SIMON STEVIN, A Quarterly Journal of Pure and Applied Mathematics
Volume 67 (1993), Number 3-4 (September-December 1993)

QUATERNIONIC REGULAR MAPS AND $\bar{\partial}$ - TYPE OPERATORS

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communicated by E. Vesentini

INTRODUCTION.

The interest in a theory of functions over the non commutative field of quaternions, **H**, began with Hamilton [Ha], Tait [Ta], and Joly [J] at the end of the nineteenth Century. The attempt to adapt the definition of a holomorphic function of a complex variable to the case of a quaternionic variable faces the fact that the functions of a quaternionic variable which have quaternionic derivatives are just the constants and linear functions (Proposition 1.1) and that the functions which can be represented by quaternionic power series are all the analytic functions in four real variables. Fueter [F], in 1938, by means of an analogue of the Cauchy-Riemann equations, identifies a special class of quaternionic functions that he calls "regular" and which play the role of holomorphic functions. More precisely, a function $f: \Omega \to \mathbf{H}$ (Ω open subset of **H**) is regular if it satisfies the equation

$$(0.1) \qquad \frac{\partial f}{\partial x_0} + i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2} + k\frac{\partial f}{\partial x_3} = 0$$

on all of Ω, the variable being $q = x_0 + ix_1 + jx_2 + kx_3$. Equation (0.1) is called the *Cauchy-Fueter* equation. Sudbery [Su], in 1978, defines the module $\varepsilon^p_{\mathbf{H}}$ of **H**-valued differential p-forms. The quaternionic exterior calculus enables him to establish rigorously the fundamental results of this theory, such as the analogue of the Cauchy's Theorem, the Cauchy's integral formula, the Laurent expansion.

A brief account concerning the theory of regular functions is given in Section 1. As Sudbery [Su] points out, there are some difficulties in this theory, such as, for example, the fact that the identity map of **H** is not regular (Proposition 1.5) and that pointwise multiplication and composition of maps do not maintain regularity (Proposition 1.6). Nevertheless, investigations in this field are carried on: it turns out, for instance, that the Cauchy-Fueter equations are strictly related to the Dirac operator [Gi]. Recently, Pertici ([P1], [P2]) studied the theory of regular maps in several quaternionic variables and obtained the analogue of the Bochner-Martinelli's formulas, Plemelj's formulas as well as Hartog's type Theorems. A generalization of the definition of a holomorphic function in the case of Clifford Algebras can be found in [B-D-S].

In this paper, we study a generalization of the Cauchy-Fueter equation (0.1) to the case of a **R**-Fréchet differentiable map f of a **H**-Banach space X into a **H**-Banach space Y. A quite interesting fact, which appears clearly in this setting, is that the generalization requires that X is a *right* **H**-Banach space and Y is a *left* **H**-Banach space. Only in this situation, in fact, the natural extension (2.10) of equation (1.6) acquires its full sense (see Lemmas 2.4 and 2.5) and leads to a definition of regularity for a map $f: X \to Y$ among **H**-spaces (Definition 2.6). Consequently, a definition of a regular map in the case of **H**-Banach spaces is given. Here, the study of a Cauchy-Fueter equation leads to a natural decomposition of the Fréchet-differential of a map in four summands, one of which is the Cauchy-Fueter operator. In the one-dimensional case, a decomposition over the module of the **H**-valued differential forms is induced.

With this new point of view, a holomorphic map g from a right **H**-space X into a left **H**-space Y (with respect to the induced right and left **C**-structures) turns out to be a regular map (Corollary 3.2).

This result seems quite natural and supports the choice of Definition 2.6: in fact, if X and Y are both left (or right) **H**-spaces, then a holomorphic map is not a regular map in general (Lemma 3.3). This fact is encountered, for example, when $X = Y = \mathbf{H}$ is considered as a left space ([Su], [P1], [P2]).
Still in Section 3, a natural decomposition of the **R**-differential of a map $f: X \to Y$ (X is a right **H**-space and Y is a left **H**-space) is investigated.
The result is that the differential df_x decomposes uniquely in four summands

(0.2) $$df_x = \overline{\partial}_0 f_x + \overline{\partial}_1 f_x + \overline{\partial}_2 f_x + \overline{\partial}_3 f_x$$

in such a way that $\overline{\partial}_0 f_x$ is **H**-antilinear, i.e. $\overline{\partial}_0 f_x(q) = \overline{q}$, and that $\overline{\partial}_s f_x$ is "ϑ_μ-antilinear", for $s = 1,2,3$ (see (2.14)).
The **H**-antilinear part of the differential of f, i.e. $\overline{\partial}_0 f_x$, corresponds to the generalized Cauchy-Fueter equation and therefore $\overline{\partial}_0$ is defined to be the generalized Cauchy-Fueter operator.

In Section 4, we use the above decomposition of the differential to define four quaternionic differential 1-forms, $d\overline{q}_0$, $d\overline{q}_1$, $d\overline{q}_2$, $d\overline{q}_3$ (see (4.3)) which span all of $\varepsilon^1_{\mathbf{H}}$. This leads to the construction of four homomorphisms (uniquely determined, Theorem 4.1)

(0.3) $$\overline{\partial}_s : \varepsilon^n_{\mathbf{H}} \to \varepsilon^{n+1}_{\mathbf{H}} \qquad (n = 0,1; s = 0,1,2,3)$$

preseving the canonical graded submodules associated to the spaces $\varepsilon^{p,q,r,s}_{\mathbf{H}}$, (4.5), of all differential (p,q,r,s)-forms and such that

(0.4) $$d = \overline{\partial}_0 + \overline{\partial}_1 + \overline{\partial}_2 + \overline{\partial}_3$$

and that

(0.5) $$\overline{\partial}_s \circ \overline{\partial}_s = 0 \qquad (s = 0,1,2,3).$$

Unfortunately, a similar result does not hold for $n \geq 2$ (Proposition 4.2). This reduces the analogy of $\overline{\partial}_s$ $(s = 0,1,2,3)$ with the classical complex operators ∂ and $\overline{\partial}$.

The authors are aware of the fact that the definition of a regular map given in this paper, as well as the results obtained, are not "intrinsic", all depending on the choice of an embedding of **C** into **H**. On the other hand, this same situation is true for the definitions and results given by Fueter [F1], [F2], Sudbery [Su], Pertici [P1], [P2] and others in the same stream of research.

The non commutative field of quaternions **H** will be endowed with the standard basis $i_0 = 1$, $i_1 = i$, $i_2 = j$, $i_3 = k$ and the bilinear product defined by $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. [G]
Each quaternion q will be expressed in the form $q = x_0 + x_1 i + x_2 j + x_3 k$ $(x_s \in \mathbf{R}; s = 0,1,2,3)$. A normed left (right) module over **H**, complete with respect to the distance induced by the norm, is called a left (right) **H**-Banach space, or simply a left (right) **H**-space. If $\Omega \subset \mathbf{H}$ is an open set, then $C^\infty_{\mathbf{H}}(\Omega)$ will denote the **H**-module of all the C^∞ functions (simply called differentiable) from Ω into **H**.

1. Preliminaries: Regular functions of a quaternionic variable.

Let $\Omega \subset \mathbf{H}$ be an open set, and let

$$f: \Omega \to \mathbf{H}$$

be a function. A first attemp to extend the definition of a holomorphic function in the case of quaternions is requiring the function f to be **H**-Fréchet differentiable at every point $q \in \Omega$, i.e. requiring that $\forall q \in \Omega$ there exists a **H**-linear function $\phi: \mathbf{H} \to \mathbf{H}$ such that

$$f(q+h) - f(q) = \phi(h) + o(\|h\|).$$

It turns out that this request is too strong:

Proposition 1.1. [Su] *Let $\Omega \subset \mathbf{H}$ be open and connected. If $f: \Omega \to \mathbf{H}$ is **H**-Fréchet differentiable, then f is an affine function on Ω.*

A second attemp is to consider the class of functions which can be expanded as quaternionic power series at any point $q \in \Omega$. That is to require that, for example, in a neighborhood of $0 \in \Omega$, f is a sum of terms of the type $a_0 q a_1 q \cdots a_{n-1} q a_n$, with q the quaternionic variable and $a_0, \cdots, a_n \in \mathbf{H}$ the coefficients. For $q = x_0 + ix_1 + jx_2 + kx_3$, it turns out that x_0, x_1, x_2, x_3 are polynomials in q. For example $x_0 = \frac{1}{4}(q - iqi - jqj - kqk)$. Therefore the space of $\mathbf{H}$-analytic functions coincides with the space of real analytic functions from $\mathbf{R}^4$ to $\mathbf{R}^4$.

In the case of a complex function $f: \Omega \to \mathbf{C}$ one has that the fact of being holomorphic can be expressed in one of the two equivalent ways:

i) $\overline{\partial} f = 0$ in Ω;

ii) $d(f dz) = 0$ in Ω.

Equation ii) leads (via the Stokes's Theorem) to the Cauchy's representation Formula.

A quaternion-valued p-form ϕ in $\Omega \subset \mathbf{H}$ is defined by:

$$\phi = \phi_0 + i\phi_1 + j\phi_2 + k\phi_3$$

where the ϕ_i are real-valued p-forms with C^∞ coefficients.

A p-form ϕ can be expressed as

$$\phi = \sum_{0 \leq i_1 < \cdots < i_p \leq 3} a_{i_0 \cdots i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p}$$

where the $a_{i_0 \cdots i_p}$ are quaternionic valued C^∞ functions.

Let $\varepsilon^p_{\mathbf{H}}(\Omega)$ be the $\mathbf{H}$ left module of the quaternion-valued p-forms. Notice that, by definition, $\varepsilon^0_{\mathbf{H}}(\Omega)$ coincides with the space $C^\infty_{\mathbf{H}}(\Omega)$ of all differentiable quaternion-valued functions on Ω, and that $\varepsilon^p_{\mathbf{H}}(\Omega)$ is actually a left-$C^\infty_{\mathbf{H}}(\Omega)$ module.

We define an exterior product $\wedge$ in a natural way

$$\wedge: \varepsilon^p_{\mathbf{H}}(\Omega) \times \varepsilon^q_{\mathbf{H}}(\Omega) \to \varepsilon^{p+q}_{\mathbf{H}}(\Omega)$$

and a differential:

$$d: \varepsilon^p_{\mathbf{H}}(\Omega) \to \varepsilon^{p+1}_{\mathbf{H}}(\Omega)$$

such that

(1.1) $$d^2 = 0$$

(1.2) $$d(\omega^p \wedge \omega^q) = d\omega^p \wedge \omega^q + (-1)^p \omega^p \wedge d\omega^q \quad \text{for} \quad \omega^p \in \varepsilon^p_{\mathbf{H}}(\Omega) \quad \text{and} \quad \omega^q \in \varepsilon^q_{\mathbf{H}}(\Omega).$$

A quaternion-valued p-form can be regarded as a mapping from $\mathbf{H}$ to the space of alternating $\mathbf{R}$-multilinear maps from $\mathbf{H} \times \cdots \times \mathbf{H}$ (p times) to $\mathbf{H}$.
Notice that $\varepsilon^p_{\mathbf{H}}(\Omega)$ can be defined, in a completely analogous way, as a right module (instead of a left-module) over $C^\infty_{\mathbf{H}}(\Omega)$. In this case formula (1.2) becomes:

$$d(\omega^p \wedge \omega^q) = \omega^p \wedge d\omega^q + (-1)^q d\omega^p \wedge \omega^q. \tag{1.2$'$}$$

Since the differential of the identity function is $dq = dx_0 + idx_1 + jdx_2 + kdx_3$, we have that

$$dq \wedge dq \neq 0.$$

If θ is the canonical volume form of $\mathbf{R}^4$, define Dq as the 3-form which satisfies

$$< h_1, Dq(h_2, h_3, h_4) >= \theta(h_1, h_2, h_3, h_4)$$

for all $h_1, h_2, h_3, h_4 \in \mathbf{H}$, where $<,>$ stands for the usual scalar product of $\mathbf{R}^4$.
Geometrically, $Dq(a,b,c)$ is a quaternion which is perpendicular to a, b, c and has magnitude equal to the volume of the 3-dimensional parallepiped whose edges are a, b, c. If f is a differentiable function from an open subset of $\mathbf{H}$ into $\mathbf{H}$ then, if we set

$$\frac{\partial f}{\partial \bar{q}} = \frac{\partial f}{\partial x_0} + i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2} + k\frac{\partial f}{\partial x_3} \tag{1.3}$$

$$\frac{f\partial}{\partial \bar{q}} = \frac{\partial f}{\partial x_0} + \frac{\partial f}{\partial x_1}i + \frac{\partial f}{\partial x_2}j + \frac{\partial f}{\partial x_3}k \tag{1.4}$$

we obtain

$$d(fDq) = \frac{f\partial}{\partial \bar{q}}\theta, \qquad d(Dqf) = \frac{\partial f}{\partial \bar{q}}\theta. \tag{1.5}$$

The preceding considerations lead to the following

Definition 1.2. *Let $\Omega \subset \mathbf{H}$ be an open set and $f: \Omega \to \mathbf{H}$ be a $\mathbf{R}$-differentiable function in Ω. The function f is left (resp. right)-regular in Ω if*

$$\frac{\partial f}{\partial \bar{q}} = \frac{\partial f}{\partial x_0} + i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2} + k\frac{\partial f}{\partial x_3} = 0 \quad \text{in} \quad \Omega;$$

$$\text{(resp.} \quad \frac{f\partial}{\partial \bar{q}} = \frac{\partial f}{\partial x_0} + \frac{\partial f}{\partial x_1}i + \frac{\partial f}{\partial x_2}j + \frac{\partial f}{\partial x_3}k = 0) \quad \text{in} \quad \Omega.$$

(f *is left (resp. right)-anti regular if*

$$\frac{\partial f}{\partial q} = \frac{\partial f}{\partial x_0} - i\frac{\partial f}{\partial x_1} - j\frac{\partial f}{\partial x_2} - k\frac{\partial f}{\partial x_3} = 0 \quad \text{in} \quad \Omega;$$

$$\text{(resp.} \quad \frac{f\partial}{\partial q} = \frac{\partial f}{\partial x_0} - \frac{\partial f}{\partial x_1}i - \frac{\partial f}{\partial x_2}j - \frac{\partial f}{\partial x_3}k = 0) \quad \text{in} \quad \Omega.)$$

Clearly, the theory of left-regular functions will be entirely equivalent to the theory of right-regular functions. For the sake of simplicity, we will only consider left-regular functions, which we will call simply *regular*. The operator $\overline{\partial}$ is called the Cauchy-Fueter operator.

The main property of the regular functions is the fact that the Cauchy-Fueter representation formula holds.

Proposition 1.3. [Su] *If* $f:\Omega \to \mathbf{H}$ *is left-regular in* Ω, *then for* $q_0 \in \Omega$ *and* $r > 0$ *such that* $\operatorname{dist}(q_0, \partial\Omega) > r$

$$f(q_0) = \frac{1}{2\pi^2} \int_{\partial B(q_0,r)} \frac{\overline{q - q_0}}{|q - q_0|^4} Dq f(q). \tag{1.7}$$

Here $G(q) = \frac{\overline{q}}{|q|^4} = \frac{q^{-1}}{|q|^2}$ is called the *Cauchy-Fueter kernel*, and corresponds to a fundamental solution of the Cauchy-Fueter operator.

Regular functions are harmonic (since $\frac{1}{4}\Delta = \partial\overline{\partial} = \overline{\partial}\partial$), therefore one can easily generalize to the space of regular functions the Lionville theorem, the identity principle, the maximum modulus principle. The Weirstrass theorem and the Morera theorem hold. Instead the open map theorem does not hold.

Proposition 1.4. [P2] *Let* $f:\Omega \subset \mathbf{H} \cong \mathbf{C} + j\mathbf{C} \to \mathbf{C}$ *be an* $\mathbf{R}$*-differentiable function. Then* f *is regular if and only if* f *is holomorphic.*

If f and $g:\Omega \to \mathbf{H}$ are two differentiable functions, the following equalities hold:

$$\frac{\partial(f+g)}{\partial \overline{q}} = \frac{\partial f}{\partial \overline{q}} + \frac{\partial g}{\partial \overline{q}} \tag{1.8}$$

$$\frac{\partial(fg)}{\partial \overline{q}} = \frac{\partial f}{\partial \overline{q}} g + \sum_{\lambda=0}^{3} i_\lambda f \frac{\partial g}{\partial x_\lambda}. \tag{1.9}$$

Hence, if a is a quaternion then

$$\frac{\partial(ga)}{\partial \overline{q}} = \frac{\partial g}{\partial \overline{q}} a \tag{1.10}$$

and if f is a real valued function then

$$\frac{\partial(fg)}{\partial\overline{q}} = \frac{\partial f}{\partial\overline{q}}g + f\frac{\partial g}{\partial\overline{q}}. \tag{1.11}$$

Let $\mathcal{R}(\Omega)$ be the set of all regular functions in Ω. It is clear, from (1.8) and (1.10) that $\mathcal{R}(\Omega)$ is a **H**-right module. We deduce from (1.9), that $\mathcal{R}(\Omega)$ is not an algebra.

Moreover, non trivial affine functions are not regular. Precisely:

Proposition 1.5. [P2] *The only* **H**-*linear regular function is the zero function.*

In particular, the identity map of **H** is not regular. Remark that in the case of complex functions, one chooses to call holomorphic the functions f which satisfy the equation $\overline{\partial} f = 0$, instead of those, having similar properties, which satisfy $\partial f = 0$.
The choice of one of the two operators corresponds to the choice of one point in the unit imaginary sphere of **C**. The situation in the case of **H** is similar, but there are many more possibilities: for every choice of a point in the imaginary sphere S^3 one can define a class of "regular functions" for the corresponding operator. Of course these classes are strictly related. In particular one of the choices corresponds to the operator

$$\overline{\partial'} = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} - j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}$$

for which the class of "regular functions" contains the identity.
The following proposition shows that the composition of regular functions is not regular, in general.

Proposition 1.6. [P1] *Let Ω be an open subset of* **H**, *and $f: \Omega \to$* **H** *be a differentiable function in Ω. The following conditions are equivalent:*

i) $f(q) = aq + b$, for some $a, b \in$ **H**;

ii) for any regular function g defined in a neighborhood of $f(\Omega)$, $g \circ f$ is regular in Ω.

If, as usual, $i_0 = 1$, $i_1 = i$, $i_2 = j$, $i_3 = k$, then, analogously to what happens for holomorphic functions, regular functions can be expanded is series with respect to the homogeneous regular polynomials

$$P_\nu(q) = \frac{1}{m!}\sum_{\lambda_1,\cdots,\lambda_m} (x_0 i_{\lambda_1} - x_{\lambda_1})\cdots(x_0 i_{\lambda_m} - x_{\lambda_m}),$$

where $\nu = [m_1, m_2, m_3]$, $m! = m_1!m_2!m_3!$ and the summation is made over all terms for which the number of λ_j's equal to h ($h = 1,2,3$) is m_h. In fact if we set

$$\sigma_m = \{\nu = [m_1, m_2, m_3]: m_i \geq 0, m_1 + m_2 + m_3 = m\}$$

$$\partial_\nu = \frac{\partial^{m_1+m_2+m_3}}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3}}$$

we have

Proposition 1.7. [Su] *Let $f: \Omega \to \mathbf{H}$ be a regular function in an open set $\Omega \subset \mathbf{H}$. Let $p \in \Omega$ and let $0 < \delta < \mathrm{dist}(p, \partial\Omega)$. Then if $|p - q| < \delta$, one has*

$$f(q) = \sum_{m=0}^{\infty} \sum_{\nu \in \sigma_m} P_\nu(q-p) a_\nu \tag{1.12}$$

where

$$a_\nu = (-1)^m \partial_\nu f(p) = \frac{1}{2\pi^2} \int_{|q-p|=\delta} (\partial_\nu G(q-p)) Dq f(q)$$

the convergence of (1.12) *being normal.*

Examples of regular functions can be obtained as follows: a) consider an entire function

$$f: \mathbf{C} \to \mathbf{C}$$

$$z \mapsto \sum_{n=0}^{\infty} a_n z^n.$$

Set

$$f(x,y) = u(x,y) + iv(x,y)$$

for $z = x + iy$, with u, v real valued functions.
Define

$$\tilde{f}(q) = u(Re q, |Im q|) + \frac{Im q}{|Im q|} v(Re q, |Im q|).$$

It follows that $\tilde{f} + \tilde{g} = \widetilde{f+g}$, $\tilde{f} \cdot \tilde{g} = \widetilde{f \cdot g}$ for any other entire function $g: \mathbf{C} \to \mathbf{C}$. In particular $\widetilde{z^n} = q^n$ and

$$\tilde{f}(q) = \sum_{n=0}^{\infty} a_n q^n.$$

Proposition 1.8. [Su] *If f is as above, then $\Delta \tilde{f}$ is a regular function.*

b) Let Ω_1, Ω_2, Ω_3 be open sets in $\mathbf{C}$ and let f, g, h be holomorphic functions

$$\begin{aligned} f\colon \Omega_1 &\to \mathbf{C} \quad , \quad f = f_0 + i f_1 \\ g\colon \Omega_2 &\to \mathbf{C} \quad , \quad g = g_0 + i g_2 \\ h\colon \Omega_3 &\to \mathbf{C} \quad , \quad h = h_0 + i h_3 \end{aligned}$$

where $f_0, f_1, g_0, g_2, h_0, h_3$ are real-valued functions. Let Ω be the open subset of $\mathbf{H}$ defined by

$$\Omega = \{q = x_0 + ix_1 + jx_2 + kx_3 \in \mathbf{H}\colon x_0 + ix_1 \in \Omega_1, x_0 + ix_2 \in \Omega_2, x_0 + ix_3 \in \Omega_3\}.$$

Then the following result holds [Ma].

Proposition 1.9. *The function $F\colon \Omega \to \mathbf{H}$ defined by*

$$F(x_0 + ix_1 + jx_2 + kx_3) =$$
$$= f_0(x_0, x_1) + g_0(x_0, x_2) + h_0(x_0, x_3) + i f_1(x_0, x_1) + j g_2(x_0, x_2) + k h_3(x_0, x_3)$$

is regular in Ω.

2. The generalized Cauchy-Fueter Equation.

Let X and Y be complex Banach spaces. If $\Lambda\colon X \to Y$ is any $\mathbf{R}$-linear map, then Λ can be uniquely decomposed as the sum of a $\mathbf{C}$-linear map L and a $\mathbf{C}$-antilinear map A,

$$\Lambda = L + A \tag{2.1}$$

where

$$L(x) = \frac{1}{2}\{\Lambda(x) - i\Lambda(ix)\} \tag{2.2}$$

and

$$A(x) = \frac{1}{2}\{\Lambda(x) + i\Lambda(ix)\}. \tag{2.3}$$

The equation

$$A(x) = \frac{1}{2}\{\Lambda(x) + i\Lambda(ix)\} = 0 \quad (\forall x \in X) \tag{2.4}$$

states that $\Lambda = L$, i.e. that Λ is a **C**-linear map.
A direct computation shows that the set of all $x \in X$ satisfaying equation (2.4) is a complex subspace of X. Therefore

Lemma 2.1. *If equation* (2.4) *holds for all the elements of a complex basis (if any) of* X*, then it holds for all the elements of* X*.*

Now, if $U \subset X$ is an open set and if $f: U \to Y$ is a Fréchet **R**-differentiable map at $x_0 \in U$, then the differential $df(x_0)$ is a **R**-linear map from X into Y. Equation (2.4) becomes the "generalized Cauchy-Riemann" equation

$$df(x_0)(h) + i df(x_0)(ih) = 0 \quad (\forall h \in X) \tag{2.5}$$

which states that $df(x_0)$ is **C**-linear, (see also [Mu], [F-V]).
When $df(x_0)$ is **C**-linear, then f is said to be **C**-*differentiable at* x_0. If f is **C**-differentiable at every point of U, then f is said to be **C**-*differentiable* (on U). A classical result states that ([Mu, Theorem 13.16])

Theorem 2.2. *Let* X *and* Y *be Banach spaces, and let* $U \subset X$ *be an open set. Then a map* $f: U \to Y$ *is* **C**-*differentiable if, and only if, it is holomorphic.*

Therefore

Corollary 2.3. *Equation* (2.5) *(the generalized Cauchy-Riemann equation) holds at every point* $x_0 \in U$ *if, and only if,* f *is holomorphic on* U*.*

In the case $X = Y = \mathbf{C}$, since $\{1\}$ is a complex basis of **C**, Lemma 2.1 implies that equation (2.5) is equivalent to the classical Cauchy-Riemann equation

$$\frac{\partial f}{\partial x}(x_0) + i\frac{\partial f}{\partial y}(x_0) = 0. \tag{2.6}$$

In general, equation (2.1) gives the classical decomposition of the real differential of f at x_0 as

$$df(x_0) = \partial f(x_0) + \overline{\partial} f(x_0) \tag{2.7}$$

when $df(x_0) = \Lambda$, $\partial f(x_0) = L$ and $\overline{\partial} f(x_0) = A$. Condition

$$\overline{\partial} f(x_0) \equiv 0 \quad \text{for all} \quad x_0 \in U \tag{2.8}$$

is equivalent to the fact that f is holomorphic on U.

In the case in which X and Y are **H**-Banach spaces, $U \subset X$ is an open set and $f: U \to Y$ is a Fréchet **R**-differentiable map on U, the requirement that $df(x_0)$ is **H**-linear is too strong as we noticed in Proposition 1.1. The classical Cauchy-Fueter equation (1.6) can be generalized in the sense of equation (2.5) in two different ways, namely, in the case of left- regularity, i.e. for a left **H**-space Y, as

$$df(x_0)(h) + i df(x_0)(ih) + j df(x_0)(jh) + k df(x_0)(kh) = 0 \quad \forall h \in X \tag{2.9}$$

or as

$$df(x_0)(h) + i df(x_0)(hi) + j df(x_0)(hj) + k df(x_0)(hk) = 0 \quad \forall h \in X \tag{2.10}$$

being X a bilateral **H**-space.
The choice of one of the above generalizations will be the main step to obtain, in the quaternionic case, a decomposition of the differential $df(x_0)$ analogous to the decomposition obtained in (2.7) in the complex case.

Let us consider a **R**-linear map $\Lambda: X \to Y$, and define

$$S(x) = \sum_{\mu=0}^{3} i_\mu \Lambda(i_\mu x) \tag{2.11}$$

$$R(x) = \sum_{\mu=0}^{3} i_\mu \Lambda(x i_\mu) \tag{2.12}$$

The maps R and S are additive and

Lemma 2.4. *The map R is* **H**-*antilinear (i.e., for all $q \in \mathbf{H}$ and $x \in X$, $R(xq) = \overline{q} R(x)$) and* $\ker R$ *is a right* **H**-*subspace of X.*

Proof.

If $q = \sum_{\mu=0}^{3} q^\mu i_\mu$, $(q^\mu \in \mathbf{R})$, then

$$R(xq) = \sum_{\mu=0}^{3} i_\mu \sum_{\nu=0}^{3} q^\nu \Lambda(x(i_\nu i_\mu)) =$$
$$= \sum_{\nu=0}^{3} q^\nu \sum_{\mu=0}^{3} i_\mu \Lambda(x(i_\nu i_\mu)) =$$
$$= \sum_{\nu=0}^{3} q^\nu \bar{i}_\nu \sum_{\mu=0}^{3} i_\nu i_\mu \Lambda(x(i_\nu i_\mu)) =$$
$$= \sum_{\nu=0}^{3} q^\nu \bar{i}_\nu \sum_{\mu=0}^{3} i_\mu \Lambda(x i_\mu) = \bar{q} R(x).$$

As a consequence of the above equality and of the additivity of Λ, it turns out that the kernel of R, $\ker R$, is a right $\mathbf{H}$-subspace of X.

QED

It is of interest to remark that, in general, $\ker R$ is not a left $\mathbf{H}$-subspace of X and that

Lemma 2.5. *If S is as defined in* (2.11), *then* $\ker S$ *is neither a left nor a right* $\mathbf{H}$*-subspace of* X.

The proof of the above Lemma is a straightforward computation. For example,

$$S(qx) = \sum_{\nu=0}^{3} q^\nu \sum_{\mu=0}^{3} i_\mu \Lambda((i_\mu i_\nu)x) =$$
$$= \sum_{\nu=0}^{3} q^\nu \bar{i}_\nu \sum_{\mu=0}^{3} i_\nu i_\mu \Lambda((i_\nu i_\mu)x) =$$

(2.13)
$$= q^0 S(x) - q^1 i_1\{i_1\Lambda(i_1 x) + i_0\Lambda(i_0 x) - i_3\Lambda(i_3 x) - i_2\Lambda(i_2 x)\} +$$
$$- q^2 i_2\{i_2\Lambda(i_2 x) - i_3\Lambda(i_3 x) + i_0\Lambda(i_0 x) - i_1\Lambda(i_1 x)\} +$$
$$- q^3 i_3\{i_3\Lambda(i_3 x) - i_2\Lambda(i_2 x) - i_1\Lambda(i_1 x) + i_0\Lambda(i_0 x)\} =$$
$$= \sum_{\nu=0}^{3} q^\nu \bar{i}_\nu \sum_{\mu=0}^{3} \bar{\vartheta}_\nu(i_\mu)\Lambda(i_\mu x)$$

where the anti-involutions ϑ_ν, $(\nu = 0, 1, 2, 3)$ are defined by

(2.14)
$$\vartheta_\nu(q) = -i_\nu \bar{q} i_\nu \quad (\text{for} \quad \nu = 1, 2, 3; q \in \mathbf{H})$$
$$\vartheta_0(q) = \bar{q} \quad (\text{conjugation})$$

and, therefore, are such that

$$\vartheta_\nu(i_\mu) = \begin{cases} -i_\mu & \text{if} \quad \nu = \mu \\ i_\mu & \text{otherwise} \end{cases} \qquad (\nu = 1, 2, 3) \tag{2.15}$$

and that

$$\vartheta_0 = \vartheta_1 \circ \vartheta_2 \circ \vartheta_3. \tag{2.16}$$

Thus, in general, $S(x) = 0$ does not imply that $S(qx) = 0$ (nor that $S(xq) = 0$).

Now Lemmas 2.4 and 2.5 point out (see Corollary 2.3 in the complex case, and Lemma 2.1) that the right generalization of the Cauchy-Fueter equation is (2.10), when Y is a left $\mathbf{H}$-space. Thus we set the following

Definition 2.6. *Let X be a right $\mathbf{H}$-Banach space and Y be a left $\mathbf{H}$-Banach space. Let $U \subset X$ be an open set, and $f: U \to Y$ a $\mathbf{R}$-differentiable map on U. Equation* (2.10). *i.e.*

$$\sum_{\mu=0}^{3} i_\mu df(x)(hi_\mu) = 0 \quad (\forall h \in X, x \in U) \tag{2.17}$$

is called the generalized Cauchy-Fueter equation. If the map f satisfies the generalized Cauchy-Fueter equation for all $x \in U$, then f is called regular (on U).

Remark 2.7. The generalized Cauchy-Fueter equation makes full sense when Y is a left $\mathbf{H}$-space and X is a right $\mathbf{H}$-space.

As it happens in the complex case (Lemma 2.1) we have that, by Lemma 2.4, the following result holds:

Corollary 2.8. *If equation* (2.17) *holds for the elements of an* $\mathbf{H}$*-basis (if any) of X, then it holds for all the elements of X.*

In the case $X = Y = \mathbf{H}$, since $\{1\}$ is a quaternionic basis of $\mathbf{H}$, equation (2.17) becomes

$$\sum_{\mu=0}^{3} i_\mu df(x)(i_\mu) = 0 \quad (x \in U) \tag{2.18}$$

i.e. the usual Cauchy-Fueter equation

$$\frac{\partial f}{\partial x_0}(x) + i\frac{\partial f}{\partial x_1}(x) + j\frac{\partial f}{\partial x_2}(x) + k\frac{\partial f}{\partial x_3}(x) = 0. \tag{2.19}$$

3. Quaternionic decomposition of the differential of a map and the generalized Cauchy-Fueter operator.

Both the left **H**-space Y and the right **H**-space X inherit, respectively, a left and a right **C**-structure from the **R**-linear inclusion of **C** into **H** defined by $x + iy \mapsto x + i_1 y$. If $\Lambda: X \to Y$ is **R**-linear, then, it can be decomposed in a sum (see (2.1), (2.2), (2.3)),

$$\Lambda = L + A \tag{3.1}$$

where L is **C**-linear

$$L(x) = \frac{1}{2}\{\Lambda(x) - i_1\Lambda(xi_1)\} \tag{3.2}$$

and A is **C**-antilinear

$$A(x) = \frac{1}{2}\{\Lambda(x) + i_1\Lambda(xi_1)\}. \tag{3.3}$$

It is worthwhile noticing that, since X is a right space and Y is a left space, the fact of being **C**-linear for a map $\Lambda: X \to Y$ is not as natural as one may think: for example if X is the right **H**-space **H** and Y is the left **H**-space **H**, then the identity map is not a **C**-linear map.

Now, by Definition of L, A and R (see (3.2), (3.3) and (2.12)) we have that

$$\begin{aligned} R(x) &= \sum_{\mu=0}^{3} i_\mu(L+A)(xi_\mu) = L(x) + A(x) + \\ &+ i_1\{L(xi_1) + A(xi_1)\} + i_2\{L(xi_2) + A(xi_2)\} + \\ &+ i_3\{L(xi_3) + A(xi_3)\} = L(x) + A(x) + \\ &+ i_1\{i_1 L(x) - i_2 A(x)\} + i_2 L(xi_2) + i_2 A(xi_2) + \\ &+ i_3\{-L(xi_2 i_1) - A(xi_2 i_1)\} = \\ &= 2A(x) + i_2 L(xi_2) + i_2 A(xi_2) + \\ &- i_3 i_1 L(xi_2) + i_3 i_1 A(xi_2) = \\ &= 2A(x) + 2i_2 A(xi_2). \end{aligned} \tag{3.4}$$

Therefore

Lemma 3.1. *If X is a right* **H**-*space, Y is a left* **H**-*space and $\Lambda: X \to Y$ is a* **R**-*linear map, then the generalized Cauchy-Fueter equation for Λ is expressed in terms of its* **C**-*antilinear part only. Namely*

$$R(x) = 2(A(x) + i_2 A(x i_2)) \qquad (\forall x \in X). \tag{3.5}$$

Since, as we stated in (2.7), $A = \overline{\partial} f$ when $\Lambda = df$, the following result holds:

Corollary 3.2. *If X is a right* **H**-*space, Y is a left* **H**-*space, $U \subset X$ is open, and if $f: U \to Y$ is a holomorphic map, then f is a regular map on U.*

Sudbery [Su], Pertici [P2] and other authors ([B-D-S]), do not obtain the result stated in Corollary 3.2. This is due to the fact that they consider the case in which $X = Y = \mathbf{H}$ (or $\mathbf{H}^n$) are both viewed as left **H**-spaces. According to us, the definition of (left)-regularity for a map $f: U \to Y$ has to require that X is a right space and Y a left space. The reasons appear in Lemmas 2.1, 2.4, and 2.5, and in the fact that, otherwise, a holomorphic map is not regular in general, as the following lemma explains.

Lemma 3.3. *If X and Y are both left* **H**-*spaces, and if $\Lambda: X \to Y$ is a* **R**-*linear map, then*

$$S(x) = 2\{A(x) + i_2 L(i_2 x)\}.$$

Proof. We have

$$
\begin{aligned}
S(x) = \sum_{\mu=0}^{3} i_\mu \Lambda(i_\mu x) = \sum_{\mu=0}^{3} i_\mu (L+A)(i_\mu x) = \\
&= L(x) + A(x) + \\
&+ i_1 L(i_1 x) + i_1 A(i_1 x) + \\
&+ i_2 L(i_2 x) + i_2 A(i_2 x) + \\
&+ i_3 L(i_1 i_2 x) + i_3 A(i_1 i_2 x) = \\
&= L(x) + A(x) + \\
&- L(x) + A(x) + \\
&+ i_2 L(i_2 x) + i_2 A(i_2 x) + \\
&+ i_3 i_1 L(i_2 x) - i_3 i_1 A(i_2 x) = \\
&= 2A(x) + 2 i_2 L(i_2 x).
\end{aligned}
\tag{3.6}
$$

QED

If, in fact, S instead of R is chosen to generalize the Cauchy-Fueter equation, then, by the above lemma, the fact that the complex Cauchy-Riemann equation is satisfied does not imply that S is zero.

Let ϑ_μ $(\mu = 0,1,2,3)$ be the anti-involutions defined in (2.14). It is easy to prove that they commute pairwise, i.e. that

$$\vartheta_\mu \circ \vartheta_\nu = \vartheta_\nu \circ \vartheta_\mu \quad (\mu, \nu = 0,1,2,3) \tag{3.7}$$

and that

$$\frac{1}{4} \sum_{k=0}^{3} \overline{\vartheta_\mu(i_k)} \vartheta_\nu(i_k) = \begin{cases} 1 & \text{if} \quad \mu = \nu \\ 0 & \text{otherwise} \end{cases} \tag{3.8}$$

Moreover, for all $q, p \in \mathbf{H}$ and any $\mu = 1,2,3$,

$$
\begin{aligned}
\overline{\vartheta_\mu(qp)} &= -i_\mu q p i_\mu = -i_\mu q i_\mu (-i_\mu) p i_\mu = \\
&= \overline{\vartheta_\mu(q)} \;\; \overline{\vartheta_\mu(p)}
\end{aligned}
\tag{3.9}
$$

i.e. $\overline{\vartheta}_\mu$ $(\mu = 0,1,2,3)$ is an automorphism of $\mathbf{H}$.

Definition 3.4. *Let X and Y be, respectively, a right $\mathbf{H}$-space and a left $\mathbf{H}$-space. A map $\Gamma: X \to Y$ is called ϑ_μ-quaternionic antilinear (for $\mu = 0,1,2,3$,) if*

$$\Gamma(xq) = \vartheta_\mu(q)\Gamma(x) \quad (\forall x \in X, \forall q \in \mathbf{H}). \tag{3.10}$$

Notice that the natural **H**-morphisms of a right **H**-space into a left **H**-space are antilinear and that ϑ_0-quaternionic antilinearity coincides with the usual antilinearity over **H**.

Proposition 3.5. *Let $\Lambda: X \to Y$ be a* **R***-linear map from a right* **H***-space X into a left* **H***-space Y. Then Λ can be decomposed in a unique way in a sum*

$$\Lambda = \sum_{\mu=0}^{3} \Lambda_\mu \tag{3.11}$$

where, for every $\mu = 0, 1, 2, 3$, the map Λ_μ is ϑ_μ-quaternionic antilinear. Furthermore, for all $x \in X$,

$$\Lambda_\mu(x) = \frac{1}{4} \sum_{k=0}^{3} \overline{\vartheta_\mu(i_k)} \Lambda(x i_k) \qquad (\mu = 0, 1, 2, 3). \tag{3.12}$$

Proof. Suppose (3.11) holds with Λ_μ a ϑ_μ-quaternionic antilinear map. Then, by using relation (3.8) we get

$$\begin{aligned}\sum_{k=0}^{3} \overline{\vartheta_\mu(i_k)} \Lambda(x i_k) &= \sum_{k=0}^{3} \overline{\vartheta_\mu(i_k)} \sum_{\nu=0}^{3} \Lambda_\nu(x i_k) = \\ &= \sum_{\nu,k=0}^{3} \overline{\vartheta_\mu(i_k)} \vartheta_\nu(i_k) \Lambda_\nu(x) = \\ &= \sum_{\nu=0}^{3} \Big(\sum_{k=0}^{3} \overline{\vartheta_\mu(i_k)} \vartheta_\nu(i_k)\Big) \Lambda_\nu(x) = 4\Lambda_\mu(x).\end{aligned}$$

Furthermore, the maps Λ_μ $(\mu = 0, 1, 2, 3)$ defined in (3.12) is ϑ_μ-quaternionic antilinear. In fact, since $\overline{\vartheta}_\mu$ is an automorphism of **H** (see (3.9)) we have

$$\begin{aligned}\Lambda_\mu(x i_\nu) &= \frac{1}{4} \sum_{k=0}^{3} \overline{\vartheta}_\mu(i_k) \Lambda(x(i_\nu i_k)) = \\ &= \frac{1}{4} \vartheta_\mu(i_\nu) \sum_{k=0}^{3} \overline{\vartheta}_\mu(i_\nu) \overline{\vartheta}_\mu(i_k) \Lambda(x(i_\nu i_k)) = \\ &= \frac{1}{4} \vartheta_\mu(i_\nu) \sum_{k=0}^{3} \overline{\vartheta_\mu(i_\nu i_k)} \Lambda(x(i_\nu i_k)) =\end{aligned}$$

$$= \vartheta_\mu(i_\nu)\frac{1}{4}\sum_{k=0}^{3}\overline{\vartheta_\mu(i_k)}\Lambda(xi_k) = \vartheta_\mu(i_\nu)\Lambda(x).$$

QED

It is easy to verify that Λ_0 and Λ_1 are **C**-antilinear and that Λ_2, Λ_3 are **C**-linear. Moreover, if L and A are the **C**-linear and the **C**-antilinear part of Λ, respectively, (see (3.1)) then $\Lambda = L + A$ and

(3.13)
$$\begin{aligned} A &= \Lambda_0 + \Lambda_1 \\ L &= \Lambda_2 + \Lambda_3 \end{aligned}.$$

Proposition 3.5 directly implies the following

Theorem 3.6. *Let X and Y be, respectively, a right and a left* **H***-space, $U \subset X$ an open set, and $f: U \to Y$ a* **R***-differentiable map on U. Then, at every point $x_0 \in U$, the differential of f can be uniquely decomposed in a sum*

(3.14)
$$df(x_0) = \overline{\partial}_0 f(x_0) + \overline{\partial}_1 f(x_0) + \overline{\partial}_2 f(x_0) + \overline{\partial}_3 f(x_0)$$

where for every $\mu = 0, 1, 2, 3,$, $\overline{\partial}_\mu f(x_0)$ is the ϑ_μ-quaternionic antilinear map expressed by

(3.15)
$$\begin{aligned} \overline{\partial}_0 f(x_0)(h) &= \frac{1}{4}\{df(x_0)(h) + i_1 df(x_0)(hi_1) + i_2 df(x_0)(hi_2) + i_3 df(x_0)(hi_3)\} \\ \overline{\partial}_1 f(x_0)(h) &= \frac{1}{4}\{df(x_0)(h) + i_1 df(x_0)(hi_1) - i_2 df(x_0)(hi_2) - i_3 df(x_0)(hi_3)\} \\ \overline{\partial}_2 f(x_0)(h) &= \frac{1}{4}\{df(x_0)(h) - i_1 df(x_0)(hi_1) + i_2 df(x_0)(hi_2) - i_3 df(x_0)(hi_3)\} \\ \overline{\partial}_3 f(x_0)(h) &= \frac{1}{4}\{df(x_0)(h) - i_1 df(x_0)(hi_1) - i_2 df(x_0)(hi_2) + i_3 df(x_0)(hi_3)\}. \end{aligned}$$

With the same notations introduced in (2.7) we have that

(3.16)
$$df(x_0) = \partial f(x_0) + \overline{\partial} f(x_0)$$

and hence, after Proposition 3.5 and Theorem 3.6

(3.17)
$$\begin{aligned} \overline{\partial} f(x_0) &= \overline{\partial}_0 f(x_0) + \overline{\partial}_1 f(x_0) \\ \partial f(x_0) &= \overline{\partial}_2 f(x_0) + \overline{\partial}_3 f(x_0). \end{aligned}$$

The $(\vartheta_0$-)quaternionic antilinear part, $\overline{\partial}_0 f$, of the differential of f corresponds to R (see (2.12)). Precisely we have that

$$\overline{\partial}_0 f(x_0) = \frac{1}{4} R \tag{3.18}$$

when $\Lambda = df(x_0)$. Therefore

Corollary 3.7. *The map f is regular on U if, and only if, the $(\vartheta_0$-)quaternionic antilinear part $\overline{\partial}_0 f(x_0)$ of the differential $df(x_0)$ vanishes for all $x_0 \in U$.*

The differential operator $\overline{\partial}_0$ defined in (3.15) is called the *generalized Cauchy-Fueter operator.*

4. Quaternionic $\overline{\partial}$ operators and right-forms.

In this section Ω will be an open subset of the *right* $\mathbf{H}$-space $\mathbf{H}$, and

$$f: \Omega \to \mathbf{H}$$

will be a differentiable function with values in the *left* $\mathbf{H}$-space $\mathbf{H}$. If $q = x_0 + x_1 i + x_2 j + x_3 k$, then set

$$\begin{aligned}
\overline{q}_0 &= \overline{q} = x_0 - x_1 i - x_2 j - x_3 k \\
\overline{q}_1 &= -i\overline{q}i = x_0 - x_1 i + x_2 j + x_3 k \\
\overline{q}_2 &= -j\overline{q}j = x_0 + x_1 i - x_2 j + x_3 k \\
\overline{q}_3 &= -k\overline{q}k = x_0 + x_1 i + x_2 j - x_3 k.
\end{aligned}$$

By Theorem 3.6, we can decompose the differential of f at $x \in \Omega$ as

$$df(x) = \overline{\partial}_0 f(x) + \overline{\partial}_1 f(x) + \overline{\partial}_2 f(x) + \overline{\partial}_3 f(x) \tag{4.2}$$

where $\overline{\partial}_i f(x)$ $(i = 0, 1, 2, 3)$, in the case of $\mathbf{H} = X = Y$, are defined by

$$\begin{aligned}
\overline{\partial}_0 f &= \frac{\partial f}{\partial \overline{q}_0} = \frac{\partial f}{\partial \overline{q}} = \frac{1}{4}\left(\frac{\partial f}{\partial x_0} + i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2} + k\frac{\partial f}{\partial x_3}\right) \\
\overline{\partial}_1 f &= \frac{\partial f}{\partial \overline{q}_1} = \frac{1}{4}\left(\frac{\partial f}{\partial x_0} + i\frac{\partial f}{\partial x_1} - j\frac{\partial f}{\partial x_2} - k\frac{\partial f}{\partial x_3}\right) \\
\overline{\partial}_2 f &= \frac{\partial f}{\partial \overline{q}_2} = \frac{1}{4}\left(\frac{\partial f}{\partial x_0} - i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2} - k\frac{\partial f}{\partial x_3}\right) \\
\overline{\partial}_3 f &= \frac{\partial f}{\partial \overline{q}_3} = \frac{1}{4}\left(\frac{\partial f}{\partial x_0} - i\frac{\partial f}{\partial x_1} - j\frac{\partial f}{\partial x_2} + k\frac{\partial f}{\partial x_3}\right).
\end{aligned}$$

In the following we will consider $\varepsilon^p_{\mathbf{H}}(\Omega)$ as a *right*-module over $C^\infty_{\mathbf{H}}(\Omega)$ (see Section 1, and (1.2)'). With this choice the subspace of $\varepsilon^p_{\mathbf{H}}(\Omega)$ whose elements are the p-forms with regular coefficients is a right $\mathbf{H}$-module.
Let us consider, now, the following quaternion-valued 1-forms (see Section 1):

$$
\begin{aligned}
d\overline{q}_0 &= d\overline{q} = dx_0 - dx_1 i - dx_2 j - dx_3 k \\
d\overline{q}_1 &= dx_0 - dx_1 i + dx_2 j + dx_3 k \\
d\overline{q}_2 &= dx_0 + dx_1 i - dx_2 j + dx_3 k \\
d\overline{q}_3 &= dx_0 + dx_1 i + dx_2 j - dx_3 k
\end{aligned} \tag{4.3}
$$

whose linear combinations with coefficients in $C^\infty_{\mathbf{H}}(\Omega)$ span all of $\varepsilon^1_{\mathbf{H}}(\Omega)$. With the above notations, following formula (4.2), the quaternionic 1-form defined on Ω

$$
df = d\overline{q}_0 \frac{\partial f}{\partial \overline{q}_0} + d\overline{q}_1 \frac{\partial f}{\partial \overline{q}_1} + d\overline{q}_2 \frac{\partial f}{\partial \overline{q}_2} + d\overline{q}_3 \frac{\partial f}{\partial \overline{q}_3} \tag{4.4}
$$

is naturally associated to the differential of the map f.
Let p, q, r, s be non-negative integers, $n = p + q + r + s$, and let $\varepsilon^{p,q,r,s}_{\mathbf{H}}(\Omega)$ be the $C^\infty_{\mathbf{H}}(\Omega)$-submodule of $\varepsilon^n_{\mathbf{H}}(\Omega)$ spanned by the n-forms

$$
d\overline{q}_{i_1} \wedge d\overline{q}_{i_2} \wedge \cdots \wedge d\overline{q}_{i_n} \tag{4.5}
$$

where $i_j \in \{0, 1, 2, 3\}$ for $j = 1, \cdots, n$ and where the number of j's such that $i_j = 0$ is p, ... , the number of j's such that $i_j = 3$ is s.
Formula (4.4) yields the decomposition

$$
\varepsilon^1_{\mathbf{H}}(\Omega) = \varepsilon^{1,0,0,0}_{\mathbf{H}}(\Omega) \oplus \varepsilon^{0,1,0,0}_{\mathbf{H}}(\Omega) \oplus \varepsilon^{0,0,1,0}_{\mathbf{H}}(\Omega) \oplus \varepsilon^{0,0,0,1}_{\mathbf{H}}(\Omega). \tag{4.6}
$$

The $\mathbf{R}$-linear homomorphisms

$$
\begin{aligned}
\overline{\partial}_0 &: C^\infty_{\mathbf{H}}(\Omega) \longrightarrow \varepsilon^{1,0,0,0}_{\mathbf{H}}(\Omega) \\
\overline{\partial}_1 &: C^\infty_{\mathbf{H}}(\Omega) \longrightarrow \varepsilon^{0,1,0,0}_{\mathbf{H}}(\Omega) \\
\overline{\partial}_2 &: C^\infty_{\mathbf{H}}(\Omega) \longrightarrow \varepsilon^{0,0,1,0}_{\mathbf{H}}(\Omega) \\
\overline{\partial}_3 &: C^\infty_{\mathbf{H}}(\Omega) \longrightarrow \varepsilon^{0,0,0,1}_{\mathbf{H}}(\Omega)
\end{aligned} \tag{4.7}
$$

defined by

$$
\overline{\partial}_i(f) = d\overline{q}_i \frac{\partial f}{\partial \overline{q}_i} \qquad (f \in C^\infty_{\mathbf{H}}(\Omega), i = 0, 1, 2, 3) \tag{4.8}
$$

are uniquely determined by the identity

$$d = \overline{\partial}_0 + \overline{\partial}_1 + \overline{\partial}_2 + \overline{\partial}_3. \tag{4.9}$$

This result can be extended for $d: \varepsilon^1_{\mathbf{H}}(\Omega) \to \varepsilon^2_{\mathbf{H}}(\Omega)$. More precisely

Theorem 4.1. *There exist four* $\mathbf{R}$*-linear homomorphisms*

$$\overline{\partial}_i: \varepsilon^1_{\mathbf{H}}(\Omega) \to \varepsilon^2_{\mathbf{H}}(\Omega) \qquad (i = 0,1,2,3)$$

uniquely determined by the following properties:

A1) *If* p,q,r,s *are non negative integers such that* $p+q+r+s=1$*, then*

$$\begin{aligned}
&\overline{\partial}_0(\varepsilon^{p,q,r,s}_{\mathbf{H}}(\Omega)) \subseteq \varepsilon^{p+1,q,r,s}_{\mathbf{H}}(\Omega)\\
&\overline{\partial}_1(\varepsilon^{p,q,r,s}_{\mathbf{H}}(\Omega)) \subseteq \varepsilon^{p,q+1,r,s}_{\mathbf{H}}(\Omega)\\
&\overline{\partial}_2(\varepsilon^{p,q,r,s}_{\mathbf{H}}(\Omega)) \subseteq \varepsilon^{p,q,r+1,s}_{\mathbf{H}}(\Omega)\\
&\overline{\partial}_3(\varepsilon^{p,q,r,s}_{\mathbf{H}}(\Omega)) \subseteq \varepsilon^{p,q,r,s+1}_{\mathbf{H}}(\Omega)
\end{aligned} \tag{4.10}$$

A2) $d = \overline{\partial}_0 + \overline{\partial}_1 + \overline{\partial}_2 + \overline{\partial}_3$

A3) $\overline{\partial}_i \circ \overline{\partial}_i = 0 \quad (i = 0,1,2,3)$.

Furthermore, for $i,j = 0,1,2,3$*, we have*

$$\overline{\partial}_i(d\overline{q}_j f) = (d\overline{q}_j \wedge d\overline{q}_i \frac{\partial f}{\partial \overline{q}_i}) - \frac{1}{2}(d\overline{q}_j \wedge d\overline{q}_i + d\overline{q}_i \wedge d\overline{q}_j)\frac{\partial f}{\partial \overline{q}_j}. \tag{4.11}$$

The proof of the above result is based on the fact that, for every fixed $i \in \{0,1,2,3\}$, the set of 2-forms

$$\{d\overline{q}_i \wedge d\overline{q}_\alpha, d\overline{q}_\alpha \wedge d\overline{q}_i : \alpha \neq i, \alpha \in \{0,1,2,3\}\} \tag{4.12}$$

is a $C^\infty_{\mathbf{H}}(\Omega)$-basis of the right-module $\varepsilon^2_{\mathbf{H}}(\Omega)$, and on the identities

$$d\overline{q}_i \wedge d\overline{q}_i = -\frac{1}{2}\sum_{\alpha \neq i}(d\overline{q}_\alpha \wedge d\overline{q}_i + d\overline{q}_i \wedge d\overline{q}_\alpha) \tag{4.13}$$

(4.14)
$$d(d\overline{q}_i) = 0.$$

Notice that the following property

A4) *For any $C^\infty_{\mathbf{H}}(\Omega)$-submodule E of $\varepsilon^n_{\mathbf{H}}(\Omega)$, the subspace $\overline{\partial}_i(E)$ of $\varepsilon^{n+1}_{\mathbf{H}}(\Omega)$ is contained in $d(E)$, for all $i = 0, 1, 2, 3$.*

is fulfilled in the cases $n = 0$ and $n = 1$. It seems to us that a reasonable definition of $\overline{\partial}_0, \ldots, \overline{\partial}_3$ should always involve property A4). This leads to the following

Proposition 4.2. *There do not exist four $\mathbf{R}$-linear homomorphisms $\overline{\partial}_0, \overline{\partial}_1, \overline{\partial}_2, \overline{\partial}_3$ from $\varepsilon^2_{\mathbf{H}}(\Omega)$ into $\varepsilon^3_{\mathbf{H}}(\Omega)$, in such a way that properties* A1), A2), A3) *and* A4) *are satisfied (for $p+q+r+s=2$).*

The proof is by contradiction and is based on several technical Lemmas, which can be stated after having assumed the existence of $\overline{\partial}_0, \overline{\partial}_1, \overline{\partial}_2, \overline{\partial}_3$ satisfying A1), A2), A3), A4).

Lemma 4.3. a) *The three forms $d\overline{q}_0 \wedge d\overline{q}_1 \wedge d\overline{q}_2$, $d\overline{q}_0 \wedge d\overline{q}_2 \wedge d\overline{q}_1$, $d\overline{q}_1 \wedge d\overline{q}_0 \wedge d\overline{q}_2$ are a basis of the module $\varepsilon^{1,1,1,0}_{\mathbf{H}}(\Omega)$.*

b) *The three forms $d\overline{q}_1 \wedge d\overline{q}_2 \wedge d\overline{q}_3$, $d\overline{q}_1 \wedge d\overline{q}_3 \wedge d\overline{q}_2$, $d\overline{q}_2 \wedge d\overline{q}_1 \wedge d\overline{q}_3$ are a basis of the module $\varepsilon^{0,1,1,1}_{\mathbf{H}}(\Omega)$.*

c) *The four forms $d\overline{q}_0 \wedge d\overline{q}_1 \wedge d\overline{q}_2$, $d\overline{q}_0 \wedge d\overline{q}_2 \wedge d\overline{q}_1$, $d\overline{q}_1 \wedge d\overline{q}_0 \wedge d\overline{q}_2$, $d\overline{q}_1 \wedge d\overline{q}_3 \wedge d\overline{q}_2$ are a basis of the module $\varepsilon^2_{\mathbf{H}}(\Omega)$.*

d) *The two forms $d\overline{q}_0 \wedge d\overline{q}_1 \wedge d\overline{q}_2$, $d\overline{q}_0 \wedge d\overline{q}_2 \wedge d\overline{q}_1 + d\overline{q}_1 \wedge d\overline{q}_3 \wedge d\overline{q}_2$ are a basis of the module $\varepsilon^{1,2,0,0}_{\mathbf{H}}(\Omega)$.*

For $\alpha_1, \cdots, \alpha_h$ elements of $\varepsilon^n_{\mathbf{H}}(\Omega)$, $(n = 2, 3)$, let us denote by $< \alpha_1, \cdots, \alpha_h >$ the $C^\infty_{\mathbf{H}}(\Omega)$-module spanned by $\alpha_1, \cdots, \alpha_h$.

Lemma 4.4. *The following facts hold:*

a)
$$d(< d\overline{q}_0 \wedge d\overline{q}_1 >) \subset < d\overline{q}_0 \wedge d\overline{q}_1 \wedge d\overline{q}_2, d\overline{q}_0 \wedge d\overline{q}_2 \wedge d\overline{q}_1 - d\overline{q}_1 \wedge d\overline{q}_0 \wedge d\overline{q}_2 + d\overline{q}_1 \wedge d\overline{q}_3 \wedge d\overline{q}_2 > .$$

b) $\quad d(< d\overline{q}_0 \wedge d\overline{q}_1 >) \not\subset < d\overline{q}_0 \wedge d\overline{q}_1 \wedge d\overline{q}_0 >.$

c) $$d(< d\overline{q}_1 \wedge d\overline{q}_0 >) \subset < d\overline{q}_1 \wedge d\overline{q}_0 \wedge d\overline{q}_2, d\overline{q}_0 \wedge d\overline{q}_1 \wedge d\overline{q}_2 - d\overline{q}_1 \wedge d\overline{q}_3 \wedge d\overline{q}_2 + \\ - d\overline{q}_0 \wedge d\overline{q}_2 \wedge d\overline{q}_1 > .$$

As a consequence of Lemmas 4.3 and 4.4 we have

$$\varepsilon_{\mathbf{H}}^{1,2,0,0}(\Omega) \cap \varepsilon_{\mathbf{H}}^{0,1,1,1}(\Omega) \cap d(< d\overline{q}_0 \wedge d\overline{q}_1 >) = < d\overline{q}_0 \wedge d\overline{q}_1 \wedge d\overline{q}_2 > \tag{4.15}$$

and

$$d(< d\overline{q}_1 \wedge d\overline{q}_0 >) \cap < d\overline{q}_0 \wedge d\overline{q}_1 \wedge d\overline{q}_2 > = \{0\}. \tag{4.16}$$

Furthermore

$$\varepsilon_{\mathbf{H}}^{2,1,0,0}(\Omega) = < d\overline{q}_1 \wedge d\overline{q}_0 \wedge d\overline{q}_2, d\overline{q}_0 \wedge d\overline{q}_1 \wedge d\overline{q}_2 + d\overline{q}_0 \wedge d\overline{q}_2 \wedge d\overline{q}_1 + \\ - d\overline{q}_1 \wedge d\overline{q}_3 \wedge d\overline{q}_2 > \tag{4.17}$$

and hence

$$\varepsilon_{\mathbf{H}}^{2,1,0,0}(\Omega) \cap d(< d\overline{q}_0 \wedge d\overline{q}_1 >) = < d\overline{q}_0 \wedge d\overline{q}_1 \wedge d\overline{q}_0 > . \tag{4.18}$$

The proof of Proposition 4.2 proceeds as follows.
Formula (4.11) and A3) imply that

$$\overline{\partial}_1(d\overline{q}_0 \wedge d\overline{q}_1 \frac{\partial f}{\partial \overline{q}_1}) = \frac{1}{2}\overline{\partial}_1[(d\overline{q}_0 \wedge d\overline{q}_1 + d\overline{q}_1 \wedge d\overline{q}_0)\frac{\partial f}{\partial \overline{q}_0}] \tag{4.19}$$

for all $f \in C^{\infty}_{\mathbf{H}}(\Omega)$.
Furthemore, the following identity holds

$$d\overline{q}_0 \wedge d\overline{q}_1 + d\overline{q}_1 \wedge d\overline{q}_0 = -(d\overline{q}_2 \wedge d\overline{q}_3 + d\overline{q}_3 \wedge d\overline{q}_2). \tag{4.20}$$

As a consequence, we have

$$\overline{\partial}_1(d\overline{q}_0 \wedge d\overline{q}_1 \frac{\partial f}{\partial \overline{q}_1}) \in \varepsilon_{\mathbf{H}}^{1,2,0,0}(\Omega) \cap \varepsilon_{\mathbf{H}}^{0,1,1,1}(\Omega) \qquad (\forall f \in C^{\infty}_{\mathbf{H}}(\Omega)) \tag{4.21}$$

and now A4) implies that

$$\overline{\partial}_1(d\overline{q}_0 \wedge d\overline{q}_1 \frac{\partial f}{\partial \overline{q}_1}) \in d(< d\overline{q}_0 \wedge d\overline{q}_1 >). \qquad (\forall f \in C^{\infty}_{\mathbf{H}}(\Omega)) \tag{4.22}$$

Hence (4.15) yields

$$\overline{\partial}_1(d\overline{q}_0 \wedge d\overline{q}_1 \frac{\partial f}{\partial \overline{q}_1}) \in < d\overline{q}_0 \wedge d\overline{q}_1 \wedge d\overline{q}_2 > \tag{4.23}$$

for all $f \in C^\infty_{\mathbf{H}}(\Omega)$.
Since for every $g \in C^\infty_{\mathbf{H}}(\Omega)$ there exists $f \in C^\infty_{\mathbf{H}}(\Omega)$ such that $\frac{\partial g}{\partial \overline{q}_1} = f$ (see [P1], [P2]) it turns out that,

$$\overline{\partial}_1(d\overline{q}_0 \wedge d\overline{q}_1 f) \in < d\overline{q}_0 \wedge d\overline{q}_1 \wedge d\overline{q}_2 > \qquad (\forall f \in C^\infty_{\mathbf{H}}(\Omega)). \tag{4.24}$$

From (4.19) it follows that

$$\overline{\partial}_1(d\overline{q}_1 \wedge d\overline{q}_0 \frac{\partial f}{\partial \overline{q}_0}) \in \overline{\partial}_1(< d\overline{q}_0 \wedge d\overline{q}_1 >) \subset < d\overline{q}_0 \wedge d\overline{q}_1 \wedge d\overline{q}_2 > \tag{4.25}$$

for every $f \in C^\infty_{\mathbf{H}}(\Omega)$. Now A4) and (4.16) imply that

$$\overline{\partial}_1(d\overline{q}_1 \wedge d\overline{q}_0 \frac{\partial f}{\partial \overline{q}_0}) \in d(< d\overline{q}_1 \wedge d\overline{q}_0 >) \cap < d\overline{q}_0 \wedge d\overline{q}_1 \wedge d\overline{q}_2 > = \{0\}. \tag{4.26}$$

It follows that

$$\overline{\partial}_1(d\overline{q}_1 \wedge d\overline{q}_0 f) = 0. \tag{4.27}$$

Now, by definition of d, we have

$$\begin{aligned} & d(d\overline{q}_0 \wedge d\overline{q}_1 f) = \\ & = d\overline{q}_0 \wedge d\overline{q}_1 \wedge d\overline{q}_0 \frac{\partial f}{\partial \overline{q}_0} + d\overline{q}_0 \wedge d\overline{q}_1 \wedge d\overline{q}_1 \frac{\partial f}{\partial \overline{q}_1} + \\ & + d\overline{q}_0 \wedge d\overline{q}_1 \wedge d\overline{q}_2 \frac{\partial f}{\partial \overline{q}_2} + d\overline{q}_0 \wedge d\overline{q}_1 \wedge d\overline{q}_3 \frac{\partial f}{\partial \overline{q}_3}. \end{aligned} \tag{4.28}$$

By applying (4.27) to (4.19) we obtain

$$\overline{\partial}_1(d\overline{q}_0 \wedge d\overline{q}_1 \frac{\partial f}{\partial \overline{q}_0}) = \frac{1}{2}\overline{\partial}_1(d\overline{q}_0 \wedge d\overline{q}_1 \frac{\partial f}{\partial \overline{q}_0}) \tag{4.29}$$

for all $f \in C^\infty_{\mathbf{H}}(\Omega)$. Then (4.28) yields

$$\overline{\partial}_1(d\overline{q}_0 \wedge d\overline{q}_1 \frac{\partial f}{\partial \overline{q}_1}) \in < d\overline{q}_0 \wedge d\overline{q}_1 \wedge d\overline{q}_0 > \tag{4.30}$$

and therefore (same argument leading from (4.23) to (4.24))

$$\overline{\partial}_1(d\overline{q}_0 \wedge d\overline{q}_1 f) \in < d\overline{q}_0 \wedge d\overline{q}_1 \wedge d\overline{q}_0 > . \tag{4.31}$$

In a similar way we obtain that

(4.32) $$\overline{\partial}_1(d\overline{q}_0 \wedge d\overline{q}_1 f) \in < d\overline{q}_0 \wedge d\overline{q}_1 \wedge d\overline{q}_1 > .$$

Since $< d\overline{q}_0 \wedge d\overline{q}_1 \wedge d\overline{q}_0 > \cap < d\overline{q}_0 \wedge d\overline{q}_1 \wedge d\overline{q}_1 >= 0$ then

(4.33) $$\overline{\partial}_1(d\overline{q}_0 \wedge d\overline{q}_1 f) = 0$$

for all $f \in C^{\infty}_{\mathbf{H}}(\Omega)$.
Similarly, we get

(4.34) $$\overline{\partial}_1(d\overline{q}_\alpha \wedge d\overline{q}_1 f) = \overline{\partial}_1(d\overline{q}_1 \wedge d\overline{q}_\alpha) = 0$$

for all $f \in C^{\infty}_{\mathbf{H}}(\Omega)$ and $\alpha = 0, 1, 2, 3$. Therefore (see (4.12)), $\overline{\partial}_1$ is identically zero. Similarly it turns out that $\overline{\partial}_2, \overline{\partial}_3$ are identically zero, and, by A2), that

(4.35) $$d = \overline{\partial}_0.$$

Then

(4.36) $$d(< d\overline{q}_0 \wedge d\overline{q}_1 >) \subset \overline{\partial}_0(< d\overline{q}_0 \wedge d\overline{q}_1 >) \subset \varepsilon^{2,1,0,0}_{\mathbf{H}}(\Omega).$$

Now, formula (4.18) implies that

(4.37) $$d(< d\overline{q}_0 \wedge d\overline{q}_1 >) \subset \varepsilon^{2,1,0,0}_{\mathbf{H}}(\Omega) \cap d(< d\overline{q}_0 \wedge d\overline{q}_1 >) =< d\overline{q}_0 \wedge d\overline{q}_1 \wedge d\overline{q}_0 >$$

and this contradicts Lemma 4.4, b).

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(received May 1992)

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This research has been supported by a grant of the Italian MURST, 40%