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Poisson structures on Hilbert schemes of points of a surface and integrable systems

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Abstract. In this paper we prove that if S is a Poisson surface, i.e., a smooth algebraic surface with a Poisson structure, the Hilbert scheme of points of S has a natural Poisson structure, induced by the one of S. This generalizes previous results obtained by A. Beauville [B1] and S. Mukai [M2] in the symplectic case, i.e., when S is an abelian or K3 surface. Finally we apply our results to give some examples of integrable Hamiltonian systems naturally defined on these Hilbert schemes. In the simple case $S = \mathbb{P}^2$ we obtain by this construction a large class of integrable systems, which includes the ones studied by P. Vanhaecke in [V1] and, more generally, in [V2].

Introduction

Hilbert schemes of points of a smooth projective surface S, and their relations with the moduli spaces of sheaves on S, have been intensively studied by many authors (e.g., [F1], [F2], [I], [B1]). In particular, A. Beauville proved in [B1] that the Hilbert scheme of points of an abelian or K3 surface carries a natural (holomorphic) symplectic structure, induced by the one present on the surface, thus giving examples of irreducible symplectic varieties of any dimension. The same result follows also from a more general fact, proved by Mukai in [M2], namely the fact that the choice of a symplectic structure on a surface S determines in a natural way a symplectic structure on the moduli space of simple sheaves on S.

In [Bo] we generalized Mukai's result to the case of Poisson structures: we proved that the choice of a Poisson structure on a surface S canonically determines a Poisson structure on the moduli space of stable vector bundles on S. It is natural then to ask if the same result holds also for the Hilbert schemes of points of S. In this paper we prove that this is actually the case. Note that we cannot rely on the results proved in [Bo], because of the technical assumption made there that sheaves are locally free, so we present here a more direct proof.

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As an application of this result, we describe some naturally defined integrable Hamiltonian systems on the Hilbert schemes of points of S. These integrable systems generalize the ones considered by A. Beauville in [B2], in the case of a symplectic (i.e., abelian or K3) surface S. Even in the simple case $S = \mathbb{P}^2$, we find in this way a large class of interesting integrable systems, which includes the ones constructed by P. Vanhaecke in [V1] and, more recently and in a more general set-up, in [V2]. We refer to these papers for a detailed description of these integrable systems and of their relations to more classical ones.

1. Hilbert schemes

In this section we shall briefly recall some basic results concerning Hilbert schemes of points.

Let X be a smooth projective variety defined over an algebraically closed field k and let us denote by $X^{[d]} = \mathrm{Hilb}^d(X)$ the Hilbert scheme parametrizing 0-dimensional subschemes of X of length d. It is well known that $X^{[d]}$ is a projective scheme; it is smooth if $d \leq 3$ or $\dim X \leq 2$.

Let us denote by $X^{(d)}$ the d-fold symmetric power of X, i.e., the geometric quotient of X^d by the symmetric group $\mathfrak{S}_d$, acting by permuting the factors. We shall denote by $\pi : X^d \to X^{(d)}$ the canonical projection and by $\Delta \subset X^d$ the 'large diagonal', i.e., the subset of X^d of elements $(x_1, \ldots, x_d)$ such that $x_i = x_j$, for some $i \neq j$. $X^{(d)}$ is a projective variety whose singular locus is the image $\Delta' = \pi(\Delta)$ of Δ; it parametrizes effective 0-cycles of degree d on X, i.e., formal linear combinations $\sum n_i[x_i]$ of points x_i in X with coefficients $n_i \in \mathbb{N}$, such that $\sum n_i = d$.

By associating to each 0-dimensional subscheme its support (with multiplicities), we obtain a natural map $X^{[d]} \to X^{(d)}$, sending a 0-dimensional subscheme Z to $\sum_{x\in X} \mathrm{length}(\mathcal{O}_{Z,x})[x]$, called the Hilbert-Chow morphism (see, for instance, [F1], [F2], [I]). When $X^{[d]}$ is smooth (e.g., when X is a smooth projective surface), this map provides a natural desingularisation of $X^{(d)}$.

2. Moduli spaces of sheaves

From now on we shall restrict to the case of a smooth projective surface S defined over $\mathbb{C}$. We shall denote by $\mathcal{M}(1, c_1, c_2)$ the moduli space of coherent rank-1 torsion-free sheaves on S with Chern classes c_1 and c_2. Since all torsion-free sheaves of rank 1 are stable, $\mathcal{M}(1, c_1, c_2)$ is a (non-empty) projective variety. The structure of a coherent rank-1 torsion-free sheaf on S is well known:

Lemma 2.1. *Let $\mathcal{F}$ be a coherent torsion-free sheaf of rank 1 on S. Then $\mathcal{F}$ is isomorphic to $\mathcal{I} \otimes L$, where $\mathcal{I}$ is a sheaf of ideals of finite colength (i.e., the sheaf of ideals of a 0-dimensional scheme of finite length of S) and L is a locally free sheaf of rank 1 on S. $\mathcal{I}$ and L are uniquely determined, up to isomorphism, by $\mathcal{F}$, and we have: $c_1(\mathcal{F}) = c_1(L)$, $c_2(\mathcal{F}) = \mathrm{colength}(\mathcal{I})$.*

Proof. Since $\mathcal{F}$ is coherent and torsion-free, its double dual $\mathcal{F}^{**}$ is locally free of rank 1. We set $L = \mathcal{F}^{**}$, and consider the natural exact sequence

$$0 \to \mathcal{F} \to L \to \mathcal{T} \to 0, \tag{2.1}$$

where $\mathcal{T} = L/\mathcal{F}$ is a torsion sheaf, supported at a finite number of points.

By tensoring (2.1) with L^{-1}, we find that $\mathcal{F} \otimes L^{-1}$ is isomorphic to the ideal sheaf $\mathcal{I}$ of a 0-dimensional subscheme of S of finite length. The uniqueness of $\mathcal{I}$ and L is now obvious. Finally, the expressions for the Chern classes of $\mathcal{F}$ in terms of L and $\mathcal{I}$ follow from basic properties of Chern classes. □

Remark 2.2. We recall that, in the case of rank-r stable sheaves, there is a map

$$\det : \mathcal{M}(r, c_1, c_2) \to \mathrm{Pic}^{c_1}(S), \tag{2.2}$$

which associates to a stable sheaf of rank r its determinant line bundle. For $r = 1$ this map is the obvious one, which sends a sheaf $\mathcal{F} \cong \mathcal{I} \otimes L$ to L. Its fibers are canonically identified with the moduli space parametrizing isomorphism classes of ideals $\mathcal{I}$ of colength c_2, i.e., with the Hilbert scheme $S^{[c_2]}$.

In what follows we shall regard a point of $S^{[d]}$ as being given either by a sheaf of ideals $\mathcal{I}$ of colength d, or by a 0-dimensional subscheme Z of length d of S. If $\mathcal{I}$ is an ideal sheaf, we denote by $Z_{\mathcal{I}}$ the corresponding subscheme (viceversa, $\mathcal{I}_Z$ will denote the ideal sheaf of the subscheme Z). From what we have seen we derive the following well known result:

Corollary 2.3. *There is a natural isomorphism*

$$\mathcal{M}(1, c_1, c_2) \cong S^{[c_2]} \times \mathrm{Pic}^{c_1}(S),$$

given by associating to $\mathcal{F} \in \mathcal{M}(1, c_1, c_2)$ the pair $(\mathcal{I}, L)$ such that $\mathcal{F} \cong \mathcal{I} \otimes L$.

To simplify the notation, we shall denote by $\mathcal{F}$ (resp. $\mathcal{I}$) either a rank-1 sheaf (resp. a sheaf of ideals) on S or the corresponding point of $\mathcal{M}(1, c_1, c_2)$ (resp. of $S^{[c_2]}$). From deformation theory it follows that the tangent spaces to the moduli space $\mathcal{M}(1, c_1, c_2)$ and to the Hilbert scheme $S^{[c_2]}$ are given by

$$T_{\mathcal{F}}\mathcal{M}(1, c_1, c_2) = \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}), \qquad T_{\mathcal{I}}S^{[c_2]} = \mathrm{Hom}(\mathcal{I}, \mathcal{O}_S/\mathcal{I}).$$

As a consequence of Corollary 2.3, we obtain a decomposition of tangent spaces

$$\mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \cong \mathrm{Hom}(\mathcal{I}, \mathcal{O}_S/\mathcal{I}) \oplus H^1(S, \mathcal{O}_S),$$

where $\mathcal{F} \cong \mathcal{I} \otimes L$. It is not difficult to give a direct proof of this fact.

Now we shall derive some useful isomorphisms. Let $\mathcal{I} \in S^{[d]}$ and denote by Z the corresponding closed subscheme of S. We set $\mathcal{O}_Z = \mathcal{O}_S/\mathcal{I}$.

Lemma 2.4. *There are canonical isomorphisms*

$$\mathrm{Hom}(\mathcal{I}, \mathcal{O}_Z) \cong \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z) \cong \mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{I}).$$

Proof. First of all we note that $\mathrm{Hom}(\mathcal{O}_S, \mathcal{O}_Z) = \mathrm{Hom}(\mathcal{O}_Z, \mathcal{O}_Z)$, because every $\mathcal{O}_S$-linear homomorphism from $\mathcal{O}_S$ to $\mathcal{O}_Z$ vanishes on $\mathcal{I}$. Hence, by applying the functor $\mathrm{Hom}(\cdot, \mathcal{O}_Z)$ to the exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_S \to \mathcal{O}_Z \to 0, \tag{2.3}$$

we obtain a long exact sequence

$$0 \to \mathrm{Hom}(\mathcal{I}, \mathcal{O}_Z) \to \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z) \to \mathrm{Ext}^1(\mathcal{O}_S, \mathcal{O}_Z).$$

Since $\mathrm{Ext}^1(\mathcal{O}_S, \mathcal{O}_Z) = H^1(S, \mathcal{O}_Z) = 0$, it follows that $\mathrm{Hom}(\mathcal{I}, \mathcal{O}_Z) \cong \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z)$.

To obtain the second isomorphism, we apply the functor $\mathrm{Hom}(\mathcal{O}_Z, \cdot)$ to (2.3) obtaining the following long exact sequence

$$\begin{aligned}\cdots \to \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_S) \to \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z) \to \mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{I})&\\ \to \mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{O}_S) \to \mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z).&\end{aligned}$$

Now, by Grothendieck–Serre duality, we have

$$\begin{aligned}\mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_S) &\cong H^1(S, \mathcal{O}_Z \otimes \omega_S)^* = 0,\\ \mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{O}_S) &\cong H^0(S, \mathcal{O}_Z \otimes \omega_S)^*\end{aligned}$$

and

$$\mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z) \cong \mathrm{Hom}(\mathcal{O}_Z, \mathcal{O}_Z \otimes \omega_S)^* \cong H^0(S, \mathcal{O}_Z \otimes \omega_S)^*.$$

The last map is then an isomorphism, and it follows that

$$\mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z) \cong \mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{I}). \qquad \square$$

Remark 2.5. For any coherent torsion-free sheaf $\mathcal{F}$ and any $i \geq 0$ there is a trace map (cf. [A2])

$$\mathrm{tr}^i : \mathrm{Ext}^i(\mathcal{F}, \mathcal{F}) \to H^i(S, \mathcal{O}_S).$$

We shall denote by $\mathrm{Ext}^i_0(\mathcal{F}, \mathcal{F})$ the trace-free part of $\mathrm{Ext}^i(\mathcal{F}, \mathcal{F})$, i.e., the kernel of tr^i. From deformation theory it follows that $\mathrm{Ext}^1_0(\mathcal{F}, \mathcal{F})$ is canonically identified with the tangent space to the fiber $\det^{-1}(\xi)$ of the determinant map (2.2) at the point $\mathcal{F}$. Actually the map

$$\mathrm{tr}^1 : \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \to H^1(S, \mathcal{O}_S)$$

coincides with the tangent map, at the point $\mathcal{F}$, of the determinant map (2.2). From this it follows that the tangent space $T_{\mathcal{I}} S^{[d]}$ is canonically isomorphic to $\mathrm{Ext}^1_0(\mathcal{I}, \mathcal{I})$, i.e., there is a canonical isomorphism

$$\mathrm{Hom}(\mathcal{I}, \mathcal{O}_Z) \cong \mathrm{Ext}^1_0(\mathcal{I}, \mathcal{I}).$$

This last result may be proved directly as in [A2, Corollary 6.4].

By what we have seen, applying Grothendieck–Serre duality, we obtain:

Corollary 2.6. *For $\mathcal{I} \in S^{[d]}$ there are canonical isomorphisms*

$$T_{\mathcal{I}} S^{[d]} \cong \mathrm{Hom}(\mathcal{I}, \mathcal{O}_Z) \cong \mathrm{Ext}^1_0(\mathcal{I}, \mathcal{I}) \cong \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z) \cong \mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{I}),$$

and

$$\begin{aligned} T^*_{\mathcal{I}} S^{[d]} &\cong \mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{I} \otimes \omega_S) \cong \mathrm{Ext}^1_0(\mathcal{I}, \mathcal{I} \otimes \omega_S) \\ &\cong \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z \otimes \omega_S) \cong \mathrm{Hom}(\mathcal{I}, \mathcal{O}_Z \otimes \omega_S). \end{aligned}$$

3. Poisson structures on $S^{[d]}$

Let us suppose now that S is a Poisson surface, i.e., a smooth algebraic surface such that $H^0(S, \omega_S^{-1}) \neq 0$ (cf. [Bo]), and let us fix a Poisson structure $s \in H^0(S, \omega_S^{-1})$. By recalling Corollary 2.6, we may define a map

$$B_s(\mathcal{I}) : T^*_{\mathcal{I}} S^{[d]} \to T_{\mathcal{I}} S^{[d]}$$

by considering the map

$$B_s(\mathcal{I}) : \mathrm{Hom}(\mathcal{I}, \mathcal{O}_Z \otimes \omega_S) \to \mathrm{Hom}(\mathcal{I}, \mathcal{O}_Z)$$

induced by the multiplication by s. The maps $B_s(\mathcal{I})$ actually determine a global map

$$B_s : T^* S^{[d]} \to T S^{[d]},$$

which is equivalent to giving a section $\theta_s \in H^0(S^{[d]}, \otimes^2 T S^{[d]})$.

By recalling the isomorphisms

$$\mathrm{Hom}(\mathcal{I}, \mathcal{O}_Z) \cong \mathrm{Ext}^1_0(\mathcal{I}, \mathcal{I})$$

and

$$\mathrm{Hom}(\mathcal{I}, \mathcal{O}_Z \otimes \omega_S) \cong \mathrm{Ext}^1_0(\mathcal{I}, \mathcal{I} \otimes \omega_S),$$

we see that the map B_s coincides with the analogous map defined in [Bo] for the moduli space of stable sheaves on S. It was then proved that B_s is antisymmetric, hence θ_s is actually a global section of $\wedge^2 T S^{[d]}$.

To prove that θ_s defines a Poisson structure on $S^{[d]}$ it remains to prove that the bracket defined by $\{f, g\}_s = \theta_s(df \wedge dg)$ satisfies the Jacobi identity. We note that in [Bo] this result was proved under the hypothesis that the sheaves involved are locally free, hence it is not directly applicable here. For this reason we shall use a different approach to the problem.

Proposition 3.1. *The bivector field θ_s defines a Poisson structure on $S^{[d]}$.*

Proof. By what we have previously seen, we have only to prove that the Poisson bracket defined by θ_s satisfies the Jacobi identity

$$\{f, \{g, h\}_s\}_s + \{g, \{h, f\}_s\}_s + \{h, \{f, g\}_s\}_s = 0.$$

In [Bo] we have defined an operator

$$\tilde{d} : H^0(S^{[d]}, \wedge^2 T S^{[d]}) \to H^0(S^{[d]}, \wedge^3 T S^{[d]}),$$

and we have proved that the Jacobi identity for $\{\cdot, \cdot\}_s$ is equivalent to the vanishing of $\tilde{d}\theta_s$ (which, in turn, is equivalent to the vanishing of the classical Schouten bracket $[\theta_s, \theta_s]$).

If we denote by S^d the d-fold product of S, the Poisson structure $s : T^*S \to TS$ of S determines a Poisson structure $s^d : T^*S^d = \bigoplus_{i=1}^d T^*S \to TS^d = \bigoplus_{i=1}^d TS$, which is invariant for the action of the symmetric group $\mathfrak{S}_d$, hence descends to the quotient $S^{(d)}$. The Hilbert-Chow morphism $\pi : S^{[d]} \to S^{(d)}$ is an isomorphism on the inverse image of the complement of Δ' in $S^{(d)}$, and the Poisson structure on $S^{(d)} \setminus \Delta'$ induced by s coincides with the restriction of θ_s on $\pi^{-1}(S^{(d)} \setminus \Delta')$, hence $\tilde{d}\theta_s|_{\pi^{-1}(S^{(d)} \setminus \Delta')} = 0$. Since this is an open dense subset of $S^{[d]}$, it follows that $\tilde{d}\theta_s$ vanishes identically on $S^{[d]}$. $\square$

We can now investigate the rank of the Poisson structure B_s, i.e., the dimension of the symplectic leaves of $S^{[d]}$.

Lemma 3.2. *Let $\mathcal{I} \in S^{[d]}$. We have*

$$\ker B_s(\mathcal{I}) = \mathrm{Hom}(\mathcal{I}, \mathcal{T}or_1^{\mathcal{O}_S}(\mathcal{O}_Z, \mathcal{O}_D)),$$

where $Z = Z_{\mathcal{I}}$ is the 0-dimensional subscheme defined by $\mathcal{I}$ and D is the divisor of s.

Proof. By tensoring by $\mathcal{O}_Z = \mathcal{O}_S/\mathcal{I}$ the exact sequence

$$0 \to \omega_S \xrightarrow{s} \mathcal{O}_S \to \mathcal{O}_D \to 0,$$

we obtain the following exact sequence:

$$0 \to \mathcal{T}or_1^{\mathcal{O}_S}(\mathcal{O}_Z, \mathcal{O}_D) \to \omega_S \otimes \mathcal{O}_Z \xrightarrow{s} \mathcal{O}_Z \to \mathcal{O}_D \otimes \mathcal{O}_Z \to 0.$$

The result follows now by applying to this sequence the functor $\mathrm{Hom}(\mathcal{I}, \cdot)$.
□

Remark 3.3. The sheaf $\mathcal{T}or_1^{\mathcal{O}_S}(\mathcal{O}_Z, \mathcal{O}_D)$ is supported at $Z \cap D$. It follows that, if $Z \cap D = \emptyset$, the map $B_s(\mathcal{I})$ is injective (hence bijective). This means that the Poisson structure θ_s is nondegenerate, hence induces a symplectic structure, on the open subset of $S^{[d]}$ consisting of points $\mathcal{I}$ such that $Z_\mathcal{I} \cap D = \emptyset$.

If the subscheme $Z_\mathcal{I}$ of $S^{[d]}$ consists of d distinct points $P_1, \ldots, P_d$, we have

$$T_\mathcal{I} S^{[d]} \cong T_{P_1} S \oplus \cdots \oplus T_{P_d} S.$$

In this case the map $B_s(\mathcal{I}) : T^*_\mathcal{I} S^{[d]} \to T_\mathcal{I} S^{[d]}$ is simply the direct sum of the maps $s(P_i) : T^*_{P_i} S \to T_{P_i} S$ defined by the section s. The map $s(P_i) : T^*_{P_i} S \to T_{P_i} S$ is obviously a bijection if $P_i \notin \mathrm{supp}(D)$, otherwise it is the zero map. This implies that $B_s(\mathcal{I})$ is a bijection if $Z_\mathcal{I} \cap D = \emptyset$, as we have seen in the preceding remark. This also shows that the rank of B_s is given by

$$\mathrm{rk}(B_s(\mathcal{I})) = 2d - 2 \cdot \#(Z_\mathcal{I} \cap D),$$

i.e., the rank of B_s decreases by 2 for each point of $Z_\mathcal{I}$ which happens to belong to the support of D.

If we set

$$H_l = \{\, \mathcal{I} \in S^{[d]} \mid \mathrm{rk}(B_s(\mathcal{I})) = 2d - 2l \,\},$$

we obtain a stratification of $S^{[d]}$ by closed subsets:

$$\emptyset \subset H_d \subset H_{d-1} \subset \cdots \subset H_1 \subset H_0 = S^{[d]}.$$

The generic point of H_l is given by a 0-dimensional subscheme Z of S consisting of d distinct points such that l of them belong to the support of D.

As previously seen, the Poisson structure θ_s induces a symplectic structure on the open subset $S^{[d]} \setminus H_1$.

4. Integrable systems

In this section we give some examples of integrable Hamiltonian systems defined on a dense open subset of the Hilbert scheme of points of S. For the definition and basic properties of integrable systems in the algebraic set-up we refer, for instance, to [V1].

Let S be a Poisson surface, with Poisson structure given by a section $s \in H^0(S, \omega_S^{-1})$, and let L be a line bundle on S. If L is sufficiently ample, we have $g \leq \dim |L|$, where $g = 1 + \frac{1}{2}(L.L + L.\omega_S)$ is the genus of a curve $C \in |L|$.

Let us choose a linear system $V \subset |L|$ of dimension g and let us denote by U the open subset of V consisting of integral curves. We also assume that, for any curve $C \in U$, C is not contained in the divisor D defined by the section s.

For any integral curve $C \in U$ and any integer d, the variety $J^d(C)$, parametrizing invertible sheaves of degree d on C, has a natural compactification $\overline{J^d(C)}$, parametrizing torsion-free sheaves of rank 1 and degree d on C. The family $(\overline{J^d(C)})|_{C \in U}$ is organized in a fibration $H : \mathcal{J}^d \to U$, called the "relative compactified Jacobian", whose fiber over a curve $C \in U$ is $H^{-1}(C) = \overline{J^d(C)}$ (the relative compactified Jacobian $\mathcal{J}^d$ was constructed in great generality by Altman and Kleiman in [AK]).

If $d = g$, there is a birational map

$$S^{[g]} \overset{\psi}{\dashrightarrow} \mathcal{J}^g$$

defined as follows: if Z is a subscheme of length g of S, consisting of g distinct points, such that there exists a unique curve $C \in U$ containing it, $\psi(Z)$ is the divisor on C determined by Z (note that this map may be extended also to some subschemes Z which do not consist of distinct points). If we denote by

$$S^{[g]} \overset{H'}{\dashrightarrow} U \tag{4.1}$$

the rational map which associates to a subscheme Z of S the curve C described above, we get a commutative diagram

$$\begin{tikzcd} S^{[g]} \arrow[rr, dashed, "\psi"] \arrow[dr, dashed, "H'"'] & & \mathcal{J}^g \arrow[dl, "H"] \\ & U. & \end{tikzcd} \tag{4.2}$$

Now we need the following result:

Lemma 4.1. *The relative compactified Jacobian $\mathcal{J}^g$ is a smooth subscheme of the moduli space of simple sheaves on S.*

Proof. Let $\mathcal{F} \in \mathcal{J}^g$. $\mathcal{F}$ is a simple sheaf on S, supported on a curve $C \in U$. It is known (cf. [A1]) that the obstruction to the smoothness of the moduli space of simple sheaves on S at the point $\mathcal{F}$ lies in $\mathrm{Ext}_0^2(\mathcal{F}, \mathcal{F})$, which denotes the trace-free part of $\mathrm{Ext}^2(\mathcal{F}, \mathcal{F})$. By Grothendieck–Serre duality, we have

$$\mathrm{Ext}_0^2(\mathcal{F}, \mathcal{F}) \cong \mathrm{Hom}_0(\mathcal{F}, \mathcal{F} \otimes \omega_S)^*.$$

Tensoring by $\mathcal{F}$ the exact sequence

$$0 \to \omega_S \xrightarrow{s} \mathcal{O}_S \to \mathcal{O}_D \to 0$$

and applying the functor $\mathrm{Hom}_0(\mathcal{F}, \cdot)$, we get an injection

$$0 \to \mathrm{Hom}_0(\mathcal{F}, \mathcal{F} \otimes \omega_S) \to \mathrm{Hom}_0(\mathcal{F}, \mathcal{F}).$$

(Note that in order for this last map to be injective, we need the hypothesis that the support of $\mathcal{F}$ and the divisor D have no common component, which is true by our assumptions on the curves in U.)

Now, by recalling that $\mathcal{F}$ is a simple sheaf, it follows that $\mathrm{Hom}_0(\mathcal{F}, \mathcal{F} \otimes \omega_S) = 0$, hence deformations of $\mathcal{F}$ are unobstructed. $\square$

By an easy generalization of the results of [M2] (cf. also [Bo]), and by the result proved in the above lemma, it follows that there is a natural bivector field $\theta_s \in H^0(\mathcal{J}^g, \wedge^2 T\mathcal{J}^g)$ on $\mathcal{J}^g$, induced by the Poisson structure of S (this actually holds for the smooth part of the moduli space of simple sheaves on S). It is now immediate to recognize that the natural Poisson structure of $S^{[g]}$ corresponds, via the birational isomorphism ψ, to this bivector field θ_s on $\mathcal{J}^g$, hence θ_s too is a Poisson structure. Now the proof of [B2, Proposition 2] can be easily adapted to prove that $H : \mathcal{J}^g \to U$ is a Lagrangian fibration. From the commutativity of the diagram (4.2), it follows that the map (4.1) defines an integrable Hamiltonian system on a dense open subset of $S^{[g]}$ whose fibers are birationally isomorphic to the Jacobians of the curves $C \in U$.

Remark 4.2. If $d \neq g$, and if we choose a d-dimensional linear system $V \subset |L|$ and denote, as before, by U the open subset of V consisting of integral curves, we may still define the rational map

$$S^{[g]} \overset{H'}{\dashrightarrow} U.$$

In this situation the fibers of H' are no longer birationally isomorphic to the Jacobians of the curves in U but, nevertheless, the component functions of H' are in involution with respect to the natural Poisson structure of $S^{[d]}$.

Remark 4.3. Note that these results do not depend on the choice of the Poisson structure on S, i.e., the map H' defines an integrable Hamiltonian system on a dense open subset of $S^{[d]}$ for any Poisson structure $s \in H^0(S, \omega_S^{-1})$.

As an example, we consider now the case $S = \mathbb{P}^2$. If we take as the anticanonical divisor D a triple line, we may choose homogeneous coordinates $(x_0 : x_1 : x_2)$ such that the section s defining D is given by $s = x_0^3$. The Poisson structure defined by s on $\mathbb{P}^2$ induces a symplectic structure on $\mathbb{C}^2 = \mathbb{P}^2 \setminus D$. If we consider the coordinates (X, Y) on $\mathbb{C}^2$ given by $X = x_1/x_0$ and $Y = x_2/x_0$, it is immediate to see that this symplectic structure is the usual one, given by the 2-form $dX \wedge dY$.

Now, for any integers n and d, with $n \geq 1$ and $d \leq \frac{1}{2}(n^2 + 3n)$, let us choose a d-dimensional family $C(h_1, \dots, h_d)$ of plane curves of degree n, depending linearly on d parameters $h_1, \dots, h_d$, i.e., let us suppose that the equation of this family of curves has the form

$$C(h_1, \dots, h_d) : \quad \sum_{i=1}^{d} h_i p_i(X, Y) = q(X, Y),$$

where p_i and q are polynomials of degree $\leq n$ (but such that $C(h_1, \dots, h_d)$ has degree n).

We may define a rational map

$$(\mathbb{C}^2)^{(d)} \overset{H}{\dashrightarrow} \mathbb{C}^d \tag{4.3}$$

by sending $P_1 + \dots + P_d$ to the d-tuple $(h_1, \dots, h_d)$ which determines the unique curve of the family $C(h_1, \dots, h_d)$ passing through the points $P_1, \dots, P_d$ (this rational map can be defined also on the Hilbert scheme $(\mathbb{C}^2)^{[d]}$).

As a corollary of the preceding results, we have:

Proposition 4.4. *The component functions $h_1, \dots, h_d$ of the rational map H in (4.3) are in involution with respect to the natural symplectic structure induced on $(\mathbb{C}^2)^{(d)}$ by the usual symplectic structure on $\mathbb{C}^2$, hence define an integrable Hamiltonian system on an open subset of $(\mathbb{C}^2)^{(d)}$. Moreover, if the general curve C of the family is smooth then, if we denote by g the genus of the smooth completion of C, and suppose that $g \neq 0$ and take $d = g$, the fiber of H over a generic curve C is birationally isomorphic to the Jacobian variety of C.*

Remark 4.5. As previously remarked (cf. Remark 4.3), the results stated in the preceding proposition hold true for any Poisson structure on $\mathbb{C}^2$, more precisely, even if the Hamiltonian vector fields of the functions $h_1, \dots, h_d$ actually depend on the Poisson structure, we always obtain an integrable Hamiltonian system on an open subset of $(\mathbb{C}^2)^{(d)}$.

Remark 4.6. We remark here that similar integrable systems on $\mathbb{C}^{2d}$ have been constructed, in a different context, by P. Vanhaecke. For a detailed description of these integrable systems we refer to [V1, Chapter III] and [V2].

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