On constrained mechanical systems: D'Alembert's and Gauss' principles

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A geometric formulation of the classical principles of D'Alembert and Gauss in analytical mechanics is given, and their equivalence for possibly non-Riemannian mechanical systems is shown, in the case of ideal holonomic constraints. This is done by means of a Gauss' function, which is defined in a natural way on the bundle of two-jets on the configuration space, and which gives the "intensity" of the "reaction forces" of the constraints. It is originated by a metric structure on the bundle of semibasic forms on the phase space determined by the Finslerian kinetic energy functions of the mechanical system.

I. INTRODUCTION

There are several reasons for a revisitation, from a geometrical point of view, of the well known Gauss' principle of "least constraint." [1]

Indeed, it is remarkable that this topic seems to not have received a great deal of attention in the formulation of analytical mechanics, within the framework of differential geometry, that took place in the last decades. For instance, we can quote, among others, the books by Godbillon,[2] Libermann and Marle,[3] or Abraham and Marsden,[4] where this principle is not treated.

However, Gauss' principle seems to be of undoubtable foundational relevance and worthy of interest, especially in a broad generality of choice for the form of the kinetic energy function of the mechanical system and for the active forces, possibly nonconservative and dependent upon the distribution of the generalized velocities in the phase space.

We wish to stress that an accurate study of Gauss' principle is also interesting from the point of view of the possible applications. Indeed, for instance, in a rather recent work by Lilov and Lorer,[5] an algorithm for a dynamical investigation of a multirigid body system is proposed on the basis of Gauss' principle. The two authors remark that "the main advantage of this approach, ..., over the derivation and investigation of the nonlinear equation of motion, ..., is that, using Gauss' principle, the accelerations can be found out from the condition for minimum of a functional, ..., and there is the possibility to use effectively the mathematical programming methods, and especially the recent iterative algorithms for constraint and unconstraint minimization of quadratic functionals."

Usually, in the analytical mechanics textbooks, Gauss' principle is stated for mechanical systems composed of a finite number of material particles under the presence of ideal constraints. Such a procedure excludes the finite-dimensional systems with an infinite number of particles, like rigid bodies, unless some limiting processes are carried out, which are sometimes lacking the necessary rigor. About this, we agree with Wang (Ref. 6, p. vii), when he states that rigid bodies should be regarded as primitive concepts like mass points and treated as such.

The present version of Gauss' principle complies with these ideas, and can be applied as soon as the finite-dimensional mechanical systems are assigned a "free" configuration manifold and a constraint manifold, together with a kinetic energy function and an "active force" field, both independent of the constraints. Mechanical systems composed of a finite number of mass points and/or rigid bodies are thus equally treated in a natural way. Indeed, the general form for the kinetic energy that we adopt gives a generalization of Gauss' principle to the case of Finslerian (possibly non-Riemannian) systems.

In order to accomplish our goal, a suitable statement of D'Alembert's principle is needed. Our approach regarding the latter is close to that of Vershik and Faddeev.[7] However, since we focus on the holonomic ideal case, constraints in this work are treated in a somewhat different way. Here the point of view and techniques of Ref. 2 are adopted, so that the present versions of the principles of D'Alembert and Gauss follow the spirit of the construction in Ref. 2.

A Gauss function is introduced in a natural way on the bundle of two-jets on the configuration space, by means of a "kinetic" metric on the bundle of semibasic forms on the velocity phase space. This norm measures the "deviation forces" that are needed for the mechanical system to undergo the motions associated with *a priori* chosen semisprays. They are compared with the only dynamically possible motion compatible with the constraints, i.e., with the motion $\mathbb{M}$ associated with semisprays determined by D'Alembert's equation. The final result, that is, the equivalence of the principles of Gauss and D'Alembert, is basically a characterization of $\mathbb{M}$ in terms of either of the following properties: (a) $\mathbb{M}$ is the unique motion along which the deviation forces are of the kind that the ideal constraints are capable of exerting; and (b) along $\mathbb{M}$ the above forces minimize, in a certain sense, the Gaussian function. Local expressions of all the definitions and results are given.

II. CONSTRAINED MECHANICAL SYSTEMS

We begin with a brief summary of some fundamental notions and results, and an introduction of the notations. We refer mainly to Refs. 2 and 3 for details.

Let M be a differentiable (C^∞) manifold of dimension n, and without boundary: $\partial M = \varnothing$ (Ref. 2, pp. 57 and 58).

To any coordinate system (x^i) on M are canonically associated natural coordinate systems $(x^i, \dot{x}^i)$ and $(x^i, \dot{x}^i, \delta x^i, \delta\dot{x}^i)$ on TM and TTM, respectively. Here and in the sequel, latin indices run from 1 to n.

The canonical tangent projections, τ_M: $TM \to M$ and T_M: $TTM \to TM$, thus have the local expressions

$$\tau_M: (x^i, \dot{x}^i) \mapsto x^i, \quad \tau_{TM}(x^i, \dot{x}^i, \delta x^i, \delta \dot{x}^i) \mapsto (x^i, \dot{x}^i), \qquad (2.1)$$

respectively. The space TTM is fibered in two ways on TM: either by means of the projection τ_{TM}: $TTM \to TM$ introduced above, or by means of $T\tau_M$: $TTM \to TM$, whose local expression is as follows:

$$T\tau_M: (x^k, \dot{x}^k, \delta x^k, \delta \dot{x}^k) \mapsto (x^k, \delta x^k). \qquad (2.2)$$

As usual, Th denotes the tangent of a mapping h.

The kernel of $T\tau_M$ is a canonical subbundle of TTM, called the *vertical tangent bundle* to TM, and denoted by VTM (see Ref. 3, p. 54). The elements of VTM are termed *vertical* and have the local expression $(x^k, \dot{x}^k, 0, \delta\dot{x}^k)$. We denote by V: $TM \to VTM$ the Liouville vertical vector field on TM, generating the one-parameter group of positive dilations of TM. In natural coordinates, it has the local expression

$$V = \dot{x}^i \frac{\partial}{\partial \dot{x}^i}, \qquad (2.3)$$

so that, for instance, $V \cdot f = \dot{x}^i (\partial f / \partial \dot{x}^i)$ for all $f \in D(TM)$, the algebra of the differentiable functions on TM.

In a similar way, we will denote by π_M: $T^*M \to M$ and π_{TM}: $T^*TM \to TM$ the cotangent projections. To (x^i), the systems of coordinates (x^i, p_i) and $(x^i, \dot{x}^i, p_i, r_i)$ on T^*M and T^*TM, respectively, are canonically associated. In this way, the above projections π_M and π_{TM} have the local expressions

$$\pi_M: (x^i, p_i) \mapsto x^i, \quad \pi_{TM}: (x^i, \dot{x}^i, p_i, r_i) \mapsto (x^i, \dot{x}^i). \qquad (2.4)$$

Following Ref. 2, we introduce the vector bundle β: $\tau_M^* T^*M \to TM$ of semibasic forms on TM. The total space $\tau_M^* T^*M$ can be identified with the subspace $\bigcup_{y \in M} \tau_M^{-1}(y) \times \pi_M^{-1}(y)$ of $TM \times T^*M$, and the projection β is the restriction of the projection of $TM \times T^*M$ onto TM (see Ref. 2, p. 166).

By Proposition 2.2 and Remark 2 in Ref. 3, pp. 55 and 56, we will identify the vector bundle β: $\tau_M^* T^*M \to TM$ of the semibasic forms on TM with the subbundle of T^*TM, $\pi_{TM}|_{(VTM)^0}$: $(VTM)^0 \to TM$, the annihilator of the vertical bundle VTM.

Hence, for simplicity, we will also use π_{TM} for semibasic forms, instead of β. Of course, in natural coordinates the elements of $(VTM)^0$ have the expression $(x^i, \dot{x}^i, p_i, 0)$. A differential one-form σ on TM is semibasic, if and only if it has, in natural coordinates, the local expression

$$\sigma = \sigma_i(x^h, \dot{x}^h) dx^i, \qquad (2.5)$$

where $\sigma_i(x^h, \dot{x}^h)$ are given functions on TM (see Ref. 2, p. 165 or Ref. 3, pp. 56–58).

The identification of the vector bundles $(VTM)^0$ and $(VTM)^*$ on TM (see Propositions 2.4, 2.5, and 3.11 in Ref. 3, pp. 55–58), allows for the definition of a vector bundle morphism v^*: $T^*TM \to (VTM)^0$ (also see Proposition 6.9 in Ref. 3, p. 70), whose local expression is

$$v^*: (x^i, \dot{x}^i, p_i, r_i) \mapsto (x^i, \dot{x}^i, r_i, 0). \qquad (2.6)$$

The morphism v^* induces an endomorphism, still denoted by v^* and called *vertical*, of the $D(TM)$-algebra $\Lambda(TM)$ of the differential forms on TM. It is locally determined by the conditions (Ref. 2, p. 161)

$$v^* f = f, \quad \text{for any } f \text{ in } D(TM), \quad v^*(dx^i) = 0,$$
$$v^*(d\dot{x}^i) = dx^i. \qquad (2.7)$$

The subalgebra $B(TM)$ of semibasic differential forms is the range and kernel of v^*. Beside the usual exterior differential d, by means of v^* the *vertical differential* d_v is also defined on $\Lambda(TM)$. It is uniquely characterized by the relations (see Ref. 2, p. 163)

$$d_v f = v^* df, \quad d_v(df) = -d(v^* df), \quad \text{for any } f \text{ in } D(TM). \qquad (2.8)$$

Locally, d_v is determined by

$$d_v f = \frac{\partial f}{\partial \dot{x}^i} dx^i, \quad d_v(dx^i) = 0, \quad d_v(d\dot{x}^i) = 0, \qquad (2.9)$$

and the relation $dd_v = -d_v d$ holds.

We now recall the following:

Definition (Ref. 2, p. 169): A mechanical system $\mathcal{M}$ is a triple (M, K, Φ) where (a) M is a differentiable manifold of dimension n, the *configuration space*; (b) K is a differentiable function on TM, the *kinetic energy*; and (c) Φ is a semibasic differential one-form on TM, the *force field*.

The differential two-form $dd_v K$ on TM is called the *fundamental form* of the mechanical system $\mathcal{M}$, which is called *regular* if $dd_v K$ is symplectic on TM. This happens if and only if locally we have (Ref. 2, p. 169)

$$\det\left(\frac{\partial^2 K}{\partial \dot{x}^i \partial \dot{x}^j}\right) \neq 0. \qquad (2.10)$$

Now, the space T^2M of two-jets of M can be defined by (see Ref. 3, p. 372)

$$T^2M = \{w \in TTM: \tau_{TM}(w) = T\tau_M(w)\}, \qquad (2.11)$$

and the canonical submersion τ_{TM}^{21}: $T^2M \to TM$ can be identified with $\tau_{TM}|_{T^2M}$ or $T\tau_M|_{T^2M}$.

Since τ_{TM} and $T\tau_M$ have the local expressions (2.1) and (2.2), the elements of T^2M are given locally by $(x^k, \dot{x}^k, \dot{x}^k, \delta\dot{x}^k)$. As usual, they will be denoted by $(x^k, \dot{x}^k, \ddot{x}^k)$, with $\ddot{x}^k$ written for $\delta\dot{x}^k$. In this way, the local expression for the canonical submersion τ_{TM}^{21}: $T^2M \to TM$ is

$$\tau_{TM}^{21}: (x^k, \dot{x}^k, \ddot{x}^k) \mapsto (x^k, \dot{x}^k). \qquad (2.12)$$

A *semispray* Y is a section of τ_{TM}^{21}. Of course, Y can be seen as a vector field Y: $TM \to TTM$, satisfying the condition

$$\tau_{TM} \circ Y = T\tau_M \circ Y. \qquad (2.13)$$

In other words, a semispray Y is a vector field on TM, which is at the same time a section of τ_{TM} and $T\tau_M$. Locally, such a Y is given by

$$Y = \dot{x}^i \frac{\partial}{\partial x^i} + b^i(x^k, \dot{x}^k) \frac{\partial}{\partial \dot{x}^i}, \qquad (2.14)$$

with $b^i(x^k, \dot{x}^k)$ given functions on TM. From this, it is immediately seen that the integral curves of $Y: TM \to T^2M$ are velocity curves of the curves on M of which Y [or $b^i(x^k, \dot{x}^k)$] is the acceleration at each point. These base curves on M are also called the *solutions* of Y, because they locally satisfy the system of equations

$$\frac{d^2x^i}{dt^2} = b^i\left(x^k, \frac{dx^k}{dt}\right). \tag{2.15}$$

For this reason, semisprays are also called second-order differential equations.

The following proposition holds (Ref. 2, p. 170). Let $\mathscr{M}$ be a regular mechanical system. Then there exists a unique vector field X on TM, such that

$$\mathbf{i}_X dd_v K = d(K - V \cdot K) + \Phi. \tag{2.16}$$

Here, the symbol $\mathbf{i}_X$ denotes as usual the interior product of a differential form by a vector field. The vector field X is called the *dynamical system* associated with $\mathscr{M}$.

Furthermore, it can be proved that the dynamical system X associated with a regular mechanical system $\mathscr{M}$ is a semispray (see Ref. 2, p. 170).

Let the local expression of the semibasic one-form Φ be $\Phi = \Phi_i(x^k, \dot{x}^k)dx^i$; then it can be proved that the solutions of the dynamical system X associated with $\mathscr{M} = (M, K, \Phi)$ locally satisfy the Lagrange equations (Ref. 2, p. 171),

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{x}^k}\right) - \frac{\partial K}{\partial x^k} = \Phi_k. \tag{2.17}$$

We now give the following:

Definition: A mechanical system with bilateral holonomic constraints is a quintuplet, $\mathscr{M}_c = (M, K, \Phi, Q, \mathscr{R})$, where (a) M, K, and Φ are as above; (b) Q, the *constraint*, is an m-dimensional ($m \leqslant n$) imbedded submanifold of M, with imbedding denoted by χ: $Q \to M$ and such that $\partial(cl_M Q) = \varnothing$; (c) $\mathscr{R} \subset (VTM)^0|_{TQ}$ is the total space of a subbundle $\pi_{TM}|_{\mathscr{R}}$: $\mathscr{R} \to TQ$ of the vector bundle $\pi_{TM}|_{(VTM)^0|_{TQ}}$: $(VTM)^0|_{TQ} \to TQ$ of the semibasic forms restricted to TQ.

Here and in the sequel, we of course identify Q with its image in M under χ, as well as TQ with its image in TM under $T\chi$, and so on. Also, for simplicity, in the sequel we will drop the restriction symbol from π_{TM}, since no confusion arises.

We explicitly notice that in (b), the condition $\partial(cl_M Q) = \varnothing$ expresses the notion that the constraints are "bilateral." The subbundle introduced in (c) describes the forces that the constraints are capable of exerting, which are called the *admissible constraint reaction forces*.

For brevity, $\mathscr{M}_c$ will be referred to as the *constrained mechanical system*; it will be called *regular* when both the systems $\mathscr{M} = (M, K, \Phi)$ and $\mathscr{Q} = (Q, \widetilde{K}, \widetilde{\Phi})$ are such, where

$$\widetilde{K} = (T\chi)^*K = K \circ T\chi, \quad \widetilde{\Phi} = (T\chi)^*\Phi. \tag{2.18}$$

Since χ is an imbedding and the fibers in TM are linear, it is easily verified that if $\mathscr{M}$ is regular, $\mathscr{Q}$ is also regular.

We will consider the case of *ideal* constraints, in which $\mathscr{R}$ is specified as follows:

$$\mathscr{R} = v^*((TTQ)^0). \tag{2.19}$$

An explicit equivalent description of $\mathscr{R}$ is the following:

$$\mathscr{R} = \{r \in (VTM)^0: \pi_{TM}(r) = u \in TQ \subset TM,$$
$$\operatorname{Ker} r = T_u TQ \subset T_{T\chi(u)} TM\}. \tag{2.20}$$

The above definitions of holonomic constraints and admissible constraint reaction forces strongly rely on the introduction of *one* assigned constraint submanifold Q of M. In Ref. 7, where anholonomic constraints are treated in a very general setting, holonomic constraints possibly emerge as foliations of M, introduced by suitable repeated integrations of distributions on TM. The simpler procedure we follow, which is closer to the classical treatments, seems more natural from a physical point of view.

A characterization of the sections of $\pi_{TM}|_{\mathscr{R}}: \mathscr{R} \to TQ$, which is important in the sequel, is given by the following:

Lemma: Let ρ be a differential semibasic one-form, i.e., a section of $\pi_{TM}|_{(VTM)^0}$: $(VTM)^0 \to TM$. Then $\rho \circ T\chi$ is a section of $\pi_{TM}|_{\mathscr{R}}: \mathscr{R} \to TQ$, if and only if

$$(T\chi)^*\rho = 0. \tag{2.21}$$

Proof: Indeed, denoting by $\tilde{\rho}$: $TQ \to T^*TQ$ the differential one-form $(T\chi)^*\rho$, it is $\tilde{\rho} = 0$, if and only if, for any arbitrarily fixed $u \in TQ$, it results that

$$\mathbf{i}_z \tilde{\rho}(u) = 0, \quad \text{for any } z \in T_u TQ. \tag{2.22}$$

But this is true, if and only if

$$\mathbf{i}_{TT\chi(z)} \rho(T\chi(u)) = 0, \quad \text{for any } z \in T_u TQ, \tag{2.23}$$

that is, if and only if $\rho(T\chi(u)) \in \mathscr{R}$ for all $u \in TQ$, or, if and only if $\rho \circ T\chi$ is a section of $\pi_{TM}|_{\mathscr{R}}: \mathscr{R} \to TQ$.

The local expressions will be useful, and we give them in detail. Let (x^i) and (q^α) be local coordinate systems on M and Q, respectively (here greek indices run from 1 to m). Furthermore, let $\rho = \rho_i(x^h, \dot{x}^h)dx^i$ be a semibasic differential one-form, and let $u = (q^\alpha, \dot{q}^\alpha)$ be a fixed arbitrary element of TQ, so that $z \in T_u TQ$ has coordinates $(q^\alpha, \dot{q}^\alpha, \delta q^\alpha, \delta \dot{q}^\alpha)$ and

$$TT\chi(z) = (\chi^k(q^\alpha), D_\sigma \chi^j(q^\alpha)\dot{q}^\sigma, D_\sigma \chi^j(q^\alpha)\delta q^\sigma,$$
$$D_\gamma D_\sigma \chi^h(q^\alpha)\dot{q}^\sigma \delta q^\gamma + D_\sigma \chi^h(q^\alpha)\delta \dot{q}^\sigma). \tag{2.24}$$

As usual, D_σ denotes the partial derivative in $\mathbb{R}^m$. Then, since

$$\rho \circ T\chi = \rho_i(\chi^k(q^\alpha), D_\sigma \chi^j(q^\alpha)\dot{q}^\sigma)dx^i \tag{2.25}$$

and

$$(T\chi)^*\rho = \rho_i(\chi^k(q^\alpha), D_\sigma \chi^j(q^\alpha)\dot{q}^\sigma)D_\alpha \chi^i(q^\alpha)dq^\alpha, \tag{2.26}$$

(2.23)–(2.25) yield

$$\rho_i(\chi^k(q^\gamma), D_\sigma \chi^j(q^\gamma)\dot{q}^\sigma)D_\alpha \chi^i(q^\gamma)\delta q^\alpha = 0,$$
$$\text{for all } \delta q^\alpha \in \mathbb{R}, \tag{2.27}$$

which of course is true if and only if

$$\rho_i(\chi^k(q^\gamma), D_\sigma \chi^j(q^\gamma)\dot{q}^\sigma)D_\alpha \chi^i(q^\gamma) = 0,$$
$$\text{for all } (q^\alpha, \dot{q}^\alpha). \tag{2.28}$$

By (2.26), we see that (2.28) is equivalent to (2.21).

Remark 1: The characterization of the constraint reaction forces given by (2.21) or (2.28), easily leads to the following, which will also be used in the sequel.

Let ρ be a semibasic differential one-form. Then $\rho\circ T\chi$ is a section of $\pi_{TM}|_{\mathscr{W}}:\mathscr{R}\to TQ$, if and only if

$$(T\chi)^*(\mathbf{i}_Y\rho)=0, \tag{2.29}$$

for all fields Y: $TM\to TTM$, such that there exists a field Z: $TQ\to TTQ$ for which the relation

$$Y\circ T\chi = TT\chi\circ Z \tag{2.30}$$

holds.

We will call the vector fields Y on TM, and Z on TQ, $T\chi$-related, when they satisfy (2.30). In this way, Y is an extension to TM of a vector field Z on TQ.

To prove the assertion of Remark 1, let us recall that we can write, for any $v\in TQ$,

$$\begin{aligned}(T\chi)^*(\mathbf{i}_Y\rho)(v) &= \mathbf{i}_{Y(T\chi(v))}[\rho(T\chi(v))]\\ &= \mathbf{i}_{TT\chi(U(v))}[\rho(T\chi(v))]=0.\end{aligned} \tag{2.31}$$

Hence the first term in (2.31) is zero for any Y, if and only if $\rho(T\chi(v))\in\mathscr{R}$ for any $v\in TQ$.

Since $\mathbf{i}_Y\rho$ obviously has the meaning of "power" of a force along a "path," Remark 1 shows that constraint forces are characterized by the fact that they do no work on vector fields on TM that extend vector fields tangent to TQ.

Remark 2: Before concluding this section, we mention without details that Remark 1 implies a further characterization of the admissible constraint forces.

Let ρ be a differential semibasic form. Then $\rho\circ T\chi$ is a section of $\pi_{TM}|_{\mathscr{W}}:\mathscr{R}\to TQ$, if and only if

$$(T\chi)^*(\mathbf{i}_{Y^c}\rho)=0, \tag{2.32}$$

for all fields Y: $M\to TM$ that are χ-related to some vector field Z: $Q\to TQ$. In (2.32), Y^c indicates the *complete lift* to TM of a vector field Y on M [see Yano and Ishihara (Ref. 8, p. 14)]. If $Y=b^i(x^k)(\partial/\partial x^i)$ locally, it is $Y^c = b^i(x^k)(\partial/\partial x^i)+D_r b^i(x^k)\dot{x}^r(\partial/\partial\dot{x}^i)$. By Remark 1, to prove the assertion of Remark 2, we just need to notice that

$$Y\circ\chi = T\chi\circ Z \Leftrightarrow Y^c\circ T\chi = TT\chi\circ Z^c,$$

which we give without proof.

Remark 2 is interesting because it clarifies the "physical meaning" of the constraint forces. Indeed, it shows that to characterize them, it is enough that they do no work just on complete lifts to TM of vector fields on M that extend fields on Q. The above local expression of the complete lift clearly shows that the latter condition basically amounts to the classical one requiring that, in the ideal case, admissible reaction forces do no work on "displacements" tangent to the constraint Q.

III. D'ALEMBERT'S PRINCIPLE

We introduce, in connection with a given mechanical system $\mathscr{M}=(M,K,\Phi)$ and with a given, arbitrary, semispray Y: $TM\to T^2M$, the following *deviation* differential one-form ρ_Y:

$$\rho_Y=\mathbf{i}_Y dd_v K - d(K-V\cdot K)-\Phi. \tag{3.1}$$

These forms ρ_Y have the meaning of "forces to be added" to the given force field Φ, in order that a semispray Y, chosen *a priori*, be the dynamical system associated with the mechanical system $(M,K,\Phi+\rho_Y)$. In fact, the following holds:

Lemma: The deviation differential one-forms are semibasic.

Proof: If $Y=\dot{x}^i(\partial/\partial x^i)+b^i(x^k,\dot{x}^k)(\partial/\partial\dot{x}^i)$ locally, with $b^i(x^k,\dot{x}^k)$ given functions, it is not difficult to show that the local expression for ρ_Y is as follows:

$$\rho_Y=\left(\frac{\partial^2K}{\partial\dot{x}^k\,\partial\dot{x}^i}b^k+\frac{\partial^2K}{\partial x^k\,\partial\dot{x}^i}\dot{x}^k-\frac{\partial K}{\partial x^i}-\Phi_i\right)dx^i. \tag{3.2}$$

Definition: Two semisprays Y and Y' are said to be equivalent, $Y\approx Y'$, when their restrictions to TQ are equal, i.e., when $Y\circ T\chi = Y'\circ T\chi$.

Now, our goal is the construction of a dynamics for the constrained mechanical system $\mathscr{M}_c=(M,K,\Phi,Q,\mathscr{R})$. In order to do this, a twofold result must be obtained. Basically, we first need to select the semisprays X that (besides defining a dynamical system on the overall manifold M, also) define a dynamical system on the constraint manifold Q, meaning that the *solutions* of X must be all on Q when the initial data are in TQ.

Furthermore, the deviation differential one-forms ρ_X, that is, the forces necessary to "maintain" the system on the constraint, must be of the kind that the constraints are capable of exerting, i.e., $\rho_X\circ T\chi$ must be a section of the bundle $\pi_{TM}|_{\mathscr{W}}:\mathscr{R}\to TQ$ of admissible constraint forces, introduced in Sec. II.

We now show that, in the case of regular systems, the two properties above characterize X in a unique way on the constraint. In fact, the following theorem holds.

D'Alembert's principle: Let $\mathscr{M}_c$ be a regular constrained mechanical system. Then, up to equivalence, there is a unique semispray X: $TM\to T^2M$, such that the following D'Alembert's equation holds:

$$\mathbf{i}_X dd_v K - d(K-V\cdot K)=\Phi+\rho_X, \tag{3.3}$$

with $\rho_X\circ T\chi$ a section of $\pi_{TM}|_{\mathscr{W}}:\mathscr{R}\to TQ$. The solutions of X have the property that their image is all on Q as soon as the initial values are in TQ.

Proof: Let us set for brevity $\omega=dd_vK$, and $\sigma=d(K-V\cdot K)$, and let us consider the following pullbacks of ω, σ, and Φ:

$$\tilde{\omega}=(T\chi)^*\omega,\quad \tilde{\sigma}=(T\chi)^*\sigma,\quad \tilde{\Phi}=(T\chi)^*\Phi. \tag{3.4}$$

Since K is a function (a zero-form), both d and d_v commute with the pull-back, so that it is $\tilde{\omega}=dd_v\tilde{K}$ [see (2.18)]; hence the hypothesis that $\mathscr{M}_c$ is regular implies that $\tilde{\omega}$ is symplectic on TQ. Then, Theorems 1.4 and 1.6 in Ref. 1 (p. 170), applied to the mechanical system $\mathscr{Q}=(Q,\tilde{K},\tilde{\Phi})$, guarantee the uniqueness of the semispray $\tilde{X}$: $TQ\to T^2Q$, such that the following equation holds:

$$\mathbf{i}_{\tilde{X}}\tilde{\omega}-\tilde{\sigma}=\tilde{\Phi}. \tag{3.5}$$

Now, let us consider an arbitrary semispray X: $TM\to T^2M$, $T\chi$-related to $\tilde{X}$:

$$X \circ T\chi = TT\chi \circ \tilde{X}. \tag{3.6}$$

Then, considering the deviation one-form ρ_X, we have [see (3.1) and (3.3)]

$$\mathrm{i}_X dd_v K - d(K - V\cdot K) = \Phi, \tag{3.7}$$

where ρ_X is such that

$$(T\chi)^*\rho_X = (T\chi)^*(\mathrm{i}_X\omega - \sigma - \Phi) \tag{3.8a}$$

$$= (T\chi)^*(\mathrm{i}_X\omega) - \tilde{\sigma} - \tilde{\Phi} \tag{3.8b}$$

$$= \mathrm{i}_{\tilde{X}}\tilde{\omega} - \tilde{\sigma} - \tilde{\Phi} \tag{3.8c}$$

$$= 0. \tag{3.8d}$$

Equality (3.8c) holds because of (3.6) and example 1.8 (iii) in Ref. 2, p. 89, whereas (3.8d) is just (3.5). By the lemma in Sec. II, (3.8) shows that $\rho_X \circ T\chi$ is a section of admissible reaction forces.

The uniqueness of X, up to equivalence, is a consequence of (3.6) and of the uniqueness of $\tilde{X}$. Finally, the last assertion in the statement is true because, again by (3.6), TQ is an integral manifold of the semispray X.

We will say that any of the equivalent semisprays X above is a *dynamical system* associated with the constrained mechanical system $\mathscr{M}_c$. Here it is worth noticing that the class of semisprays X, $T\chi$-related to $\tilde{X}$, is not empty.

We also remark that the theorem above shows that, for fixed $\mathscr{M}$, the deviation forms ρ_Y connected with a semispray Y, are not, in general, forces that the constraints are capable of exerting. Indeed, they are such, only when Y is a dynamical system associated with $\mathscr{M}_c$. This is the reason why we refrained altogether from calling the ρ_Y "constraint reaction forms."

The local expressions will be useful in Sec. IV. If, locally, the field $\tilde{X}$, uniquely determined by Eq. (3.5), is given by

$$\tilde{X} = \dot{q}^\alpha \frac{\partial}{\partial q^\alpha} + \tilde{a}^\sigma(q^\beta, \dot{q}^\beta) \frac{\partial}{\partial \dot{q}^\sigma},$$

let us consider a semispray

$$X = \dot{x}^i \frac{\partial}{\partial x^i} + a^i(x^k, \dot{x}^k) \frac{\partial}{\partial \dot{x}^i},$$

such that Eq. (3.6) holds, that is, such that

$$a^i(\chi^k(q^\alpha), D_\sigma\chi^j(q^\alpha)\dot{q}^\sigma) = \tilde{a}^\beta(q^\alpha, \dot{q}^\sigma) D_\beta \chi^i(q^\alpha) + D_\gamma D_\sigma \chi^i(q^\alpha)\dot{q}^\sigma \dot{q}^\gamma. \tag{3.9}$$

Then, writing $\rho_X = \rho_{X_i}(x^h, \dot{x}^h)dx^i$, with

$$\rho_{X_i} = \frac{\partial^2 K}{\partial \dot{x}^k \partial \dot{x}^i} a^k + \frac{\partial^2 K}{\partial x^k \partial \dot{x}^i} \dot{x}^k - \frac{\partial K}{\partial x^i} - \Phi_i \tag{3.10}$$

[see (3.2)], by (3.3), we conclude that the equation

$$\rho_{X_i}(\chi^k(q^\alpha), D_\sigma \chi^j(q^\alpha)\dot{q}^\sigma) D_\beta \chi^i(q^\alpha) = 0 \tag{3.11}$$

holds, if and only if the functions $a^i(x^k, \dot{x}^k)$ satisfy (3.9).

It is worth noticing explicitly that the semispray $\tilde{X}$: $TQ \to T^2Q$, introduced in the proof of D'Alembert's principle, is such that

$$\mathrm{i}_{\tilde{X}} dd_v \tilde{K} - d(\tilde{K} - \tilde{V}\cdot\tilde{K}) = \tilde{\Phi}, \tag{3.12}$$

where $\tilde{V}$ is the canonical Liouville field on TQ. Hence, by (3.12), the solutions of $\tilde{X}$ locally satisfy the equations of Lagrange relative to the mechanical system $\mathscr{Q} = (Q, \tilde{K}, \tilde{\Phi})$,

$$\frac{d}{dt}\left(\frac{\partial \tilde{K}}{\partial \dot{q}^\alpha}\right) - \frac{\partial \tilde{K}}{\partial q^\alpha} = \tilde{\Phi}_\alpha. \tag{3.13}$$

Equation (3.12) is easily seen to hold because $\tilde{\omega} = dd_v \tilde{K}$ (see above) and $(T\chi)^*(V\cdot K) = \tilde{V}\cdot\tilde{K}$. The latter equality is a consequence of the fact that the fields V and $\tilde{V}$ are $T\chi$-related. Thus (3.12) is immediately derived from (3.5).

We conclude this section with a statement of the classical "energy theorem."

Let X: $TM \to T^2M$ be a semispray solving the equation of D'Alembert (3.3) and let $\tilde{X}$ be the $T\chi$-related semispray $TQ \to T^2Q$, solving Eq. (3.12). Let η: $[a,b] \to TQ$ be an integral curve of $\tilde{X}$, so that $T\chi\circ\eta$ is an integral curve of X. Then

$$\int_a^b \eta^* \circ (T\chi)^*\Phi = [V\cdot K - K]_{T\chi[\eta(a)]}^{T\chi[\eta(b)]}$$

$$\equiv [(\tilde{V}\cdot\tilde{K} - \tilde{K})]_{\eta(a)}^{\eta(b)}. \tag{3.14}$$

To prove that Eq. (3.14) holds, we recall that, being the semisprays $\tilde{X}$ and X, $T\chi$-related, Remark 1 of Sec. II gives

$$(T\chi)^*(\mathrm{i}_Y \rho_Y) = 0. \tag{3.15}$$

Then, since X solves D'Alembert's equation (3.3), and taking into account that $dd_v K$ is symplectic, evaluation on X of the forms appearing in (3.3), gives

$$(T\chi)^*[X\cdot(V\cdot K - K)] = (T\chi)^*(\mathrm{i}_X \Phi). \tag{3.16}$$

Equation (3.16) immediately yields (3.14).

IV. GAUSS' PRINCIPLE

As in the preceding sections, we do not necessarily consider K to be quadratic on the fibers $\tau_M{}^{-1}(x)$ of TM. This is the case, for instance, of Newtonian classical mechanics. Rather, we have in mind certain generalizations, such as Finslerian mechanics (see, for example, Ref. 2, p. 130, and also Rund,[9] Ruiz,[10] and Eringen[11]); we do not require here that K be a Riemannian metric.

Let $\mathscr{M}_c$ be a regular constrained mechanical system. Following Ref. 7, we introduce the (2,0)-tensor field Π on TM, defined by means of the relation

$$dd_v K[\Pi(\sigma), H] = \mathrm{i}_H \theta, \tag{4.1}$$

which is to hold for every differential one-form σ, and vector field H, on TM. Also, a new (2,0)-tensor field Γ on TM is defined, such that

$$\Gamma(\alpha, \gamma) = \Pi(\alpha, v^*\gamma), \tag{4.2}$$

for any differential one-forms α and γ on TM. Of course, we denote as usual with the same symbol both the morphism and the bilinear form induced by the (2,0)-tensor field Π.

A straightforward calculation shows that, setting for brevity,

$$K_{ij}(x^r, \dot{x}^s) = \frac{\partial^2 K}{\partial \dot{x}^i \partial \dot{x}^j}, \tag{4.3}$$

the local expression of Γ is

$$\Gamma = K^{ij}(x^r, \dot{x}^s) \frac{\partial}{\partial \dot{x}^i} \otimes \frac{\partial}{\partial \dot{x}^j}, \tag{4.4}$$

where

$$K_{ij}K^{jh} = \delta_i^{\ h}. \tag{4.5}$$

Hence Γ is connected to the Hessian of the function K along the fibers of TM (see Spivak, Ref. 12, Vol, 2, pp. 206 and 207).

From now on, we assume that K is the "energy" of a Finslerian structure on M; explicitly, we suppose that there exists a function F on TM such that $K = F^2$, with the following properties: (a) $F(v) \neq 0$, $F(\lambda v) = |\lambda| F(v)$, if $v \neq 0$, $v \in TM$, and $\lambda \in \mathbb{R}$; (b) the functions $K_{ij}(x^r, \dot{x}^s)$ define a positive-definite quadratic form on $V_v TM$, at every point $v = (x^r, \dot{x}^s)$ of TM with $\dot{x}^s \neq 0$.

It is clear that the (2,0)-tensor field Γ introduced in (4.2)–(4.5) is but the dual to the tensor $K_{ij}(x^r, \dot{x}^s) \times d\dot{x}^i \otimes d\dot{x}^j$ on TM, canonically associated with the Finslerian metric F on M (see Ref. 12, Vol. 2, p. 208).

Owing to its structure, Γ generates a metric for the quotient bundle $T^*TM/(VTM)^0$, which we will continue to indicate by Γ. Indeed, in natural local coordinates, letting $[\alpha] = [x^r, \dot{x}^s, r_i, p_j]$ be the equivalence class in $T^*TM/(VTM)^0$ of an element $\alpha = (x^r, \dot{x}^s, r_i, p_j) \in T^*TM$, we have

$$\Gamma([\alpha],[\alpha]) = \Gamma(\alpha, \alpha) = K^{ij}(x^r, \dot{x}^s) p_i p_j, \tag{4.6}$$

The number $\Gamma([\alpha],[\alpha]) = K^{ij}(x^r, \dot{x}^s) p_i p_j$ does not depend on the coordinates r_i of the element α, that is, it does not depend on the particular representative chosen for the equivalence class $[\alpha]$, so that the function Γ is well defined on equivalence classes.

By means of v^* (see Sec. II), we now construct the vector bundle isomorphism ν: $T^*TM/(VTM)^0 \to (VTM)^0$, such that $\nu^* = \nu \circ pr$, where pr: $T^*TM \to T^*TM/(VTM)^0$ is the usual quotient projection. As mentioned in Sec. II, $(VTM)^0$ is identified with the vector bundle of semibasic forms on TM by means of the results of Proposition 2.2 in Ref. 3 (p. 55).

Using natural coordinates, ν^{-1} has the local expression

$$\nu^{-1}: (x^r, \dot{x}^s, p_i, 0) \mapsto [x^r, \dot{x}^s, r_j, p_i], \tag{4.7}$$

where the r_j are arbitrarily fixed real numbers that label the elements in an equivalence class.

It is now clear that, through ν^{-1}, Γ determines a well-defined metric on the bundle $(VTM)^0$ of semibasic forms on TM. Let $w = (x^k, \dot{x}^k, \ddot{x}^k)$ be a given element of T^2M, with $\tau^{21}_{TM}(w) = v = (x^k, \dot{x}^k) \in TM$. In correspondence with w, let us consider the following element of $(V_v TM)^0$:

$$r_w = \mathbf{i}_w [dd_v K(v)] - d(K - V \cdot K)(v) - \Phi(v). \tag{4.8}$$

The coordinates of r_w are easily seen to be $(x^k, \dot{x}^k, p_i, 0)$, see (3.2), with

$$p_i = \frac{\partial^2 K}{\partial \dot{x}^k \partial \dot{x}^i} \ddot{x}^k + \frac{\partial^2 K}{\partial x^k \partial \dot{x}^i} \dot{x}^k - \frac{\partial K}{\partial x^i} - \Phi_i. \tag{4.9}$$

Then the following Gauss function:

$$\mathrm{G}: T^2M \to \mathbb{R}$$

$$w \mapsto \tfrac{1}{2} \Gamma[\nu^{-1}(r_w), \nu^{-1}(r_w)], \tag{4.10}$$

is well defined on T^2M.

To the element w in T^2M (describing the configuration and the distribution of velocities and accelerations of the system), G associates the Γ norm of the deviation force r_w "excited" by w. We recall that, by D'Alembert's principle, $r_w \in \mathscr{R}$, if and only if $w = X(v)$, with X a semispray associated with $\mathscr{M}_c$.

In the case of a system of N mass points, G has the classical expression

$$\mathrm{G} = \sum_1^N {}_i \frac{|m_i \mathbf{a}_i - \mathbf{F}_i(\mathbf{x}_1, \ldots, \mathbf{x}_N, \mathbf{v}_1, \ldots, \mathbf{v}_N)|^2}{m_i}, \tag{4.11}$$

with a clear meaning of the symbols.

Let us now fix an arbitrary element u in TQ, and let us introduce the following restricted pull-back of Gauss' function (4.10):

$$G_u = [(T^2\chi)^* \mathrm{G}]|_{T^2_u Q}, \tag{4.12}$$

where $T^2\chi$ is the map $T^2Q \to T^2M$, canonically induced by χ, and $T^2_u Q = (\tau^{21}_{TQ})^{-1}(u)$ is the fiber in T^2Q over u. Let us notice that, when natural local coordinates are used in T^2Q, associated with a local coordinate system (q^α) in Q, then the elements of the m-dimensional vector space $T^2_u Q$ have the expression $(\bar{q}^\alpha, \bar{\dot{q}}^\alpha, \ddot{q}^\alpha)$, where $(\bar{q}^\alpha, \bar{\dot{q}}^\alpha)$ are the (fixed) coordinates of $u \in TQ$.

The following proposition holds.

Gauss' principle: The semispray $\tilde{X}$, associated with the mechanical system $\mathscr{Q} = (Q, \tilde{K}, \tilde{\Phi})$, is characterized by the following property for any fixed $u \in TQ$:

$$G_u[\tilde{X}(u)] < G_u(z), \quad \text{for all } z \in T^2_u Q \setminus \{\tilde{X}(u)\}. \tag{4.13}$$

Proof: Let an arbitrary element $z \in T^2_u Q$ be chosen; in natural local coordinates, we have $z = (\bar{q}^\alpha, \bar{\dot{q}}^\alpha, \ddot{q}^\alpha)$, so that

$$T^2\chi(z) = (\chi^k(\bar{q}^\alpha), D_\sigma \chi^j(\bar{q}^\alpha)\, \bar{\dot{q}}^\sigma, D_\gamma D_\sigma \chi^h(\bar{q}^\alpha)\, \dot{q}^\sigma \dot{q}^\gamma + D_\sigma \chi^h(\bar{q}^\alpha) \ddot{q}^\sigma). \tag{4.14}$$

Hence, by (4.9), the local expression of $r_{T^2\chi(z)} \in (V_{T\chi(u)} TM)^0$ is

$$r_{T^2\chi(z)} = (\chi^k(\bar{q}^\alpha), D_\sigma \chi^j(\bar{q}^\alpha)\, \bar{\dot{q}}^\sigma, p_i(\ddot{q}^\alpha), 0), \tag{4.15}$$

where, for brevity we have set

$$p_i(\ddot{q}^\alpha) = \bar{K}_{ih}\{D_\gamma D_\sigma \chi^h(\bar{q}^\alpha)\, \bar{\dot{q}}^\sigma \bar{\dot{q}}^\gamma + D_\sigma \chi^h(\bar{q}^\alpha) \ddot{q}^\sigma\} + \bar{K}'_{ki} D_\sigma \chi^k(\bar{q}^\alpha)\, \bar{\dot{q}}^\sigma - \bar{K}_i - \bar{\Phi}_i, \tag{4.16}$$

with [see (4.4)]

$$\bar{K}_{is} = K_{is}(\chi^k(\bar{q}^\alpha), D_\sigma \chi^j(\bar{q}^\alpha)\, \bar{\dot{q}}^\sigma), \tag{4.17}$$

$$\bar{K}'_{ki} = \frac{\partial^2 K}{\partial x^k \partial \dot{x}^i}(\chi^h(\bar{q}^\alpha), D_\gamma \chi^j(\bar{q}^\alpha)\, \dot{q}^\gamma), \tag{4.18}$$

$$\bar{K}_i = \frac{\partial K}{\partial x^i}(\chi^k(\bar{q}^\alpha), D_\sigma \chi^j(\bar{q}^\alpha)\, \dot{q}^\sigma), \tag{4.19}$$

$$\bar{\Phi}_i = \Phi_i(\chi^k, D_\sigma \chi^j(\bar{q}^\alpha)\, \bar{\dot{q}}^\sigma). \tag{4.20}$$

Then, from (4.12), (4.10), (4.4)–(4.7), and (4.15), we explicitly get, for G_u,

$$G_u(\bar{q}^\alpha, \bar{\dot{q}}^\alpha, \ddot{q}^\alpha) = \tfrac{1}{2}\bar{K}^{ih} p_i(\ddot{q}^\alpha) p_h(\ddot{q}^\alpha). \quad (4.21)$$

From this we see that G_u is trivially differentiable, and, by (4.16) and (4.21), that it is indeed a quadratic polynomial in $\ddot{q}^\alpha$. Hence we only need to prove that its differential dG_u vanishes at $\widetilde{X}(u)$, and only there. It is sufficient to show this locally. A direct calculation yields the local expression for the differential dG_u at the point $z = (\bar{q}^\alpha, \bar{\dot{q}}^\alpha, \ddot{q}^\alpha) \in T_u^2 Q$, as follows:

$$dG_u(z) = \bar{K}^{ij} \bar{K}_{jh} p_i(\ddot{q}^\alpha) D_\alpha \chi^h(\bar{q}^\alpha) d\ddot{q}^\alpha \quad (4.22a)$$

$$= p_i(\ddot{q}^\alpha) D_\sigma \chi^i(\bar{q}^\alpha) d\ddot{q}^\sigma, \quad (4.22b)$$

where (4.22b) holds because of (4.5).

Now, let $\widetilde{X} = \dot{q}^\alpha(\partial/\partial q^\alpha) + \tilde{a}^\beta(q^\sigma, \dot{q}^\sigma)(\partial/\partial \dot{q}^\beta)$ be the local expression for the dynamical system $\widetilde{X}$: $TQ \to T^2Q$, associated with $\mathscr{D} = (Q, \widetilde{K}, \widetilde{\Phi})$, so that $\widetilde{X}(u) = (\bar{q}^\alpha, \bar{\dot{q}}^\alpha, \tilde{a}^\beta(\bar{q}^\alpha, \overline{\dot{q}^\sigma}))$.

The local expressions (3.10)–(3.12) of D'Alembert's principle show that, for the deviation semibasic form $r_{T^2\chi(z)}$ [see (4.16)],

$$p_i(\ddot{q}^\alpha) D_\sigma \chi^i(\bar{q}^\alpha) = 0 \quad (4.23)$$

holds, if and only if $\ddot{q}^\beta = \tilde{a}^\beta(\bar{q}^\alpha, \overline{\dot{q}^\sigma})$, i.e., if and only if $z = (\bar{q}^\alpha, \bar{\dot{q}}^\alpha, \tilde{a}^\beta(\bar{q}^\alpha, \overline{\dot{q}^\sigma})) = \widetilde{X}(u)$. Hence, by (4.22b) and (4.23), $dG_u(z) = 0$, if and only if $z = \widetilde{X}(u)$.

To conclude that $\widetilde{X}(u)$ is indeed a minimizing point for G_u, we recall that, as already noticed above, G_u is a quadratic polynomial in $\ddot{q}^\alpha$, whose leading term is

$$\bar{K}_{ih} D_\alpha \chi^i(\bar{q}^\sigma) D_\beta \chi^h(\bar{q}^\sigma) \ddot{q}^\alpha \ddot{q}^\beta, \quad (4.24)$$

which is positive-definite because of condition (b) following (4.5) and because χ is an imbedding.

Remark: Gauss' principle can be stated, in an equivalent way, directly in terms of Gauss' function G above, rather than in terms of its pull-back G. In this case, the wording turns out to be closer to the classical statements of the principle that can be found in the literature. Nevertheless, the statement itself becomes more involved and we omit the details here.

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