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Invariant measures for the Musiela equation with deterministic diffusion term*

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Abstract. In this article the forward rates equation of the Musiela model is analysed. The equation is studied in the Sobolev spaces $H^1_\gamma(\mathbb{R}^+)$ and $H^1(\mathbb{R}^+)$. Explicit mild solutions and equivalent conditions for the existence and uniqueness of invariant measures are presented.

Key words: term structure of interest rates, stochastic partial differential equations, mild solutions, invariant measures, C^0-semigroups in Hilbert spaces

JEL classification: E43

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1. Introduction

The aim of this work is to study in a rigorous way some asymptotic properties of the Musiela equation in the Gaussian case. Our methods are based on the theory of stochastic equations in separable Hilbert spaces by Da Prato and Zabczyk [5], and we study invariant measures for the equation and weak convergence of the solution to such measures.

The Musiela model, developed by Musiela himself and other authors in [3], [4], [8], [9] and [12], is based on the family of rates $(r(t,x))_{t,x\geq 0}$, where $r(t,x)$ represents the forward rate prevailing in t for the time $t+x$. This model is

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a reparametrization of the well-known Heath-Jarrow-Morton (HJM) model (see [10]), such that $r(t,x) = f(t,t+x)$, where $f(t,T)$ are the forward rates analysed in [10]. The Musiela reparametrization is coherent with other forward rates models (see [3] and [14]) and allows us to consider the forward curve $r(t,\cdot)$ as a Markov process in a suitable function space, while in the HJM model the state space changes with time.

Let us now introduce more in detail the Musiela model. We suppose that we have a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ endowed with the filtration $(\mathscr{F}_t)_{t\geq 0}$, and that the price at time t of a bond expiring at time T is represented by the process $(B(t,T))_{t\in[0,T]}$. Furthermore, we suppose that we have a random field $(r(t,x))_{t,x\geq 0}$ such that $r(t,x)$ is $\mathscr{F}_t$-measurable and $B(t,T) = \exp(-\int_0^{T-t} r(t,u)\,du)$. The quantity $r(t,x)$ is called **instantaneous forward rate** at time t for the maturity $t+x$ and represents the rate at time t at which one can enter a forward contract for the time $t+x$ for a short (infinitesimal) period of time. The **actualized price** at time t of a bond expiring at time T is given by $\tilde{B}(t,T) = B(t,T)/\beta(t)$, where $\beta(t) = \exp(\int_0^t r(u,0)\,du)$ is the actualizing factor. Now we add the hypotheses that there exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$, called **risk-neutral probability** under which the processes $(\tilde{B}(t,T))_{t\in[0,T]}$ are martingales for all $T > 0$, and that there exists a k-dimensional standard $\mathbb{Q}$-Brownian motion $(W_t)_{t\geq 0}$ adapted to $(\mathscr{F}_t)_{t\geq 0}$ such that $(r(t,x))_{t,x\geq 0}$ satisfies the stochastic partial differential equation

$$\begin{cases} dr(t,x) = \left(\dfrac{\partial}{\partial x} r(t,x) + \displaystyle\sum_{n=1}^{k} \tau_n(t,x) \int_0^x \tau_n(t,u)\,du \right) dt + \displaystyle\sum_{n=1}^{k} \tau_n(t,x)\,dW_t^n , \\ r(0,\cdot) \in AC(\mathbb{R}^+) , \end{cases} \tag{1}$$

where the $(\tau_n(t,x))_{t,x\geq 0}$, $n = 1,\dots,k$ are random fields such that $\tau_n(t,x)$ is $\mathscr{F}_t$-measurable and such that Eq. (1) is well defined for all maturities $x \geq 0$. This model is justified by the following result that can be found both in [4] as in [12].

Theorem 1. *If $\forall t \in \mathbb{R}^+$ the application $x \to r(t,x)$ belongs to $AC(\mathbb{R}^+)$ $\mathbb{Q}$-a.s. and r satisfies Eq. (1), then for all $T > 0$ the process $(\tilde{B}(t,T))_{t\in[0,T]}$ is a local martingale with respect to $\mathbb{Q}$.*

Our aim is to study asymptotic properties of Eq. (1) in the simple case when the diffusion term τ is deterministic and the process r takes values in a separable Hilbert space H contained in $AC(\mathbb{R}^+)$. This leads to a model that is Gaussian in the forward rates $r(t,x)$ and it is analytically tractable. In fact, in this case we can find an explicit solution to Eq. (1), given by Eq. (5) and usually called mild solution. First we show that this mild solution, under technical conditions, is well defined in some particular Hilbert spaces $H \subseteq AC(\mathbb{R}^+)$, and is unique in the class of weak solutions of Eq. (1) (see [5]). Then we will study (when possible) the characterization of the invariant measures for the process of the solution. The study of invariant measures is a first step in studying asymptotic properties of a system; in this case, the existence of invariant measures means that the distribution of the forward curve converges, as $t \to +\infty$, to an invariant measure on H (see Theorem 7).

Let us restate the problem in a more systematic way. We rewrite Eq. (1) as a Langevin equation in a separable Hilbert space H (that we will specify later):

$$\begin{cases} dr_t = (Ar_t + c)\ dt + dW_t\ , \\ r_0 \in L^2(\Omega, \mathscr{F}_0, \mathbb{Q}; H)\ , \end{cases} \tag{2}$$

where $A = \frac{\partial}{\partial x}$, $c(x) = \sum_{n=1}^{\infty} \tau_n(x) \int_0^x \tau_n(u)\ du$ and dW_t stands for $\sum_{n=1}^{\infty} \tau_n\ dW_t^n$ and is an H-valued Brownian motion. This means that, if we indicate with $Q = \sum_{n=1}^{\infty} \tau_n \otimes \tau_n$ the covariance operator of W, we have that Tr $Q = \sum_{n=1}^{\infty} \|\tau_n\|_H^2 < +\infty$. In practical terms, this situation corresponds to the well accepted idea that there are infinitely many sources of randomness in the model, but only a few "principal components" are significant, because the intensity of the noises decreases rather quickly (see [11] for another kind of genuine infinite dimensional dynamics of the forward rates). In order to obtain the usual case of a k-dimensional driving Brownian motion (see for examples [1], [2], [3], [4]), it is sufficient to suppose that $\tau_n = 0$ for all $n > k$. By a solution of Eq. (1) (or better of Eq. (2)), we will mean, as in [4] and [12], a so called **mild solution** (see [5]), given by

$$r_t = S_t r_0 + \int_0^t S_{t-u} c\ du + \int_0^t S_{t-u}\ dW_u\ , \tag{3}$$

where $(S_t)_t$ is the C^0-semigroup in H generated by A (see [13]), which in this case is the translation semigroup defined by

$$(S_t f)(x) = f(x+t) \quad \forall t, x \geq 0\ .$$

To specify exactly in which space H we will study Eq. (1), we define the Sobolev spaces

$$H_\gamma^k(\mathbb{R}^+) = \left\{ u : \mathbb{R}^+ \to \mathbb{R}\ \middle|\ \exists u', \ldots, u^{(k)} \text{ and } u, u', \ldots, u^{(k)} \in L_\gamma^2(\mathbb{R}^+) \right\}\ ,$$

where u', u'', ..., $u^{(k)}$ indicate respectively the first, second, ..., k-th weak derivative, and

$$L_\gamma^p(\mathbb{R}^+) = \left\{ u : \mathbb{R}^+ \to \mathbb{R}\ \middle|\ u \text{ meas. and s.t. } \int_0^{+\infty} u^p(x) \mathrm{e}^{-\gamma x}\ dx < +\infty \right\}\ .$$

We notice that H_0^k and L_0^p coincide with the usual definitions of H^k and L^p, and we recall that H_γ^k is a Hilbert space with respect to the scalar product

$$\langle f, g \rangle_{H_\gamma^k} = \sum_{i=0}^{k} \langle f^{(i)}, g^{(i)} \rangle_{L_\gamma^2} = \sum_{i=0}^{k} \int_0^{+\infty} f^{(i)}(x) g^{(i)}(x) \mathrm{e}^{-\gamma x}\ dx\ . \tag{4}$$

Furthermore $H_\gamma^1 \subseteq H_{\gamma'}^1$, $\forall \gamma < \gamma'$, and $H_\gamma^1 \subseteq AC(\mathbb{R}^+)$ $\forall \gamma \geq 0$.

In view of Theorem 1, we will study Eq. (1) in the spaces $H_\gamma^1(\mathbb{R}^+)$ for $\gamma \geq 0$, which are contained in the space $AC(\mathbb{R}^+)$. If we studied the equation in $L_\gamma^2(\mathbb{R}^+)$ (as

[8] and [9] do), we would obtain an invariant measure in the whole space $L^2_\gamma(\mathbb{R}^+)$, and it would be difficult to prove that the measure is concentrated on $AC(\mathbb{R}^+)$. We start studying the Musiela equation in the spaces $H^1_\gamma(\mathbb{R}^+)$ for $\gamma > 0$. In these spaces we find infinitely many Gaussian invariant measures (among which there are the ones found in [12]), but also other non-Gaussian ones, and we show that if the initial r_0 is deterministic, then the solution r_t converges weakly as $t \to +\infty$ to a Gaussian invariant measure, which depends on the initial forward curve r_0. The properties of the equation in $H^1(\mathbb{R}^+)$ are completely different, and for this reason we study this specific situation separately. In $H^1(\mathbb{R}^+)$ flat term structures $r(x) = \text{const.} \neq 0$ are not allowed and the forward curves must converge to 0 for $x \to +\infty$. This means that $H^1(\mathbb{R}^+)$ does not contain the simplest forward curves. Moreover, in this space we find only one invariant measure, that is Gaussian. This is another drawback, because the invariant measure does not depend on the initial r_0. We conclude our work by summarizing these and other financial remarks and giving some final comments in a specific section.

Our work is organized as follows. In Section 2 we study the Musiela equation in the space $H^1_\gamma(\mathbb{R}^+)$ for $\gamma > 0$. Section 3 is specifically devoted to the Musiela equation in the space $H^1(\mathbb{R}^+)$. Section 4 contains the conclusions of our work.

2. The forward rate equation in the space H^1_γ

We first study the case when the Hilbert space is $H^1_\gamma(\mathbb{R}^+)$, and start by presenting some sufficient conditions under which the mild solution of Eq. (1) is well defined in $H^1_\gamma(\mathbb{R}^+)$. These conditions will be also necessary to obtain the existence of invariant measures in the same space.

Theorem 2. *If $\sum_{n=1}^\infty \|\tau_n\|^2_{H^1_\gamma} < +\infty$, $\sum_{n=1}^\infty \|\tau_n\|^4_{H^1} < +\infty$, $\sum_{n=1}^\infty \|\tau_n\|^4_{L^4_\gamma} < +\infty$, and $r_0 \in L^2(\Omega, \mathscr{F}_0, \mathbb{Q}; H)$, then the mild solution of Eq. (1) in $H^1_\gamma(\mathbb{R}^+)$, given by*

$$\begin{aligned} r_t(x) &= r_0(x+t) + \sum_{n=1}^\infty \int_0^t \tau_n(x+t-u) \left(\int_0^{x+t-u} \tau_n(v)\, dv \right) du + \\ &\quad + \int_0^t \tau_n(x+t-u)\, dW^n_u = \\ &= r_0(x+t) + \frac{1}{2} \sum_{n=1}^\infty \left(\left(\int_0^{x+t} \tau_n(u)\, du \right)^2 - \left(\int_0^x \tau_n(u)\, du \right)^2 \right) + \\ &\quad + \sum_{n=1}^\infty \int_0^t \tau_n(x+t-u)\, dW^n_u\ , \end{aligned} \tag{5}$$

is well defined. Moreover, it is the unique solution in the class of weak solutions (see [5]). The mild solution is a Gaussian process with mean

$$\mathbb{E}[r_t(x)] = \mathbb{E}[r_0(x+t)] + \frac{1}{2} \sum_{n=1}^\infty \left(\left(\int_0^{x+t} \tau_n(u)\, du \right)^2 - \left(\int_0^x \tau_n(u)\, du \right)^2 \right) \tag{6}$$

and covariance

$$\begin{aligned}\mathrm{Cov}\,(r_t(x), r_v(y)) &= \mathrm{Cov}\,(r_0(t+x), r_0(v+y)) + \\ &+ \sum_{n=1}^{\infty} \int_0^t \tau_n(x+t-u)\tau_n(y+v-u)\,du\ . \end{aligned} \tag{7}$$

Proof. Following [5], Theorem 5.4, p. 121, it is sufficient to check that Tr $Q < +\infty$, $c \in H$, $r_0 \in L^2(\Omega, \mathscr{F}_0, \mathbb{Q}; H)$ and that A generates a C^0-semigroup $(S_t)_t$ in H^1_γ. The first condition is equivalent to $\sum_{n=1}^{\infty} \|\tau_n\|^2_{H^1_\gamma} < +\infty$. Now we observe that the weak derivative of c is

$$c'(x) = \sum_{n=1}^{\infty} \left(\tau_n'(x) \int_0^x \tau_n(u)\,du + \tau_n^2(x) \right)\ . \tag{8}$$

If we impose $\sum_{n=1}^{\infty} \|\tau_n\|^4_{L^2} < +\infty$, then, from Jensen's and Hölder's inequalities, it follows that $c \in H^1_\gamma$. To prove the last condition, we will use the following well known fact, whose proof we omit:

Lemma 3. *For all $\gamma \geq 0$, the translation semigroup $(S_t)_t$ is a C^0-semigroup in $H^1_\gamma(\mathbb{R}^+)$, having as infinitesimal generator A, which has domain equal to $H^2_\gamma(\mathbb{R}^+)$.*

In view of Lemma 3 and Eq. (3), the theorem is proved. □

Now we analyse the problem of finding invariant measures. In the following, $N(b, Q)$ will indicate the Gaussian measure in $H^1_\gamma(\mathbb{R}^+)$ having as mean the function $b \in H^1_\gamma(\mathbb{R}^+)$ and as covariance the operator $Q : H^1_\gamma(\mathbb{R}^+) \times H^1_\gamma(\mathbb{R}^+) \to \mathbb{R}$.

Theorem 4. *There exist invariant measures for Eq. (2) in the Hilbert space H^1_γ if and only if $\sum_{n=1}^{\infty} \|\tau_n\|^2_{H^1} < +\infty$ and $\sum_{n=1}^{\infty} \|\tau_n\|^4_{L^4_\gamma} < +\infty$. If these conditions hold there exists an infinite number of invariant measures. In particular, the measures $N(b^*(\cdot) + b_0, Q_\infty)$ are invariant, where b_0 is an arbitrary real number, b^* is the function given by Eq. (11), and*

$$Q_\infty = \sum_{n=1}^{\infty} \int_0^{+\infty} \tau_n(\cdot + u) \otimes \tau_n(\cdot + u)\,du\ . \tag{9}$$

In the proof we will need the following result from [6]:

Theorem 5. *Let the following conditions be satisfied:*

(i) $\sup_{t\geq 0} \int_0^t \mathrm{Tr}\, S_u Q S_u^*\,du < +\infty$
(ii) $\exists b \in D(A)$ *such that* $Ab + c = 0$*, and there exists an invariant measure ν for the equation*

$$dZ_t = AZ_t\,dt\ . \tag{10}$$

Then there exists an invariant measure for Eq. (2). In particular every invariant measure is of the form $\nu * N(b, Q_\infty)$, where $Q_\infty = \int_0^{+\infty} S_u Q S_u^*\,du$.

Now we prove Theorem 4.

Proof. It is sufficient to check points (i) and (ii) of Theorem 5. First of all we calculate

$$\begin{aligned}\sup_{t\geq 0}\int_0^t \mathrm{Tr}\ Q_u\ du &= \int_0^{+\infty}\sum_{n=1}^{\infty}\int_0^{+\infty}\left(\tau_n^2(x+u)+\tau_n'^2(x+u)\right)\mathrm{e}^{-\gamma x}\ dx\ du = \\ &= \int_0^{+\infty}\sum_{n=1}^{\infty}\int_u^{+\infty}\left(\tau_n^2(x)+\tau_n'^2(x)\right)\mathrm{e}^{-\gamma(x-u)}\ dx\ du = \\ &= \int_0^{+\infty}\sum_{n=1}^{\infty}\left(\tau_n^2(x)+\tau_n'^2(x)\right)\mathrm{e}^{-\gamma x}\int_0^x \mathrm{e}^{\gamma u}\ du\ dx = \\ &= \int_0^{+\infty}\sum_{n=1}^{\infty}\left(\tau_n^2(x)+\tau_n'^2(x)\right)\frac{1-\mathrm{e}^{-\gamma x}}{\gamma}\ dx\ .\end{aligned}$$

We notice that the condition $\sum_{n=1}^{\infty}\|\tau_n\|_{H^1}^2 < +\infty$ is necessary and sufficient to guarantee that point (i) in Theorem 5 holds. We now check (ii). An invariant measure for Eq. (10) is given by the measure δ_0. Moreover, a solution of the equation $Ab + c = 0$ is:

$$b^*(x) = -\int_0^x c(u)\ du = -\frac{1}{2}\sum_{n=1}^{\infty}\left(\int_0^x \tau_n(u)\ du\right)^2\ . \tag{11}$$

To check that $b^*(x) \in D(A)$, we first verify that $b^*(x) \in L^2_\gamma$:

$$\begin{aligned}|b^*(x)| &\leq \sum_{n=1}^{\infty}\int_0^x |\tau_n(u)|\int_0^u |\tau_n(v)|\ dv\ du = \\ &= \frac{1}{2}\sum_{n=1}^{\infty}\left(\int_0^x |\tau_n(u)|\ du\right)^2 \leq \frac{1}{2}x\sum_{n=1}^{\infty}\int_0^x |\tau_n(u)|^2\ du \leq x\sum_{n=1}^{\infty}\|\tau_n\|_{L^2}^2\ ,\end{aligned}$$

so $\|b^*\|_{L^2_\gamma} \leq \|x\|_{L^2_\gamma}\sum_{n=1}^{\infty}\|\tau_n\|_{L^2}^2$, and $b^* \in L^2_\gamma$. Since b^* has the weak derivative equal to $-c$, which is in H^1, then $b^* \in H^2_\gamma$. We also notice that constant functions belong to H^1_γ, so every function $b^*(\cdot) + b_0$, $b_0 \in \mathbb{R}$ is a solution of $Ab + c = 0$ and we have the result. □

Remark 6. There exist also non-Gaussian invariant measures for Eq. (1). In fact, if in Eq. (10) we choose an initial datum $Z_0 \equiv f \in H^1_\gamma$ periodical with period T, then the solution would be $Z_t(x) \equiv f(t+x)$. Now we define the random variable $\pi : [0,T] \to H^1_\gamma$ such that $\pi(t) = f(\cdot + t)$ and we set the uniform density on $[0,T]$; then π induces a measure μ_f on H^1_γ. So, if in Eq. (10) we choose an initial datum Z_0 with law μ_f, then we obtain a solution $(Z_t)_t$ with Z_t having law μ_f $\forall t \geq 0$. This means that $\mu_f * N(b^*(\cdot) + b_0, Q_\infty)$ is an invariant measure for Eq. (1), and it is not Gaussian.

We can see that the ergodic behaviour of Eq. (1) could be more intricate than a simply Gaussian one, and a characterization of all the invariant measures would be very difficult. However, in the next theorem we see that if we start from a deterministic initial datum r_0, then the asymptotic law is one of the Gaussian invariant measures found in Theorem 4. Besides, the theorem will show also that all the invariant measures found before can be reached starting from a suitable r_0.

Theorem 7. *If $\sum_{n=1}^{\infty} \|\tau_n\|_{L^1}^2 < +\infty$, the assumptions of Theorem 4 hold, and the initial datum is a deterministic $r_0 \in H_\gamma^1$ such that $\lim_{x\to+\infty} r_0(x) = r_0(\infty) \in \mathbb{R}$, $\lim_{x\to+\infty} r_0'(x) = 0$, then r_t converges weakly for $t \to +\infty$ to the invariant measure $N(r_0(\infty) + b_0 + b(x),\ Q_\infty)$, where b and Q_∞ are given respectively by (11) and (9), and b_0 is given by*

$$b_0 = \int_0^{+\infty} c(u)\, du = \frac{1}{2} \sum_{n=1}^{\infty} \left(\int_0^{+\infty} \tau_n(u)\, du \right)^2 . \tag{12}$$

Proof. Let us consider an initial datum with a degenerate law concentrated on r_0, and see what happens to the marginal law of the solution r_t. The solution r_t has functional mean

$$\mathbb{E}[r_t](\cdot) \quad = \quad r_0(\cdot + t) + \int_0^t c(\cdot + u)\, du$$

and functional variance

$$\mathrm{Var}\,[r_t] = \sum_{n=1}^{\infty} \int_0^t \tau_n(\cdot + u) \otimes \tau_n(\cdot + u)\, du\ .$$

Then for all $G \in C_b(H_\gamma^1)$ we have that

$$\begin{aligned} \lim_{t\to+\infty} \mathbb{E}[G(r_t)] \quad &= \quad \lim_{t\to+\infty} \mathbb{E}\left[G\left(\mathbb{E}[r_t] + \sum_{n=1}^{\infty} \int_0^t \tau_n(\cdot + t - s)\, dW_s \right) \right] = \\ &= \quad \int_{H_\gamma^1} G(r) N(r_0(\infty) + b_0 + b(\cdot), Q_\infty)(dr) \end{aligned}$$

In fact:

$$\begin{aligned} &\lim_{t\to+\infty} \|\mathbb{E}[r_t] - r_0(\infty) - b_0 - b(\cdot)\|_{H_\gamma^1} \le \lim_{t\to+\infty} \|r_0(t + \cdot) - r_0(\infty)\|_{H_\gamma^1} + \\ &\quad + \lim_{t\to+\infty} \left\| \int_0^{\cdot+t} c(u)\, du - \int_0^{+\infty} c(u)\, du + \int_0^{\cdot} c(u)\, du \right\|_{H_\gamma^1} = \\ &= \quad 0 + \lim_{t\to+\infty} \left\| \int_{\cdot+t}^{+\infty} c(u)\, du \right\|_{H_\gamma^1} = 0 \end{aligned}$$

where the two limits follow from Lebesgue's theorem. Besides

$$\sum_{n=1}^{\infty} \int_0^t \tau_n(\cdot + t - s)\, dW_s$$

has law $N(0, Q_t)$, where $Q_t = \sum_{n=1}^{\infty} \int_0^t \tau_n(\cdot + s) \otimes \tau_n(\cdot + s)\, ds$. Since $\mathrm{Tr}\, Q_\infty < +\infty$, then $\lim_{t\to+\infty}(Q_\infty - Q_t) = 0$, so $N(0, Q_t)$ converges weakly to $N(0, Q_\infty)$. □

Theorem 7 says that the solution r_t starting from a rather general deterministic initial forward curve r_0, under some regularity conditions, converges weakly to a Gaussian invariant measure. Moreover, the invariant measure to which the solution converges is determined by $r_0(\infty)$. From this we deduce that if we take a suitable initial datum r_0, it is possible to reach any invariant Gaussian measure we found in the last theorem. From a mathematical point of view this means that there are no "privileged" invariant measures, at least in the purely Gaussian case. From a practical point of view, this non-uniqueness is related to the deterministic part of the forward curve r_0 known at time 0, and corresponds to the limit of the forward curve at infinity. This is intuitively quite appealing.

Remark 8. The quantity $r_t(\infty)$ is usually called **long forward rate**. In our case, it is easy to see from Eq. (5) that if $r_0(\infty)$ exists and $\tau_n(\infty)$ exists and is equal to 0 for all $n \in \mathbb{N}$, then $r_t(\infty) = r_0(\infty)$ $\mathbb{Q}$-almost surely. This is a common behaviour of the long forward rate: in fact, while it can be shown by no-arbitrage techniques that this quantity is $\mathbb{Q}$-almost surely increasing with t (see [7]), in our model (as in many models used in practice) the long forward rate is constant.

3. The forward rate equation in the space H^1

Now we study Eq. (1) in the space $H^1(\mathbb{R}^+)$. As before, we first present some sufficient conditions for the mild solution of Eq. (1) to be well defined in this space. These conditions will be also necessary in order to obtain the existence of an invariant measure.

Theorem 9. *If $\sum_{n=1}^{\infty} \|\tau_n\|_{H^1}^2 < +\infty$, $\sum_{n=1}^{\infty} \|\tau_n\|_{L^4}^4 < +\infty$, the functions $\sqrt{x}\tau_n(x)$ and $\sqrt{x}\tau_n'(x)$ are uniformly bounded in $L^2(\mathbb{R}^+)$ and $r_0 \in H^1(\mathbb{R}^+)$, then the mild solution of Eq. (2) in $H^1(\mathbb{R}^+)$, given by (5), is well defined. Moreover, it is the unique solution in the class of weak solutions. The mild solution is a Gaussian process with mean (6) and covariance (7).*

Proof. The proof is similar to the one of Theorem 2. □

Theorem 10. *Given Eq. (2) in the Hilbert space $H^1(\mathbb{R}^+)$, necessary and sufficient conditions to have an invariant measure are the following: $\sum_{n=1}^{\infty} \|\tau_n\|_{H^1}^2 < +\infty$, $\sum_{n=1}^{\infty} \|\tau_n\|_{L^4}^4 < +\infty$, $\sum_{n=1}^{\infty} \|\sqrt{x}\tau_n(x)\|_{L^2}^2 < +\infty$, $\sum_{n=1}^{\infty} \|\sqrt{x}\tau_n'(x)\|_{L^2}^2 < +\infty$, $\sum_{n=1}^{\infty} \|\tau_n\|_{L^1}^2 < +\infty$, and that the functions $\int_{\cdot}^{+\infty} \tau_n(u)\, du$ are uniformly bounded in $L^2(\mathbb{R}^+)$. Under these hypotheses $N(b^*(\cdot) + b_0, Q_\infty)$ is the only invariant measure, where b^* is given by Eq. (11), b_0 is given by Eq. (12), and Q_∞ is given by Eq. (9).*

Proof. As before, we check if the conditions of Theorem 5 are satisfied. Since (i) must hold, and

$$\sup_{t\geq 0} \int_0^t \mathrm{Tr}\, Q_u\, du \quad = \quad \int_0^{+\infty} \sum_{n=1}^{\infty} \int_u^{+\infty} \left(\tau_n^2(x) + \tau_n'^2(x)\right)\, dx\, du =$$

$$= \sum_{n=1}^{\infty} \int_0^{+\infty} \left(\tau_n^2(x) + \tau_n'^2(x)\right) \int_0^x du\ dx =$$
$$= \sum_{n=1}^{\infty} \left(\left\| \sqrt{x} \tau_n(x) \right\|_{L^2}^2 + \left\| \sqrt{x} \tau_n'(x) \right\|_{L^2}^2 \right) ,$$

the conditions $\sum_{n=1}^{\infty} \|\sqrt{x}\tau_n(x)\|_{L^2}^2 < +\infty$ and $\sum_{n=1}^{\infty} \|\sqrt{x}\tau_n'(x)\|_{L^2}^2 < +\infty$ are necessary to have an invariant measure. We now check point (ii) of Theorem 5, and see if there exists $b \in D(A)$ such that $Ab + c = 0$. A solution is $b(x) = b^*(x) + b_0$, where b^* is defined by Eq. (11) and $b_0 \in \mathbb{R}$. Since b is a decreasing function, then $b \in L^2$ only if $\lim_{x\to+\infty} b(x) = 0$. This determines b_0:

$$0 = \lim_{x\to+\infty} b(x) = b_0 - \lim_{x\to+\infty} \frac{1}{2} \sum_{n=1}^{\infty} \left(\int_0^x \tau_n(u)\ du \right)^2 .$$

The necessary conditions for this to happen are that $\tau_n \in L^1$, $\sum_{n=1}^{\infty} \|\tau_n\|_{L^1}^2 < +\infty$ and that b_0 is given by Eq. (12). Now let us see under which conditions $b \in L^2$. If we define $T_n(x) = \int_x^{+\infty} \tau_n(u)\ du$ then, applying the algebraic identity $(c^2 - d^2)^2 = (c - d)^2(c^2 + 2cd + d^2)$ to the function b, we can see that the conditions $T_n \in L^2$ and T_n uniformly bounded in L^2 are necessary and sufficient to have $b \in L^2$. Since $b' = -c \in H^1$, we have $b \in H^2$, so b is a solution of $Ab + c = 0$.

Now we search for invariant measures for Eq. (10) of the kind δ_z with $z \in H^1$. This problem is equivalent to find solutions for the equation $z' = Az$ in H^1 which are constant in t. This means that $Az = 0$ must hold, so the solutions must have the form $z(x) = \text{const}$. Since the only constant function in H^1 is the null function, the only invariant measure of the kind δ_z is δ_0. Thus we have proved that $N(b^*(\cdot) + b_0, Q_\infty)$ is an invariant measure.

We claim that this is the only invariant measure for Eq. (1). To this aim, we refer to various results on asymptotic behaviour of solutions contained in [5]. We call $Z_t(X)$ the solution of Eq. (10) at time t having initial datum $Z_0 = X$, and $\mathscr{L}(Y)$ the law of a generic random variable Y under $\mathbb{Q}$. Since $\lim_{t\to+\infty} S_t f = 0$ for all $f \in H^1$, then $\mathscr{L}(Z_t(f)) = \delta_{S_t f}$ converges weakly to δ_0 for all $f \in H^1$. This means that δ_0 is the only invariant measure for Eq. (10). Hence $N(b^*(\cdot)+b_0, Q_\infty)$ is the unique invariant measure for Eq. (1) in $H^1(\mathbb{R}^+)$. □

We have just shown that in $H^1(\mathbb{R}^+)$ there exists only one invariant measure. This means that the invariant measure is independent of the initial forward curve r_0. Moreover, a result like Theorem 6 in Section 2 would be useless in this case, because $r_0(\infty)$ is always equal to 0, and this is clearly inconsistent with the way the market extrapolates the yield curve to get an idea of levels of interest rates beyond the longest observable maturities. The conclusion seems to be that $H^1(\mathbb{R}^+)$ is too poor to allow for a good financial interpretation of the results.

4. Concluding remarks

We analysed the Musiela forward rates equation in the Sobolev spaces $H^1_\gamma(\mathbb{R}^+)$ for all $\gamma \geq 0$ in the Gaussian case. For each of the spaces, we presented an explicit solution, called mild solution, and analysed its asymptotic behaviour. From our work it is clear that the asymptotic evolution of the solutions depends on the choice of the particular space H, and in our case it seems that working on $H^1_\gamma(\mathbb{R}^+)$ for $\gamma > 0$ is better than working on $H^1(\mathbb{R}^+)$ for the practical applications. This is because for $\gamma > 0$ we find a wide range of invariant measures, and the Gaussian ones are significant in the sense that if we start from a forward curve r_0 that is known today, then the law of r_t converges to a Gaussian invariant measure for $t \to +\infty$. Moreover, the particular invariant measure to which the law of r_t converges is determined by the level of the long forward rate $r_0(\infty)$. Conversely, in $H^1(\mathbb{R}^+)$ there exists only one invariant measure, that is independent of the particular initial forward curve r_0. Moreover in H^1 flat term structures are not allowed and all the forward curves converge to 0 as $x \to +\infty$. These facts let us believe that H^1 is not a suitable choice for practical purposes.

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