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GENERAL TOPOLOGY

Kuratowski's Index of a Decomposable Set

by

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Summary. We prove that Kuratowski's index alpha of a decomposable set is the integral of the diameter of the associated multifunction.

1. Introduction. In this paper we consider a decomposable subset S of $\mathscr{L}^1(I)$, or the set of integrable selections of a multifunction. We prove that Kuratowski's index α of the set S is the integral of the diameter of the associated multifunction. In particular, the index α of the decomposable hull of two functions f and g is $\|f-g\|_1$. This result expresses in a precise way the difference between decomposable hulls and convex hulls. From a further corollary it follows that any decomposable set S has subsets whose α equals $\lambda\alpha(S)$ for any prescribed λ in $[0, 1]$. This property is common to convex sets.

2. Notations and preliminary results. In this paper we consider a measure space $(I, \mathscr{M}, \mu)$, where $\mathscr{M}$ is a σ-algebra of subsets of I and μ — a positive measure; E is a separable Banach space with norm $|\ldots|$. By $\mathscr{L}^1(I)$ we mean the space of Bochner integrable functions with values in E and by $\|h\|_1$ — the integral $\int_I |h|\,d\mu$.

DEFINITION 1. A nonempty subset $S \subset \mathscr{L}^1(I)$ is called *decomposable* if f and g in S and A in $\mathscr{M}$ imply $f1_A + g1_{I\setminus A} \in S$.

DEFINITION 2. Let f and g in $\mathscr{L}^1(I)$. The decomposable hull of f and g, denoted by $S(f, g)$, is the set of functions $f1_A + g1_{I\setminus A}$ as A ranges in $\mathscr{M}$.

PROPOSITION 1. *The closure of a decomposable set is decomposable.*

For further properties of decomposable sets we refer to [5].

DEFINITION 3. Let $F\colon I \to 2^E$ be a multivalued function. F is called *measurable* if for each closed set C in E, $F^{-1}(C) \in \mathscr{M}$. The set of integrable selections is denoted by S_F.

The relationship between decomposable sets and selections of a multivalued function is clarified by the following statements (see for example [4]):

PROPOSITION 2. *Let S be a nonempty closed subset of $\mathscr{L}^1(I)$. Then there exists a measurable multifunction $F\colon I \to 2^E$, $F(w)$ nonempty and closed for every w, such that $S = S_F$, if and only if S is decomposable. Moreover, when S is bounded, there exists $h \in \mathscr{L}^1(I)$ such that, for every w in I, $|F(w)| \leqslant h(w)$, i.e. is F integrably bounded.*

PROPOSITION 3. *Let S be a closed, bounded, decomposable set in $\mathscr{L}^1(I)$, and $F\colon I \to 2^E$ be a measurable multifunction such that $S = S_F$. Then:*

(a) *the function $w \to diam\ (F(w))$ is measurable;*

(b) $\int_I diam\ (F(w))d\mu = sup\{\|f-g\|_1 \colon f,\ g \in S\} = diam(S)$.

Proof. Let $(g_n)_{n\in N}$ be such that, for every w, $F(w) = \mathrm{cl}\{g_n(w)\}_{n\in N}$ ([4], lemma 1.1).

(a) For w in I and x in E set $\varphi(w, x) = \sup\{|x-a|\colon a \in F(w)\}$, i.e. $\varphi(w, x) = \sup\{|x-g_n(w)|\colon n \in N\}$. We observe that φ is a measurable function in $I \times E$. In fact, for x fixed in E, the function $w \to |x-g_n(w)|$ is measurable, and so is $w \to \sup\{|x-g_n(w)|\colon n \in N\} = \varphi(w, x)$; for w fixed in I, the function $x \to \varphi(w, x)$ is Lipschitz. The claim follows by (Theorem 6.1 of [3]). Furthermore, diam $(F(w)) = \sup\{\varphi(w, x)\colon x \in F(w)\}$, and by ([4] Lemma 2.1), the function $w \to \mathrm{diam}\ (F(w))$ is measurable.

(b) Set I_φ to be: $I_\varphi(f) = \int_I \varphi(w, f(w))d\mu = \int_I \sup\{|g_n(w)-f(w)|\colon n \in N\}\, d\mu$ and let h in $\mathscr{L}^1(I)$ be such that, for every w in I, $|F(w)| \leqslant h(w)$. Then I_φ is defined for every f in S. From ([4] Theorem 2.2) it follows that $\sup\{I_\varphi(f)\colon f \in S\} = \int_I \sup\{\varphi(w, x)\colon x \in F(w)\}\, d\mu$, i.e. $\sup\{\int_1 \varphi(w, f(w))d\mu\colon f \in S\} = \int_I \mathrm{diam}(F(w))d\mu$.

It is obvious that $\sup\{\|f-g\|_1\colon f,\ g \in S\} \leqslant \int_1 \mathrm{diam}(F(w))d\mu$. Hence, we have to show that $\sup\{\|f-g\|_1\colon f,\ g \in S\} \geqslant \sup\{\int_I \varphi(w, f(w))d\mu\colon f \in S\}$. Fix $\varepsilon > 0$, and let f^* in $\mathscr{L}^1(I)$ be such that $\sup\{\int_I \varphi(w, f(w))d\mu\colon f \in S\} \leqslant \int_I \varphi(w, f^*(w))d\mu + \varepsilon/3$. Let ϱ in $\mathscr{L}^1(I)$ be such that for every w in I, $\varrho(w) \neq 0$ and $\int_I \varrho(w)d\mu \leqslant \varepsilon/3$.

Set A_i in $\mathscr{M}$ to be $A_i = \{w \in I\colon |g_i(w)-f^*(w)| \geqslant \sup\{|x-f^*(w)|\colon x \in F(w)\} - \varrho(w)\}$, i.e.

$$A_i = \{w \in I\colon \varphi(w, f^*(w)) - |g_i(w)-f^*(w)| \leqslant \varrho(w)\}, \quad \text{and}$$

$$E_1 = A_1;\ E_{N+1} = A_{N+1} \setminus \left(\bigcup_1^N A_i\right).$$

The family $(E_m)_{m\in N}$ is a countable partition of I. Let N be such that $\int_I h 1_{\cup\{E_i: i\geq N+1\}}\, d\mu \leq \varepsilon/6$. Set $E_0 = \cup\{E_i\colon i \geq N+1\}$, and let g^* to be $g^* = f^* 1_{E_0} + \sum_1^N g_i 1_{E_i}$. Then

$$\int_I |g^*-f^*|\,d\mu = \int_{E_0} |g^*-f^*|\,d\mu + \int_{\cup_1^N E_i} |g^*-f^*|\,d\mu = \int_{\cup_1^N E_i} |g^*-f^*|\,d\mu.$$

It is clear that

$$\begin{aligned}\int_{\cup_1^N E_i} |g^*-f^*|\,d\mu &\geq \int_{\cup_1^N E_i} \big(\varphi(w,f^*(w))-\varrho(w)\big)\,d\mu\\ &\geq \int_I \varphi(w,f^*(w))\,d\mu - \int_I \varrho(w)\,d\mu - 2\int_{E_0} h\,d\mu\\ &\geq \int_I \varphi(w,f^*(w))\,d\mu - \varepsilon/3 - 2\varepsilon/6\\ &\geq \sup\Big\{\int_I \varphi(w,f(w))\,d\mu\colon f\in S\Big\} - \varepsilon/3-\varepsilon/3-\varepsilon/3\\ &= \int_I \operatorname{diam}(F(w))\,d\mu - \varepsilon.\end{aligned}$$

Since $\operatorname{diam}(S) \geq \|f^*-g^*\|_1$, we have $\operatorname{diam}(S) \geq \int_I \operatorname{diam}(F(w))\,d\mu - \varepsilon$. This proves (b). □

We shall need the following corollary to the well-known Liapunov's theorem on the range of a nonatomic vector measure (see for example [2]).

PROPOSITION 4. *In $(I, \mathcal{M}, \mu)$ let be nonatomic and let $f_1, \ldots, f_N$ be in $\mathcal{L}^1(I)$. Then there exists a family $(A(\alpha))_{\alpha\in[0,1]}$, $A(\alpha)\in\mathcal{M}$, with the properties:*
(a) *$A(0) = \emptyset$, $A(1) = I$ and $A(\alpha)\subset A(\beta)$ when $\alpha\leq\beta$;*
(b) *for every α: $\int_{A(\alpha)} |f_i|\,d\mu = \alpha\int_I |f_i|\,d\mu$, for every i in $\{1, \ldots, N\}$.*

We also recall the definition to *Kuratowski's index* α:

DEFINITION 4. Let X be a metric space, $S\subset X$ be bounded. We set

$$\alpha(S) = \inf\Big\{\varepsilon > 0\colon S = \bigcup_1^N S_i \quad \text{and } \operatorname{diam}(S_i)\leq\varepsilon\Big\}.$$

In the following by S^n we will denote the set defined by $S^n = \{x\in \mathbf{R}^{n+1}\colon \sum_i |x_i| = 1\}$. About S^n we have the following result of Lusternik-Schnirelman-Borsuk.

THEOREM 1 [1]. *In any covering $M_1, \ldots, M_{n+1}$ of S^n by $(n+1)$ closed sets, at least one set M_i must contain a pair of antipodal points.*

3. Main results. Our purpose is to prove the following:

THEOREM 2. *Let S be a bounded decomposable subset of $\mathscr{L}^1(I)$. Then*

$$\alpha(S) = diam(S).$$

Proof. It is clear that $\alpha(S) \leqslant \operatorname{diam}(S)$. To prove the opposite inequality, fix $\varepsilon > 0$. Let f and g in S be such that $\|f-g\|_1 \geqslant \operatorname{diam}(S)-\varepsilon$. Since $S(f, g)$ is contained in S, it is enough to prove that $\alpha(S(f, g)) \geqslant \|f-g\|_1$.

(i) Consider the map

$$T: S(f, g) \to S(f, g) \text{ to be } T(f1_A+g1_{I\setminus A}) = f1_{I\setminus A}+g1_A.$$

To prove that T is well defined, we need to show that whenever $f1_A+g1_{I\setminus A} = f1_B+g1_{I\setminus B}$ for some A and B in $\mathscr{M}$, then $f1_{I\setminus A}+g1_A = f1_{I\setminus B}+g1_B$.

Set C to be $\{x \in I: f(x) = g(x)\}$. Remark that, for x in C, the last equality holds. Hence consider $x \notin C$. Then either $x \in A$ or $x \notin A$. In the first case, when $x \notin B$, we have $f(x) = g(x)$, which is a contradiction. So x is in B, and the claim follows. Analogously for the second case. As it is evident, the map T has the following property:

(1) $$\text{for every } h = f1_A+g1_{I\setminus A} \quad \|h-T(h)\|_1 = \|f-g\|_1.$$

(ii) We wish to define a sequence of maps φ_n from S^n to $S(f, g)$ satisfying:

(a) $\varphi_n(-x) = T(\varphi_n(x))$;

(b) φ_n is continuous.

For this purpose consider a family $(A(\alpha))_{\alpha\in[0,1]}$ in $\mathscr{M}$ (existing by Proposition 4) such that

$$A(0) = \emptyset;\ A(1) = I; \quad A(\alpha) \subset A(\beta)$$

if $\alpha \leqslant \beta$ and for every

$$\alpha \in [0, 1]: \int_{A(\alpha)} |f|\,d\mu = \alpha \int_I |f|\,d\mu; \quad \int_{A(\alpha)} |g|\,d\mu = \alpha \int_I |g|\,d\mu.$$

For $j \in \{0, 1, \ldots, n\}$ and $z \in S^n$, set $p_j(z) = |z_0|+\ldots+|z_j|$. Given $x = (x_0, \ldots, x_n)$ in S^n, set

$$N_0(x) = A(|x_0|) \text{ and } N_i(x) = A(p_i(x))\setminus A(p_{i-1}(x)) \quad \text{for } i \geqslant 1.$$

Remark that the $N_i(x)$ are a partition of I and that the following properties hold:

$$\text{for every } i, \int_{N_i(x)} |f|\,d\mu = |x_i| \int_I |f|\,d\mu, \quad \int_{N_i(x)} |g|\,d\mu = |x_i| \int_I |g|\,d\mu.$$

For the same x consider also the index sets $I_x^+ = \{i: x_i > 0\}$ and $I_x^- = \{i: x_i < 0\}$. Set

$$\varphi_n: S^n \to S(f, g) \text{ to be } \varphi_n(x) = \sum_{i\in I_x^+} f1_{N_i(x)} + \sum_{i\in I_x^-} g1_{N_i}(x).$$

(iii) We claim that (a) and (b) above hold.

(*a*). Since $I^+_{-x} = I^-_x$ and $I^-_{-x} = I^+_x$, it is clear that

$$\varphi_n(-x) = \sum_{i\in I_x^-} f1_{N_i(x)} + \sum_{i\in I_x^+} g1_{N_i(x)} = T\Big(\sum_{i\in I_x^+} f1_{N_i(x)} + \sum_{i\in I_x^-} g1_{N_i(x)}\Big) = T(\varphi_n(x)).$$

(*b*). Fix x in S^n and $\varepsilon > 0$. Set ϱ, ε' and δ to be

$$\varrho = \min\{|x_i|: x_i \neq 0\}, \quad \varepsilon' = \min\{\varrho, \varepsilon/(\|f\|_1 + \|g\|_1)\}, \quad \delta = \varepsilon'/2(n+1)^2.$$

Let y in S^n be such that $|x_i - y_i| \leqslant \delta$ for every i. Remark then that $x_i \neq 0$ implies $\operatorname{sgn}(y_i) = \operatorname{sgn}(x_i)$; hence

$$|x_i - y_i| = \big||x_i| - |y_i|\big| \tag{2}$$

for every i. Set

$$J^+ = \{i: x_i = 0 \text{ and } y_i > 0\}; \quad J^- = \{i: x_i = 0 \text{ and } y_i < 0\}.$$

From (2), $I_y^+ = I_x^+ \cup J^+$ and $I_y^- = I_x^- \cup J^-$. Hence

$$\varphi_n(y) - \varphi_n(x) = \sum_{i\in I_x^+} (f1_{N_i(y)} - f1_{N_i(x)}) +$$
$$+ \sum_{i\in I_x^-} (g1_{N_i(y)} - g1_{N_i(x)}) + \sum_{i\in J^+} f1_{N_i(y)} + \sum_{i\in J^-} g1_{N_i(y)}.$$

Let i be any index in $\{1, \ldots, n\}$. As a consequence of (2) and of the choice of δ we have:

$$p_{i-1}(x) - \varepsilon'/2(n+1) \leqslant p_{i-1}(y) \quad \text{and } p_i(y) \leqslant p_i(x) + \varepsilon'/2(n+1).$$

It follows that

$$N_i(y) = A(p_i(y)) \backslash A(p_{i-1}(y)) \subset A(p_i(x) + \varepsilon'/2(n+1)) \backslash A(p_{i-1}(x) - \varepsilon'/2(n+1)).$$

Hence

$$\|f1_{N_i(y)} - f1_{N_i(x)}\|_1 \leqslant 2\varepsilon' \|f\|_1/2(n+1) = \varepsilon' \|f\|_1/(n+1).$$

Similarly

$$\|g1_{N_i(y)} - g1_{N_i(x)}\|_1 \leqslant \|g\|_1 \varepsilon'/(n+1).$$

Furthermore, it is clear that, if $x_i = 0$,

$$\|f1_{N_i(y)}\|_1 = |y_i| \, \|f\|_1 \leqslant \delta \|f\|_1 = \|f\|_1 \varepsilon'/2(n+1)^2 \leqslant \|f\|_1 \varepsilon'/(n+1),$$
$$\text{and } \|g1_{N_i(y)}\|_1 \leqslant \|g\|_1 \varepsilon'/(n+1).$$

From these inequalities it follows that

$$\|\varphi_n(y) - \varphi_n(x)\|_1 \leqslant [(n+1)\|f\|_1 \varepsilon'/(n+1)] + [(n+1)\|g\|_1 \varepsilon'/(n+1)]$$
$$= \varepsilon'(\|f\|_1 + \|g\|_1) \leqslant \varepsilon,$$

hence our claim is proved. □

(iv) To conclude the proof, suppose that there exists a finite covering of $S(f, g)$ by sets (that we can assume to be closed) $M_1, \ldots, M_{n+1}$, having diameter strictly less than $\|f-g\|_1$. Consider φ_n. We have:

$$S^n = \varphi_n^{-1}(S(f, g)) = \varphi_n^{-1}(M_1) \cup \ldots \cup \varphi_n^{-1}(M_{n+1}).$$

Since φ_n is continuous, each $\varphi_n^{-1}(M_i)$ is a closed subset of S^n. By Theorem 1, there exist x and i such that

$$x \in \varphi_n^{-1}(M_i) \text{ and } -x \in \varphi_n^{-1}(M_i) \quad \text{i.e. } \varphi_n(x) \in M_i \text{ and } \varphi_n(-x) \in M_i.$$

By (a) of (iii), $\varphi_n(-x) = T(\varphi_n(x))$ so that $\varphi_n(x) \in M_i$ and $T(\varphi_n(x)) \in M_i$. Finally, from (1), $\|\varphi_n(x) - T(\varphi_n(x))\|_1 = \|f-g\|_1$, thus contradicting the assumptions on M_i. □

COROLLARY 1. *Let $F: I \to 2^E$ be a nonempty valued integrably bounded measurable multifunction with closed values. Let S_F in $\mathscr{L}^1(I)$ be the set of integrable selection of F.* Then

$$\alpha(S_F) = \int_I diam(F(w))d\mu.$$

COROLLARY 2. *Let $S \subset \mathscr{L}^1(I)$ be a bounded decomposable set. Then there exists a family $(S(\lambda))_{\lambda \in [0,1]}$ of bounded decomposable subsets of $\mathscr{L}^1(I)$ such that*

(a) $S(0) = \{f^*\}$, $S(1) = S$ *and* $S(\alpha) \subset S(\beta)$ *when* $\alpha \leqslant \beta$

(b) *for every* $\lambda \in [0, 1]$: $\alpha(S(\lambda)) = \lambda\alpha(S)$.

Proof. Let F be such that cl $S = S_F$ and let $(A(\lambda))_\lambda$, so that (a) and (b) of Proposition 4 hold with $f_i = $ diam F. Consider the set of restrictions to $A(\lambda)$, denoted by $S|_{A(\lambda)}$, a subset of $\mathscr{L}^1(A(\lambda))$. We claim that $\alpha(S|_{A(\lambda)}) = \lambda\alpha(S)$. In fact, it is easy to see that $\mathrm{cl}(S|_{A(\lambda)}) = (\mathrm{cl}(S))|_{A(\lambda)}$. Hence $\alpha(S|_{A(\lambda)}) = \alpha(\mathrm{cl}(S|_{A(\lambda)})) = \alpha(\mathrm{cl}(S)|_{A(\lambda)}) = \int_{A(\lambda)} \mathrm{diam}(F(w))d\mu = \lambda\alpha(S)$. Fix f^* in S. Let $S(\lambda)$ be $S(\lambda) = S|_{A(\lambda)} + f^* 1_{I \setminus A(\lambda)}$. Clearly, $S(\lambda)$ is a decomposable subset of S so that (a) and (b) hold. □

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А. Челлина, К. Мариконда, **Индекс Куратовского разлагаемого множества**

В настоящей статье доказывается, что индекс альфа разлагаемого множества Куратовского является интегралом диаметра ассоцированной мультифункции.