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Non smooth Lagrangian sets and estimations of micro-support

By Andrea D'AGNOLO, Corrado MARASTONI and Giuseppe ZAMPIERI

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1. Notation and review.

Let X be a real C^1 manifold and let $Y \subset X$ be a closed submanifold. One denotes by $\pi: T^*X \to X$ the cotangent bundle to X and by T_Y^*X the conormal bundle to Y in X.

One denotes by $\mathrm{D}^b(X)$ the derived category of the category of bounded complexes of sheaves of $\boldsymbol{C}$-vector spaces on X. For F an object of $\mathrm{D}^b(X)$, one denotes by $\mathrm{SS}(F)$ its micro-support, a closed, conic, involutive subset of T^*X.

Let $A \subset X$ be a closed C^1-convex subset at $x_0 \in A$ (i.e., A is convex for a choice of local C^1 coordinates at x_0). One denotes by $\boldsymbol{C}_A$ the sheaf which is zero on $X \setminus A$ and the constant sheaf with fiber $\boldsymbol{C}$ on A. In order to describe $\mathrm{SS}(\boldsymbol{C}_A)$ fix a local system of coordinates $(x)=(x', x'')$ at x_0 so that A is convex and $Y=\{x \in X ; x''=0\}$ is its linear hull. Denote by $j: Y \to X$ the embedding and by ${}^t j': Y \times_X T^*X \to T^*Y$ the associated projection. One has

$$\mathrm{SS}(\boldsymbol{C}_A) = {}^t j'(N_Y^*(A)),$$

where $N_Y^*(A)$ denotes the conormal cone to A in Y. In other words, $(x ; \xi) \in \mathrm{SS}(\boldsymbol{C}_A)$ if and only if $x \in A$ and the half space $\{y \in X ; \langle y-x, \xi \rangle \geqq 0\}$ contains A. By analogy with the smooth case, we set $T_A^*X = \mathrm{SS}(\boldsymbol{C}_A)$.

For $p \in T^*X$, $\mathrm{D}^b(X ; p)$ denotes the localization of $\mathrm{D}^b(X)$ with respect to the null system $\{F \in \mathrm{D}^b(X) ; p \notin \mathrm{SS}(F)\}$. One also considers the microlocalization bifunctor $\mu hom(\cdot, \cdot)$ which is defined in [**K-S**].

REMARK 1.1. In [**K-S**] the bifunctor μhom is considered only for C^2 manifolds but it is clear that its definition is possible for a C^1 manifold as well. Roughly speaking, this functor is the composition of the specialization functor (which is defined as long as the normal deformation is defined, i.e., for C^1 manifolds) and the Fourier-Sato transform which is defined for vector bundles over any locally compact space.

If X is of class C^2 one has the following estimate:

(1.1) $$\mathrm{SS}(\mu hom(F, G)) \subset C(\mathrm{SS}(F), \mathrm{SS}(G)),$$

where $C(\cdot, \cdot) \subset TT^*X \cong T^*T^*X$ denotes the strict normal cone.

Assume X of class C^2 and let $\chi: T^*X \to T^*X$ be a germ of homogeneous contact transformation at $p \in T^*X$, i.e., a diffeomorphism at p preserving the canonical one-form. Set $\Lambda_\chi^a = \{(x, y ; \xi, \eta) ; \chi(x ; \xi) = (y ; -\eta)\}$, the antipodal of the graph of χ. It is possible to consider "quantizations" of χ in order to make contact transformations operate on sheaves.

THEOREM 1.2 (cf. [**K-S**, Chapter 7]). *There exists* $K \in \mathrm{D}^b(X \times X)$ *with* $\mathrm{SS}(K) \subset \Lambda_\chi^a$ *in a neighborhood of* $(p, \chi(p)^a)$, *which induces an equivalence of categories* $\Phi_K : \mathrm{D}^b(X ; p) \to \mathrm{D}^b(X ; \chi(p))$ *defined by* $\Phi_K(F) = \mathrm{R}q_{2*}(K \otimes q_1^{-1}F)$ *where* q_i *is the i-th projection from* $X \times X$ *to* X. *Moreover one has the relations*

(1.2) $$\mathrm{SS}(\Phi_K(F)) = \chi(\mathrm{SS}(F)),$$

(1.3) $$\chi_* \mu hom(F, G) \cong \mu hom(\Phi_K(F), \Phi_K(G)) \quad \textit{near } \chi(p).$$

2. The main theorem.

The characterization of those sheaves whose microsupport is contained in a smooth Lagrangian is given by the following theorem.

THEOREM 2.1 (cf. [**K-S**, Theorem 6.6.1]). *Let X be a real C^2 manifold, let $Y \subset X$ be a closed submanifold and take $p \in T_Y^*X$. Let F be an object of* $\mathrm{D}^b(X)$ *such that*

$$\mathrm{SS}(F) \subset T_Y^*X \quad \textit{in a neighborhood of } p.$$

Then one has $F \cong M_Y$ in $\mathrm{D}^b(X ; p)$ *for a complex M of $\mathbf{C}$-vector spaces.*

REMARK 2.2. The extension from the C^2 to the C^1 frame has already been given in the paper [**D'A-Z**]. Concerning this extension, we point out the following fact. Let $Y \subset X$ be a hypersurface of regularity $C^1 \setminus C^2$ and let Y^+ be the closed half space with boundary Y such that $p \in \mathrm{SS}(A_{Y^+})$. The crucial point here is that, even though $T_Y^*X \hat{+} T_Y^*X \supset \pi^{-1}\pi(p)$, nevertheless $N^*(Y^+) \hat{+} N^*(Y^+) \subset N^*(Y^+)$.

Here we give the following extension of this result.

THEOREM 2.3. *Let X be a real C^1 manifold, let $A \subset X$ be a closed C^1-convex subset at x_0 and take $p \in (T_A^*X)_{x_0}$. Let F and G be objects of* $\mathrm{D}^b(X)$ *such that*

$$\mathrm{SS}(F), \mathrm{SS}(G) \subset T_A^*X \quad \textit{in a neighborhood of } p.$$

Then

(i) $\mu hom(F, G) \cong N_{T_A^*X}$ *for a complex N of $\mathbf{C}$-vector spaces;*

(ii) *$F\cong M_A$ in $D^b(X;p)$ for a complex M of $\boldsymbol{C}$-vector spaces;*
(iii) *for M as in* (ii), *one has $M\cong\mu hom(\boldsymbol{C}_A, F)_p$.*

REMARK 2.4. Let X be a real C^2 manifold and $Y\subset X$ a closed submanifold. In this context, the assertion (ii) already appears in [**U-Z**] for any closed subset $A\subset Y$ satisfying $N_Y^*(A)_{x_0}\neq T_{x_0}^*Y$ (which holds true, in particular, for C^1-convex subsets at x_0), but only for $p\in T_Y^*X\cap T_A^*X$.

PROOF OF THEOREM 2.3. The problem being local, fix a system of local coordinates at x_0 so that A is convex in $X\subset\boldsymbol{R}^n$ with coordinates $(x)=(x_1, \cdots, x_n)$. Let $(x;\xi)$ be the associated symplectic coordinates of T^*X and consider the contact transformation

$$\chi: T^*X \longrightarrow T^*X$$
$$(x;\xi)\longmapsto\left(x-\varepsilon\frac{\xi}{|\xi|};\xi\right).$$

The set $A_\varepsilon=\{x\in X; \operatorname{dist}(x, A)\leqq\varepsilon\}$ has a C^1 boundary for $0<\varepsilon\ll 1$ and one has $\chi(T_A^*X)=T_{A_\varepsilon}^*X$ near $\chi(p)$. It is also easy to verify that, setting

$$S=\{(x, y)\in X\times X; \operatorname{dist}(x, y)\leqq\varepsilon\},$$

the complex $K=\boldsymbol{C}_S$ verifies the hypothesis of Theorem 1.2 for such a χ.

Let $\phi: X\to X$ be a C^1 diffeomorphism so that $\phi(A_\varepsilon)=\{x\in X; x_1\leqq 0\}$ and set $Z=\{x\in X; x_1=0\}$. By (1.2) one has that $SS(\phi_*(\Phi_K(\cdot)))\subset T_Z^*X$ near ${}^t\phi'(\chi(p))$ for $(\cdot=F, G)$ and hence, by (1.1),

$$SS(\mu hom(\phi_*(\Phi_K(F)), \phi_*(\Phi_K(G))))\subset C(T_Z^*X, T_Z^*X)$$
$$\cong T_{T_Z^*X}^*T^*X.$$

By Theorem 2.1 one then has

$$\mu hom(\phi_*(\Phi_K(F)), \phi_*(\Phi_K(G)))\cong N_{T_Z^*X}$$

for a complex N of $\boldsymbol{C}$-vector spaces. It follows by (1.3) that

$$\mu hom(F, G)\cong {}^t\phi'^{-1}(\chi^{-1}(N_{T_Z^*X}))\cong N_{T_A^*X},$$

which proves (i).

For any complex M of $\boldsymbol{C}$-vector spaces let us now compute $\Phi_K(M_A)$. There is an isomorphism $(Rq_{2!}M_{S\cap(A\times X)})_x\cong R\Gamma_c(q_2^{-1}(x); M_{S\cap(A\times X)})$. Since $q_2^{-1}(x)\cap S\cap(A\times X)$ is either empty if $x\notin A_\varepsilon$ or compact convex if $x\in A_\varepsilon$, one has:

$$\Phi_K(M_A)\cong M_{A_\varepsilon}. \tag{2.1}$$

Moreover notice that

$$\begin{aligned}\phi_*(\Phi_K(F)) &\cong M_Z \\ &\cong \phi_*(M_{A_\varepsilon}) \\ &\cong \phi_*(\Phi_K(M_A)),\end{aligned}$$

where the first isomorphism follows from Theorem 2.1 and the third from (2.1). Since $\phi_* \circ \Phi_K$ is an equivalence of categories, assertion (ii) follows.

As for (iii), one has the chain of isomorphisms:

$$\begin{aligned}\mu hom(\boldsymbol{C}_A, F)_p &\cong \mu hom(\boldsymbol{C}_Z, M_Z)_{t\phi'(\chi(p))} \\ &\cong \mu hom(\boldsymbol{C}_{\{x_1\le 0\}}, M_{\{x_1\le 0\}})_{t\phi'(\chi(p))} \\ &\cong \mathrm{R}\Gamma_{\{x_1\le 0\}}(M_{\{x_1\le 0\}})_{\pi(t\phi'(\chi(p)))} \\ &\cong M.\end{aligned}$$

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Andrea D'AGNOLO
Institut de Mathématiques
Analyse Algébrique (Case 247)
Université Pierre et Marie Curie
4, place Jussieu
F-75252 Paris Cedex 05
France

Corrado MARASTONI
Dipartimento di Matematica
Università di Padova
via Belzoni 7
I-35131 Padova
Italy

Giuseppe ZAMPIERI
Dipartimento di Matematica
Università di Padova
via Belzoni 7
I-35131 Padova
Italy