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SOME RESULTS ON GOLDBACH'S PROBLEM

Abstract. In Section 1 we introduce the Goldbach Conjecture and give a brief account on the main contribution to this subject. In the next sections we sketch the proofs of some results on the existence of Goldbach numbers in short intervals and the exceptional set for Goldbach's problem.

1. Introduction

In 1742, in two letters to Euler, Goldbach conjectured that

every even integer $n > 2$ is a sum of two primes.

This statement is known as the Goldbach Conjecture. Sometimes by the Goldbach Conjecture one means also the weaker statement

every sufficiently large even integer n is a sum of two primes.

The previous statements are still unproven. In the following we will call "G-number" an even integer satisfying the Goldbach conjecture.

A short history of the main results on this topic is the following.

In 1919, Brun [3], using a combinatorial variant of the Eratosthenes sieve, proved that every sufficiently large even integer can be written as a sum of two integers with at most 9 prime divisors. In the 1920's Hardy and Littlewood [14,15] applied their method (the circle method) to this problem. They proved, under the Generalized Riemann Hypothesis (GRH), the following two results:

(i) every sufficiently large odd integer is a sum of three primes (ternary Goldbach problem);

(ii) writing $E = \{2n : 2n \text{ is not a } G\text{-number}\}$ for the exceptional set for Goldbach's problem and letting $E(N) = E \cap [1, N]$, one has for every $\varepsilon > 0$ that

$$|E(N)| \ll N^{1/2+\varepsilon},$$

i.e., "almost all" even integers are G-numbers. In 1937, I.M. Vinogradov [35,36] removed the dependence on GRH in (i) and so he solved unconditionally the ternary Goldbach problem. For the proof see Vaughan [34], ch. 3, or Davenport [6], ch. 26.

The problem of the distribution of G-numbers in short intervals goes back to Linnik [25]. In 1952, using the circle method, he proved, under the assumption of the Riemann Hypothesis (RH), that every interval $[N, N+H]$, with N sufficiently large and $H \geq \log^{3+\epsilon} N$, contains a G-number.

In the latest 40 years many results have been proved. The state of the art is the following.

Concerning "approximations" of Goldbach's problem, in 1966 Chen [4], [5] proved that every sufficiently large even integer can be written as a sum of a prime and an "almost-prime" number (an "almost-prime" number is an integer with at most two prime factors).

THEOREM (Chen [4], [5], 1966). *Every sufficiently large even integer N can be represented as*

$$N = p + a,$$

where p is a prime and $a \in P_2 := \{a \in \mathbb{N} : a$ has at most two prime divisors $\}$.

To prove Chen's theorem one can use the modern sieve techniques and some analytic arguments developed to study the distribution of primes in arithmetic progressions. For a proof see e.g. Halberstam-Richert [13], ch. 11.

In 1975 Montgomery-Vaughan [29] obtained an unconditional result on the size of the exceptional set.

THEOREM (Montgomery-Vaughan [29], 1975). *There exists an effectively computable constant $\delta > 0$ such that*

$$|E(N)| \ll N^{1-\delta}.$$

The best conditional (under GRH) result in this direction was proved by Goldston [11] and Kaczorowski-Perelli-Pintz [18] (see the remark in the next page):

THEOREM (Goldston [11], 1992; Kaczorowski-Perelli-Pintz [18], 1993). *Assume GRH. Then*

$$|E(N)| \ll N^{1/2} \log^3 N .$$

Writing $E(N, H) = E \cap [N, N+H]$ for the exceptional set in short intervals, the best unconditional result concerning the size of $E(N, H)$ is

THEOREM (Mikawa [26], 1992). *Let $\varepsilon > 0$, $A > 0$ be arbitrary constants and $N^{7/48+\varepsilon} \leq H \leq N$. Then*

$$|E(N, H)| \ll_{\varepsilon, A} HL^{-A}.$$

Mikawa's result does not give information on the asymptotic behavior of the counting function $R(2n) = \sum_{h+k=2n} \Lambda(h)\Lambda(k)$ of the number of representations of a G-number as a sum of two primes. In this direction the best unconditional result in short intervals is

THEOREM (Perelli-Pintz [30], 1992). *Let $\varepsilon > 0$, $A, B > 0$ be arbitrary constants and $N^{1/3+\varepsilon} \leq H \leq N$. Then for all $2n \in [N, N+H]$, with at most $O(HL^{-B})$ exceptions, one has*

$$R(2n) = 2n\mathfrak{S}(2n) + O(NL^{-A}),$$

where $\mathfrak{S}(2n) = 2\prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right)\prod_{\substack{p|2n\\p>2}}\left(\frac{p-1}{p-2}\right)$ is the "singular series" of Goldbach's problem.

A similar result holds for a certain restricted counting function $R^*(2n)$ in the shorter interval $N^{7/36+\varepsilon} \leq H \leq N$, see Perelli-Pintz [30].

In the class of conditional results on the exceptional set in short intervals the best one (under GRH) is the following

THEOREM (Kaczorowski-Perelli-Pintz [18], 1993). *Assume GRH and let $H\log^{-6} N \to \infty$ for $N \to \infty$. Then*

$$|E(N,H)| \ll H^{1/2}\log^3 N.$$

Actually, Kaczorowski-Perelli-Pintz's proof was not totally correct. In fact their technique yields only $|E(N,H)| \ll H^{1/2}\log^5 N$. However the result can be saved by a technical device, see Languasco-Perelli [23] and [21].

Assuming further a pair correlation type hypothesis on the zeros of the Dirichlet L-functions, A. Perelli and the author have recently proved the following

THEOREM (Languasco-Perelli [23], 1995). *Assume GRH and a certain pair correlation hypothesis on the zeros of the Dirichlet L-series. Let $\varepsilon > 0$ and $N^\varepsilon \leq H \leq N$. Then*

$$|E(N,H)| \ll N^\varepsilon.$$

In Section 5 we will explain, with some details, the basic argument in the proof of the previous theorem.

Another problem is to study the existence of G-numbers in short intervals. We have the following

THEOREM (Ramachandra [31,32], 1973 and 1976; Montgomery-Vaughan [29], 1975). *Let $\varepsilon > 0$ be an arbitrary constant and let N be sufficiently large. Then*

$$\sum_{n\in[N,N+N^\theta]} r(n) \gg N^\theta ,$$

where $r(n) = \sum_{n=p_1+p_2} 1$, $\theta > \theta_1\theta_2$ and θ_1,θ_2 are two positive real numbers such that

(i) *every interval $[x, x+H_1]$, with $H_1 \geq x^{\theta_1+\varepsilon}$, contains $\gg \frac{H_1}{\log x}$ prime numbers;*

(ii) *all but $o(\frac{X}{\log X})$ intervals of the type $[x, x+H_2]$, $x \in \mathbb{N} \cap [1, X]$ and $H_2 \geq x^{\theta_2+\varepsilon}$, contain $\gg \frac{H_2}{\log x}$ prime numbers.*

This means that the existence of G-numbers in short intervals is connected with the distribution level (θ_1) of primes in short intervals and with the distribution level (θ_2) of primes in "almost all" short intervals. The best results are $\theta_1 = 0.535$ (Baker-Harman [2]) and $\theta_2 = \frac{1}{14}$ (Watt [37]), hence the technique of Ramachandra and Montgomery-Vaughan can prove that there exist G-numbers in $[N, N+H]$, provided that $H \gg N^{\theta+\varepsilon}$, $\theta = 0.535\frac{1}{14} = 0.03821\ldots$. We observe that from the above result one can get also that a positive proportion of the numbers in $[N, N+H]$ are G-numbers, provided that $H \gg N^{\theta+\varepsilon}$.

Assuming suitable hypotheses on the distribution of the zeros of the Riemann zeta function, one can obtain better results. Indeed we have

THEOREM (Kátai [19], 1967; Montgomery-Vaughan [29], 1975). *Assume RH. Then there exists $C > 0$ such that, for N sufficiently large, the interval $[N, N + C\log^2 N]$ contains a G-number.*

Other proofs, based on the circle method, have been given by Goldston [10] (1990) and by Languasco-Perelli [22] (1994), see Section 3.

The previous result can be sharpened using a stronger hypothesis. Assuming Montgomery's conjecture (MC), see [28], one has

THEOREM (Goldston [10], 1990). *Assume RH and MC. Then there exists $C > 0$ such that, for N sufficiently large, the interval $[N, N + C\log N]$ contains a G-number.*

Recently the author [21] has proved, in analogy with Ramachandra and Montgomery-Vaughan's result, that, under the assumption of RH and MC, a positive proportion of the numbers in $[N, N+H]$ are G-numbers, provided that $H \gg \log N$.

The methods used to obtain the last two results apparently do not give intervals shorter than $\log N$. One can obtain shorter intervals assuming further the Elliott-Halberstam conjecture (EH), see Elliott-Halberstam [7]. We have:

THEOREM (Friedlander-Goldston [8], 1995). *Assume RH, MC and EH. Then there exist* $C > 0$ *such that, for* N *sufficiently large, the interval* $[N, N + C\frac{(\log\log N)^2}{\log\log\log N}]$ *contains a* G*-number.*

For a more exhaustive presentation of these results see the author's Ph. D. thesis [20]. We finally remark that some improvements of the above results by Mikawa, Baker-Harman and Watt have been recently obtained by Chinese researchers.

2. The Circle Method

An important tool to approach additive problems and, hence, to prove some of the previously quoted results, is the circle (or Hardy-Littlewood) method. Since Zaccagnini [38], also collected in this volume, furnishes a general introduction to this method, in the following we use his notation, specialized in the case of Goldbach's problem.

We write

$$\mathcal{A} = \mathcal{B} = \mathfrak{P} = \{p \in \mathbb{N} : p \text{ prime }\}$$

and, by technical reasons, we take

$$R(N) = \sum_{h+k=N} \Lambda(h)\Lambda(k),$$

where $\Lambda(n)$ is the von Mangoldt function, as the weighted counting function of the set of the G-numbers. Then the associated Fourier polynomial becomes

$$S(\alpha) = \sum_{n \le 2N} \Lambda(n)e(n\alpha).$$

To obtain information on the asymptotic formula for $R(N)$ and on the size of the exceptional set we have to distinguish between major and minor arcs, while for the existence of G-numbers in short intervals, as we will see in the next section, it suffices to study the behavior of $S(\alpha)$ in a neighborhood of 0.

3. Circle Method and G-numbers in short intervals

In what follows, we will present in more detail our result on G-numbers in short intervals.

Since the explicit formula for $S(\alpha)$ is not a direct one (see, e.g., Baker-Harman [1]), for the problem in short intervals it is more convenient to use a "smooth" version of $S(\alpha)$, i.e.,

$$\widetilde{S}(\alpha) = \sum_{n=1}^{\infty} \Lambda(n)e(n\alpha)e^{-n/N}.$$

$\widetilde{S}(\alpha)$ is the original Hardy-Littlewood function. We have the following explicit formula

$$\widetilde{S}(\alpha) = \frac{1}{z} - \sum_{\rho} z^{-\rho}\Gamma(\rho) + O((\log N)^3)$$

where $z = \frac{1}{N} - 2\pi i\alpha$, ρ runs over the non trivial zeros of $\zeta(s)$ and Γ is the gamma function.

The result is

THEOREM 1 (Languasco-Perelli [22]). *Assume RH, let* $z = \frac{1}{N} - 2\pi i\alpha$ *and* $L = \log N$. *For* N *sufficiently large and* $0 \leq \xi \leq \frac{1}{2}$ *we have*

$$\int_{-\xi}^{\xi} |\widetilde{S}(\alpha)^2 - \frac{1}{z^2}| d\alpha \ll N\xi L^2 + N\xi^{1/2} L \ . \tag{1}$$

Theorem 1 sharpens (by a $\log N$ factor) an analogous result due to Linnik [25]. This follows by using the ingenious averaging technique of Saffari-Vaughan (see [33], Lemmas 5 and 6) which makes the use of the explicit formula for $\widetilde{S}(\alpha)$ more efficient.

From Theorem 1 we deduce, by a pure circle method technique, the result of Kátai and Montgomery-Vaughan:

COROLLARY 1. *Assume RH. There exists a constants* $C > 0$ *such that, for* N *sufficiently large, the interval* $[N, N + CL^2]$ *contains G-numbers.*

We now sketch the proof given in [22], Corollary 1.

Let

$$L(\alpha) = \left| \sum_{m=1}^{H} e(-m\alpha) \right|^2 = \sum_{m=-H}^{H} a(m) e(-m\alpha) \ ,$$

where $a(m) = H - |m|$ and

$$\widetilde{E}(\alpha) = \widetilde{S}(\alpha)^2 - \frac{1}{z^2} \ .$$

Hence we get

$$\begin{aligned} \sum_{n=N-H}^{N+H} a(n-N) e^{-n/N} R(N) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \widetilde{S}(\alpha)^2 L(\alpha) e(-N\alpha) d\alpha \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{L(\alpha)}{z^2} e(-N\alpha) d\alpha + \int_{-\frac{1}{2}}^{\frac{1}{2}} \widetilde{E}(\alpha) L(\alpha) e(-N\alpha) d\alpha = I_1 + I_2 \ . \end{aligned} \tag{2}$$

I_1 is computed using the residue theorem. We have

$$\begin{aligned} I_1 &= \sum_{n=N-H}^{N+H} a(n-N) n e^{-n/N} + O(H^2) = \frac{N}{e} \sum_{n=N-H}^{N+H} a(n-N) + O(H^3) \\ &= \frac{H^2 N}{e} + O(H^3) \ . \end{aligned} \tag{3}$$

The estimation of I_2 follows from (1) and

$$L(\alpha) \ll \min(H^2, \frac{1}{|\alpha|^2}).$$

By partial integration we obtain

(4) $$I_2 \ll HNL^2 + H^{3/2}NL.$$

Finally, from (2), (3) and (4), it follows that

(5) $$\sum_{n=N-H}^{N+H} a(n-N)e^{-n/N}R(N) = \frac{H^2N}{e} + O(H^3 + HNL^2 + H^{3/2}NL) .$$

Choosing $H = CL^2$, $C > 0$ sufficiently large, from (5) we get

$$\sum_{n=N-H}^{N+H} a(n-N)e^{-n/N}R(N) \gg H^2N$$

and then Corollary 1 follows.

4. Parseval identity for $\widetilde{S}(\alpha)$

In this section we discuss a problem related to the topics we have seen in the previous section: the truncated Parseval identity for $\widetilde{S}(\alpha)$. It is easy to prove, using the Parseval identity and the Prime Number Theorem, that

(6) $$\int_{-\frac{1}{2}}^{\frac{1}{2}} |\widetilde{S}(\alpha)|^2 d\alpha \sim \frac{NL}{2} .$$

Using Theorem 1 and

(7) $$\int_{-\xi}^{\xi} \frac{1}{|z|^2} d\alpha = \frac{N}{\pi} \arctan(2\pi N\xi),$$

which is easily obtained, we have

$$\int_{-\xi}^{\xi} |\widetilde{S}(\alpha)|^2 d\alpha = \frac{N}{\pi} \arctan(2\pi N\xi) + O(N\xi L^2 + N\xi^{1/2}L),$$

i.e., a conditional truncated version of (6). However we can observe that taking $\xi = \frac{1}{2}$ in the previous formula, we obtain only the result, weaker than (6),

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |\widetilde{S}(\alpha)|^2 d\alpha \ll NL^2 .$$

In 1959, Lavrik [24] proved that

$$\int_a^b |\widetilde{S}(\alpha)|^2 d\alpha = \frac{b-a}{2} NL + O(N \log^2 L)$$

if $0 \le b - a \le 1$. An unconditional result concerning truncations of Parseval's identity, which improves Lavrik's result, is the following

THEOREM 2 (Languasco-Perelli [22]). *Let* $0 \le b-a \le 1$ *and* N *sufficiently large. Then*

$$\int_a^b |\widetilde{S}(\alpha)|^2 d\alpha = \frac{b-a}{2} NL + O(N(L(b-a))^{1/3}) + O(N) .$$

We remark that Theorem 2 is essentially the best possible, in the sense that one cannot replace the term $O(N)$ by $o(N)$. We finally remark that Theorem 2 enables one to deduce the order of magnitude of $\int_{-\xi}^{\xi} |\widetilde{S}(\alpha)|^2 d\alpha$ in the whole range $0 \le \xi \le \frac{1}{2}$. In fact we have

COROLLARY 2 (Languasco-Perelli [22]). *Let* N *be sufficiently large. Then*

$$\int_{-\xi}^{\xi} |\widetilde{S}(\alpha)|^2 d\alpha \asymp \begin{cases} N^2\xi & \text{if } 0 \le \xi \le \frac{1}{N} \\ N & \text{if } \frac{1}{N} \le \xi \le \frac{1}{L} \\ N\xi L & \text{if } \frac{1}{L} \le \xi \le \frac{1}{2}. \end{cases}$$

The proof of Corollary 2 runs as follows.

If $0 \le \xi \le \frac{1}{N}$, by Stirling's formula and the zero-free region of $\zeta(s)$ we have

$$\sum_{\rho} z^{-\rho}\Gamma(\rho) \ll \sum_{\rho} |z|^{-\beta} |\gamma|^{\beta-\frac{1}{2}} \exp(\gamma \arctan 2\pi N\alpha - \frac{\pi}{2}|\gamma|) = o(N) ,$$

where $\rho = \beta + i\gamma$ runs over the non-trivial zeros of $\zeta(s)$. Using (7) we have

$$\int_{-\xi}^{\xi} |\widetilde{S}(\alpha)|^2 d\alpha \asymp N^2\xi .$$

Since $\int_{-\xi}^{\xi} |\widetilde{S}(\alpha)|^2 d\alpha$ is an increasing function of ξ, from the previous result we have that

$$\int_{-\xi}^{\xi} |\widetilde{S}(\alpha)|^2 d\alpha \gg N$$

for $\frac{1}{N} \le \xi \le \frac{1}{L}$. The corresponding upper bound follows from Theorem 2. Then Corollary 2 follows arguing in a similar way in the range $\frac{1}{L} \le \xi \le \frac{1}{2}$.

5. The exceptional set

The classical method to study the size of the exceptional set for Goldbach's problem is based on the following remark: if

$$\sum_{N \le 2n \le N+H} |R(2n) - 2n\mathfrak{S}(2n)|^2 \ll N^2 f(N,H) \tag{8}$$

then, since $\mathfrak{S}(2n) \gg 1$, one has

$$E(N,H) \le \sum_{N \le 2n \le N+H} (2n)^{-2} |R(2n) - 2n\mathfrak{S}(2n)|^2 \ll f(N,H). \tag{9}$$

Obviously (8) and (9) are of interest only if $f(N,H) = o(H)$. In the following we will sketch a conditional proof of (8) with $f(N,H) = N^{\varepsilon}$.

Writing in this case

$$S(\frac{a}{q} + \eta) = \frac{\mu(q)}{\varphi(q)} T(\eta) + R(\eta, q, a),$$

where $T(\eta) = \sum_{n \le 2N} e(n\eta)$, it is not difficult to prove, applying the circle method, see [18], with parameters P and Q, see [38], that

$$\begin{aligned} \sum_{N \le 2n \le N+H} &|R(2n) - 2n\mathfrak{S}(2n) + F(n,N,H)|^2 \\ &\le \sum_{N \le 2n \le N+H} \left| \int_{\mathfrak{m}} S(\alpha)^2 e(-2n\alpha) d\alpha \right|^2 \\ &+ \sum_{N \le 2n \le N+H} \left| 2n \sum_{q > P} \frac{\mu(q)^2}{\varphi(q)^2} c_q(-2n) \right|^2, \end{aligned} \tag{10}$$

where $F(n,N,H)$ is a certain function satisfying

$$F(n,N,H) \ll N^{1/2} \sum_{q \le P} \varphi(q)^{-1/2} \Big(\sum_{a=1}^{q}{}^{*} \int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} |R(\eta,q,a)|^2 d\eta \Big)^{1/2} + PQ. \tag{11}$$

Since, under GRH, we have, see [18] and [22],

$$\sum_{a=1}^{q}{}^{*} \int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} |R(\eta,q,a)|^2 d\eta \ll \frac{NL^4}{Q}, \tag{12}$$

we obtain, choosing $P = Q \log^{-10} N$, that (11) becomes

$$F(n,N,H) \ll N \log^{-3} N, \tag{13}$$

uniformly for $2n \in [N, N+H]$.

We remark that (12) is the corrected version of Lemma 1 of [18], see [22], [21].

The second term on the right hand side of (10) is of arithmetical nature, and its estimation (see Lemma 2 of [18]) leads to a negligible error term.

Hence, the key point is the estimation of the first term on the right hand side of (10). Squaring out and interchanging summation and integration we get

$$\sum_{N\le 2n\le N+H}\left|\int_{\mathfrak{m}} S(\alpha)^2 e(-2n\alpha)d\alpha\right|^2 \ll HNL \max_{\substack{P<q\le Q\\ (a,q)=1}} \int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} |S(\frac{a}{q}+\eta)|^2 d\eta, \tag{14}$$

i.e., the problem is now to estimate the mean square of $|S(\frac{a}{q}+\eta)|$ over a single minor arc. Since $T(\eta) \ll \min(N, \frac{1}{|\eta|})$, we have

$$\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} |S(\frac{a}{q}+\eta)|^2 d\eta \ll \frac{N}{\varphi(q)^2} + \int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} |R(\eta,q,a)|^2 d\eta, \tag{15}$$

so it is clear that the quantity defined by

$$I(X,Q;q,a) = \int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} |R(\eta,q,a)|^2 d\eta$$

plays an important role in (14). By (12), we could expect a bound of the type

$$I(X,Q;q,a) \ll \frac{NL^4}{qQ}. \tag{16}$$

In a recent paper, A. Perelli and the author proved that a form of Montgomery's pair correlation conjecture implies a weaker version of (16) where the factor L^4 is replaced by N^ε. The result is the following

THEOREM 3 (Languasco-Perelli [22]). *Assume GRH. Let*

$$F(N,T;q,a) = \sum_{\chi_1,\chi_2 (\mathrm{mod}\ q)} \chi_1(a)\overline{\chi}_2(a)\tau(\overline{\chi}_1)\tau(\chi_2) \sum_{|\gamma_1|,|\gamma_2|\le T} N^{i(\gamma_1-\gamma_2)} w(\gamma_1-\gamma_2),$$

where $\tau(\chi)$ denotes the Gauss sum and γ_j, $j=1,2$, run over the imaginary parts of the non trivial zeros of $L(s,\chi_j)$. If $F(N,T;q,a)$ verifies

$$F(N,T;q,a) \ll_\varepsilon q^2 T N^\varepsilon \tag{17}$$

uniformly for $\frac{N^{1/2}}{q} \le T \le N$, $q \le \frac{N^{1/2}}{2} = Q$ and $(a,q)=1$ then

$$I(X,Q;a,a) \ll \frac{N^{1+\varepsilon}}{qQ}$$

holds for $q \le Q$ and $(a,q)=1$.

Actually, the result in [22] is more precise and depends on the choice $Q = N^{\theta}$, $\theta \in (0, \frac{1}{2}]$. We also remark that the trivial upper bound for $F(N,T;q,a)$, as $T \to \infty$, is clearly

$$F(N,T;q,a) \ll q\varphi(q)^2 T \log^2 qT$$

uniformly in N, q and a. Moreover, by adapting Montgomery's method in [28] we can prove, see [22], that

$$F(N,T;q,a) \sim \frac{1}{\pi}\varphi(q)^2 T \log N, \tag{18}$$

uniformly for $N \log N \le T \le N^A$ and $q \le N \log^{-4} N$, and that

$$F(N,T;q,a) \ll_A q^2 T \log N, \tag{19}$$

uniformly for $N \le T \le N^A$ and $q \le N \log^{-3} N$.

Now, using Theorem 3 and (10)-(15), we have that (8) holds with $f(N,H) = N^{\epsilon}$ and then, by (9), we have

$$E(N,H) \ll N^{\epsilon}.$$

Finally observe that one can repeat the previous argument replacing in (17) the factor N^{ϵ} with $\log N$, i.e., using in the conjecture the expected order of magnitude for $F(N,T;q,a)$. So, choosing $P = QL^{-1-\epsilon}$, we could expect that

$$E(N,H) \ll_{\epsilon} L^{3+\epsilon}(\log H)^2,$$

but the estimation of the tails, produced by an application of Gallagher's lemma, see [21], [23], allows us to choose only $P = QL^{-3-\epsilon}$ and then to obtain only the weaker estimate

$$E(N,H) \ll_{\epsilon} L^{7+\epsilon}(\log H)^2.$$

If we assume only GRH the tails problem can be avoided using the function $\widetilde{S}(\alpha)$ instead of $S(\alpha)$, see [21], [23].

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