Acta Math. Hungar.
79 (3) (1998), 191–206.

A NOTE ON PRIMES AND GOLDBACH NUMBERS IN SHORT INTERVALS

A. LANGUASCO* (Genova)

Abstract. Let $J(N,H)$ be the Selberg integral and $E(x,T)$ the error term in Kaczorowski–Perelli's weighted form of the classical explicit formula. We prove that the estimate $J(N,H)=o(H^2N)$ is connected with an appropriate estimate of $\int_N^{2N}\left|E(x,T)\right|^2dx$, uniformly for H and T in some ranges. Moreover, assuming a suitable bound for $\int_N^{2N}\left|E(x,T)\right|^2dx$, we also obtain, for all sufficiently large N and $H\gg(\log N)^{11/2}$, that every interval $[N,N+H]$ contains $\gg H$ Goldbach numbers.

1. Introduction

In 1993 Kaczorowski–Perelli [10] showed that an estimate of the form

$$J(N,H)=o(H^2N) \quad \text{for} \quad N^\varepsilon \leqq H \leqq N^{1-\varepsilon}, \tag{1}$$

where $0<\varepsilon<1$ and

$$J(N,H)=\int_N^{2N}\big(\psi(x+H)-\psi(x)-H\big)^2\,dx$$

is Selberg's integral, follows from an estimate of the form

$$\int_N^{2N}\left|E(x,T)\right|^2dx=o\left(\frac{N^3}{T^2L}\right) \quad \text{for} \quad N^\varepsilon\leqq T\leqq N^{1-\varepsilon}.$$

Here $L=\log N$ and $E(x,T)$ denotes the remainder term in Kaczorowski–Perelli's [9] weighted form of the classical explicit formula

$$\psi(x)=x-\sum_{|\gamma|\leqq T}w\left(\frac{|\gamma|}{T}\right)\frac{x^\rho}{\rho}+E(x,T),$$

* Research supported by CNR Grant No. 201.01.121.

0236-5294/98/$5.00 © 1998 Akadémiai Kiadó, Budapest

where $\rho = \beta + i\gamma$ runs over the non trivial zeros of the Riemann zeta function $\zeta(s)$ and

$$w(u) = \begin{cases} 1 & \text{if } 0 \leq u \leq \frac{1}{2} \\ 2(1-u) & \text{if } \frac{1}{2} \leqq u \leqq 1. \end{cases}$$

Moreover, they proved that

$$\int_N^{2N} \left| E(x,T) \right|^2 dx = o\left(\frac{N^3}{T^2}\right) \quad \text{for} \quad N^{\varepsilon} \leq T \leq N^{1-\varepsilon} \tag{2}$$

follows from (1).

We recall that, from an unconditional viewpoint, $J(N,H) = o(H^2N)$ holds for $H \geqq N^{1/6+\varepsilon}$, see e.g. Heath-Brown [6], and hence that $I(N,T) = o\left(\frac{N^3}{T^2}\right)$ holds for $T \leqq N^{5/6-\varepsilon}$.

From a conditional viewpoint we recall that, under the assumption of the Riemann Hypothesis (RH), $J(N,H) = o(H^2N)$ holds for $H = \infty(\log^2 N)$, where we write $f = \infty(g)$ to denote $g = o(f)$, see Selberg [15], and that, under the assumption of RH and the Montgomery's pair correlation conjecture (MC), $J(N,H) = o(H^2N)$ holds for $H = \infty(\log N)$, see Goldston–Montgomery [5].

The first aim of this paper is to show that the connection between (1) and (2) holds for H and T in wider ranges. The second aim is to apply these extended results to the study of the distribution of Goldbach numbers, i.e. even numbers representable as a sum of two primes, in short intervals.

We will need the following slight modification of Kaczorowski–Perelli's explicit formula [9]. Let

$$\operatorname{sgn}(u) = \begin{cases} 1 & \text{if } u > 0 \\ 0 & \text{if } u = 0 \\ -1 & \text{if } u < 0 \end{cases}, \quad G(x,T,n) = \frac{2}{T}\int_{T/2}^{T}\left(\int_{\tau|\log\frac{x}{n}|}^{\infty} \frac{\sin u}{u}\, du\right) d\tau$$

and

$$N(\sigma,T) = \left|\left\{\rho \,:\, \rho = \beta + i\gamma \text{ zero of } \zeta(s) \text{ with } \beta \geqq \sigma,\ |\gamma| \leqq T\right\}\right|.$$

We have

THEOREM 1. *Let* $16 \leqq N \leqq x \leqq 2N$, $4 \leqq T \leqq \frac{N}{4}$ *and* $1 \leqq M \leqq \frac{T}{4}$. *Then*

$$\psi(x) = x - \sum_{|\gamma| \leqq T} w\left(\frac{|\gamma|}{T}\right)\frac{x^{\rho}}{\rho} + E(x,T),$$

where $\rho = \beta + i\gamma$ runs over the non-trivial zeros of the Riemann zeta function $\zeta(s)$ and

$$E(x,T) = E_1(x,T) + E_2(x,T) + E_3(x,T) \tag{3}$$

with

$$E_1(x,T) = \frac{1}{\pi} \sum_{x-\frac{MN}{T} < n \leqq x+\frac{MN}{T}} \Lambda(n) \operatorname{sgn}(x-n) G(x,T,n), \tag{4}$$

$$E_2(x,T) \ll \frac{MNL}{T^2 \log \frac{MN}{T}} + \frac{N^{\frac{\sigma+3}{4}} L^4}{T} + \frac{N}{T^{1+\alpha}} + \frac{NL^4}{T^{2-\alpha}} N(\sigma, T) \tag{5}$$

for every $\alpha \in (0,1]$ and $\sigma \in \left[\frac{1}{2}, 1\right)$, and

$$E_3(x,T) \ll \frac{NL}{TM \log \frac{N}{T}}. \tag{6}$$

Moreover,

$$\int_N^{2N} \left| E_3(x,T) \right|^2 dx \ll M^{-2} J\left(N, \frac{2N}{T}\right) + \frac{N^3}{T^2 M^2}. \tag{7}$$

The proof of Theorem 1 follows closely the argument in [9] and hence will be omitted. The bound in (7) can be obtained from (7) of [9] by straightforward computations. In the same way we also obtain that

$$\sum_{m=N}^{2N} \left| E_3(m,T) \right|^2 dx \ll M^{-2} \tilde{J}\left(N, \frac{2N}{T}\right) + \frac{N^3}{T^2 M^2}, \tag{8}$$

where

$$\tilde{J}(N,H) = \sum_{n=N}^{2N} \left(\psi(n+H) - \psi(n) - H \right)^2$$

is the discrete version of Selberg's integral.

From the Corollary in Kaczorowski–Perelli [9] we quote the following

COROLLARY 1. *Let $16 \leq N \leqq x \leqq 2N$, $4 \leqq T \leqq \frac{N}{4}$. Then*

$$E(x,T) \ll \frac{NL}{T \log \frac{N}{T}}.$$

We denote by $I(N,T)$ the quantity $\int_N^{2N} \left| E(x,T) \right|^2 dx$ and by $\tilde{I}(N,T)$ the quantity $\sum_{N \leqq m \leqq 2N} \left| E(m,T) \right|^2$.

Our first result about the relations between $J(N,H)$ and $I(N,T)$ is

THEOREM 2. *Let* $16 \leqq N \leqq x \leqq 2N$, $M^4 \leqq T \leqq \frac{N}{M^2}$ *and* $1 \leqq M \leqq \min\left(\frac{N^{1/16}}{L^4}; \frac{T^{1/5}}{L^9}\right)$. *Then*

$$I(N,T) \ll M^2 J\left(N, \frac{N}{TM}\right) + M^{-2} J\left(N, \frac{2N}{T}\right) + \frac{N^3}{T^2} \frac{\log^2(M+1)}{M^2}.$$

Since the proof of Theorem 2 can be obtained following step by step the argument used in Theorem 1 of [10], we will only give a brief sketch of it. In fact, Theorem 2 is a sharpened version of Theorem 1 of [10].

From Theorem 2 we obtain

COROLLARY 2. *Let* k *be a parameter such that* $k = \infty(1)$ *for* $N \to \infty$. *Assume that*

$$J(N,H) = o(H^2 N) \quad \textit{uniformly for} \quad kL \leqq H \leqq o(N).$$

Then

$$I(N,T) = o\left(\frac{N^3}{T^2}\right) \quad \textit{uniformly for} \quad \infty(1) \leqq T \leqq o\left(\frac{N}{kL}\right).$$

Corollary 2 allows us to connect directly the non-trivial bound $J(N,H) = o(H^2 N)$ with the non-trivial bound $I(N,T) = o\left(\frac{N^3}{T^2}\right)$. We remark that Theorem 2 and Corollary 2 sharpen Theorem 1 and Corollary 1 of Coppola–Vitolo [3].

Concerning $\tilde{I}(N,T)$ we have

THEOREM 3. *Let* $16 \leqq N \leqq x \leqq 2N$, $4 \leqq T \leqq \frac{N}{4}$ *and* $1 \leqq M \leqq \min\left(\frac{N^{1/16}}{L^4}; \frac{T^{1/5}}{L^9}\right)$. *Then*

$$\tilde{I}(N,T) \ll I(N,T) + \frac{T^2}{N^2} J\left(N, \frac{NM}{T}\right)$$
$$+ M^{-2}\left(J\left(N, \frac{2N}{T}\right) + \tilde{J}\left(N, \frac{2N}{T}\right)\right) + \frac{N^3}{T^2 M^2} + M^2 N.$$

We use Theorem 3 to obtain a sort of "converse" to Theorem 2. To this end we will need also the following

LEMMA. *Let* $1 \leqq H \leqq N$. *Then*

$$J(N,H) \ll H^2N + HNL \quad \textit{and} \quad \tilde{J}(N,H) \ll H^2N + HNL.$$

Since $I(N,T)$ is related to the "second" difference of primes in short intervals (for more details see the Introduction of [10]), we cannot hope to obtain a "direct" converse of Theorem 2. However, following [10], we can prove a "partial" converse to Theorem 2.

THEOREM 4. *Let* $\delta > 0$ *be a sufficiently small constant and* k_1, k_2 *be two parameters. Let*

$$k_2L^{11/2} \leqq H \leqq N^{1-\delta} \quad \textit{and} \quad J = \left[\frac{\log\left(N^{1-\delta}(Lk_1)^{1/2}H^{-1}\right)}{\log 2}\right],$$

where $1 \leqq k_2 \leqq L^A$, $A > 0$ *absolute constant,* $1 \leqq k_1 \leqq \dfrac{H^2}{k_2^2L^{11}}$ *and* $N \to \infty$. *Let further*

$$H_j = \frac{2^{j-2}H}{(Lk_1)^{1/2}} \in \left[\frac{k_2L^5}{2}, \frac{N^{1-\delta}}{4}\right] \quad \textit{and} \quad T_j = \frac{N}{100H_j} \in \left[\frac{N^\delta}{25}, \frac{N}{50k_2L^5}\right]$$

for every $j = 1, \dots, J$.

Then

$$J(N,H) \ll H^2 \sum_{j=1}^{J} H_j^{-2} I(N,T_j)$$

$$+H^2N\left(k_1^{-1} + \left(\frac{L}{H}k_1\right)^2 + k_2^{-1} + \exp\left(-cL^{1/4}\right)\right),$$

where $c > 0$ *is a small absolute constant.*

Again, we will only give a sketch of the proof of Theorem 4. The "natural" lower bound on H would be $H \gg L$. The limit of our method is given by the available density estimate near $\sigma = \frac{1}{2}$ and it appears to be $H \gg L^4$. The further loss of a factor $L^{3/2}$ follows from the dissection method used in the proof.

Coppola–Vitolo [3] contains a slightly sharper statement than our Theorem 4. However, it appears that their treatment of the quantity $E_1(j)$ in the proof of our Theorem 4 contains a mistake which affects the final result. After correction of that mistake their result coincides with ours.

Choosing $k_2^2L^{11/2} \leqq H \leqq N^{1-\delta}$, $k_1 = k_2^2$ and $\infty(1) \leqq k_2 \leqq o\left(\left(\frac{H}{L}\right)^{1/2}\right)$ in Theorem 4 we easily obtain

COROLLARY 3. *Let δ, N be as in Theorem 4. Assume that*

$$I(N,T) = o\left(\frac{N^3}{T^2L}\right) \quad \textit{uniformly for} \quad \frac{N^\delta}{25} \leqq T \leqq o\left(\frac{N}{L^5}\right).$$

Then

$$J(N,H) = o(H^2N) \quad \textit{uniformly for} \quad \infty(L^{11/2}) \leqq H \leqq N^{1-\delta}.$$

Corollary 3 is, in some sense, a "converse" to Corollary 2. Unfortunately, to obtain the non-trivial bound $J(N,H) = o(H^2N)$, Corollary 3 needs the stronger hypothesis $I(N,T) = o\left(\frac{N^3}{T^2L}\right)$. This is due to the dissection argument used in the proof of Theorem 4.

Corollary 4 below furnishes a conditional result on the distribution of Goldbach numbers in short intervals.

COROLLARY 4. *Let δ, N be as in Theorem 4. Let further $H \geqq CL^{11/2}$, where $C > 0$ is a sufficiently large constant. Assume that there exists a sufficiently small constant $c_1 > 0$ such that*

$$I(N,T) \leqq c_1\frac{N^3}{T^2L} \quad \textit{uniformly for} \quad \frac{N^\delta}{25} \leqq T \leqq \frac{N}{50C^{1/2}L^5}.$$

Then a positive proportion of the even integers in the interval $[N, N+H]$ are Goldbach numbers.

We recall that the best unconditional result on the positive proportion of Goldbach numbers in short intervals is $H \gg N^{0.535/20}$, see e.g. Baker–Harman–Pintz [1] and Jia [8]. From a conditional viewpoint we have, under the assumption of RH, that $H \gg \log^2 N$, see Kátai [11], Montgomery–Vaughan [14], Goldston [4] and Languasco–Perelli [12], and, assuming RH and MC, that $H \geqq (\log N)^{1+\varepsilon}$, see Goldston [4].

2. Proof of Theorem 2 and Corollary 2

We divide the interval $\left(x - \frac{MN}{T}, x + \frac{MN}{T}\right]$ into $P \ll M^2$ subintervals of the form

$$I_j = (n_j, n_j + K], \quad K = \frac{N}{TM}, \quad n_j = x \pm jK, \quad j = 1, \ldots, P.$$

We may suppose also that either $I_j \subset (0,x]$ or $I_j \subset [x, +\infty)$ for every j, hence $\operatorname{sgn}(x-n)$ is constant on each I_j.

Hence, by Lemma 1 of [10], (3)–(5) with $\alpha = \frac{1}{5}$ and $\sigma = \frac{3}{4}$, we have

$$E(x,T) \ll \sum\nolimits_1 + \sum\nolimits_2 + E_3(x,T) + \frac{N}{TM}, \tag{9}$$

where

$$\sum\nolimits_1 = \sum_{j=1}^{P} \left|G(x,T,n_j)\right| \left|\sum_{n \in I_j} \left(\Lambda(n) - 1\right)\right|$$

and

$$\sum\nolimits_2 = \sum_{j=1}^{P} \sum_{n \in I_j} \Lambda(n) \left|G(x,T,n) - G(x,T,n_j)\right|.$$

The estimation of the mean-square of $\sum_1$ and $\sum_2$ can be performed as in [10] and hence we obtain

$$\int_N^{2N} \left|\sum\nolimits_1\right|^2 dx \ll M^2 J\left(N, \frac{N}{TM}\right) \tag{10}$$

and

$$\int_N^{2N} \left|\sum\nolimits_2\right|^2 dx \ll \log^2(M+1) J\left(N, \frac{N}{TM}\right) + \frac{N^3}{T^2}\frac{\log^2(M+1)}{M^2}. \tag{11}$$

Theorem 2 now follows from (7), (9), (10) and (11).

To prove Corollary 2 we choose $M \leqq kL$ and $M = \infty(1)$ for $N \to \infty$. So Theorem 2 implies

$$I(N,T) \ll M^2 J\left(N, \frac{N}{TM}\right) + M^{-2} J\left(N, \frac{2N}{T}\right) + o\left(\frac{N^3}{T^2}\right)$$

uniformly for $M^4 \leqq T \leqq \frac{N}{M^2}$.

Now, using the hypothesis $J(N,H) = o(H^2 N)$ uniformly for $kL \leqq H \leqq o(N)$, we get

$$I(N,T) = o\left(\frac{N^3}{T^2}\right) \quad \text{uniformly for} \quad \infty(1) \leqq T \leqq \frac{N}{MkL}$$

and then Corollary 2 follows.

3. Proof of Theorem 3

We have, for $n \neq [x]$, that $\operatorname{sgn}(x-n) = \operatorname{sgn}\big([x]-n\big)$ and so the intervals $\big(x - \frac{MN}{T}, x + \frac{MN}{T}\big]$ and $\big([x] - \frac{MN}{T}, [x] + \frac{MN}{T}\big]$ differ at most for the two endpoints. By (4)–(5) with $\alpha = \frac{1}{5}$ and $\sigma = \frac{3}{4}$, we obtain

$$
\begin{aligned}
(12) \qquad & E(x,T) - E\big([x],T\big) \\
&= \frac{1}{\pi} \sum_{2<|n-x|<\frac{MN}{T}-2} \Lambda(n) \operatorname{sgn}(x - n)\big(G(x,T,n) - G\big([x],T,n\big)\big) \\
&\qquad + E_3(x,T) - E_3\big([x],T\big) + O\left(\frac{N}{TM}\right).
\end{aligned}
$$

Arguing as in Lemma 2 of [10], we have

$$
\begin{aligned}
(13) \qquad & \frac{1}{\pi} \sum_{2<|n-x|<\frac{MN}{T}-2} \Lambda(n) \operatorname{sgn}(x-n)\big(G(x,T,n) - G\big([x],T,n\big)\big) \\
&\ll \frac{1}{T} \sum_{|n-x| \leqq \frac{MN}{T}} \Lambda(n) \int_{T/2}^{T} \left(\int_{\tau|\log \frac{[x]}{n}|}^{\tau|\log \frac{x}{n}|} \frac{\sin u}{u}\, du \right) d\tau, \\
&\ll T \sum_{|n-x| \leqq \frac{MN}{T}} \Lambda(n) \left| \left|\log \frac{x}{n}\right| - \left|\log \frac{[x]}{n}\right| \right| \ll \frac{T}{N} \sum_{|n-x| \leqq \frac{MN}{T}} \Lambda(n)
\end{aligned}
$$

and hence, by (12)–(13), we have

(14)
$$
E(x,T) - E\big([x],T\big) \ll \frac{T}{N} \sum_{|n-x| \leqq \frac{MN}{T}} \Lambda(n) + E_3(x,T) + E_3\big([x],T\big) + \frac{N}{TM}.
$$

By (14), for any $m \in [N, 2N]$, we obtain

$$
\begin{aligned}
\big|E(m,T)\big|^2 &\ll \int_{m-1}^{m} \big|E(x,T)\big|^2\, dx + \frac{T^2}{N^2} \int_{m-1}^{m} \left| \sum_{|n-x| \leqq \frac{MN}{T}} \Lambda(n) \right|^2 dx \\
&\qquad + \int_{m-1}^{m} \big|E_3(x,T)\big|^2\, dx + \big|E_3(m,T)\big|^2 + \frac{N^2}{T^2 M^2}.
\end{aligned}
$$

Theorem 3 now follows summing over m, using $(a+b)^2 \leqq 2a^2 + 2b^2$ and (7)–(8).

4. Proof of the Lemma

If, for any fixed $\varepsilon > 0$, $H \geqq N^\varepsilon$ we get, by the Brun–Titchmarsh theorem, that $\psi(x+H) - \psi(x) - H \ll H$ and hence

$$J(N,H) \ll H^2 N. \tag{15}$$

Let now $1 \leqq H \leqq N^\varepsilon$. By Gallagher's lemma, see e.g. Montgomery [13], Lemma 1.9, the Brun–Titchmarsh theorem and Parseval's identity, see e.g. Kaczorowski–Perelli [10], we have

$$J(N,H) = \int_{-1/2}^{1/2} \left| S(\alpha) - T(\alpha) \right|^2 L(\alpha)\, d\alpha + O\left(\frac{H^3 L^2}{\log^2 H} \right) + O'(HN), \tag{16}$$

where

$$S(\alpha) = \sum_{N \leqq n \leqq 2N} \Lambda(n) e(n\alpha), \quad T(\alpha) = \sum_{N \leqq n \leqq 2N} e(n\alpha),$$

$$L(\alpha) = \left| \sum_{m=1}^{H} e(-m\alpha) \right|^2, \quad e(x) = \exp(2\pi i x)$$

and O' means that the error term is present only if $H \notin \mathbf{N}$.

By

$$L(\alpha) \ll \min\left(H^2; |\alpha|^{-2} \right) \tag{17}$$

and partial integration we get

$$\int_{-1/2}^{1/2} \left| S(\alpha) - T(\alpha) \right|^2 L(\alpha)\, d\alpha \tag{18}$$

$$\ll H^2 \int_{-1/H}^{1/H} \left| S(\alpha) - T(\alpha) \right|^2 d\alpha + \int_{-1/2}^{1/2} \left| S(\alpha) - T(\alpha) \right|^2 d\alpha$$

$$+ \left(\int_{-1/2}^{-1/H} + \int_{1/H}^{1/2} \right) \left(\int_{-t}^{t} \left| S(\alpha) - T(\alpha) \right|^2 d\alpha \right) t^{-3}\, dt.$$

By suitable modifications of the technique of Languasco Perelli [12], we can get

$$\int_{-\xi}^{\xi} |S(\alpha)|^2 \, d\alpha \asymp \begin{cases} N^2\xi & \text{if } 0 \leqq \xi \leqq \frac{1}{N} \\ N & \text{if } \frac{1}{N} \leqq \xi \leqq \frac{1}{L} \\ N\xi L & \text{if } \frac{1}{L} \leqq \xi \leqq \frac{1}{2}, \end{cases} \tag{19}$$

where $\xi \in \left(0, \frac{1}{2}\right]$ and $f \asymp g$ means that $g \ll f \ll g$.

Now, using $T(\alpha) \ll \min\left(N; |\alpha|^{-1}\right)$, we obtain

$$\int_{-\xi}^{\xi} |T(\alpha)|^2 \, d\alpha \begin{cases} \ll N^2\xi & \text{if } 0 < \xi < \frac{1}{N} \\ = N + O\left(\frac{1}{\xi}\right) & \text{if } \frac{1}{N} \leqq \xi \leqq \frac{1}{2}. \end{cases} \tag{20}$$

Hence, by (19)–(20) and the Cauchy–Schwarz inequality, we get

$$\int_{-\xi}^{\xi} |S(\alpha) - T(\alpha)|^2 \, d\alpha \ll \begin{cases} N^2\xi & \text{if } 0 \leqq \xi \leqq \frac{1}{N} \\ N & \text{if } \frac{1}{N} \leqq \xi \leqq \frac{1}{L} \\ N\xi L & \text{if } \frac{1}{L} \leqq \xi \leqq \frac{1}{2}, \end{cases} \tag{21}$$

and then, by (16), (18) and (21), we have

$$J(N,H) \ll H^2 N + HNL. \tag{22}$$

From (15) and (22) we obtain the first inequality in the Lemma. The second inequality follows easily using $\tilde{J}(N,H) = J(N,H) + O(HNL)$.

5. Proof of Theorem 4

By (16) we have

$$J(N,H) = S(N,H) + O\left(\frac{H^3 L^2}{\log^2 H}\right) + O'(HN), \tag{23}$$

where

$$S(N,H) = \int_{-1/2}^{1/2} |S(\alpha) - T(\alpha)|^2 L(\alpha) \, d\alpha$$

and, as before, O' means that the error term is present only if $H \notin \mathbf{N}$.

Hence we study $S(N,H)$. Let $\xi \in \left(0, \frac{1}{2}\right]$ to be chosen later on. Then, by Parseval identity, the Prime Number Theorem and (17), we have

$$\left(\int_{-1/2}^{-\xi} + \int_{\xi}^{1/2}\right) \left|S(\alpha) - T(\alpha)\right|^2 L(\alpha)\, d\alpha \ll \frac{NL}{\xi^2} \tag{24}$$

and

$$\int_{-\xi}^{\xi} \left|S(\alpha) - T(\alpha)\right|^2 L(\alpha)\, d\alpha \ll H^2 \int_{-\xi}^{\xi} \left|S(\alpha) - T(\alpha)\right|^2 d\alpha. \tag{25}$$

Now we dissect $(-\xi, \xi)$ into $2J + 1 = O(L)$ subintervals of the form

$$A_0 = (-N^{\delta-1}, N^{\delta-1}) \quad \text{and} \quad A_j = \left(\pm\frac{\xi}{2^j}, \pm\frac{\xi}{2^{j-1}}\right), \quad j = 1, \ldots, J,$$

where $J = \left[\frac{\xi N^{1-\delta}}{\log 2}\right]$.

Moreover, for every non-trivial zero $\rho = \beta + i\gamma$ of $\zeta(s)$, we define

$$T_\rho(\alpha) = \sum_{n=N}^{2N} a_{n,\rho} e(n\alpha) \quad \text{with} \quad a_{n,\rho} = \int_n^{n+1} t^{\rho-1}\, dt.$$

Let now $T_j \in \left[\frac{N^\delta}{25}, \frac{N}{50k_2L^5}\right]$, $j = 1, \ldots, J$, to be chosen later on.

For $\alpha \in A_j$ we write

$$S(\alpha) - T(\alpha) = -\sum_{|\gamma| \leqq T_j} w\left(\frac{|\gamma|}{T_j}\right) T_\rho(\alpha) + R_j(\alpha),$$

where

$$R_j(\alpha) = \sum_{n=N}^{2N} a_j(n) e(n\alpha) \quad \text{and} \quad a_j(n) = \Lambda(n) - 1 + \sum_{|\gamma| \leqq T_j} w\left(\frac{|\gamma|}{T_j}\right) a_{n,\rho}.$$

By the Cauchy–Schwarz inequality, we have

$$\int_{A_j} \left|S(\alpha) - T(\alpha)\right|^2 d\alpha \ll \left(\sum_{|\gamma| \leqq T_j} \left(\int_{A_j} \left|T_\rho(\alpha)\right|^2 d\alpha\right)^{1/2}\right)^2 + \int_{A_j} \left|R_j(\alpha)\right|^2 d\alpha$$

$$= E_1(j) + E_2(j),$$

say, and hence, using the same argument as in [10], we obtain

$$\int_{-\xi}^{\xi} |S(\alpha) - T(\alpha)|^2 \, d\alpha \ll \sum_{j=1}^{J} \big(E_1(j) + E_2(j) \big) + N \exp(-cL^{1/4}), \tag{26}$$

where $c > 0$ is an absolute constant (not necessarily the same at each occurrence).

Inserting Theorem 3, the Lemma and Corollary 1 in the technique of [10], we obtain

$$E_2(j) \ll H_j^{-2} I(N, T_j) + H_j^{-2}(NL + M^2 N) + \frac{N^3}{(H_j T M)^2} + H_j^{-2} T_j M L + \frac{N^2 L^2}{H_j T_j^2 \log^2 \frac{N}{T_j}} \tag{27}$$

provided that $1 \leqq M \leqq \min\big(N^{1/16} L^{-4}; T_j^{1/5} L^{-9}\big)$.

Now we proceed to estimate $E_1(j)$. Arguing as in Theorem 2 of [10], we have

$$T_\rho(\alpha) \ll \frac{N^{\beta-1}}{|\alpha|} \quad \text{for every} \quad \alpha \in A_j$$

and hence

$$E_1(j) \ll L^2 H_j N^{-2} \Big(\sup_{0<\sigma<1} N^\sigma N(\sigma, T_j) \Big)^2,$$

where $T_j = \frac{N}{100 H_j}$.

Let $0 < \varepsilon' < \frac{1}{4}$ be fixed. If $H_j \geqq N^{\varepsilon'}$ we can use the density estimate (see, e.g., Ivić [7])

$$N(\sigma, T) \ll T^{3/2-\sigma} \log^5 T$$

to obtain

$$E_1(j) \ll N \exp(-cL^{1/4}). \tag{28}$$

If $\frac{1}{2} k_2 L^5 \leqq H_j \leqq N^{\varepsilon'}$ we use the density estimate (see Conrey [2] and, e.g., Ivić [7])

$$N(\sigma, T) \ll \begin{cases} T^{1-(8/7-\theta)(\sigma-1/2)} \log T & \text{if } \frac{1}{2} \leqq \sigma \leqq \frac{1}{2} + \frac{84}{4+21\theta} \frac{\log\log T}{\log T} \\ T^{3(1-\sigma)/(2-\sigma)} \log^5 T & \text{if } \frac{1}{2} + \frac{84}{4+21\theta} \frac{\log\log T}{\log T} \leqq \sigma \leqq \frac{3}{4} \\ T^{3(1-\sigma)/(3\sigma-1)} \log^{44} T & \text{if } \frac{3}{4} \leqq \sigma \leqq 1. \end{cases}$$

Since the maximum of $N^\sigma N(\sigma, T_j)$ is attained at $\sigma = \frac{1}{2}$ we have

$$E_1(j) \ll NH_j^{-1}L^4 \leqq \frac{N}{k_2L}, \tag{29}$$

provided that θ is sufficiently small and N is sufficiently large.

Hence, by (28)–(29), we get

$$E_1(j) \ll \frac{N}{k_2L} + N\exp(-cL^{1/4}) \tag{30}$$

for every $\frac{1}{2}k_2L^5 \leqq H_j \leqq \frac{N^{1-\delta}}{4}$.

Now, by (24)–(27) and (30), we obtain

$$S(N,H) \ll H^2\sum_{j=1}^{J} H_j^{-2}I(N,T_j) + H^2\xi^2(NL + M^2N) + \frac{H^2NL}{M^2} \tag{31}$$

$$+H^2\xi^3NML + H^2N^{1-\delta}L^3 + \frac{H^2N}{k_2} + \frac{NL}{\xi^2} + H^2N\exp(-cL^{1/4}).$$

Theorem 4 follows by choosing $M = (Lk_1)^{1/2}$ and $\xi = \frac{M}{H}$ in (31) and using (23).

6. Proof of Corollary 4

Let $R(n) = \sum_{m_1+m_2=n} \Lambda(m_1)\Lambda(m_2)$. From its definition (see Section 4) we get that $L(\alpha) = \sum_{m=-H}^{H} a(m)e(-m\alpha)$, where $a(m) = H - |m|$.

A sufficient condition to prove that a positive proportion of the even integers in the interval in $[N - H, N + H]$ are Goldbach numbers is

$$\sum_{n=N-H}^{N+H} a(n-N)R(n) \gg H^2N, \tag{32}$$

see e.g. Goldston [4].

It is easy to prove that

$$\sum_{n=N-H}^{N+H} a(n-N)R(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} S(\alpha)^2L(\alpha)e(-N\alpha)\,d\alpha \tag{33}$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} T(\alpha)^2 L(\alpha) e(-N\alpha)\, d\alpha + \int_{-\frac{1}{2}}^{\frac{1}{2}} E(\alpha) L(\alpha) e(-N\alpha)\, d\alpha,$$

where $E(\alpha) = S(\alpha)^2 - T(\alpha)^2$.

By straightforward computations we get

(34)

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} T(\alpha)^2 L(\alpha) e(-N\alpha)\, d\alpha = \sum_{n=N-H}^{N+H} a(n-N) \sum_{h+k=n} 1 = H^2 N + O(H^3).$$

and, using (17) and (20),

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |T(\alpha)|^2 L(\alpha)\, d\alpha \ll H^2 N. \tag{35}$$

Using the identity $f^2 - g^2 = 2f(f-g) - (f-g)^2$, the Cauchy–Schwarz inequality and (35) we have

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} E(\alpha) L(\alpha) e(-N\alpha)\, d\alpha \ll \left(H^2 N \int_{-\frac{1}{2}}^{\frac{1}{2}} |S(\alpha) - T(\alpha)|^2 L(\alpha)\, d\alpha \right)^{1/2} + \int_{-\frac{1}{2}}^{\frac{1}{2}} |S(\alpha) - T(\alpha)|^2 L(\alpha)\, d\alpha. \tag{36}$$

Hence, by (33)–(34) and (36), to obtain (32) it is sufficient to prove that *there exists a sufficiently small constant $c_2 > 0$ such that*

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |S(\alpha) - T(\alpha)|^2 L(\alpha)\, d\alpha \leqq c_2 H^2 N \tag{37}$$

holds.

By (16) we have

$$\int_{-1/2}^{1/2} |S(\alpha) - T(\alpha)|^2 L(\alpha)\, d\alpha = J(N,H) + O\left(\frac{H^3 L^2}{\log^2 H}\right) + O'(HN). \tag{38}$$

Choosing $H \geqq C L^{11/2}$, where $C > 0$ is a sufficiently large constant, and using Theorem 4 with $k_1 = k_2^2 = C$, we obtain by (38)

$$\int_{-1/2}^{1/2} |S(\alpha) - T(\alpha)|^2 L(\alpha)\, d\alpha \tag{39}$$

$$\ll H^2 \sum_{j=1}^{J} H_j^{-2} I\left(N, \frac{N}{100H_j}\right) + \frac{H^2N}{C^{1/2}} + o(H^2N).$$

Since $J \leqq c_3 L$, choosing $c_1 \leqq \left(10000 c_3 C^{1/2}\right)^{-1}$ we get by (39) and the hypothesis on $I(N,T)$ that (37) holds with $c_2 = C^{-1/2}$.

Hence (32) holds and Corollary 4 follows arguing as in Goldston [4].

Acknowledgement. *The author wishes to thank Professor A. Perelli for some useful discussions.*

References

[1] R. Baker, G. Harman and J. Pintz, *The exceptional set for Goldbach's problem in short intervals*, in *Sieve Methods, Exponential Sums and their Applications in Number Theory* (ed. G. R. H. Greaves et al.) Cambridge U. P. (1996), pp. 1–54.

[2] J. B. Conrey, At least two fifths of the zeros of the Riemann zeta function are on the critical line, *Bull. A.M.S.*, **20** (1989), 79–81.

[3] G. Coppola and A. Vitolo, On the distribution of primes in intervals of length $\log^\theta N$, *Acta Math. Hungar.*, **70** (1996), 151–166.

[4] D. A. Goldston, Linnik's theorem on Goldbach numbers in short intervals, *Glasgow Math. J.*, **32** (1990), 285–297.

[5] D. A. Goldston and H. L. Montgomery, Pair correlation of zeros and primes in short intervals, in *Analytic Number Theory and Dioph. Probl.* (ed. by A. C. Adolphson et al.), Birkhäuser Verlag (1987), pp. 183–203.

[6] D. R. Heath-Brown, The differences between consecutive primes, *J. London Math. Soc.*, **18** (1978), 7–13.

[7] A. Ivić, *The Riemann Zeta-function*, J. Wiley (1985).

[8] C. Jia, Almost all short intervals contain prime numbers, *Acta Arith.*, **76** (1996), 21–84.

[9] J. Kaczorowski and A. Perelli, A new form of the Riemann–von Mangoldt explicit formula, *Boll. U.M.I.*, **10**-B (1996), 51–66.

[10] J. Kaczorowski and A. Perelli, On the distribution of primes in short intervals, *J. Math. Soc. Japan*, **45** (1993), 447–458.

[11] I. Kátai, A remark on a paper of Ju. V. Linnik, *Magyar Tud. Akad. Mat. Fiz. Oszt. Közl*, **17** (1967), 99–100 (in Hungarian).

[12] A. Languasco and A. Perelli, On Linnik's theorem on Goldbach numbers in short intervals and related problems, *Ann. Inst. Fourier*, **44** (1994), 307–322.

[13] H. L. Montgomery, *Topics in Multiplicative Number Theory*, Springer L.N. 227 (1971).

[14] H. L. Montgomery and R. C. Vaughan, The exceptional set in Goldbach's problem, *Acta Arith.*, **27** (1975), 353–370.

[15] A. Selberg, On the normal density of primes in small intervals, and the difference between consecutive primes, *Arch. Math. Naturvid.*, **47** (1943), 87–105.

(Received October 14, 1996)

DIPARTIMENTO DI MATEMATICA
UNIVERSITÀ DI GENOVA
VIA DODECANESO 35
16146 GENOVA
ITALY
E-MAIL: LANGUASCO@DIMA.UNIGE.IT