambda \subset T^*_M X$ of p such that*

$$\varinjlim_{U \in \mathcal{U}, W \in \mathcal{W}, B \ni x_\circ} \frac{\Gamma(U \cap B, \mathcal{O}_X)}{\Gamma(W \cap B, \mathcal{O}_X)} = 0,$$

where $B \subset X$ ranges through an open neighborhood system of $x_\circ$ and $\mathcal{U}$ (resp. $\mathcal{W}$) is the family of Ω-tuboids with a profile γ such that $\gamma^{\circ a} \subset \Lambda$ (resp. with profile $\overline{\Omega} \times_M T_M X$).

PROOF: One may find an open conic neighborhood $\Lambda \subset T^*_M X$ of p such that $s^-(M,p') \geq 1$ for every $p' \in \Lambda$.

We already noticed that, by Theorem 2.2, the complex $(\mathcal{C}_{\Omega/X})_\Lambda$ is concentrated in degree > 0. Denoting by $\mathcal{G}$ the family of open convex cones $\gamma \subset \overline{\Omega} \times_M T_M X$ such that $\tau(\gamma) \supset \overline{\Omega}$ in a neighborhood of $x_\circ$ and $\gamma^{\circ a} \subset V$, it then follows from the injective morphism:

$$\varinjlim_{B \ni x, \gamma \in \mathcal{G}} \Gamma_{\gamma^{\circ a}}(\dot{\pi}^{-1}(B); H^0(\mathcal{C}_{\Omega/X})_{T^*_M X}) \hookrightarrow H^0(\mathcal{C}_{\Omega/X})_p. \tag{4.2}$$

that the limit on the left hand side of (4.2) vanishes.

Applying the functor $H^0 \mathrm{R}\Gamma(B; \cdot)$ to the distinguished triangle

$$\begin{aligned} \mathrm{R}\Gamma_{\pi(\gamma^{\circ a})}((\mathcal{C}_{\Omega/X})_{T^*_M X}) &\longrightarrow \mathrm{R}\pi_* \mathrm{R}\Gamma_{\gamma^{\circ a}}((\mathcal{C}_{\Omega/X})_{T^*_M X}) \longrightarrow \\ &\longrightarrow \mathrm{R}\dot{\pi}_* \mathrm{R}\Gamma_{\gamma^{\circ a}}((\mathcal{C}_{\Omega/X})_{T^*_M X}) \xrightarrow{+1}, \end{aligned}$$

one gets an injection

$$\frac{\Gamma_{\gamma^{\circ a}}(\pi^{-1}(B); H^0(\mathcal{C}_{\Omega/X})_{T^*_M X})}{\Gamma_{\pi(\gamma^{\circ a})}(\pi^{-1}(B); H^0(\mathcal{C}_{\Omega/X})_{T^*_M X})} \hookrightarrow \Gamma_{\gamma^{\circ a}}(\dot{\pi}^{-1}(B); H^0(\mathcal{C}_{\Omega/X})_{T^*_M X}),$$

and hence, taking injective limit:

$$\varinjlim_{B \ni x, \gamma \in \mathcal{G}} \frac{\Gamma_{\gamma^{\circ a}}(\pi^{-1}(B); H^0(\mathcal{C}_{\Omega/X})_{T^*_M X})}{\Gamma_{\pi(\gamma^{\circ a})}(\pi^{-1}(B); H^0(\mathcal{C}_{\Omega/X})_{T^*_M X})} = 0.$$

The result then follows from Proposition 4.1. Q.E.D.

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