Computation of caustics related to geometrical solutions with arbitrary initial data of the eikonal equation

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Summary. — In this paper we wish to show how to compute the support of caustics related to geometrical solutions (Lagrangian submanifolds) of the geometrical Cauchy problem for the eikonal equation, a special case of Hamilton-Jacobi equation. Although the computation is carried out for the simple Hamiltonian $H(q,p) = (1/2)p^2$ on $T^*\mathbb{R}^2$, we can treat cases with *arbitrary* C^2 initial data functions $\sigma : \Sigma \longrightarrow \mathbb{R}$, assigned on the initial manifold (curve) Σ immersed in $\mathbb{R}^2$. From the physical point of view, the caustics in such a way generated are closely linked to the caustics (in the sense of geometrical optics) determined by the curve Σ emitting light with intensity varying according to the differential of σ.

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1. – Introduction

The Hamilton-Jacobi method is a well-established tool for integrating (at least locally) the canonical systems of Hamilton equations. In most cases the Hamilton-Jacobi method is introduced starting from the search of a canonical transformation of coordinates which transforms the original Hamiltonian in a simpler one. By following the line of thought of W. M. Tulczyjew, it is now well understood (see [1] and [2]) that, from the geometrical point of view, this method is completely equivalent to find out a particular foliation inside the phase space of the dynamical system considered. In fact, it is well known that given a differentiable function $H : T^*Q \longrightarrow \mathbb{R}$ (where T^*Q is the cotangent bundle over

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a differentiable manifold Q of dimension n), the canonical system

$$\dot{q}^i = \frac{\partial H(q^j, p_j)}{\partial p_i}\,, \qquad \dot{p}_i = -\frac{\partial H(q^j, p_j)}{\partial q^i}$$

can be solved using a *complete solution* of the corresponding Hamilton-Jacobi equation (Jacobi theorem), that is, a family of differentiable functions $S(q^i, a^L)$ depending on $n-1$ parameters a^L such that

$$H\left(q^i, \frac{\partial S(q^j, a^L)}{\partial q^i}\right) = e \tag{1}$$

and

$$\mathrm{rk}\left(\frac{\partial^2 S(q,a)}{\partial q^i \partial a^L}\right) = \max = n-1. \tag{2}$$

For every a fixed, one can consider the following submanifold of T^*Q:

$$\Lambda_a := \left\{(q,p) \in T^*Q : p_i = \frac{\partial S(q,a)}{\partial q^i}\right\}.$$

By the condition (1) one has that $\Lambda_a \subseteq H^{-1}(e)$; moreover Λ_a satisfies two important properties:

1) $\dim\Lambda_a = \dim Q = n$ and

2) $\omega|_{\Lambda_a} = 0$, where ω is the canonical symplectic form on T^*Q, which in fibered coordinates (q^i, p_i) assumes the form $\omega = \mathrm{d}p_i \wedge \mathrm{d}q^i$.

Submanifolds of T^*Q characterized by the two conditions above are called *Lagrangian submanifolds*. In view of (2), it is obvious that if $a \neq b$, then $\Lambda_a \cap \Lambda_b = \emptyset$, that is, the corresponding Lagrangian submanifolds are disjointed. By using this geometric framework, one can say that the Hamilton-Jacobi method amounts to finding out a Lagrangian foliation (that is a foliation by Lagrangian submanifolds) inside $H^{-1}(e)$. This geometric setting is meaningful in that it leads to consider *generic* Lagrangian submanifolds contained in $H^{-1}(e)$, as geometrical solutions of the Hamilton-Jacobi equation. Indeed, the submanifolds Λ_a, previously obtained via the function $S(q,a)$, have the property to be *completely* parameterized by the configuration manifold Q, that is, if $j : \Lambda_a \hookrightarrow T^*Q$ is an embedding, it is always true that

$$\mathrm{rk}\,[d(\pi_Q \circ j)] = \max = n \tag{3}$$

on Λ_a ($\pi_Q : T^*Q \longrightarrow Q$ is the natural cotangent fibration). In this sense, the Lagrangian submanifolds generated via $S(q,a)$ are not generic. On the other hand, on a generic Lagrangian submanifold Λ contained in $H^{-1}(e)$ can happen that

$$\mathrm{rk}\,[d(\pi_Q \circ j(\lambda))] < \max \tag{4}$$

at some $\lambda \in \Lambda$. (The most simple example of this phenomenology is the phase curve of a harmonic oscillator, which is obviously a Lagrangian submanifold of $T^*\mathbb{R}^1$ and in which the condition (3) fails at the turning points.) The locus on Λ in which the condition (3) fails is called *singular cycle* or *Maslov cycle* $Z(\Lambda)$ of the Lagrangian embedding determined by the pair (Λ, j).

The projection on Q of $Z(\Lambda)$ is called the *caustic* $C(\Lambda) \equiv \pi_Q(Z(\Lambda))$ related to Λ.

Arnol'd in appendix 12 of [3] has given a complete description of the normal forms for Lagrangian singularities in the case where $\dim\Lambda \leq 5$. However, only rarely explicit computation of the caustic associated to a Lagrangian embedding has be done, exploiting simple elements of differential geometry of curves (see for example [4]) or the theory of Morse family for Lagrangian submanifold (see, for example, [2]). In this paper, by using the projective duality between $P^2(\mathbb{R})$ and $P^2(\mathbb{R})^*$, we restrict the computation of the caustics to a particular (but large enough) class of Lagrangian submanifolds. (For a brief account of projective duality see the appendix at the end of this paper; more exhaustive descriptions can be found for example in [5], [6] and [7].) One of our main aims is to give an explicit use of projective duality in studying unidimensional fronts, as suggested for example by Arnol'd in [8] and more implicitly in the textbook [9]. This method has also been developed for investigating other problems such as Legendre singularities (see [10]). We would like to underline, however, that the computation here presented can equally well be developed using the framework of Morse families, as showed explicitly for the case $\sigma = \text{const}$ in [2], p. 301.

Before showing how to do this, it is necessary to recall briefly a geometrical setting for the Cauchy problem of Hamilton-Jacobi equation. This will be done in the following section (for a more exhaustive account of the problem see [11]).

2. – Lagrangian submanifolds as geometrical solutions of Hamilton-Jacobi equation

Let j be an embedding:

$$j: \begin{array}{ccc} \Sigma & \hookrightarrow & Q \\ s^\mu & \longmapsto & \tilde{q}^i(s^\mu) \end{array}$$

of a connected submanifold Σ of codimension 1 in Q (*initial manifold*). Then let $\sigma : \Sigma \longrightarrow \mathbb{R}$ be a smooth function on Σ (*initial datum*). Using these elements we construct the *submanifold of initial data* $\Lambda_{(\Sigma,\sigma)}$:

$$\Lambda_{(\Sigma,\sigma)} := \{(q,p) \in T^*Q : \langle v, d\sigma\rangle = \langle Tj(v), p\rangle, \forall v \in T\Sigma, \pi_Q(p) = \tau_Q(v)\},$$

where $\tau_Q : TQ \longrightarrow Q$ is the canonical projection associated to the tangent fibration.

Locally, this means that $(q,p) \in \Lambda_{(\Sigma,\sigma)}$ iff

$$\frac{\partial\sigma}{\partial s^\mu}(s)v^\mu = \frac{\tilde{q}^i}{\partial s^\mu}(s)v^\mu p_i, \qquad q^i = \tilde{q}^i(s), \qquad \forall s \in \Sigma, \quad \forall v \in T\Sigma \tag{5}$$

or more explicitly

$$q^i = \tilde{q}^i(s), \qquad \frac{\partial\sigma}{\partial s^\mu}(s) = \frac{\partial\tilde{q}^i}{\partial s^\mu}(s)p_i, \quad \forall s \in \Sigma. \tag{6}$$

Having built up the submanifold $\Lambda_{(\Sigma,\sigma)}$, we can state the following *geometric Cauchy problem* for the Hamilton-Jacobi equation related to a time-independent Hamiltonian function H: given $\Lambda_{(\Sigma,\sigma)}$, you have to determine a connected Lagrangian submanifold $\Lambda \subset T^*Q$ such that

1) $\Lambda \subset H^{-1}(e)$ and

2) $\pi_Q(\Lambda \cap \Lambda_{(\Sigma,\sigma)})$ is contained in Σ, as an open subset.

Let

$$\Lambda_0^{(n-1)} := H^{-1}(e) \cap \Lambda_{(\Sigma,\sigma)}$$

be a submanifold of dimension $n-1$ and we assume that it is transversal to the Hamiltonian vector field X_H, that is

$$\forall x \in \Lambda_0^{(n-1)}: \quad X_H(x) \notin T_x\Lambda_0^{(n-1)}.$$

Then it is possible to prove (see [12]) that the solution of the geometrical Cauchy problem is given by

$$\Lambda := \bigcup_{t\in\mathbb{R}} \Phi^t_{X_H}(\Lambda_0^{(n-1)}),$$

where $\Phi^t_{X_H}$ is the Hamiltonian flow corresponding to the Hamiltonian function H.

It is obvious that in a general setting one has to expect that the caustic related to corresponding geometric solution Λ has to be quite complicate. Moreover, the initial data Σ and σ play a crucial part in determining certain topological properties of the corresponding geometric solutions Λ, in particular in controlling singular loci $Z(\Lambda)$ and consequently caustics $C(\Lambda)$. Indeed, even in dynamical systems governed by a very simple Hamiltonian function, the structure of caustics related to geometrical solutions changes drastically varying initial data.

For example, if $H : T^*\mathbb{R}^n \longrightarrow \mathbb{R}$ is given by $H = (1/2)p^2$ and if we choose Σ to be an $(n-1)$-dimensional vector subspace of $\mathbb{R}^n$ and σ to be a constant, then the corresponding geometrical solution Λ has no singular locus, so $C(\Lambda) = \emptyset$, as is readily verified. Indeed, from eq. (6), the impulses p_i have to be orthogonal to Σ; then, from the form of the Hamiltonian function, it is clear that the projection on $Q = \mathbb{R}^n$ of the corresponding phase curves is a family of straight lines orthogonal to Σ, so their envelope is empty. But the envelope of the projection of the phase curves on the configuration manifold is exactly the caustic of the related geometrical solution Λ, as is proved in [13].

This simple situation changes drastically if one chooses more complicate initial data: as far as we know the problem of the computation of the support of $C(\Lambda)$ is still unsolved even in the case Σ is a generic 1-codimensional submanifold in $\mathbb{R}^n$ and σ is a constant.

3. – Computation of caustics via projective duality

The aim of this section is to determine the support of the caustic corresponding to a geometric solution Λ of a geometric Cauchy problem for the Hamilton-Jacobi equation, when

1) $H : T^*\mathbb{R}^2 \longrightarrow \mathbb{R}$ is given by $H = (1/2)p^2$;

2) the plane curve Σ is represented by

$$\begin{array}{ccc} \Sigma & \overset{j}{\hookrightarrow} & Q \equiv \mathbb{R}^2 \\ s & \longmapsto & (x(s), y(s)) \, , \end{array}$$

where $s \in I = (a, b) \subseteq \mathbb{R}$ and Σ is regular, that is the functions $x(s), y(s)$ are smooth and

$$\dot{x}(s)^2 + \dot{y}(s)^2 \neq 0 \quad \forall s \in I;$$

3) the initial datum σ is an *arbitrary* smooth function on Σ, which can be identified with a function:

$$\sigma : I \ni s \longmapsto \sigma(s) \in \mathbb{R}.$$

It is already known that in the case σ is constant the corresponding caustic is exactly the locus determined by the centers of the circles osculating the given curve Σ, that is the *evolute*. This result is achieved using simple elements of differential geometry of plane curves (see, for example, [4]) or the theory of generating functions for Lagrangian submanifolds (see, for example, [2]). Instead, our construction is essentially based on projective duality and mainly consists in determining the envelope of a family of straight lines which intersect Σ at different angles, according to the differential of the initial datum σ.

Now we can state our main result:

Proposition: *Under the hypotheses already described on H, σ, Σ, (above conditions* 1, 2 *and* 3) *the homogeneous coordinates* (x_0, x_1, x_2) (*in* $P^2(\mathbb{R})$) *of the caustic associated to the corresponding geometric solution of Hamilton-Jacobi problem* (*for a fixed value e of the Hamiltonian function*) *are given, respectively, by*

$$x_{0,} = \Big[2e(\dot{x}\ddot{y} - \ddot{x}\dot{y})\mathrm{sgn}(\dot{x})(A\dot{B} - B\dot{A}) \Big] \, , \tag{7}$$

$$x_{1,} = \Big[AB|\dot{x}| + 2ex(\dot{x}\ddot{y} - \ddot{x}\dot{y}) - B^2\dot{y}x\,\mathrm{sgn}(\dot{x})(A\dot{B} - B\dot{A}) \Big] \, , \tag{8}$$

$$x_{2,} = \Big[AB\dot{y} + 2ey(\dot{x}\ddot{y} - \ddot{x}\dot{y}) + B^2\dot{x}y\,\mathrm{sgn}(\dot{x})(A\dot{B} - B\dot{A}) \Big] \, , \tag{9}$$

where A and B are given by

$$A = \dot{\sigma}, \tag{10}$$

$$B = \sqrt{2e(\dot{x}^2 + \dot{y}^2) - \dot{\sigma}^2}, \tag{11}$$

while $\mathrm{sgn}(\dot{x}) = \frac{\dot{x}}{|\dot{x}|}$.

Some comments: In all formulas above, dots denote derivative with respect to s. From the homogeneous coordinate, the usual coordinates $(x_{\mathrm{caus}}(s), y_{\mathrm{caus}}(s))$ in $\mathbb{R}^2$ of the caustic are given by

$$x_{\mathrm{caus}}(s) = \frac{x_{1,}(s)}{x_{0,}(s)}\,, \tag{12}$$

$$y_{\mathrm{caus}}(s) = \frac{x_{2,}(s)}{x_{0,}(s)} \tag{13}$$

(provided $x_{0,}(s) \neq 0$). The sign in the relations which define the homogeneous coordinates of the caustic is due to the fact that, from eq. (6) we can obtain only the cosine of the angle between the tangent to Σ and the impulse p, which defines the direction of the line escaping from Σ. Therefore we have only two possibilities according to the ambiguity inherent in the cosine.

Proof: It is based on projective duality in the following sense. (See the appendix for definitions, basic results and their proof.) Suppose we have to determine in $P^2(\mathbb{R})$ a curve C (the caustic) and of this curve we know only the family of straight lines tangent to C; that is, we assume that a set of three functions is given

$$f(s) = (f_0(s), f_1(s), f_2(s)) \tag{14}$$

depending on the parameter s of the curve C in such a way that, for $\bar{s}$ fixed the expression (14) gives the Plückerian coordinates of the line τ_Q tangent to the curve C at the point $Q = [x_0(\bar{s}), x_1(\bar{s}), x_2(\bar{s})]$. As showed in the appendix, it is possible to think of these Plückerian coordinates as homogeneous coordinates of points belonging to the dual projective plane $P^2(\mathbb{R})^*$: as Q varies in C, the Plückerian coordinates of τ_Q, seen as homogeneous coordinates, define a new curve C^* in $P^2(\mathbb{R})^*$, called the *dual curve*. By hypothesis, we know the support of this new curve C^*, since the homogeneous coordinates of its points are simply given by the Plückerian coordinates of the tangent lines to C, thought as homogeneous coordinates, and so they are given by $[f_0(s), f_1(s), f_2(s)]$. Hence it is easy to determine the Plückerian coordinates of the tangent lines to C^* at each of its points (this time in the dual projective plane): see the procedure described in the appendix and compare the following expression to eq. (A.8):

$$\Big(f_1(s)\dot{f}_2(s) - f_2(s)\dot{f}_1(s), f_2(s)\dot{f}_0(s) - f_0(s)\dot{f}_2(s), f_0(s)\dot{f}_1(s) - f_1(s)\dot{f}_0(s)\Big)\,. \tag{15}$$

Due to projective duality, we can think of expression (15) not only as Plückerian coordinates of the family of straight lines tangent to C^*, but also as homogeneous coordinates of points in the ordinary projective plane $P^2(\mathbb{R})$. These points in $P^2(\mathbb{R})$ define a curve, which, due to projective duality, is exactly the unknown curve C, as shown in the appendix. By using simple elements of projective geometry, it is then possible to determine the homogeneous coordinates of a curve C (expression (15), interpreted as homogeneous coordinates), starting from the Plückerian coordinates of its tangent straight lines (expression (14)), that is to reconstruct a curve as the envelope of its tangent lines.

Now we come back to our original problem. We fix a positive value e for the energy and we consider the submanifold determined by $H^{-1}(e)$ in $T^*\mathbb{R}^2$. Associating to each point of the curve $\Sigma = \{(x(s), y(s))\}_{s\in I}$ the tangent vector:

$$(\dot{x}(s), \dot{y}(s)),$$

from the equations which describe $\Lambda_{(\Sigma,\sigma)}$ (in particular from (6)), we obtain that it has to be satisfied the following equation:

$$\dot{x}p_x + \dot{y}p_y = \dot{\sigma}. \tag{16}$$

Equation (16) means that the differential of σ is equal to the scalar product between the tangent vector and p. Obviously, because of the fact that it has to be $\Lambda_{(\Sigma,\sigma)} \cap H^{-1}(e) \neq \emptyset$, it has to be true that $p^2 = 2e$ and so it has to be satisfied not only (16), but at the same time the equation

$$p_x^2 + p_y^2 = 2e. \tag{17}$$

To determine the direction between the straight line escaping from Σ in the point P (this line is the projection on $Q = \mathbb{R}^2$ of the corresponding phase curve) and the tangent τ_P to Σ in P we use eqs. (16) and (17). Notice that it is useless to determine the angle between the *unit* tangent vector and p, in that we are interested only in the homogeneous coordinates $(0, p_x, p_y)$ which gives the direction of the straight line escaping from Σ. Indeed, solving (16) with respect to p_x and substituting in (17), we find the following equation for p_y:

$$p_y^2(\dot{x}^2 + \dot{y}^2) - 2\dot{\sigma}\dot{y}p_y + \dot{\sigma}^2 - 2e\dot{x}^2 = 0\,. \tag{18}$$

Thus, by using definitions (10) and (11), we find immediately the following expressions:

$$p_x = \frac{A - \dot{y}p_y}{\dot{x}} \tag{19}$$

and

$$p_y = \frac{A\dot{y}|\dot{x}|B}{\dot{x}^2 + \dot{y}^2}\,. \tag{20}$$

It is interesting to observe that it is necessary that

$$-\sqrt{2e(\dot{x}^2 + \dot{y}^2)} \leq \dot{\sigma} \leq \sqrt{2e(\dot{x}^2 + \dot{y}^2)}$$

and so only some parts of Σ are involved in generating a Lagrangian submanifold contained in $H^{-1}(e)$ (those parts for which the derivative of σ satisfy the previous inequality). Let us also observe that this does not happen when the initial datum σ is constant.

To exploit the projective duality we immerse the plane $\mathbb{R}^2$, as the configuration manifold, in the projective plane $P^2(\mathbb{R})$ in the ordinary way: that is, if $(x(s), y(s))$ are the parametrical equations of Σ in $\mathbb{R}^2$, then the corresponding homogeneous coordinates will be

$$(x_0(s), x_1(s), x_2(s)) = (1, x(s), y(s)),$$

while the homogeneous coordinates which define the direction of the straight line escaping from Σ in the point $(1, x(s), y(s))$ are given by (see eqs. (20) and (19) and assume $\dot{x} \neq 0$)

$$(0, A - \dot{y}p_y, \dot{x}p_y), \tag{21}$$

or more explicitly

$$(0, A\dot{x}^2 \mp |\dot{x}|\dot{y}B, \dot{x}\dot{y}A\dot{x}|\dot{x}|B). \tag{22}$$

(To obtain relation (22) we have multiplied by $\dot{x}^2+\dot{y}^2$ which is never zero by hypothesis.) Now we can find the Plückerian coordinates of the corresponding straight lines (in the ordinary projective plane) which define the wanted caustic as their envelope. Following the method previously described in this proof and in more detail in the appendix (computation of the Plückerian coordinates corresponding to a straight line through two given points in $P^2(\mathbb{R})$, whose homogeneous coordinates are given in this case by $[1, x, y]$ and by expression (22)) we find

$$\left((A\dot{x}^2 \mp |\dot{x}|\dot{y}B)y - x(\dot{x}\dot{y}A\dot{x}|\dot{x}|B), \dot{x}\dot{y}A\dot{x}|\dot{x}|B, |\dot{x}|\dot{y}B - A\dot{x}^2\right) . \tag{23}$$

Now to prove eqs. (7), (8) (9) we have simply to identify (23) with (14) and then to apply relation (15), which gives the homogeneous coordinates in $P^2(\mathbb{R})$ of the envelope curve (which is the caustic in our case). Computations are rather lengthy, but trivial in substance. □

Remark 1: There is a common factor $\dot{x}^2$ in the homogeneous coordinates of the caustic computed in this way. The coordinates given by relations (7), (8) and (9) are obtained by dividing by the common factor, assuming it different from zero. There is also the common factor $\dot{x}^2 + \dot{y}^2$ which is never zero by hypothesis, so that we can divide by this factor without any further assumption.

Remark 2: From the physical point of view, the problem solved by the previous proposition is exactly equivalent to determine the caustic (in the sense of geometric optics) produced by a plane curve, which gives out light with intensity varying from point to point.

Remark 3: In the case the initial datum σ is constant, from eqs. (7), (8) and (9) we can find out immediately the already known expressions of the coordinates (in $\mathbb{R}^2$) for the wanted caustic:

$$x_{\text{caus}} = x + \frac{\dot{y}(\dot{x}^2+\dot{y}^2)}{\ddot{x}\dot{y} - \ddot{y}\dot{x}}\,, \tag{24}$$

$$y_{\text{caus}} = y - \frac{\dot{x}(\dot{y}^2+\dot{x}^2)}{\ddot{x}\dot{y} - \dot{x}\ddot{y}}\,. \tag{25}$$

Equations (24) and (25) describe exactly the locus given by the centers of the circles osculating Σ (see [4]).

Remark 4: It is worthwhile to observe that the "high-energy limit" is related to the case $\sigma = \text{const}$. In fact, no matter what the initial datum σ is, as long as it is bounded on $I \ni s$, it is very easy to see that

$$\lim_{e\to+\infty} \frac{x_{1,}}{x_{0,}} = x + \frac{\dot{y}(\dot{x})^2+(\dot{y})^3}{\ddot{x}\dot{y} - \ddot{y}\dot{x}}\,, \tag{26}$$

$$\lim_{e\to+\infty} \frac{x_{2,}}{x_{0,}} = y - \frac{(\dot{x})^3+\dot{x}(\dot{y})^2}{\ddot{x}\dot{y} - \dot{x}\ddot{y}}\,. \tag{27}$$

Equations (26) and (27) prove that in the limit of high-energy caustics "tend" to the evolute of Σ. This means that tha case σ = constant is *exactly* equivalent to the high energy limit.

Appendix A.

Projective duality and Plückerian coordinates

Let us review for completeness some basic facts about projective geometry. We recall that $P^2(\mathbb{R})$ can be identified with $(\mathbb{R}^3 - 0)/\sim$, where the equivalence relation $\sim$ means that $(x, y, z) \sim (x', y', z')$ iff $\exists\lambda \in \mathbb{R}^*$ such that $(x', y', z') = (\lambda x, \lambda y, \lambda z)$. Under this relation, a point Q in $P^2(\mathbb{R})$ is identified with a straight line through the origin in $\mathbb{R}^3$.

Given two points Q and Q' in $P^2(\mathbb{R})$ $(Q \neq Q')$, which correspond to the lines l and l' in $\mathbb{R}^3$, we construct the line $\pi_{QQ'}$ in $P^2(\mathbb{R})$, joining Q and Q'. Obviously, $\pi_{QQ'}$ is the image under $\sim$ of the lines in $\mathbb{R}^3$, which are contained in the plane through the origin generated by l and l'. More explicitely, the lines $l = (\bar{x}, \bar{y}, \bar{z}) = (\lambda x_0, \lambda x_1, \lambda x_2)$ and $l' = (x', y', z') = (\mu x_0', \mu x_1', \mu x_2')$ correspond to $Q = [x_0, x_1, x_2]$ and $Q' = [x_0', x_1', x_2']$; then the equation of the plane Π in $\mathbb{R}^3$ containing l and l' is of the form

$$ax + by + cz = 0,$$

where the coefficients (a, b, c) are obtained imposing $l \subset \Pi$ and $l' \subset \Pi$. This implies that

$$(a, b, c) = (x_0, x_1, x_2) \wedge (x_0', x_1', x_2'), \tag{A.1}$$

where $\wedge$ means vector product (compare eq. (15)). Thus a vector $v = (\bar{x}, \bar{y}, \bar{z}) \in (\mathbb{R}^3 - 0)$ belongs to Π iff $a\bar{x} + b\bar{y} + c\bar{z} = 0$. This equation can be read also as an equation in $P^2(\mathbb{R})$, remembering that v identifies a unique line l through the origin, and so a unique point Q in $P^2(\mathbb{R})$. Under this identification we have that a point $S = [\tilde{x}_0, \tilde{x}_1, \tilde{x}_2] \in P^2(\mathbb{R})$ belongs to $\pi_{QQ'}$ iff

$$a\tilde{x}_0 + b\tilde{x}_1 + c\tilde{x}_2 = 0\,, \quad (a, b, c) \overset{(A.1)}{=} (x_1x_2' - x_2x_1', x_2x_0' - x_0x_2', x_0x_1' - x_1x_0'). \tag{A.2}$$

In the literature (see [5], [6] and [7], for example) the coefficients (a, b, c) are called *Plückerian coordinates* of the line $\pi_{QQ'}$. By construction these coordinates are defined up to multiplication by a common nonzero constant, that is (a, b, c) and $(\mu a, \mu b, \mu c)$ identify, as Plückerian coordinates the *same* line in $P^2(\mathbb{R})$. In this way the coordinates (a, b, c) can be thought as *homogeneous coordinates* $[a, b, c]$ on a different projective space: under the identification between lines and their Plückerian coordinates, we obtain that lines in $P^2(\mathbb{R})$ become the elements (points) of a two-dimensional projective space, called *dual projective space* $P^2(\mathbb{R})^*$. Therefore, associated to $P^2(\mathbb{R})$, there is $P^2(\mathbb{R})^*$, whose points are lines (hyperplanes) of $P^2(\mathbb{R})$.

It is natural to ask what are the lines of $P^2(\mathbb{R})^*$ (points in the double dual $P^2(\mathbb{R})^{**}$). *Projective duality* states that $P^2(\mathbb{R})^{**} \equiv P^2(\mathbb{R})$, so that lines in $P^2(\mathbb{R})^*$ can be identified with points in $P^2(\mathbb{R})$, in the following sense. In complete analogy with $P^2(\mathbb{R})$, we have that a line $\pi^* \in P^2(\mathbb{R})^*$ can be described by an equation of the form

$$\alpha x_0^* + \beta x_1^* + \gamma x_2^* = 0\,, \tag{A.3}$$

where $[x_0^*, x_1^*, x_2^*]$ are homogeneous coordinates of points in $P^2(\mathbb{R})^*$. So $P = [x_0^*, x_1^*, x_2^*] \in \pi^*$ iff (A.3) is satisfied. But now observe that the homogeneous coordinates of $P =$

$[x_0^*, x_1^*, x_2^*]$ can be thought also as Plückerian coordinates of the corresponding line π_P in $P^2(\mathbb{R})$, whose equation is

$$x_0^* x_0 + x_1^* x_1 + x_2^* x_2 = 0.$$

Now we compute

$$\bigcap_{P \in \pi^*} \pi_P \subset P^2(\mathbb{R})$$

and prove that this intersection corresponds to a unique point $Q_{\pi^*} \in P^2(\mathbb{R})$ (the subscript π^* in Q means that this point is uniquely identified by the line π^*). Computing the previous intersection is the same as solving simultaneously the equations

$$x_0^* x_0 + x_1^* x_1 + x_2^* x_2 = 0\,, \tag{A.4}$$

$$\bar{x}_0^* x_0 + \bar{x}_1^* x_1 + \bar{x}_2^* x_2 = 0 \tag{A.5}$$

for two distinct points $P = [x_0^*, x_1^*, x_2^*]$ and $P' = [\bar{x}_0^*, \bar{x}_1^*, \bar{x}_2^*]$ belonging to π^*. Simple linear algebra shows that (x_0, x_1, x_2) is a solution of (A.4) and (A.5) iff $[x_0, x_1, x_2] = [\alpha, \beta, \gamma]$. So the Plückerian coordinates (α, β, γ) of the line π^* in $P^2(\mathbb{R})^*$ correspond to the homogeneous coordinates $[\alpha, \beta, \gamma]$ of the point $Q_{\pi^*} \in P^2(\mathbb{R})$. So we have proved that $P^2(\mathbb{R})^{**} \equiv P^2(\mathbb{R})$. Thus Plückerian coordinates of lines can always be identified with homogeneous coordinates of points in the dual. The situation is summarized in the following table:

$P^2(\mathbb{R})$	$P^2(\mathbb{R})^*$
points	lines
lines	points

Suppose it is given a parametric curve C in $P^2(\mathbb{R})$, whose homogeneous coordinates are described by $[x_0(s), x_1(s), x_2(s)] \in P^2(\mathbb{R})$. Fix $Q = [x_0(\bar{s}), x_1(\bar{s}), x_2(\bar{s})]$ in C: we want to compute the equation of the tangent line τ_Q to C in Q. Let Q' be a different point belonging to C. By the previous results developed in this appendix we know how to compute the equation in $P^2(\mathbb{R})$ of the line $\pi_{QQ'}$ through Q and Q':

$$\pi_{QQ'}: \quad h(x_0, x_1, x_2) := x_0(x_1(\bar{s})x_2(s') - x_2(\bar{s})x_1(s')) +$$

$$+ x_1(x_2(\bar{s})x_0(s') - x_0(\bar{s})x_2(s')) + x_2(x_0(\bar{s})x_1(s') - x_1(\bar{s})x_0(s')) = 0.$$

Then the tangent line τ_Q can be thought as obtained "by letting Q' going to Q", so that

$$\tau_Q: \quad \lim_{s' \to \bar{s}} \frac{h(x_0, x_1, x_2)}{s' - \bar{s}}, \tag{A.6}$$

that is

$$\tau_Q: \quad x_0(x_1(\bar{s})\dot{x}_2(\bar{s}) - x_2(\bar{s})\dot{x}_1(\bar{s})) + x_1(x_2(\bar{s})\dot{x}_0(\bar{s}) - x_0(\bar{s})\dot{x}_2(\bar{s})) + \tag{A.7}$$

$$+x_2(x_0(\bar{s})\dot{x}_1(\bar{s}) - x_1(\bar{s})\dot{x}_0(\bar{s})) = 0.$$

By (A.7), we have obtained that the Plückerian coordinates $(a, b, c)_{\tau_Q}$ of the tangent line to a curve C to a point $Q = [x_0(\bar{s}), x_1(\bar{s}), x_2(\bar{s})]$ are simply given by (compare with eq. (15))

$$(a, b, c)_{\tau_Q} := (x_0(\bar{s}), x_1(\bar{s}), x_2(\bar{s})) \wedge (\dot{x}_0(\bar{s}), \dot{x}_1(\bar{s}), \dot{x}_2(\bar{s})). \tag{A.8}$$

As we already observed, these Plückerian coordinates define homogeneous coordinates in the dual projective space; repeating this construction for every point of the curve C, we construct a new curve C^* in $P^2(\mathbb{R})^*$, called the *dual curve* to C. The homogeneous coordinates of points belonging to C^* are exactly Plückerian coordinates of tangent lines to C, so the homogeneous parametric equations of C^* are

$$[x_0^*(s), x_1^*(s), x_2^*(s)] = [(x_0(s), x_1(s), x_2(s)) \wedge (\dot{x}_0(s), \dot{x}_1(s), \dot{x}_2(s))].$$

As expected, if now we compute the dual curve to C^*, we come back to our original curve C, as a straightforward calculation can prove. In fact, the homogeneous coordinates of points of C^* are collectively given by an expression like

$$[x^*(s)] = [x(s) \wedge \dot{x}(s)], \tag{A.9}$$

thus the homogeneous coordinates of points of C^{**} are given by

$$[x^{**}(s)] = [x^*(s) \wedge \dot{x}^*(s)] \overset{(A.9)}{=} [(x(s) \wedge \dot{x}(s)) \wedge \frac{\mathrm{d}}{\mathrm{d}s}(x(s) \wedge \dot{x}(s))]. \tag{A.10}$$

Now, exploiting the properties of vector product, it follows from eq. (A.10) that

$$[x^{**}(s)] = [\big(x(s) \wedge \dot{x}(s) \big| \ddot{x}(s)\big) x(s)] = [g(s)x(s)], \tag{A.11}$$

where $(.|.)$ denotes the standard scalar product in $\mathbb{R}^3$ and $g(s)$ is the scalar function $(x(s) \wedge \dot{x}(s)|\ddot{x}(s))$. Thus, the homogeneous coordinates of points of C^{**} are *exactly* the homogeneous coordinates of points of C, provided $g(s)$ is never vanishing (or at most vanishing at some isolated points). In fact, in our framework, starting from a *regular* curve C in $\mathbb{R}^2$ and embedding it in $P^2(\mathbb{R})$, we can think that the curve C is sitting on the plane $x_0 = 1$ in $\mathbb{R}^3$ as a plane curve (the embedding of $\mathbb{R}^2$ in $P^2(\mathbb{R})$ is equivalent to identify $\mathbb{R}^2$ to the plane $x_0 = 1$ in $\mathbb{R}^3$). Under this representation it is obvious that $x(s) \wedge \dot{x}(s)$ is never vanishing. Moreover, if we assume that C is not a line, then also $\ddot{x}(s)$ is generically different from zero (except eventually at the inflection points). So in our case we have that

$$g(s) = \Big((1, x_1(s), x_2(s)) \wedge (0, \dot{x}_1(s), \dot{x}_2(s)) \Big| (0, \ddot{x}_1(s), \ddot{x}_2(s))\Big),$$

and then

$$g(s) = \dot{x}_1(s)\ddot{x}_2(s) - \ddot{x}_1(s)\dot{x}_2(s).$$

The last expression of $g(s)$ is nothing else that the scalar product in $\mathbb{R}^2$ of the vectors $v = (-\dot{x}_2(s), \dot{x}_1(s))$ and $w = (\ddot{x}_1(s), \ddot{x}_2(s))$. Then we have that $g(s)$ is never vanishing (except eventually at the inflection points) because v and w are never orthogonal, since v spans the direction orthogonal to the curve C (thought in $\mathbb{R}^2$), and w represents the acceleration vector.

REFERENCES

[1] TULCZYJEW W. M., *Geometric Formulations of Physical Theories* (Bibliopolis, Napoli) 1989, ISBN 88-7088-216-0.

[2] BENENTI S., *Relazioni Simplettiche: La trasformazione di Legendre e la teoria di Hamilton-Jacobi, Quaderni U.M.I.*, **33** (Pitagora Editrice, Bologna) 1988.

[3] ARNOL'D V. I., *Mathematical methods of Classical Mechanics*, *Springer Graduate Texts in Math.*, No. **60** (Springer Verlag, New York) 1978.

[4] CONTI F., *Calcolo, fondamenti e applicazioni* (McGraw-Hill Libri Italia) 1993.

[5] ROSENBAUM R. A., *Introduction to Projective Geometry and Modern Algebra* (Addison-Wesley Pub. Comp.) 1963.

[6] SAMUEL P., *Projective Geometry, Undergraduate Texts in Mathematics* (Springer-Verlag) 1988.

[7] VEBLEN O. and YOUNG J. W., *Projective Geometry*, Vol. **1** (Blaisdell Pub. Comp.) 1936.

[8] ARNOL'D V. I., *Symplectic Geometry and Topology*, Cahiers Mathématique de la Décision, n. 9410, 24/02/94, CEREMADE, Université Paris IX Dauphine.

[9] ARNOL'D V. I., *Geometrical Methods in the Theory of Ordinary Differential Equations* (Springer Verlag, New York) 1983.

[10] SHCHERBAK, *Sel. Math. Sov.*, **5** (1986) 391.

[11] CARDIN F., *Nuovo Cimento B*, **104** (1989) 525.

[12] VINOGRADOV A. M. and KUPERSCHMIDT B. A., *Russ. Math. Surveys*, **32** (1977) 177.

[13] ARNOL'D V. I. and GIVENTAL' A. B., *Symplectic Geometry*, in *Dynamical Systems 4* (Springer Verlag) 1990.