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A conditional result on Goldbach numbers in short intervals

by

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1. Introduction. Define a *Goldbach number* (G-number) to be an even number representable as a sum of two primes, and write $L = \log X$. The first result concerning the existence of G-numbers in short intervals is due to Linnik [11] who proved, assuming the Riemann Hypothesis (RH), that for any $\varepsilon > 0$ and X sufficiently large, the interval $[X, X + L^{3+\varepsilon}]$ contains a G-number. Linnik's result was improved by Kátai [8] and, independently, by Montgomery–Vaughan [14] who showed that the interval $[X, X + CL^2]$ contains a G-number for some constant C and X sufficiently large. Other proofs of the Kátai and Montgomery–Vaughan result have recently been obtained by Goldston [3] and Languasco–Perelli [9].

The main aim of this paper is to study the distribution of G-numbers in short intervals under the assumption of RH and Montgomery's pair correlation conjecture, a form of which asserts that

$$F(X,T) \sim \frac{1}{2\pi} T \log T \quad \text{for } X \to \infty \tag{1}$$

uniformly for $X^{\varepsilon} \le T \le X$, for every fixed $\varepsilon > 0$, where

$$F(X,T) = 4 \sum_{0<\gamma_1,\gamma_2\le T} \frac{X^{i(\gamma_1-\gamma_2)}}{4+(\gamma_1-\gamma_2)^2}$$

and γ_j, $j = 1, 2$, run over the imaginary part of the nontrivial zeros of the Riemann zeta-function $\zeta(s)$.

It is easy to see that

$$F(X,T) \ll T \log^2 T \tag{2}$$

uniformly in X. Moreover, Montgomery [13] (see also Goldston–Montgomery

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[4]) proved, again under RH, that

$$F(X,T) \sim \frac{1}{2\pi} T \log X \quad \text{for } X \to \infty \tag{3}$$

holds uniformly for $X \le T$.

In the following we will denote by MC the hypothesis (1) and by WMC a weaker form of it where $\ll$ replaces $\sim$. As a slight generalization of WMC, we state the following

Hypothesis. *Let $\theta \in [1,2)$. For any $\varepsilon > 0$ the estimate*

$$F(X,T) \ll T(\log T)^{\theta} \tag{4}$$

holds uniformly for $X^{\varepsilon} \le T \le X$.

We will denote the hypothesis above by WMC(θ). Observe that WMC = WMC(1) and that, for $\theta \ge 2$, (4) is implied by (2). Our result is

Theorem. *Let $\theta \in [1,2)$ be fixed and $1/(2X) \le \xi \le 1/2$. Assume RH and WMC(θ) uniformly in the range $2X\xi \le T \le X$. Then*

$$\int_{-\xi}^{\xi} \Big| \sum_{X \le n \le 2X} (\Lambda(n) - 1) e(n\alpha) \Big|^2 d\alpha \ll X\xi L^{\theta} + \min\left(\frac{L^2}{\xi(\log 2\xi)^2}; X\xi L^{\theta+2} \right),$$

where Λ is the von Mangoldt function and $e(x) = \exp(2\pi i x)$.

We remark that this theorem is an analogue of Theorem 3 of Languasco–Perelli [10]. As an application we can obtain the following result on the distribution of Goldbach numbers in short intervals.

Corollary. *Let $\theta \in [1,2)$ and $H = CL^{\theta}$, where $C > 0$ is a sufficiently large constant. Assume RH and WMC(θ) uniformly in the range $X/H \le T \le X$. Then, for all sufficiently large X, the interval $[X, X+H]$ contains a G-number.*

We remark that our Corollary can be obtained using the method of Goldston [3].

We also recall that, under RH and MC, Goldston [3] proved that the interval $[X, X+CL]$ contains a G-number and that Friedlander–Goldston [2] proved that the interval $\big[X, X+C\frac{(\log\log X)^2}{\log\log\log X}\big]$ contains a G-number assuming RH together with a strong form of MC and a suitable form of the Elliott–Halberstam conjecture.

Our second aim is to prove the following result, which may have some independent interest, on the mean-square of the singular series of the Goldbach problem.

PROPOSITION. *Let*

$$\mathfrak{S}(n) = \begin{cases} 2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\prod_{\substack{p\mid n\\ p>2}}\left(\frac{p-1}{p-2}\right) & \text{if } n \text{ is even,}\\ 0 & \text{if } n \text{ is odd.}\end{cases}$$

Then

$$\sum_{n\le X}\mathfrak{S}(n)^2 = 2X\prod_{p>2}\left(1+\frac{1}{(p-1)^3}\right) - \frac{1}{4}L^2 + O(L^{5/3}).$$

The Proposition is a sharper version of Lemma 2 of Goldston [3], and its proof is based on the argument in Friedlander–Goldston [2].

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2. Some lemmas. Now we state two lemmas whose proofs follow the lines of Languasco–Perelli [10] (see also Heath-Brown [6]).

LEMMA 1. *Writing*

$$\Sigma(X,T,v) = \sum_{0<\gamma\le T} X^{i\gamma}e^{iv\gamma}$$

we have

$$F(X,T) = \int_{-\infty}^{\infty} |\Sigma(X,T,v)|^2 e^{-2|v|}\,dv.$$

LEMMA 2. *Let $\alpha=\alpha(X)$ and $\beta=\beta(X)$ be real numbers satisfying $c\le\alpha<\beta\le C$ for some absolute constants $c, C>0$. Let $T>U\ge 0$. Then*

$$\int_{\alpha X}^{\beta X}\Big|\sum_{0<\gamma\le T} y^{i\gamma}\Big|^2\,dy \ll XF(X,T)$$

and

$$\sum_{U<\gamma\le T} X^{i\gamma} \ll T^{1/2}(\max_{U\le u\le T} F(X,u))^{1/2}.$$

The next lemma is an analogue of Lemma 2 in which we insert a factor $1/\varrho$ in the summation on the γ.

LEMMA 3. *Let $\varrho = 1/2+i\gamma$, $\alpha=\alpha(X)$ and $\beta=\beta(X)$ be real numbers satisfying $c\le\alpha<\beta\le C$ for some absolute constants $c, C>0$. Let $T>U\ge 0$. Then*

$$\int_{\alpha X}^{\beta X}\Big|\sum_{U<\gamma\le T}\frac{y^{i\gamma}}{\varrho}\Big|^2\,dy \ll X\bigg(\frac{F(X,T)}{T^2}+\frac{F(X,U)}{U^2}+\frac{1}{U^{1/2}}\int_U^T F(X,u)\frac{du}{u^{5/2}}\bigg)$$

and

$$\sum_{U<\gamma\le T}\frac{X^{i\gamma}}{\varrho} \ll \Big(\frac{1}{T}\max_{U\le u\le T}F(X,u)\Big)^{1/2}+\int_U^T(\max_{U\le v\le u}F(X,v))^{1/2}\frac{du}{u^{3/2}}.$$

Proof. By partial summation

$$\sum_{U<\gamma\le T}\frac{X^{i\gamma}}{\varrho} \ll \frac{1}{T}\Big|\sum_{U<\gamma\le T}X^{i\gamma}\Big|+\int_U^T\Big|\sum_{U<\gamma\le u}X^{i\gamma}\Big|\frac{du}{u^2}.$$

The second estimate above follows immediately from the second estimate in Lemma 2. For the first estimate we have, by the Cauchy–Schwarz inequality and Lemma 2,

$$\begin{aligned}
&\int_{\alpha X}^{\beta X}\Big|\sum_{U<\gamma\le T}\frac{y^{i\gamma}}{\varrho}\Big|^2\,dy\\
&\quad\ll \frac{1}{T^2}\int_{\alpha X}^{\beta X}\Big|\sum_{U<\gamma\le T}y^{i\gamma}\Big|^2\,dy+\int_{\alpha X}^{\beta X}\Big|\int_U^T\Big|\sum_{U<\gamma\le u}y^{i\gamma}\Big|\frac{du}{u^2}\Big|^2\,dy\\
&\quad\ll \frac{X}{T^2}(F(X,T)+F(X,U))+\int_{\alpha X}^{\beta X}\bigg(\int_U^T\Big|\sum_{U<\gamma\le u}y^{i\gamma}\Big|^2\frac{du}{u^{5/2}}\bigg)\bigg(\int_U^T\frac{du}{u^{3/2}}\bigg)\,dy\\
&\quad\ll \frac{X}{T^2}(F(X,T)+F(X,U))+\frac{1}{U^{1/2}}\int_U^T X(F(X,u)+F(X,U))\frac{du}{u^{5/2}}\\
&\quad\ll X\bigg(\frac{F(X,T)}{T^2}+\frac{F(X,U)}{U^2}+\frac{1}{U^{1/2}}\int_U^T F(X,u)\frac{du}{u^{5/2}}\bigg).
\end{aligned}$$

3. Proof of the Theorem and of the Corollary. Writing Selberg's integral

$$J(X,2X,H)=\int_X^{2X}|\psi(t+H)-\psi(t)-H|^2\,dt,$$

where $\psi(t)=\sum_{n\le t}\Lambda(n)$, we get, by Gallagher's lemma (see, e.g., Montgomery [12], Lemma 1.9),

$$(5)\quad \int_{-\xi}^{\xi}\Big|\sum_{X\le n\le 2X}(\Lambda(n)-1)e(n\alpha)\Big|^2\,d\alpha$$
$$\ll \xi^2\Big[\int_{X-1/(2\xi)}^{X}\Big|\psi\Big(t+\frac{1}{2\xi}\Big)-\psi(X)-\Big(t-X+\frac{1}{2\xi}\Big)\Big|^2\,dt$$
$$+J\Big(X,2X-\frac{1}{2\xi},\frac{1}{2\xi}\Big)+\int_{2X-1/(2\xi)}^{2X}|\psi(2X)-\psi(t)-(2X-t)|^2\,dt\Big]$$
$$+X\xi^2+\xi$$
$$=\xi^2(I_1+I_2+I_3)+O(X\xi^2+\xi),$$

say. We remark that the term $O(X\xi^2+\xi)$ in (5) arises from the $O(1)$ term in the estimate $\sum_{a<n<b}1=b-a+O(1)$ applied in the above integrals.

Using the explicit formula (see Davenport [1], Ch. 17)

$$\psi(x)=x-\sum_{|\gamma|\le K}\frac{x^\varrho}{\varrho}+O\Big(\frac{x(\log xK)^2}{K}\Big)+O\Big(\log x\min\Big(1;\frac{x}{K\|x\|}\Big)\Big),$$

where $\|x\|=\min_{n\in\mathbb{N}}|x-n|$, with $K=XL^2\xi^{1/2}$, we get, by (5),

$$(6)\quad \int_{-\xi}^{\xi}\Big|\sum_{X\le n\le 2X}(\Lambda(n)-1)e(n\alpha)\Big|^2\,d\alpha\ll\xi^2(J_1+J_2+J_3)+O(X\xi),$$

where

$$J_1=\int_{X-1/(2\xi)}^{X}\Big|\sum_{0<\gamma\le K}\frac{(y+1/(2\xi))^\varrho-X^\varrho}{\varrho}\Big|^2\,dy,$$
$$J_2=\int_{X}^{2X-1/(2\xi)}\Big|\sum_{0<\gamma\le K}\frac{(y+1/(2\xi))^\varrho-y^\varrho}{\varrho}\Big|^2\,dy,$$
$$J_3=\int_{2X-1/(2\xi)}^{2X}\Big|\sum_{0<\gamma\le K}\frac{(2X)^\varrho-y^\varrho}{\varrho}\Big|^2\,dy.$$

We first consider the terms in $J_1+J_2+J_3$ where $0<\gamma\le 2X\xi$, and show that they make a contribution

$$(7)\qquad \ll\frac{1}{X\xi^2}\int_{\alpha X}^{\beta X}\Big|\sum_{0<\gamma\le 2X\xi}y^{i\gamma}\Big|^2\,dy.$$

Using the Cauchy–Schwarz inequality with $0 < V < W$ we get

$$\Big|\sum_{0<\gamma\le U}\frac{W^{\varrho}-V^{\varrho}}{\varrho}\Big|^2=\Big|\int_V^W\sum_{0<\gamma\le U}u^{\varrho-1}\,du\Big|^2\le|W-V|\int_V^W\Big|\sum_{0<\gamma\le U}u^{i\gamma}\Big|^2\frac{du}{u}.$$

Applying this estimate in J_1, J_2 and J_3 we obtain (7). By Lemma 2 and WMC(θ) the right hand side of (7) is $\ll F(X, 2X\xi)/\xi^2 \ll XL^\theta/\xi$, which, by (6), makes the contribution $X\xi L^\theta$.

Now we consider the contribution from the terms $2X\xi < \gamma \le K$ in $J_1 + J_2 + J_3$. We see immediately that this contribution is

$$\ll X\int_{\alpha X}^{\beta X}\Big|\sum_{2X\xi<\gamma\le K}\frac{y^{i\gamma}}{\varrho}\Big|^2dy+\frac{X}{\xi}\bigg(\Big|\sum_{2X\xi<\gamma\le K}\frac{X^{i\gamma}}{\varrho}\Big|^2+\Big|\sum_{2X\xi<\gamma\le K}\frac{(2X)^{i\gamma}}{\varrho}\Big|^2\bigg).$$

By Lemma 3 and WMC(θ) the first term is $\ll XL^\theta/\xi$. The other terms on the right come from J_1 and J_3. By Lemma 3 and WMC(θ) they contribute $\ll XL^{2+\theta}/\xi$; alternatively they are bounded by $I_1 + I_3$, which by the Brun–Titchmarsh theorem is $\ll L^2/(\xi^3(\log 2\xi)^2)$. These estimates, combined with (6), give the second error term in the Theorem.

Now we prove the Corollary. Let $H = [CL^\theta]$, where $C \ge 1$ is a constant. Define

$$L(\alpha)=\Big|\sum_{m=1}^{H}e(-m\alpha)\Big|^2=\sum_{m=-H}^{H}a(m)e(-m\alpha),$$

where $a(m) = H - |m|$,

$$R(n)=\sum_{h+k=n}\Lambda(h)\Lambda(k),\qquad S(\alpha)=\sum_{X\le n\le 2X}\Lambda(n)e(n\alpha),$$

$$T(\alpha)=\sum_{X\le n\le 2X}e(n\alpha),\qquad E(\alpha)=S(\alpha)^2-T(\alpha)^2.$$

We have

$$\sum_{n=X-H}^{X+H}a(n-X)R(n)=\int_{-1/2}^{1/2}S(\alpha)^2L(\alpha)e(-X\alpha)\,d\alpha \tag{8}$$

$$=\int_{-1/2}^{1/2}T(\alpha)^2L(\alpha)e(-X\alpha)\,d\alpha+\int_{-1/2}^{1/2}E(\alpha)L(\alpha)e(-X\alpha)\,d\alpha=A+B,$$

say. It is easy to prove that

$$A=\sum_{n=X-H}^{X+H} a(n-X)\sum_{h+k=n} 1 = H^2X+O(H^3). \tag{9}$$

Now we proceed to estimate B. Using

$$T(\alpha)\ll \min(X;1/|\alpha|)\quad \text{for } |\alpha|\le 1/2$$

we get

$$\int_{-\xi}^{\xi} |T(\alpha)|^2\,d\alpha \begin{cases} \ll X^2\xi & \text{if } 0<\xi<1/X,\\ = X+O(1/\xi) & \text{if } 1/X\le\xi\le 1/2.\end{cases} \tag{10}$$

Hence, using the identity

$$f^2-g^2=2f(f-g)-(f-g)^2,$$

the Cauchy–Schwarz inequality and (10), we have

$$\int_{-\xi}^{\xi} |E(\alpha)|\,d\alpha \ll \Big(X\int_{-\xi}^{\xi} |S(\alpha)-T(\alpha)|^2\,d\alpha\Big)^{1/2}+\int_{-\xi}^{\xi} |S(\alpha)-T(\alpha)|^2\,d\alpha \tag{11}$$

provided $1/X\le\xi\le 1/2$.

Since

$$L(\alpha)\ll \min(H^2;1/|\alpha|^2)\quad \text{for } |\alpha|\le 1/2, \tag{12}$$

we have

$$B\ll H^2\int_{-1/H}^{1/H} |E(\alpha)|\,d\alpha+\int_{1/H}^{1/2} |E(\alpha)|\frac{d\alpha}{\alpha^2}. \tag{13}$$

From the Theorem and (11) we get

$$H^2\int_{-1/H}^{1/H} |E(\alpha)|\,d\alpha \ll H^{3/2}XL^{\theta/2}. \tag{14}$$

By partial integration, the Theorem and (11) we obtain

$$\int_{1/H}^{1/2} |E(\alpha)|\frac{d\alpha}{\alpha^2}\ll H^{3/2}XL^{\theta/2}. \tag{15}$$

Hence from (13)–(15) we have

$$B\ll H^{3/2}XL^{\theta/2}, \tag{16}$$

and from (8), (9) and (16) we get

$$\sum_{n=X-H}^{X+H} a(n-X)R(n) \gg H^2 X \tag{17}$$

provided that C is sufficiently large. Thus the Corollary follows.

4. Proof of the Proposition. Let

$$\mathfrak{S} = \prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right).$$

We have

$$\begin{aligned}\sum_{n\le X}\mathfrak{S}(n)^2 &= 4\mathfrak{S}^2\sum_{2n\le X}\prod_{\substack{p\,|\,n\\ p>2}}\left(\frac{p-1}{p-2}\right)^2\\ &= 4\mathfrak{S}^2\sum_{2n\le X}\prod_{\substack{p|n\\ p>2}}\left(1+\frac{2p-3}{(p-2)^2}\right)\\ &= 4\mathfrak{S}^2\sum_{n\le X/2}\sum_{j|n} f(j),\end{aligned} \tag{18}$$

where

$$f(j) = \begin{cases}\mu^2(j)\prod_{p|j}\dfrac{2p-3}{(p-2)^2} & \text{if } j \text{ is odd},\\ 0 & \text{if } j \text{ is even}.\end{cases}$$

Then, changing the order of summation in (18), we obtain

$$\begin{aligned}\sum_{n\le X}\mathfrak{S}(n)^2 &= 4\mathfrak{S}^2\sum_{j\le X/2} f(j)\left[\frac{X}{2j}\right]\\ &= 2\mathfrak{S}^2 X\sum_{j=1}^{\infty}\frac{f(j)}{j} - 2\mathfrak{S}^2 X\sum_{j>X/2}\frac{f(j)}{j}\\ &\quad - 2\mathfrak{S}^2\sum_{j\le X/2} f(j) - 4\mathfrak{S}^2\sum_{j\le X/2} f(j)P\left(\frac{X}{2j}\right),\end{aligned} \tag{19}$$

where $P(u) = u - [u] - 1/2$.

By straightforward computations we get

$$\begin{aligned}2\mathfrak{S}^2\sum_{j=1}^{\infty}\frac{f(j)}{j} &= 2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)^2\left(1+\frac{2p-3}{p(p-2)^2}\right)\\ &= 2\prod_{p>2}\left(1+\frac{1}{(p-1)^3}\right).\end{aligned} \tag{20}$$

Next, we will show

$$\sum_{j\le U} f(j) = \frac{1}{8\mathfrak{S}^2}\log^2 U + B\log U + O(1), \tag{21}$$

where B is a constant. By partial summation this implies

$$\sum_{j>U}\frac{f(j)}{j} = \frac{1}{4\mathfrak{S}^2}\cdot\frac{\log U}{U} + O\Big(\frac{1}{U}\Big). \tag{22}$$

The Proposition follows from (19)–(22) together with the estimate

$$\sum_{j\le X} f(j)P(X/j) \ll L^{5/3}. \tag{23}$$

Now we prove (21). Writing

$$H(s) = \sum_{m=1}^{\infty} f(m)m^{-s} = \prod_{p>2}\Big(1+\frac{2p-3}{p^s(p-2)^2}\Big)$$

we see that $H(s)$ is an analytic function for $\operatorname{Re} s = \sigma > 0$ and, using the Perron formula with error term (see, e.g., Lemma 3.12 of Titchmarsh [15]), we obtain

$$\sum_{j\le U} f(j) = \frac{1}{2\pi i}\int_{c-iZ}^{c+iZ} H(s)\frac{U^s}{s}\,ds + O\Big(U^c\sum_{m=1}^{\infty}\frac{|f(m)|}{m^c(1+Z|\log(U/m)|)}\Big),$$

where $\varepsilon > 0$ is a fixed constant, $c = \varepsilon + 1/\log U < 1/4$ and Z will be chosen later on.

The error term can be estimated using

$$f(m) \ll \frac{d(m)}{m}\sum_{h|m}\frac{d^2(h)}{h}$$

(see (30) of Goldston [3]), where $d(m)$ is the divisor function, and the classical estimates $\sum_{v\le m} d(v)^q \ll m(\log m)^{2^q-1}$ (see, e.g., Theorem 5.3 of Hua [7]) and $d(m) \ll m^{\varepsilon}$ (see, e.g., Theorem 315 of Hardy–Wright [5]). So we have

$$\sum_{j\le U} f(j) = \frac{1}{2\pi i}\int_{c-iZ}^{c+iZ} H(s)\frac{U^s}{s}\,ds + O\Big(\frac{U^c}{Z}\Big). \tag{24}$$

Now we observe that

$$H(s) = \Big(1-\frac{1}{2^{s+1}}\Big)^2\zeta(s+1)^2 g(s),$$

where

$$g(s) = \prod_{p>2}\Big(1-\frac{1}{p^{s+1}}\Big)^2\Big(1+\frac{2p-3}{p^s(p-2)^2}\Big),$$

which converges absolutely and is analytic for $\sigma > -1/2$. So $H(s)U^s/s$ has a triple pole at $s=0$ with residue

$$\frac{1}{8\mathfrak{S}^2}\log^2 U + B\log U + O(1).$$

Consider a rectangular contour with right side $s = c+it$, $t\in[-Z,Z]$, and left side $-1/4+it$, $t\in[-Z,Z]$. The contribution of the top, bottom and left sides of the contour can be estimated using $\zeta(\sigma+it) \ll t^{1/6}$ for $\sigma \ge 1/2$ (see, e.g., Titchmarsh [15], p. 115). Hence we have

$$\sum_{j\le U} f(j) = \frac{1}{8\mathfrak{S}^2}\log^2 U + B\log U + O(Z^{-2/3}U^c + Z^{1/3}U^{-1/4}) + O(1).$$

Choosing $Z = U^{3c}$, we obtain (21).

Now we prove (23). For j odd we have

$$\begin{aligned} f(j) &= \mu^2(j)\prod_{p\mid j}\Big(\frac{2}{p-2}\Big)\Big(\frac{p-3/2}{p-2}\Big) = \frac{\mu^2(j)d(j)}{\varphi_2(j)}\prod_{p\mid j}\Big(1+\frac{1}{2(p-2)}\Big)\\ &= \frac{\mu^2(j)d(j)}{\varphi_2(j)}\sum_{\delta\mid j}\frac{\mu^2(\delta)}{2^{\omega(\delta)}\varphi_2(\delta)}, \end{aligned}$$

where $\omega(n)$ is the number of distinct prime factors of n, $\varphi_2(p) = p-2$ and φ_2 is extended to square-free numbers by multiplicativity.

Hence, interchanging the order of summation, we obtain

$$\sum_{j\le X} f(j)P\Big(\frac{X}{j}\Big) = \sum_{\substack{\delta\le X\\ (\delta,2)=1}}\frac{\mu^2(\delta)d(\delta)}{2^{\omega(\delta)}(\varphi_2(\delta))^2}\Big(\sum_{\substack{k\le X/\delta\\ (k,2)=1}}\frac{\mu^2(k)d(k)}{\varphi_2(k)}P\Big(\frac{X/\delta}{k}\Big)\Big).$$

Using the argument in (2.9)–(2.13) of Friedlander–Goldston [2], we find that the inner sum can be estimated by

$$\sum_{n\le X}\frac{d(n)}{n}P(X/n),$$

which is $\ll L^{5/3}$ by the remark at the end of Section 2 of [2]. Using this estimate we obtain

$$\sum_{j\le X} f(j)P\Big(\frac{X}{j}\Big) \ll L^{5/3}\sum_{\substack{\delta\le X\\ (\delta,2)=1}}\frac{\mu^2(\delta)d(\delta)}{(\varphi_2(\delta))^2}$$

and hence (23) follows from the convergence of the series

$$\sum_{\substack{\delta=1\\ (\delta,2)=1}}^{\infty}\frac{\mu^2(\delta)d(\delta)}{(\varphi_2(\delta))^2}.$$

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