

## NONZERO-SUM STOCHASTIC DIFFERENTIAL GAMES WITH DISCONTINUOUS FEEDBACK\*

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**Abstract.** The existence of a Nash equilibrium feedback is established for a two-player nonzero-sum stochastic differential game with discontinuous feedback. This is obtained by studying a parabolic system strongly coupled by discontinuous terms.

**Key words.** nonzero-sum stochastic games, Nash point, strongly coupled parabolic systems, discontinuous feedbacks

**AMS subject classifications.** 35K50, 49K30, 49N35, 91A10, 91A15

**DOI.** 10.1137/S0363012903423715

**1. Introduction.** The aim of this paper is to study the existence of Nash equilibrium points for a two-player nonzero-sum stochastic differential game. The game is governed by a stochastic differential equation with two controls and two payoffs.

This problem can be found, for instance, in Friedman [7] and in a series of papers by Bensoussan and Frehse [2], [3], [4]. All these papers make the assumption that feedback is continuous.

We are interested in studying the problem assuming that the controls take values in compact sets. In this case one cannot expect a Nash equilibrium among continuous feedback, and the Hamiltonian functions associated with the game are nonsmooth.

We consider a simple multidimensional model problem taking two players, affine dynamics, affine payoff functions, and compact control sets.

The loss of continuity of the feedback, due to the hard constraints, leads us to consider a parabolic system strongly coupled by discontinuous terms. In fact, from the usual necessary condition satisfied by the value of the Nash equilibrium feedback in terms of the Hamilton–Jacobi theory, we reduce ourselves to studying the existence of a sufficiently regular solution to a system of nonlinear parabolic equations which contains the Heaviside graph. By this regularity result, we are able to construct Nash equilibrium feedback whose optimality is proved by using the verification approach in the sense of [2], [3], [4].

The motivation for studying games in compact control sets comes from standard nonlinear control theory; this seems a natural assumption in many applications. In particular, Nash equilibria for nonzero-sum deterministic differential games were recently studied by Olsder [12] and Cardaliaguet and Plaskacz [5].

**2. Statement of the problem.** Let  $\Omega$  be a bounded smooth domain in  $\mathbf{R}^N$ . Let  $X$  be a process which satisfies the following stochastic differential equation

$$(2.1) \quad dX(s) = f(s, X(s), u_1(s, X(s)), u_2(s, X(s)))ds + \sigma(s, X(s))dw,$$

$$(2.2) \quad \begin{aligned} X(t) &= x, \\ s &\in [t, T], \quad x \in \Omega \subset \mathbf{R}^N. \end{aligned}$$

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\*Received by the editors February 26, 2003; accepted for publication (in revised form) March 8, 2004; published electronically December 1, 2004. This work was partially supported by the Italian MURST national project “Analisi e controllo di equazioni di evoluzione deterministiche e stocastiche.”

<http://www.siam.org/journals/sicon/43-4/42371.html>

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For each  $s$ ,  $X(s)$  represents the state evolution of a system controlled by two players. The  $i$ th player acts by means of a feedback control function  $u_i : (t, T) \times \mathbf{R}^N \rightarrow U_i \subset \mathbf{R}^{k_i}$ , where  $i = 1, 2$ .

Let  $\mathcal{U}_i := \{u_i \text{ Borel-measurable applications } (t, T) \times \mathbf{R}^N \rightarrow U_i \subset \mathbf{R}^{k_i}\}$ ,  $i = 1, 2$ , be the set of the control functions  $u_i$  with values  $u_i(s, X)$  in  $U_i$ . The term  $\sigma(s, X(s))dw$  represents the “noise,” where  $w$  is an  $N$ -dimensional standard Brownian motion and  $\sigma$  is an  $N \times N$  matrix. We assume that  $\sigma$  does not depend on the control variables  $u_1, u_2$  and that  $\sigma$  and  $\sigma^{-1}$  are bounded and Lipschitz on  $X$ . The function  $f(s, X, u_1, u_2)$  is called the dynamic of the game (2.1).

We refer to [7] for the definitions about stochastic processes, stochastic differential equations and functional spaces.

A control function  $u_i \in \mathcal{U}_i$  will be called *admissible* if it is adapted to the filtration defined on the probability space.

It is possible to prove, using Girsanov’s theorem, that, under convenient assumptions on  $f$ , for all  $(u_1, u_2) \in \mathcal{U}_1 \times \mathcal{U}_2$  admissible controls, there exists a unique weak solution to the problem (2.1), (2.2) (see, for example, [4], [9], [6, Chapter 4]).

For any choice of admissible controls  $u_1, u_2$  we have the following *payoff* functions:

$$J_i(t, x, u_1, u_2) = E_{tx} \left\{ \int_t^\tau l_i(s, X(s), u_1(s, X(s)), u_2(s, X(s))) ds + g_i(T, X(T)) \right\},$$

(2.3)  $i = 1, 2,$

where  $\tau \equiv T \wedge \inf\{s \geq t, X(s) \notin \Omega\}$ ,  $E_{tx}$  is the expectation under the probability  $P_{tx}$ ,  $l_i$  and  $g_i$  are prescribed functions (the assumptions will be specified later), and  $X = X(s)$  is the unique weak solution of (2.1)–(2.2) corresponding to  $(u_1, u_2) \in \mathcal{U}_1 \times \mathcal{U}_2$  admissible controls.

Each player wants to maximize his own payoff.

DEFINITION 2.1. A pair of admissible controls  $(\bar{u}_1, \bar{u}_2) \in \mathcal{U}_1 \times \mathcal{U}_2$  is called the Nash equilibrium point of the differential game (2.1)–(2.2), with payoff (2.3), if

$$(2.4) \quad J_1(t, x, \bar{u}_1, \bar{u}_2) \geq J_1(t, x, u_1, \bar{u}_2),$$

$$(2.5) \quad J_2(t, x, \bar{u}_1, \bar{u}_2) \geq J_2(t, x, \bar{u}_1, u_2),$$

for all  $(t, x) \in (0, T) \times \Omega$  and for all  $(u_1, u_2) \in \mathcal{U}_1 \times \mathcal{U}_2$  admissible controls.

The functions

$$(2.6) \quad V_1(t, x) := J_1(t, x, \bar{u}_1, \bar{u}_2), \quad V_2(t, x) := J_2(t, x, \bar{u}_1, \bar{u}_2)$$

are a *value* of the Nash equilibrium point  $(\bar{u}_1, \bar{u}_2)$ .

We define the pre-Hamiltonians  $H_i(t, x, p, u_1, u_2) : (0, T) \times \mathbf{R}^N \times \mathbf{R}^N \times U_1 \times U_2 \rightarrow \mathbf{R}$ ,  $i = 1, 2$ :

$$(2.7) \quad \begin{aligned} H_1(t, x, p, u_1(t, x), u_2(t, x)) &:= p \cdot f(t, x, u_1(t, x), u_2(t, x)) \\ &\quad + l_1(t, x, u_1(t, x), u_2(t, x)), \\ H_2(t, x, p, u_1(t, x), u_2(t, x)) &:= p \cdot f(t, x, u_1(t, x), u_2(t, x)) \\ &\quad + l_2(t, x, u_1(t, x), u_2(t, x)). \end{aligned}$$

We set

$$a = \frac{1}{2} \sigma \sigma^*$$

( $\sigma^*$  is the transpose of  $\sigma$ ) to be the matrix with elements  $a_{h,k}$ ,  $h, k = 1, \dots, N$ .

If the value functions  $V_1, V_2 \in C^{1,2}$ , we can apply Itô's formula; changing the time variable ( $T - t \rightarrow t$ ), we get that  $V_1, V_2$  solve, in  $\Omega_T := (0, T) \times \Omega$ , the following nonlinear parabolic system coupled by the Nash equilibrium problem:

$$\begin{aligned}
 (2.8) \quad & \frac{\partial V_1(t, x)}{\partial t} - \sum_{h,k=1}^N a_{hk}(t, x) \frac{\partial^2 V_1(t, x)}{\partial x_h \partial x_k} \\
 & = \max_{\{u_1 \in U_1\}} H_1(t, x, \nabla_x V_1(t, x), u_1(t, x), \bar{u}_2(t, x)) \\
 & = H_1(t, x, \nabla_x V_1(t, x), \bar{u}_1(t, x), \bar{u}_2(t, x)),
 \end{aligned}$$

$$\begin{aligned}
 (2.9) \quad & \frac{\partial V_2(t, x)}{\partial t} - \sum_{h,k=1}^N a_{hk}(t, x) \frac{\partial^2 V_2(t, x)}{\partial x_h \partial x_k} \\
 & = \max_{\{u_2 \in U_2\}} H_2(t, x, \nabla_x V_2(t, x), \bar{u}_1(t, x), u_2(t, x)) \\
 & = H_2(t, x, \nabla_x V_2(t, x), \bar{u}_1(t, x), \bar{u}_2(t, x)),
 \end{aligned}$$

$$(2.10) \quad \bar{u}_1(t, x) \in \operatorname{argmax}_{\{u_1 \in U_1\}} H_1(t, x, \nabla_x V_1(t, x), u_1(t, x), \bar{u}_2(t, x)),$$

$$(2.11) \quad \bar{u}_2(t, x) \in \operatorname{argmax}_{\{u_2 \in U_2\}} H_2(t, x, \nabla_x V_2(t, x), \bar{u}_1(t, x), u_2(t, x)),$$

$$(2.12) \quad V_1 = g_1(t, x), \quad V_2 = g_2(t, x) \quad \text{on } \partial_p \Omega_T,$$

where  $\partial_p \Omega_T := ((0, T) \times \partial \Omega) \cup (\{t = 0\} \times \Omega)$ . (Here and in the following we write, for the sake of brevity,  $\operatorname{argmax}_{u_i \in U_i} H_i$ , which means  $\operatorname{argmax}_{u_i(t,x) \in U_i} H_i$ ).

The functions

$$\begin{aligned}
 & H_1(t, x, \nabla_x V_1(t, x), \bar{u}_1(t, x), \bar{u}_2(t, x)) \\
 & = \max_{\{u_1 \in U_1\}} H_1(t, x, \nabla_x V_1(t, x), u_1(t, x), \bar{u}_2(t, x)), \\
 & H_2(t, x, \nabla_x V_1(t, x), \bar{u}_1(t, x), \bar{u}_2(t, x)) \\
 & = \max_{\{u_2 \in U_2\}} H_2(t, x, \nabla_x V_2(t, x), \bar{u}_1(t, x), u_2(t, x))
 \end{aligned}$$

are called the Hamiltonian functions associated with the game (2.1)–(2.3).

We want to outline here the classical procedure used in Friedman's book [7] to prove the existence of a Nash equilibrium point  $\bar{u}_1, \bar{u}_2$ .

1. Suppose that, for any fixed  $p \in \mathbf{R}^N$ , there exist  $u_1^*, u_2^*$  such that

$$\begin{aligned}
 & u_1^*(t, x, p) \in \operatorname{argmax}_{\{u_1 \in U_1\}} H_1(t, x, p, u_1(t, x), u_2^*(t, x)), \\
 & u_2^*(t, x, p) \in \operatorname{argmax}_{\{u_2 \in U_2\}} H_2(t, x, p, u_1^*(t, x), u_2(t, x)) \\
 (2.13) \quad & \text{are measurable in } (t, x) \in \Omega_T \text{ and continuous in } p.
 \end{aligned}$$

2. Solve the parabolic system

$$\begin{aligned}
 (2.14) \quad & \frac{\partial V_1(t, x)}{\partial t} - \sum_{h,k=1}^N a_{hk}(t, x) \frac{\partial^2 V_1(t, x)}{\partial x_h \partial x_k} \\
 & = H_1(t, x, \nabla_x V_1, u_1^*(t, x, \nabla_x V_1), u_2^*(t, x, \nabla_x V_2)), \\
 & \frac{\partial V_2(t, x)}{\partial t} - \sum_{h,k=1}^N a_{hk}(t, x) \frac{\partial^2 V_2(t, x)}{\partial x_h \partial x_k} \\
 & = H_2(t, x, \nabla_x V_2, u_1^*(t, x, \nabla_x V_1), u_2^*(t, x, \nabla_x V_2)), \\
 & V_1 = g_1(t, x), \quad V_2 = g_2(t, x) \quad \text{on } \partial_p \Omega_T.
 \end{aligned}$$

3. Prove that the pair of functions  $(\bar{u}_1, \bar{u}_2)$  with values

$$\bar{u}_1(t, x) := u_1^*(t, x, \nabla_x V_1(t, x)),$$

$$\bar{u}_2(t, x) := u_2^*(t, x, \nabla_x V_2(t, x))$$

is a Nash equilibrium point (see Definition 2.1).

Therefore to obtain Nash equilibrium points for the associated stochastic differential game, we look for a “sufficiently regular” solution of system (2.14).

A similar procedure is used, in the elliptic case, by Bensoussan and J. Frehse [2], [3], [4] to study systems of Bellman equations.

We want to emphasize that the results of Friedman and Bensoussan and J. Frehse on the existence of classical solutions and of Nash equilibrium points are obtained under the assumption that there exist some feedback

$$u_1^* \in \operatorname{argmax}_{\{u_1 \in U_1\}} H_1(t, x, p, u_1, u_2^*), \quad u_2^* \in \operatorname{argmax}_{\{u_2 \in U_2\}} H_2(t, x, p, u_1^*, u_2)$$

that are continuous in  $p$  (see, for example, assumption (D) [7, section 17, p. 497]). If we assume that the sets  $U_i, i = 1, 2$ , are compact, the assumption on the continuity of the feedback can be too restrictive.

Weaker assumptions on the regularity of the feedback can be found in [8] and [9].

In this paper we consider a model problem with  $U_1, U_2$  compact sets in  $\mathbf{R}$ , affine dynamics of the game, and affine payoff.

Let us list the assumptions:

$$(2.15) \quad U_1 = U_2 = [0, 1],$$

$$(2.16) \quad \begin{aligned} f(x, u_1(t, x), u_2(t, x)) &: \Omega \times U_1 \times U_2 \rightarrow \mathbf{R}^N, \\ f(x, u_1(t, x), u_2(t, x)) &= f_1(x)u_1(t, x) + f_2(x)u_2(t, x), \\ f_i(x) &: \Omega \rightarrow \mathbf{R}^N, f_i(x) \in C^1(\bar{\Omega}), \quad i = 1, 2, \end{aligned}$$

$$(2.17) \quad \begin{aligned} l_i(x, u_1(t, x), u_2(t, x)) &: \Omega \times U_1 \times U_2 \rightarrow \mathbf{R}, \\ l_i(x, u_1(t, x), u_2(t, x)) &= l_i(x)u_i(t, x), \quad i = 1, 2, \\ l_i(x) &: \Omega \rightarrow \mathbf{R}^N, l_i(x) \in C^1(\bar{\Omega}), \quad i = 1, 2, \end{aligned}$$

$$(2.18) \quad \begin{aligned} g_i(t, x) &\in H^{1+\alpha}(\bar{\Omega}_T), \quad \alpha \in (0, 1), \quad i = 1, 2, \\ a_{hk}(t, x) &\in C^2(\bar{\Omega}_T), \end{aligned}$$

$$(2.19) \quad \begin{aligned} \nu|\xi|^2 &\leq \sum_{h,k=1}^N a_{hk}(t, x)\xi_h\xi_k \leq \mu|\xi|^2, \quad \nu, \mu > 0, \\ &\text{for all } (t, x) \in \Omega_T \text{ and for all } \xi \in \mathbf{R}^N. \end{aligned}$$

Taking into account the affine structure of  $f$  and  $l$  in (2.16)–(2.17), the functions  $H_1$  and  $H_2$  in (2.7) become

$$(2.20) \quad \begin{aligned} H_1(x, p, u_1(t, x), u_2(t, x)) &= (p \cdot f_1(x) + l_1(x))u_1(t, x) + p \cdot f_2(x)u_2(t, x), \\ H_2(x, p, u_1(t, x), u_2(t, x)) &= (p \cdot f_2(x) + l_2(x))u_2(t, x) + p \cdot f_1(x)u_1(t, x). \end{aligned}$$

From (2.15) and (2.20), for any fixed  $p$ , we have

$$\begin{aligned}
 \bar{u}_1(t, x) &\in \operatorname{argmax}_{\{u_1 \in U_1\}} H_1(x, p, u_1(t, x), \bar{u}_2(t, x)) \\
 &= \operatorname{argmax}_{\{u_1(t, x) \in [0, 1]\}} (p \cdot f_1(x) + l_1(x))u_1(t, x) + p \cdot f_2(x)\bar{u}_2(t, x) \\
 (2.21) \quad &= \operatorname{Heav}(p \cdot f_1(x) + l_1(x)),
 \end{aligned}$$

where  $\operatorname{Heav}(\eta)$  is the Heaviside graph defined as  $\operatorname{Heav}(\eta) = 1$  if  $\eta > 0$ ,  $\operatorname{Heav}(\eta) = 0$  if  $\eta < 0$ , and  $\operatorname{Heav}(0) = [0, 1]$ .

Analogously

$$(2.22) \quad \bar{u}_2(t, x) \in \operatorname{Heav}(p \cdot f_2(x) + l_2(x)).$$

From (2.20), (2.21), (2.22), the Hamiltonian functions assume the form

$$\begin{aligned}
 H_1(x, p, \bar{u}_1(t, x), \bar{u}_2(t, x)) &\equiv \max_{\{u_1 \in U_1\}} H_1(x, p, u_1(t, x), \bar{u}_2(t, x)) \\
 (2.23) \quad &= (p \cdot f_1(x) + l_1(x))_+ + p \cdot f_2(x)\bar{u}_2(t, x),
 \end{aligned}$$

$$\begin{aligned}
 H_2(x, p, \bar{u}_1, \bar{u}_2) &\equiv \max_{\{u_2 \in U_2\}} H_2(x, p, \bar{u}_1(t, x), u_2(t, x)) \\
 (2.24) \quad &= (p \cdot f_2(x) + l_2(x))_+ + p \cdot f_1(x)\bar{u}_1(t, x),
 \end{aligned}$$

where we denoted by  $(h)_+$  the positive part of the function  $h$ .

Taking, for any fixed  $p$ ,

$$(2.25) \quad u_1^*(t, x, p) \in \operatorname{Heav}(p \cdot f_1(x) + l_1(x)),$$

$$(2.26) \quad u_2^*(t, x, p) \in \operatorname{Heav}(p \cdot f_2(x) + l_2(x)),$$

we have that  $u_1^*, u_2^*$  do not satisfy (2.13) because they are not continuous in  $p$ . Hence, in this case, we cannot use the results of [7].

From (2.25), (2.26), the parabolic system (2.8), (2.9) becomes

$$\begin{aligned}
 \frac{\partial V_1}{\partial t} - \sum_{h,k=1}^N a_{hk} \frac{\partial^2 V_1}{\partial x_h \partial x_k} &\in \left( \nabla_x V_1 \cdot f_1(x) + l_1(x) \right) \operatorname{Heav}(\nabla_x V_1 \cdot f_1(x) + l_1(x)) \\
 (2.27) \quad &+ \nabla_x V_1 \cdot f_2(x) \operatorname{Heav}(\nabla_x V_2 \cdot f_2(x) + l_2(x)) \text{ in } \Omega_T,
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial V_2}{\partial t} - \sum_{h,k=1}^N a_{hk} \frac{\partial^2 V_2}{\partial x_h \partial x_k} &\in \left( \nabla_x V_2 \cdot f_2(x) + l_2(x) \right) \operatorname{Heav}(\nabla_x V_2 \cdot f_2(x) + l_2(x)) \\
 (2.28) \quad &+ \nabla_x V_2 \cdot f_1(x) \operatorname{Heav}(\nabla_x V_1 \cdot f_1(x) + l_1(x)) \text{ in } \Omega_T,
 \end{aligned}$$

$$(2.29) \quad V_1(t, x) = g_1(t, x), \quad V_2(t, x) = g_2(t, x) \quad \text{on } \partial_p \Omega_T.$$

This is a uniformly parabolic system strongly coupled by the Heaviside graph containing the first order derivatives of the unknown functions.

Equations (2.27) and (2.28) are to be interpreted in the following way:

$$\frac{\partial V_1}{\partial t} - \sum_{h,k=1}^N a_{hk} \frac{\partial^2 V_1}{\partial x_h \partial x_k} = \left( \nabla_x V_1 \cdot f_1 + l_1 \right) h_1(t, x) + \nabla_x V_1 \cdot f_2 h_2(t, x),$$

$$\frac{\partial V_2}{\partial t} - \sum_{h,k=1}^N a_{hk} \frac{\partial^2 V_2}{\partial x_h \partial x_k} = \left( \nabla_x V_2 \cdot f_2 + l_2 \right) h_2(t, x) + \nabla_x V_2 \cdot f_1 h_1(t, x),$$

$$h_1(t, x) \in \operatorname{Heav}(\nabla_x V_1 \cdot f_1(x) + l_1(x)), \quad h_2(t, x) \in \operatorname{Heav}(\nabla_x V_2 \cdot f_2(x) + l_2(x)).$$

Following the previous scheme, first we investigate the existence of a solution  $V_1, V_2$  of (2.27)–(2.29). Next, if we find sufficient regularity, it will be possible to prove the existence of a Nash equilibrium point.

In section 3 we provide an existence result for a solution  $V_1, V_2$  in  $H^{1+\alpha}(\overline{\Omega}_T) \cap W_q^{1,2}(\Omega_T)$  of the system (2.27)–(2.29), and in section 4 we prove the existence of a Nash equilibrium point.

**3. Existence of a solution to the parabolic system.** We give the following definition.

DEFINITION 3.1.  $(V_1, V_2)$  is a strong solution of the system (2.27)–(2.29) if

- (a)  $V_1(t, x), V_2(t, x) \in H^{1+\alpha}(\overline{\Omega}_T) \cap W_q^{1,2}(\Omega_T)$  for some  $\alpha \in (0, 1), q > N + 2$ ;
- (b) equations (2.27)–(2.28) hold almost everywhere and (2.29) holds.

THEOREM 3.2. Under assumptions (2.15)–(2.19), taking  $\text{Heav}(\eta) = 1$  if  $\eta > 0$ ,  $\text{Heav}(\eta) = 0$  if  $\eta < 0$ , and  $\text{Heav}(0) = [0, 1]$ , there exists at least a strong solution  $(V_1, V_2)$  of the parabolic system (2.27)–(2.29).

*Proof.* Let us consider the approximating problems obtained by replacing the Heaviside graph  $\text{Heav}(\eta)$  with smooth functions  $H_n$ :

$$\begin{aligned}
 &H_n(\eta) \in C^\infty(\mathbf{R}), \quad H_n(\eta) \in L_\infty(\mathbf{R}), \\
 &H_n(\eta) = 0 \text{ if } \eta \leq 0, \quad H_n(\eta) = 1 \text{ if } \eta \geq \frac{1}{n}, \\
 (3.1) \quad &H'_n \geq 0, \\
 &H_n(\eta) \rightarrow \text{Heav}(\eta) \text{ in } L_p(K), \quad p > 1, \quad K \subset \mathbf{R} \text{ is any compact of } \mathbf{R}, \\
 &H_n(\eta) \rightarrow \text{Heav}(\eta) \text{ in } C^0 \text{ outside a neighbourhood of } \eta = 0.
 \end{aligned}$$

We denote by  $V_{1n}, V_{2n}$  the solution of the problem

$$\begin{aligned}
 &\frac{\partial V_{1n}}{\partial t} - \sum_{h,k=1}^N a_{hk} \frac{\partial^2 V_{1n}}{\partial x_h \partial x_k} = \left( \nabla_x V_{1n} \cdot f_1 + l_1 \right) H_n(\nabla_x V_{1n} \cdot f_1 + l_1) \\
 (3.2) \quad &+ \nabla_x V_{1n} \cdot f_2 H_n(\nabla_x V_{2n} \cdot f_2 + l_2) \text{ in } \Omega_T,
 \end{aligned}$$

$$\begin{aligned}
 &\frac{\partial V_{2n}}{\partial t} - \sum_{h,k=1}^N a_{hk} \frac{\partial^2 V_{2n}}{\partial x_h \partial x_k} = \left( \nabla_x V_{2n} \cdot f_2 + l_2 \right) H_n(\nabla_x V_{2n} \cdot f_2 + l_2) \\
 (3.3) \quad &+ \nabla_x V_{2n} \cdot f_1 H_n(\nabla_x V_{1n} \cdot f_1 + l_1) \text{ in } \Omega_T,
 \end{aligned}$$

$$(3.4) \quad V_{1n} = g_1, \quad V_{2n} = g_2 \text{ in } \partial_p \Omega_T.$$

From [11, Theorem 7.1, p. 596] on quasi-linear parabolic systems with smooth coefficients, there exists a unique solution of problem (3.2)–(3.4),  $V_{1n}(t, x), V_{2n}(t, x) \in H^{2+\alpha}(\Omega_T)$ .

At this point, regarding the terms  $H_n(\nabla_x V_{1n} \cdot f_1 + l_1) + f_2 H_n(\nabla_x V_{2n} \cdot f_2 + l_2)$  and  $H_n(\nabla_x V_{2n} \cdot f_2 + l_2) + f_1 H_n(\nabla_x V_{1n} \cdot f_1 + l_1)$  in (3.2)–(3.3) as bounded uniformly on  $n$ , from [11, Theorem 9.1, p. 341], we find a uniform estimate in  $W_q^{1,2}$ :

$$(3.5) \quad \|V_{1n}\|_{q,\Omega_T}^{(2)} + \|V_{2n}\|_{q,\Omega_T}^{(2)} \leq C(\|H_n\|_{q,\Omega_T}, \|g_1\|_{q,\partial_p \Omega_T}^{(2-1/q)}, \|g_2\|_{q,\partial_p \Omega_T}^{(2-1/q)}) \leq C,$$

where  $C$  is independent of  $n$  and  $q > 1$ .

By means of an embedding theorem (see, for example, [11, Chapter 2, Lemma 3.3]), taking  $q > N + 2$ , we obtain

$$(3.6) \quad |V_{1n}|_{\Omega_T}^{(1+\alpha)} + |V_{2n}|_{\Omega_T}^{(1+\alpha)} \leq C, \quad \alpha = 1 - \frac{N + 2}{q},$$

where  $C$  is independent of  $n$ .

We can now extract two subsequences, which we denote again by  $V_{1n}, V_{2n}$  such that

$$(3.7) \quad \begin{aligned} V_{in} &\rightarrow V_i \text{ in } C^0(\Omega_T), \quad i = 1, 2, \\ \frac{\partial V_{in}}{\partial x_h} &\rightarrow \frac{\partial V_i}{\partial x_h} \text{ in } C^0(\Omega_T), \quad i = 1, 2, \quad h = 1, \dots, N. \end{aligned}$$

From the weak precompactness of the unit ball of  $W_q^{2,1}$ , we have

$$(3.8) \quad \begin{aligned} \frac{\partial V_{in}}{\partial t} &\rightharpoonup \frac{\partial V_i}{\partial t} \text{ weakly in } L_2(\Omega_T), \quad i = 1, 2, \\ \frac{\partial^2 V_{in}}{\partial x_h \partial x_k} &\rightharpoonup \frac{\partial^2 V_i}{\partial x_h \partial x_k} \text{ weakly in } L_2(\Omega_T), \quad i = 1, 2, \quad h, k = 1, \dots, N. \end{aligned}$$

From (3.7), (3.8)

$$(3.9) \quad V_1(t, x), V_2(t, x) \in H^{1+\alpha}(\Omega_T) \cap W_q^{1,2}(\Omega_T), \text{ with } \alpha = 1 - \frac{N+2}{q}.$$

Now we have to prove that  $V_1, V_2$  solve (2.27)–(2.28) almost everywhere in  $\Omega_T$ .

From assumptions (3.1) and (3.8), the two sequences  $H_n(\nabla_x V_{1n} \cdot f_1 + l_1), H_n(\nabla_x V_{2n} \cdot f_2 + l_2)$  are uniformly bounded in  $L_2(\Omega_T)$ , and hence we can extract two subsequences such that

$$(3.10) \quad \begin{aligned} H_n(\nabla_x V_{1n} \cdot f_1 + l_1) &\rightharpoonup h_1(t, x) \text{ weakly in } L_2(\Omega_T), \\ H_n(\nabla_x V_{2n} \cdot f_2 + l_2) &\rightharpoonup h_2(t, x) \text{ weakly in } L_2(\Omega_T). \end{aligned}$$

We now show that  $h_i(t, x) \in \text{Heav}(\nabla_x V_i \cdot f_i + l_i)$  almost everywhere  $i = 1, 2$ . To do this let us consider the following sets:

$$\mathcal{P}_i := \{(t, x) \in \Omega_T : \nabla_x V_i(t, x) \cdot f_i(x) + l_i(x) > 0\},$$

$$\mathcal{N}_i := \{(t, x) \in \Omega_T : \nabla_x V_i(t, x) \cdot f_i(x) + l_i(x) < 0\},$$

$$\mathcal{Z}_i := \{(t, x) \in \Omega_T : \nabla_x V_i(t, x) \cdot f_i(x) + l_i(x) = 0\},$$

$$i = 1, 2.$$

From (3.7), we have that, for a sufficiently large  $n$ ,  $\nabla_x V_{in}(t, x) \cdot f_i(x) + l_i(x) > 0$  for all  $(t, x) \in \mathcal{P}_i$ . Hence, from (3.1), we obtain that

$$H_n(\nabla_x V_{in}(t, x) \cdot f_i(x) + l_i(x)) \rightarrow 1 = \text{Heav}(\nabla_x V_i(t, x) \cdot f_i(x) + l_i(x))$$

for all  $(t, x) \in \mathcal{P}_i, i = 1, 2$ .

Analogously, in  $\mathcal{N}_i$ , for a sufficiently large  $n$ ,  $\nabla_x V_{in}(x, t) \cdot f_i(x) + l_i(x) < 0$ , and hence

$$H_n(\nabla_x V_{in}(x, t) \cdot f_i(x) + l_i(x)) \equiv 0 = \text{Heav}(\nabla_x V_i(x, t) \cdot f_i(x) + l_i(x))$$

for all  $(t, x) \in \mathcal{N}_i, i = 1, 2$ .

In  $\mathcal{Z}_i$ , from the assumptions on  $H_n$  (see (3.1)), we have that  $0 \leq h_i \leq 1$  almost everywhere, and hence

$$(3.11) \quad h_1(t, x) \in \text{Heav}(\nabla_x V_1(t, x) \cdot f_1(x) + l_1(x)) \text{ almost everywhere in } \Omega_T,$$

$$(3.12) \quad h_2(t, x) \in \text{Heav}(\nabla_x V_2(t, x) \cdot f_2(x) + l_2(x)) \text{ almost everywhere in } \Omega_T.$$

At this point, from (3.7), (3.8), (3.10), (3.11), (3.12), we obtain that  $V_1, V_2$  satisfy (2.27)–(2.28) almost everywhere in  $\Omega_T$  and from the regularity of the functions  $V_1, V_2$ , we have that  $V_1(t, x) = g_1(t, x), V_2(t, x) = g_2(t, x)$  for all  $(t, x) \in \partial_p \Omega_T$ .  $\square$

REMARK 3.1. *If we choose  $W(\eta) \in \text{Heav}(\eta)$ , we are not able to solve the problem*

$$(3.13) \quad \begin{aligned} \frac{\partial V_1}{\partial t} - \sum_{h,k=1}^N a_{hk} \frac{\partial^2 V_1}{\partial x_h \partial x_k} &= (\nabla_x V_1 \cdot f_1 + l_1)W(\nabla_x V_1 \cdot f_1 + l_1) \\ &+ \nabla_x V_1 \cdot f_2 W(\nabla_x V_2 \cdot f_2 + l_2), \end{aligned}$$

$$(3.14) \quad \begin{aligned} \frac{\partial V_2}{\partial t} - \sum_{h,k=1}^N a_{hk} \frac{\partial^2 V_2}{\partial x_h \partial x_k} &= (\nabla_x V_2 \cdot f_2 + l_2)W(\nabla_x V_2 \cdot f_2 + l_2) \\ &+ \nabla_x V_2 \cdot f_1 W(\nabla_x V_1 \cdot f_1 + l_1), \end{aligned}$$

because we cannot exclude that  $\text{meas}\{(t, x) \in \Omega_T : \nabla_x V_i \cdot f_i(x) + l_i(x) = 0\} > 0, i = 1, 2$ . Hence we cannot prove that

$$H_n(\nabla_x V_{in} \cdot f_i + l_i) \rightharpoonup W(\nabla_x V_i \cdot f_i + l_i), \quad i = 1, 2,$$

but only that

$$H_n(\nabla_x V_{in} \cdot f_i + l_i) \rightharpoonup h_i \in \text{Heav}(\nabla_x V_i \cdot f_i + l_i), \quad i = 1, 2.$$

**4. Existence of a Nash equilibrium point.** We now prove the following.

THEOREM 4.1. *Suppose that the assumptions of Theorem 3.2 hold. Let  $(V_1, V_2)$  be a strong solution of the parabolic system (2.27)–(2.29); then any admissible control  $(\bar{u}_1, \bar{u}_2)$  such that*

$$(4.1) \quad \bar{u}_1(t, x) \in \text{Heav}(\nabla_x V_1(t, x) \cdot f_1(x) + l_1(x)),$$

$$(4.2) \quad \bar{u}_2(t, x) \in \text{Heav}(\nabla_x V_2(t, x) \cdot f_2(x) + l_2(x))$$

is a Nash equilibrium point for the stochastic differential game (2.1)–(2.2) with payoff (2.3).

*Proof.* The existence of a strong solution  $(V_1, V_2)$  of the parabolic system (2.27)–(2.29) is stated by Theorem 3.2.

To prove that  $\bar{u}_i(t, x) \in \text{Heav}(\nabla_x V_i(t, x) \cdot f_i(x) + l_i(x)), i = 1, 2$ , are the values of a Nash equilibrium point, as in Definition 2.1, we have to show that

$$(4.3) \quad \begin{aligned} J_1(t, x, \bar{u}_1, \bar{u}_2) &\geq J_1(t, x, u_1, \bar{u}_2), \\ J_2(t, x, \bar{u}_1, \bar{u}_2) &\geq J_2(t, x, \bar{u}_1, u_2) \end{aligned}$$

for all  $(u_1, u_2) \in \mathcal{U}_1 \times \mathcal{U}_2$  admissible controls.

Let us denote

$$(4.4) \quad \begin{aligned} v_1(t, x) &:\equiv J_1(t, x, \bar{u}_1, \bar{u}_2), \\ v_2(t, x) &:\equiv J_2(t, x, \bar{u}_1, \bar{u}_2). \end{aligned}$$



Using a generalization of Itô's formula applied to functions in  $W_q^{1,2}$  (see, for example, [1, Theorem 4.1, p. 126]), we have that the couple  $(v_1, v_2)$  solves the following parabolic system (here and in the following we omit, for the sake of brevity, the dependence on the variables  $(t, x)$ ):

$$(4.5) \quad \frac{\partial v_1}{\partial t} - \sum_{h,k=1}^N a_{hk} \frac{\partial^2 v_1}{\partial x_h \partial x_k} = H_1(x, t, \nabla_x v_1, \bar{u}_1, \bar{u}_2) \\ = (\nabla_x v_1 f_1 + l_1) \bar{u}_1 + \nabla_x v_1 f_2 \bar{u}_2 \quad \text{in } \Omega_T,$$

$$(4.6) \quad \frac{\partial v_2}{\partial t} - \sum_{h,k=1}^N a_{hk} \frac{\partial^2 v_2}{\partial x_h \partial x_k} = H_2(x, t, \nabla_x v_2, \bar{u}_1, \bar{u}_2) \\ = (\nabla_x v_2 \cdot f_2 + l_2) \bar{u}_2 + \nabla_x v_2 f_1 \bar{u}_1 \quad \text{in } \Omega_T,$$

$$(4.7) \quad v_1 = g_1(t, x), \quad v_2 = g_2(t, x) \quad \text{on } \partial_p \Omega_T.$$

From (4.1), (4.2), we have

$$(4.8) \quad \frac{\partial v_1}{\partial t} - \sum_{h,k=1}^N a_{hk} \frac{\partial^2 v_1}{\partial x_h \partial x_k} \in (\nabla_x v_1 \cdot f_1 + l_1) \text{Heav}(\nabla_x V_1 \cdot f_1 + l_1) \\ + \nabla_x v_1 \cdot f_2 \text{Heav}(\nabla_x V_2 \cdot f_2 + l_2) \quad \text{in } \Omega_T,$$

$$(4.9) \quad \frac{\partial v_2}{\partial t} - \sum_{h,k=1}^N a_{hk} \frac{\partial^2 v_2}{\partial x_h \partial x_k} \in (\nabla_x v_2 \cdot f_2 + l_2) \text{Heav}(\nabla_x V_2 \cdot f_2 + l_2) \\ + \nabla_x v_2 \cdot f_1 \text{Heav}(\nabla_x V_1 \cdot f_1 + l_1) \quad \text{in } \Omega_T, \\ v_1 = g_1(t, x), \quad v_2 = g_2(t, x) \quad \text{on } \partial_p \Omega_T.$$

Let us now fix  $(u_1, u_2) \in \mathcal{U}_1 \times \mathcal{U}_2$  admissible controls and denote

$$(4.10) \quad w_1(t, x) := J_1(t, x, u_1, \bar{u}_2), \\ w_2(t, x) := J_2(t, x, \bar{u}_1, u_2).$$

The couple  $(w_1, w_2)$  solves the following parabolic system:

$$(4.11) \quad \frac{\partial w_1}{\partial t} - \sum_{h,k=1}^N a_{hk} \frac{\partial^2 w_1}{\partial x_h \partial x_k} = H_1(x, t, \nabla_x w_1, u_1, \bar{u}_2) \\ = (\nabla_x w_1 f_1 + l_1) u_1 + \nabla_x w_1 f_2 \bar{u}_2 \quad \text{in } \Omega_T,$$

$$(4.12) \quad \frac{\partial w_2}{\partial t} - \sum_{h,k=1}^N a_{hk} \frac{\partial^2 w_2}{\partial x_h \partial x_k} = H_2(x, t, \nabla_x w_2, \bar{u}_1, u_2) \\ = (\nabla_x w_2 \cdot f_2 + l_2) u_2 + \nabla_x w_2 f_1 \bar{u}_1 \quad \text{in } \Omega_T,$$

$$(4.13) \quad w_1 = g_1(t, x), \quad w_2 = g_2(t, x) \quad \text{on } \partial_p \Omega_T, \\ w_1, w_2 \in W_q^{1,2}(\Omega_T).$$

From the expressions (2.20) of  $H_1, H_2$ , taking into account (2.10), (2.11), we have that, for any  $p$  fixed,

$$(4.14) \quad (pf_1 + l_1)u_1(t, x) \leq (pf_1 + l_1)\bar{u}_1(t, x), \\ (pf_2 + l_2)u_2(t, x) \leq (pf_2 + l_2)\bar{u}_2(t, x).$$

Consider now the functions  $z_1 := v_1 - w_1$ ,  $z_2 := v_2 - w_2$ . From systems (4.5)–(4.7) and (4.11)–(4.13) we have

$$\begin{aligned}
 (4.15) \quad \frac{\partial z_1}{\partial t} - \sum_{h,k=1}^N a_{hk} \frac{\partial^2 z_1}{\partial x_h \partial x_k} &= (\nabla_x v_1 \cdot f_1 + l_1) \bar{u}_1 + \nabla_x v_1 \cdot f_2 \bar{u}_2 \\
 &\quad - (\nabla_x w_1 \cdot f_1 + l_1) u_1 - \nabla_x w_1 \cdot f_2 \bar{u}_2 \quad \text{in } \Omega_T, \\
 \frac{\partial z_2}{\partial t} - \sum_{h,k=1}^N a_{hk} \frac{\partial^2 z_2}{\partial x_h \partial x_k} &= (\nabla_x v_2 \cdot f_2 + l_2) \bar{u}_2 + \nabla_x v_2 \cdot f_1 \bar{u}_1 \\
 &\quad - (\nabla_x w_2 \cdot f_2 + l_2) u_2 - \nabla_x w_2 \cdot f_1 \bar{u}_1 \quad \text{in } \Omega_T, \\
 z_1 = z_2 = 0 &\quad \text{on } \partial_p \Omega_T.
 \end{aligned}$$

Taking into account (4.14), we obtain

$$\begin{aligned}
 (4.16) \quad \frac{\partial z_1}{\partial t} - \sum_{h,k=1}^N a_{hk} \frac{\partial^2 z_1}{\partial x_h \partial x_k} &\geq (\nabla_x v_1 \cdot f_1 + l_1) u_1 + \nabla_x v_1 \cdot f_2 \bar{u}_2 \\
 &\quad - (\nabla_x w_1 \cdot f_1 + l_1) u_1 - \nabla_x w_1 \cdot f_2 \bar{u}_2 \\
 &= \nabla_x z_1 \cdot \left( f_1 u_1 + f_2 \bar{u}_2 \right) \quad \text{in } \Omega_T,
 \end{aligned}$$

$$\begin{aligned}
 (4.17) \quad \frac{\partial z_2}{\partial t} - \sum_{h,k=1}^N a_{hk} \frac{\partial^2 z_2}{\partial x_h \partial x_k} &\geq (\nabla_x v_2 \cdot f_2 + l_2) u_2 + \nabla_x v_2 \cdot f_1 \bar{u}_1 \\
 &\quad - (\nabla_x w_2 \cdot f_2 + l_2) u_2 - \nabla_x w_2 \cdot f_1 \bar{u}_1 \\
 &= \nabla_x z_2 \cdot \left( f_2 u_2 + f_1 \bar{u}_1 \right) \quad \text{in } \Omega_T, \\
 z_1 = z_2 = 0 &\quad \text{on } \partial_p \Omega_T.
 \end{aligned}$$

Equations (4.16), (4.17) are no longer coupled and the terms  $f_1 u_1 + f_2 \bar{u}_2$ ,  $f_2 u_2 + f_1 \bar{u}_1$  are known and bounded. Hence we can apply an extension of the maximum principle to parabolic equations whose coefficients are in  $L^\infty$  (see, for example, [10, Chapter 7]), obtaining

$$(4.18) \quad z_1(t, x) \geq 0, \quad z_2(t, x) \geq 0 \quad \text{in } \Omega_T.$$

Taking into account (4.4) and (4.10), from (4.18) we obtain (4.3), i.e., the result.  $\square$

REMARK 4.1. *The results proved in section 3 and 4 hold true even if we take  $M > 2$  players and if we take the functions  $f$  and  $l$  linear and dependent explicitly on  $t$ , i.e.,*

$$\begin{aligned}
 f(t, x, u_1, u_2) &= f_1(t, x)u_1 + f_2(t, x)u_2 + f_3(t, x), \\
 l_1(t, x, u_1, u_2) &= l_1(t, x)u_1 + h_1(t, x), \\
 l_2(t, x, u_1, u_2) &= l_2(t, x)u_2 + h_2(t, x).
 \end{aligned}$$

*The only difference is the appearance in (2.27)–(2.28), as source terms, of the functions  $f_3(t, x) + h_1(t, x)$  and  $f_3(t, x) + h_2(t, x)$ , respectively.*

REMARK 4.2. In [12], Olsder studied the Nash equilibria of the following nonzero-sum deterministic differential game with open-loop bang-bang controls:

$$\dot{x} = (1 - x)u_1 - xu_2,$$

with payoff

$$J_1 = \int_t^T (c_1x - u_1) ds, \quad J_2 = \int_t^T (c_2(1 - x) - u_2) ds,$$

and controls subject to

$$0 \leq u_i(t) \leq 1.$$

Because of the hard constraints, also in this case, the optimal controls contain Heaviside functions.

REMARK 4.3. If we change the control sets taking  $U_1 = U_2 = [-1, 1]$ , as done in [5] in the deterministic case, the optimal feedback equilibria are

$$\begin{aligned} \bar{u}_1 &\in \text{sign}(p \cdot f_1 + l_1), \\ \bar{u}_2 &\in \text{sign}(p \cdot f_2 + l_2), \end{aligned}$$

where  $\text{sign}(\eta) = 1$  if  $\eta > 0$ ,  $\text{sign}(\eta) = -1$  if  $\eta < 0$ , and  $\text{sign}(0) = [-1, 1]$ .

The Hamiltonian functions take the form

$$\begin{aligned} H_1(x, p, \bar{u}_1, \bar{u}_2) &= |p \cdot f_1(x) + l_1(x)| + p \cdot f_2(x)\bar{u}_2, \\ H_2(x, p, \bar{u}_1, \bar{u}_2) &= |p \cdot f_2(x) + l_2(x)| + p \cdot f_1(x)\bar{u}_1, \end{aligned}$$

and also in this case our method can be applied.

**Acknowledgments.** The author would like to thank Martino Bardi, who stated the problem, for his helpful comments and suggestions. The author thanks the referees for their useful remarks.

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