Nonholomorphic terms in $N=2$ supersymmetric Wilsonian actions and the renormalization group equation

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In this paper we first investigate the ansatz of one of the present authors for $K(\Psi,\overline{\Psi})$, the adimensional, modular-invariant, nonholomorphic correction to the Wilsonian effective Lagrangian of an $N=2$ globally supersymmetric gauge theory. The renormalization group β function of the theory crucially allows us to express $K(\Psi,\overline{\Psi})$ in a form that easily generalizes to the case in which the theory is coupled to N_F hypermultiplets. $K(\Psi,\overline{\Psi})$ satisfies an equation which should be viewed as a fully nonperturbative ''nonchiral superconformal Ward identity.'' We also determine its renormalization group equation. Furthermore, as a first step towards checking the validity of this ansatz, we compute the contribution to $K(\Psi,\overline{\Psi})$ from multi-instanton configurations of winding number $k=1$ and $k=2$. As a by-product of our analysis we check a nonrenormalization theorem for $N_F=4$. [S0556-2821(97)07520-6]

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I. INTRODUCTION

In a celebrated paper, Seiberg and Witten studied a globally $N=2$ supersymmetric Yang-Mills theory (SYM) with the SU(2) gauge group [1]. They also extended their analysis to theories with additional hypermultiplets (SQCD) [2], and they were able to exactly determine the Wilsonian effective action up to two derivatives and four fermions. In terms of an $N=2$ chiral superfield Ψ, these leading terms are encoded in a holomorphic function $\mathcal{F}(\Psi)$ called the effective prepotential. From a physical point of view, the Wilsonian effective action describes the low-energy degrees of freedom of the $N=2$ microscopic supersymmetric gauge theory. These results were achieved thanks to a certain number of conjectures which were suggested by the physics of the problem. It was later shown in [3] that, in the case of $N=2$ SYM, these assumptions follow from the symmetries of the theory and from the inversion formula first derived in [4] (subsequently generalized to SQCD in [5]). They are consistent with microscopic instanton computations in the cases of SYM and SQCD [6–12].

Since the moduli space of vacua of the theory is a thrice-punctured Riemann sphere, one can study the transformation properties of $\mathcal{F}(\Psi)$ under the modular group $\Gamma(2)$. The result of such an exercise is the inversion formula in [4], which relates $\mathcal{F}(\Psi)$ and its first derivative to a modular invariant function. The entire physical content of the theory can now be extracted from this differential equation [3,4], which was also derived as an anomalous superconformal Ward identity in [13].

Recently a great deal of work has been devoted to the study of nonholomorphic (i.e., higher-derivative) corrections to the $N=2$ SYM and SQCD low-energy effective actions [14–22]. Indeed, as it is well known, a Wilsonian effective action can be expanded in powers of the external momentum over some subtraction scale. The first nonholomorphic correction is a term with four derivatives or 8 fermions, and it is given by the full superspace integral of a real, adimensional function $K(\Psi,\overline{\Psi})$ [14]. Much in the same vein of the previous analysis of the symmetries of the effective prepotential, the investigation of the transformation properties under the modular group of the complete Wilsonian action leads to the conclusion that $K(\Psi,\overline{\Psi})$ is a modular invariant [14]. However, it seems that also the higher-order terms are modular invariant. Indeed, let us denote the nonholomorphic part of the Wilsonian effective action by $\hat{S}[\Psi,\overline{\Psi}]$; furthermore, let S,T be the SL$(2,\mathbb{Z})$ generators with $S^2=1$ and $(ST)^3=1$. In [14] it was shown that $\hat{S}[\Psi,\overline{\Psi}]$ does not transform under the action of T while, under duality, $\mathcal{F}(\Psi)\rightarrow\mathcal{F}_D(\Psi_D)=\mathcal{F}(\Psi)+\Psi_D\Psi$, where $\Psi_D=\partial\mathcal{F}/\partial\Psi$. Now, if the action of T on $\hat{S}[\Psi,\overline{\Psi}]$ is trivial and the group has only two generators, the action of S must be trivial too, since

$$\hat{S}[\Psi,\overline{\Psi}]=(ST)^3\circ\hat{S}[\Psi,\overline{\Psi}]=S^3\circ\hat{S}[\Psi,\overline{\Psi}]=S\circ\hat{S}[\Psi,\overline{\Psi}]. \tag{1.1}$$

However, we observe that the modular invariance described above is considered with respect to the S and T action defined in [14] whereas, strictly speaking, a function $G(\Psi,\overline{\Psi})$ is said to be modular invariant if $G(\gamma(\Psi),\overline{\gamma(\Psi)})=G(\Psi,\overline{\Psi})$, $\gamma\in SL(2,\mathbb{Z})$.

Let us now leave this argument on the side and let us remind that the perturbative 1-loop term and the contribution of instantons of winding number $k=1$ to $K(\Psi,\overline{\Psi})$ were computed in [15,16]; on the basis of these results, and by using uniformization theory, one of the present authors was able to write a modular invariant function which satisfies the constraints imposed by perturbative and instanton calculations and which has no other singularities but the one at weak coupling [18]. This function satisfies the physical requirements of the theory, for example, it vanishes at those points of the moduli space where monopoles or dyons become massless: we consider it to be a candidate for the expression of $K(\Psi,\overline{\Psi})$. Its actual form will be reviewed in Sec. III where we also write it in terms of the β function of the theory, and work out the renormalization group equation satisfied by K. We also find that this function satisfies an equation which should be viewed as a fully nonperturbative "nonchiral superconformal Ward identity." In the same section we also extend the ansatz to the case of SQCD with N_F hypermultiplets. Furthermore, we study the higher-derivative corrections to the SYM and SQCD effective Lagrangians, and in particular we focus our attention on the contributions of instantons of winding number $k=1,2$ to the real adimensional function $K(\Psi,\overline{\Psi})$. This is a first step in the direction of checking the proposal in [18] and that of Sec. III. As we shall discuss in Sec. IV, the situation is more involved than in the case of the holomorphic part of the effective Lagrangian, and we cannot provide here a check for the expression of $K(\Psi,\overline{\Psi})$. We plan to come back on this point in a future publication.

The plan of the paper is the following: in Sec. II we review the solution of [1] to fix the notations and compute the relationship between the Pauli-Villars renormalization group invariant scale and that appearing in [1]. We do this in great detail because we shall need it in the following and because the literature is plagued with inconsistent notations. The content of Sec. III has been discussed above. We start Sec. IV by computing the $k=1$ contribution to $K(\Psi,\overline{\Psi})$. It turns out to be in agreement with the result of [16], which was derived by using different methods. In the second part of the same section we compute the $k=2$ contribution, for $N=2$ SYM and SQCD. Furthermore, we check a recent result concerning a nonrenormalization theorem in the case of four flavors [19]. While we were writing this paper a work by Dorey *et al.* [21] has appeared in which computations partly similar to ours, in the case of winding number $k=1$, are carried out and the nonrenormalization theorem for $N_F=4$ is checked by using scaling arguments. Our results agree with theirs.

II. A REVIEW OF THE SEIBERG-WITTEN MODEL

The Lagrangian density for the microscopic $N=2$ SYM theory, in the $N=2$ supersymmetric formalism is given by

$$L=\frac{1}{16\pi}\,\mathrm{Im}\int d^2\theta d^2\tilde{\theta}\mathcal{F}(\Psi). \tag{2.1}$$

The chiral superfield Ψ, which describes the vector multiplet of the $N=2$ SUSY, transforms in the adjoint representation of the gauge group G [which will be SU(2) from now on]. Reexpressing the Lagrangian density in the $N=1$ formalism, we have

$$L=\frac{1}{16\pi}\,\mathrm{Im}\left[\int d^2\theta d^2\overline{\theta}K(\Phi,\overline{\Phi},V)+\int d^2\theta f_{ab}(\Phi)W^aW^b\right], \tag{2.2}$$

where a, b are indices of the adjoint representation of G. The Kähler potential $K(\Phi,\overline{\Phi},V)$ and the holomorphic function $f_{ab}(\Phi)$ are given, in terms of $\mathcal{F}$, by

$$K(\Phi,\overline{\Phi},V)=(\overline{\Phi}e^{-2V})^a\,\frac{\partial\mathcal{F}}{\partial\Phi^a}, \tag{2.3}$$

$$f_{ab}(\Phi)=\frac{\partial^2\mathcal{F}}{\partial\Phi^a\partial\Phi^b}. \tag{2.4}$$

The classical action for the $N=2$ SYM theory is obtained by choosing for the holomorphic prepotential $\mathcal{F}$ the functional form

$$\mathcal{F}_{\mathrm{cl}}(\Psi)=\frac{\tau_{\mathrm{cl}}}{2}\,(\Psi^a\Psi^a), \tag{2.5}$$

where we conventionally define τ_{cl} as

$$\tau_{\mathrm{cl}}=\frac{\theta}{2\pi}+\frac{4\pi i}{g^2}. \tag{2.6}$$

Our normalizations are the same as in [1]. The choice (2.5) is dictated by renormalizability requirements. After eliminating the auxiliary fields, the classical action of the theory is given by

$$S=S_G+S_H+S_F+S_Y+S_{\mathrm{pot}}. \tag{2.7}$$

S_G is the usual gauge field action; the kinetic terms for the Fermi and Bose fields minimally coupled to the gauge field A_μ are

$$S_F[\lambda,\overline{\lambda},A]=\int d^4x\overline{\lambda}^{\dot{A}a}[\not{D}(A)\lambda_{\dot{A}}]^a, \tag{2.8}$$

where $\lambda_{\dot{A}}$ are the two gauginos, $\dot{A}=1,2$, and

$$S_H[\phi,\phi^\dagger,A]=\int d^4x(D\phi)^{\dagger a}(D\phi)^a. \tag{2.9}$$

The Yukawa interactions are given by

$$S_Y[\phi,\phi^\dagger,\lambda,\overline{\lambda}]=\sqrt{2}g\epsilon^{abc}\int d^4x\phi^{a\dagger}(\lambda_1^b\lambda_2^c)+\mathrm{H.c.} \tag{2.10}$$

and finally $S_{\mathrm{pot}}=\int d^4xV(\phi,\phi^\dagger)$ comes from the potential term

$$V(\phi,\phi^\dagger)=\mathrm{Tr}[\phi,\phi^\dagger]^2, \tag{2.11}$$

for the complex scalar field. As required by supersymmetry, one has $V(\phi,\phi^\dagger)\geqslant 0$. The condition $V(\phi,\phi^\dagger)=0$ implies that $[\phi,\phi^\dagger]=0$: ϕ is then a normal operator, and can be diagonalized by a unitary matrix: that is, a color rotation. The most general (supersymmetric) classical vacuum configuration is then

$$\phi_0=a\left(\Omega\,\frac{\sigma_3}{2}\,\Omega^\dagger\right),\quad a\in\mathbb{C},\quad \Omega\in\mathrm{SU}(2). \tag{2.12}$$

When $a\neq 0$ the SU(2) gauge symmetry is spontaneously broken to U(1). The classical vacuum "degeneracy" for the $N=2$ SYM theory is lifted neither by perturbative nor by nonperturbative quantum corrections [23,24]. In fact any nonzero superpotential would explicitly break the extended supersymmetry of the model; however, the Witten index of the theory is nonzero [25], so supersymmetry stays unbroken. We then have a fully quantum moduli space, $\mathcal{M}_{SU(2)}$, for the low-energy theory. The effective Lagrangian for the massless U(1) fields $\Phi=\{\phi^3,\lambda^3_{\alpha 1},F^3\}$, $W_\alpha=\{A^3_\mu,\lambda^3_{\alpha 2},D^3\}$ will be the U(1) version of Eq. (2.1) and reads, in $N=1$ notation,

$$L_{\rm eff}=\frac{1}{16\pi}\,{\rm Im}\left[\int d^2\theta\mathcal{F}''(\Phi)W^\alpha W_\alpha+\int d^2\theta d^2\bar\theta\bar\Phi\mathcal{F}'(\Phi)\right]. \tag{2.13}$$

The low-energy dynamics are then governed by a unique function $\mathcal{F}(\Phi)$, the effective prepotential, whose functional form is not restricted by supersymmetry. The crucial property of $\mathcal{F}(\Phi)$, first proved in [26], is its holomorphicity. In analogy with Eq. (2.6) we can also define an effective coupling constant as

$$\tau(\Phi)=\mathcal{F}''(\Phi). \tag{2.14}$$

It is now a simple exercise to rewrite Eq. (2.13) in the component field formalism.[1] This way we obtain

$$L_{\rm eff}=\frac{1}{4\pi}\,{\rm Im}\Bigg[-\mathcal{F}''(\phi)\left(|\partial_\mu\phi|^2+i\bar\lambda_{\dot A}\partial\!\!\!/\lambda^{\dot A}+\frac{1}{4}F_{\mu\nu}F_{\mu\nu}\right)+\frac{1}{\sqrt2}\mathcal{F}'''(\phi)\lambda_1\sigma^{\mu\nu}\lambda_2F_{\mu\nu}+\frac{1}{4}\mathcal{F}^{IV}(\phi)\lambda_1^2\lambda_2^2\Bigg]+\cdots, \tag{2.15}$$

where the dots stand for terms of higher order in the coupling constant. The effective description of the low-energy dynamics in terms of the U(1) superfields Φ and W_α is not appropriate for all vacuum configurations. In particular, the quantum moduli space $\mathcal{M}_{SU(2)}$ is better described in terms of the variable a and its dual $a_D=\partial_a\mathcal{F}$. When the gauge group is SU(2), we can describe $\mathcal{M}_{SU(2)}$ in terms of the gauge-invariant coordinate $u=\langle{\rm Tr}\phi^2\rangle$. Then $\mathcal{M}_{SU(2)}$ is the Riemann sphere with punctures at $u=\infty$ and $u=\pm\Lambda^2$, where Λ is the renormalization group invariant scale (RGI) in the normalization of [1].

At the classical level

$$\mathcal{F}_{\rm cl}(a)=\frac{\tau_{\rm cl}}{2}a^2; \tag{2.16}$$

however, perturbative as well as nonperturbative effects modify the expression of the prepotential. We shall then write

$$\mathcal{F}(a)=\mathcal{F}_{\rm pert}(a)+\mathcal{F}_{\rm np}(a), \tag{2.17}$$

including the classical contribution in the first term. The perturbative term has been calculated by Seiberg [29] and is exactly determined thanks to the holomorphicity requirements on $\mathcal{F}(a)$ and to the $U(1)_R$ symmetry:

$$U(1)_R:\lambda_{\dot A}\to e^{i\alpha}\lambda_{\dot A},\quad \phi\to e^{2i\alpha}\phi. \tag{2.18}$$

The associated current J^μ_R is anomalous;

$$J_{\mu R}=\bar\lambda_1\bar\sigma_\mu\lambda_1+\bar\lambda_2\bar\sigma_\mu\lambda_2+2i\phi^\dagger\overleftrightarrow{\partial}_\mu\phi,$$

$$\partial_\mu J^\mu_R=-\frac{i}{32\pi^2}(F^a_{\mu\nu}\tilde F^a_{\mu\nu})(4N_c) \tag{2.19}$$

(in our case the number of colors is taken to be $N_c=2$). The discrete subgroup $\mathbb{Z}_8\subset U(1)_R$ generated by the transformations (2.18) with $\alpha_m=(2\pi/8)m$, $m\in\mathbb{Z}$ is a symmetry of the full quantum theory, since in this case the action functional S transforms as

$$S\to S+i8k\alpha_m=S+2\pi im. \tag{2.20}$$

At a given point in the u-moduli space the $\mathbb{Z}_8$ symmetry spontaneously breaks down to $\mathbb{Z}_4$, since the $U(1)_R$ charge of u is $+4$. However, (2.20) tells us that the points u and $-u$ correspond to physically equivalent theories. We now immediately rewrite (2.18) in terms of the U(1) superfield Ψ of the $N=2$ supersymmetry as

$$U(1)_R:\Psi(x,\theta)\to\Psi'(x,\theta)=e^{2i\alpha}\Psi(x,\theta e^{-i\alpha}): \tag{2.21}$$

if we now assign a charge of $+1$ to θ, the charge of Ψ is $+2$, in such a way that the classical prepotential (2.5) is invariant. Then the perturbative effective Lagrangian

$$L_{\rm pert}[\Psi]=\frac{1}{16\pi}\,{\rm Im}\int d^2\theta d^2\tilde\theta\mathcal{F}_{\rm pert}[\Psi(x,\theta)] \tag{2.22}$$

transforms into

$$\begin{aligned}L^{(\alpha)}_{\rm pert}[\Psi']&=\frac{1}{16\pi}\,{\rm Im}\int d^2\theta d^2\tilde\theta\mathcal{F}_{\rm pert}[e^{2i\alpha}\Psi(x,\theta e^{-i\alpha})]\\&=\frac{1}{16\pi}\,{\rm Im}\int d^4\theta e^{-4i\alpha}\mathcal{F}_{\rm pert}[e^{2i\alpha}\Psi(x,\theta)],\end{aligned} \tag{2.23}$$

where $d^4\theta=d^2\theta d^2\tilde\theta$. After a little algebra we get

$$L_{\rm pert}+\delta_\alpha L_{\rm pert}=\frac{1}{16\pi}\,{\rm Im}\int d^4\theta\left[1+4i\alpha\left(-1+\Psi^2\frac{\partial}{\partial\Psi^2}\right)\right]\mathcal{F}_{\rm pert}(\Psi). \tag{2.24}$$

Furthermore, we know that under a $U(1)_R$ transformation

$$\delta_\alpha L_{\rm pert}=-(4N_c\alpha)\left(\frac{1}{32\pi^2}F^a_{\mu\nu}\tilde F^a_{\mu\nu}\right), \tag{2.25}$$

(with $N_c=2$), so that

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[1]Throughout the article we shall use the conventions of Wess and Bagger [27] for the product of Weyl spinors and integration on superspace. We also define the Euclidean $\sigma_\mu,\bar\sigma_\mu$ matrices as $\sigma_\mu=(\mathbb{1},i\sigma^a)$, $\bar\sigma_\mu=(\mathbb{1},-i\sigma^a)$, σ^a, $a=1,2,3$ being the usual Pauli matrices, and the (anti)self-dual matrices $(\bar\sigma_{\mu\nu})\sigma_{\mu\nu}$ are $\sigma_{\mu\nu}=\frac12(\sigma_\mu\bar\sigma_\nu-\sigma_\nu\bar\sigma_\mu)=i\eta^a_{\mu\nu}\sigma^a$, $\bar\sigma_{\mu\nu}=\frac12(\bar\sigma_\mu\sigma_\nu-\bar\sigma_\nu\sigma_\mu)=i\bar\eta^a_{\mu\nu}\sigma^a$, where $\eta^a_{\mu\nu},\bar\eta^a_{\mu\nu}$, are the 't Hooft symbols defined in [28].

$$L_{\rm pert}+\delta_\alpha L_{\rm pert}=\frac{1}{16\pi}\,{\rm Im}\int d^4\theta\left[\mathcal{F}_{\rm pert}(\Psi)-\frac{2\alpha}{\pi}\,\Psi^2\right], \tag{2.26}$$

from which it immediately follows that[2]

$$\left(\Psi^2\,\frac{\partial}{\partial\Psi^2}-1\right)\mathcal{F}_{\rm pert}(\Psi)=\frac{i}{2\pi}\,\Psi^2. \tag{2.27}$$

This is the semiclassical version [13,30,31] of the nonperturbative relation

$$i\pi\left(\mathcal{F}-\frac{a}{2}\,\frac{\partial\mathcal{F}}{\partial a}\right)=\langle{\rm Tr}\phi^2\rangle, \tag{2.28}$$

obtained in [4] and subsequently rederived in [13]. The solution to Eq. (2.27) is

$$\mathcal{F}_{\rm pert}(\Psi)=\frac{i}{2\pi}\,\Psi^2\,\ln\frac{\Psi^2}{\mu^2}, \tag{2.29}$$

where μ can be fixed by the value of the coupling constant at some subtraction point. The normalization of the (one-loop) perturbative contribution must be fixed together with the nonperturbative contributions and the definition of the RGI scale Λ. To this end, we first write the nonperturbative prepotential as

$$\mathcal{F}_{\rm np}(a)=\sum_{k=1}^{\infty}\,\mathcal{F}_k\left(\frac{\Lambda}{a}\right)^{4k}a^2, \tag{2.30}$$

and similarly

$$u(a)=\frac{1}{2}\,a^2+\sum_{k=1}^{\infty}\,\mathcal{G}_k\left(\frac{\Lambda}{a}\right)^{4k}a^2. \tag{2.31}$$

It is easy to check that the expressions (2.30), (2.31) possess the correct invariance properties under the $\mathbb{Z}_8$ symmetry. The values of the $\mathcal{F}_k$'s and the $\mathcal{G}_k$'s are meaningful only if one specifies the choice of the renormalization-group-invariant (RGI) scale Λ, and can be obtained via a k-instanton calculation [6,9,12]. In the following we shall need the expressions for the 1-instanton contribution to $u(a)$, which was found to be [6,12]

$$\langle{\rm Tr}\phi^2\rangle_{k=1}=\frac{\Lambda_{\rm PV}^4}{a^2}. \tag{2.32}$$

Here $\Lambda_{\rm PV}$ is the Pauli-Villars RGI scale, which naturally arises when performing instanton calculations after the cancellation of the determinants of the kinetic operators of the various fields [32]. We shall fix in a moment its relationship with the scale employed in [1]. Note that the relation (2.28) gives the $\mathcal{F}_k$'s as functions of the $\mathcal{G}_k$'s;

$$2i\pi k\mathcal{F}_k=\mathcal{G}_k\,. \tag{2.33}$$

[2]Disregarding terms which vanish when integrated in $d^4\theta$.

By making some hypotheses on the structure of the moduli space and on the monodromies of τ around its singularities, Seiberg and Witten obtained the expressions of $a(u)$ and $a_D(u)$, which are given by

$$a(u)=\frac{\sqrt{2}}{\pi}\int_{-\Lambda^2}^{\Lambda^2}dx\,\frac{\sqrt{x-u}}{\sqrt{x^2-\Lambda^4}}, \tag{2.34}$$

$$a_D(u)=\frac{\sqrt{2}}{\pi}\int_{\Lambda^2}^{u}dx\,\frac{\sqrt{x-u}}{\sqrt{x^2-\Lambda^4}}, \tag{2.35}$$

where Λ is the Seiberg-Witten RGI scale (to be matched against the Pauli-Villars one). We now put

$$a_D(u)=\frac{\sqrt{2u}}{\pi}\,g(1/u), \tag{2.36}$$

where

$$g(1/u)=\int_{\Lambda^2/u}^{1}dz\,\frac{\sqrt{z-1}}{\sqrt{z^2-\Lambda^4/u^2}} \tag{2.37}$$

$$=\int_{\Lambda^2/u}^{1}dz\left[\frac{\sqrt{z-1}}{\sqrt{z^2-\Lambda^4/u^2}}-\frac{i}{z}\right]+\int_{\Lambda^2/u}^{1}dz\,\frac{i}{z}. \tag{2.38}$$

The perturbative constant ($u\gg\Lambda^2$) contribution to $g(1/u)$ is

$$\int_0^1 dz\left[\frac{\sqrt{z-1}}{z}-\frac{i}{z}\right]=2i\,\ln\frac{2}{e}, \tag{2.39}$$

so that

$$a_D(u)\to i\,\frac{\sqrt{2u}}{\pi}\,\ln\frac{4u}{(e\Lambda)^2}. \tag{2.40}$$

Using the asymptotic expansion (2.31) we finally obtain an expression for a_D as a function of a in the perturbative regime,

$$a_D(a)\to\frac{i}{\pi}\,a\,\ln\frac{2a^2}{(e\Lambda)^2}, \tag{2.41}$$

that is, in the same limit,

$$\mathcal{F}_{\rm pert}(a)=\frac{i}{2\pi}\,a^2\,\ln\frac{2a^2}{e^3\Lambda^2}. \tag{2.42}$$

This sets the normalization of the classical and perturbative contributions. From Eq. (2.42) it follows that

$$\tau_{\rm pert}(a)=\mathcal{F}''_{\rm pert}(a)=\frac{i}{\pi}\,\ln\frac{2a^2}{\Lambda^2}. \tag{2.43}$$

We now examine the first instanton correction to $u(a)$; via the relation (2.33) we shall then fix the normalization of the $\mathcal{F}_k$'s. Expanding the expression (2.34) for $u\gg\Lambda^2$ we get

$$a(u)=\frac{\sqrt{2}}{\pi}\left[\int_{-\Lambda^2}^{\Lambda^2}dx\,\frac{1}{\sqrt{\Lambda^4-x^2}}-\frac{1}{8u^2}\times\int_{-\Lambda^2}^{\Lambda^2}dx\,\frac{x^2}{\sqrt{\Lambda^4-x^2}}+O(\Lambda^8/u^4)\right]$$
$$=\sqrt{2u}\left[1-\frac{\Lambda^4}{16u^2}+O(\Lambda^8/u^4)\right]. \quad (2.44)$$

In the same approximation we also have that

$$u(a)=\frac{a^2}{2}\left[1+2\mathcal{G}_1\left(\frac{\Lambda}{a}\right)^4+O\left(\frac{\Lambda^8}{a^8}\right)\right]; \quad (2.45)$$

substituting into Eq. (2.44) we get

$$a=a\left[1+\mathcal{G}_1\left(\frac{\Lambda}{a}\right)^4+O\left(\frac{\Lambda^8}{a^8}\right)\right]\left\{1-\frac{1}{4}\left(\frac{\Lambda}{a}\right)^4\left[1+O\left(\frac{\Lambda^8}{a^8}\right)\right]\right\}, \quad (2.46)$$

and, for consistency, we must impose

$$\mathcal{G}_1=\frac{1}{4}, \quad (2.47)$$

with respect to the RGI scale Λ in [1]. Comparing Eq. (2.47) with Eq. (2.32) we find

$$\Lambda_{PV}=\frac{\Lambda}{\sqrt{2}}. \quad (2.48)$$

The holomorphic prepotential $\mathcal{F}(a)$ is then given by

$$\mathcal{F}(a)=\frac{i}{2\pi}\,a^2\,\ln\frac{2a^2}{e^3\Lambda^2}+a^2\sum_{k=1}^{\infty}\mathcal{F}_k\left(\frac{\Lambda}{a}\right)^{4k}$$
$$=\frac{i}{2\pi}\,a^2\,\ln\frac{a^2}{e^3\Lambda_{PV}^2}+a^2\sum_{k=1}^{\infty}\mathcal{F}_k 2^{2k}\left(\frac{\Lambda_{PV}}{a}\right)^{4k}, \quad (2.49)$$

where $\mathcal{F}_1=\mathcal{G}_1/2\pi i=1/8\pi i$.

Finally, when we add N_F hypermultiplets, the holomorphic prepotential $\mathcal{F}^{(N_F)}(a)$ becomes

$$\mathcal{F}^{(N_F)}(a)=\frac{i}{8\pi}\,(4-N_F)a^2\,\ln\frac{a^2}{e^3(\Lambda_{PV}^{(N_F)})^2}+a^2\sum_{k=1}^{\infty}\mathcal{F}_k^{(N_F)}2^{2k}\left(\frac{\Lambda_{PV}^{(N_F)}}{a}\right)^{k(4-N_F)}, \quad (2.50)$$

where $\mathcal{F}_{2k+1}^{(N_F)}=0$ in the presence of massless hypermultiplets. This is a consequence of the $\mathbb{Z}_{4(4-N_F)}$ chiral symmetry group of the quantum theory [2].

III. NONHOLOMORPHIC CORRECTIONS AND THE β FUNCTION

Let us briefly describe the general form of the nonholomorphic (higher-derivative) corrections to the Lagrangian (2.2). Since an effective Lagrangian is written as an expansion in the space of momenta, the higher-order contributions will come out of terms with four or more derivatives or eight or more fermions. In the case of $N=2$ SYM theory, they will be written as an expansion in spinor derivatives:

$$S_{\rm NH}(\Psi,\overline{\Psi})=\int d^4x d^4\theta d^4\overline{\theta}[K(\Psi,\overline{\Psi})+G(\Psi,\overline{\Psi})D\Psi D\Psi \overline{D\Psi D\Psi}+\cdots+O(D^4,\overline{D^4})]. \quad (3.1)$$

If we assign the scaling dimensions $[dx]=1$, $[d\theta]=-1/2$ and $[D]=1/2$, as a consequence of $N=2$ supersymmetry, the expansion will contain only terms with even dimension. Furthermore, the $U(1)_R$ anomaly and the nonperturbative corrections are completely encoded in the analytic prepotential $\mathcal{F}$, which is the only holomorphic term that appears in the effective Lagrangian. Therefore Eq. (3.1) is integrated over the whole superspace. From now on we shall restrict our attention to the first term $K(\Psi,\overline{\Psi})$ in Eq. (3.1), which is adimensional and does not contain spinor derivatives of Ψ and $\overline{\Psi}$.

We now consider the derivation of K proposed in [18]. Let $H=\{w|\mathrm{Im}\,w>0\}$ be the upper half plane endowed with the Poincaré metric $ds_P^2=(\mathrm{Im}\,w)^{-2}|dw|^2$. Since $\tau=\partial_a^2\mathcal{F}$ is the inverse of the map uniformizing $\mathcal{M}_{SU(2)}$, it follows that the positive-definite metric

$$ds_P^2=\frac{|\partial_a^3\mathcal{F}|^2}{(\mathrm{Im}\tau)^2}\,|da|^2=\frac{|\partial_u\tau|^2}{(\mathrm{Im}\tau)^2}\,|du|^2=e^{\varphi}|du|^2 \quad (3.2)$$

is the Poincaré metric on $\mathcal{M}_{SU(2)}$. This implies that φ satisfies the Liouville equation

$$\partial_{\bar{u}}\partial_u\varphi=\frac{e^{\varphi}}{2}. \quad (3.3)$$

Observe that this equation is satisfied since, for any fundamental domain F in H, $\tau(u)$ is a *univalent* (i.e., one-to-one) map between $\mathcal{M}_{SU(2)}$ and F. In this context we stress that $\tau(u)$ is not properly a function; rather, it is a *polymorphic* function (i.e., it is Möbius transformed after going around nontrivial cycles). Therefore, classical theorems concerning standard meromorphic functions do not hold. In particular, Im $\tau(u)$ is a zero mode of the Laplacian. Observe that on the moduli space $\tau(u)$ is holomorphic as zeroes and poles are at the punctures (that is, missing points). Zeroes and poles are manifest on the compactified moduli space. However, these critical points are absent in the case of higher-genus Riemann surfaces without punctures. This follows from the fact that punctures correspond to points $\tau\in\mathbb{R}=\partial H$. In particular, as the fundamental domains of negatively curved Riemann surfaces without punctures Σ belong to H, it follows that in these cases τ is a holomorphic nowhere vanishing function on Σ. In particular, Δ Im$\tau=0$. In [4,33,3] it was shown how the results of [1] are naturally described in the framework of uniformization theory. We now show how the function $K(\Psi,\overline{\Psi})$ derived in [18] naturally arises in this context.

To see this let us first recall some asymptotics for the Poincaré metric. Let us consider the Riemann sphere with elliptic or parabolic points (punctures) at $u_1,\ldots,u_{n-1}$, $u_n=\infty$. Near an elliptic point the behavior of the Poincaré metric is

$$e^{\varphi}\sim\frac{4q_k^2 r_k^{2q_k-2}}{(1-r_k^{2q_k})^2}, \qquad (3.4)$$

where q_k^{-1} is the ramification index of u_k and $r_k=|u-u_k|$, $k=1,\ldots,n-1$, $r_n=|u|$. Taking the $q_k\to 0$ limit, we get the parabolic singularity (puncture)

$$e^{\varphi}\sim\frac{1}{r_k^2\ln^2 r_k}. \qquad (3.5)$$

It follows that in the case of $\mathcal{M}_{\mathrm{SU}(2)}$ the Poincaré metric e^{φ} vanishes only at the puncture $u=\infty$, where $\varphi\sim -2\ln(|u|\ln|u|)$. Furthermore, e^{φ} is divergent only at the punctures $u=\pm\Lambda^2$, where $\varphi\sim-2\ln(|u\mp\Lambda^2|\ln|u\mp\Lambda^2|)$.

Let us now gather some known results on $K(\Psi,\bar\Psi)$. First observe that in [15] it was proved that, to the one-loop order

$$K(\Psi,\bar\Psi)\sim c\,\ln\frac{\Psi}{\Lambda}\ln\frac{\bar\Psi}{\Lambda}, \qquad (3.6)$$

where c is a constant which was recently calculated [22] in the formalism of harmonic superspace for $0\leqslant N_F\leqslant 4$. The nonholomorphic terms in the effective Lagrangian are $\mathrm{U}(1)_R$ invariant. If we follow the arguments used for $\mathcal{F}$ [which eventually led to Eq. (2.27)] we get, in particular,

$$\int d^4\theta d^4\bar\theta\left\{\Psi\frac{\partial}{\partial\Psi}-\bar\Psi\frac{\partial}{\partial\bar\Psi}\right\}K(\Psi,\bar\Psi)=0, \qquad (3.7)$$

which should be considered as a semiclassical Ward identity for K. The solution of this equation is simply given, modulo Kähler transformations, by

$$K(y,\bar y)=P(y+\bar y)+y\overline{g(y)}+\bar y g(y), \qquad (3.8)$$

where $y=\ln(\Psi/\Lambda)$, g is an arbitrary function and $P=\bar P$.[3] In particular, the term found in [15] is a solution to this equation, but it seems that, in principle, no nonrenormalization theorem prevents us from considering solutions with higher-order polynomials in $\ln(\Psi\bar\Psi/\Lambda^2)$. These terms would represent higher-loop contributions to K. However, in the case of SQCD with $N_F=4$ massless hypermultiplets and gauge group SU(2), we know that the β function vanishes, so that no scale can arise in the theory. In this case the only possible function of Ψ/Λ which can appear in the solution (3.8) is linear in the product $\ln(\Psi/\Lambda)\ln(\bar\Psi/\Lambda)$ [or, up to purely chiral or antichiral terms, quadratic in $\ln(\Psi\bar\Psi/\Lambda^2)$] [19]; indeed, only in this case the scale Λ is a fake (it does not multiply nonholomorphic terms in the Lagrangian), as it should be for a scale-invariant theory.

Let us go back to the $N_F=0$ case. Besides Eq. (3.6) we know that K is a modular invariant [14] and that the one-instanton contribution is [16]

$$K(\Psi,\bar\Psi)|_{k=1}=\frac{1}{32\pi^2}\left(\frac{\Lambda}{\Psi}\right)^4\ln\frac{\Psi\bar\Psi}{\Lambda^2}+\mathrm{H.c.} \qquad (3.9)$$

Strictly speaking, a function $G(\Psi,\bar\Psi)$ is said to be modular invariant if $G(\gamma(\Psi),\overline{\gamma(\Psi)})=G(\Psi,\bar\Psi)$, $\gamma\in SL(2,\mathbb{Z})$. However, $K(\Psi,\bar\Psi)$ has the invariance $T\circ K(\Psi,\bar\Psi)=K(\Psi,\bar\Psi)$ and $S\circ K(\Psi,\bar\Psi)=K(\Psi,\bar\Psi)$. While in the former case there is no change in the functional structure of K, in the latter, according to the S-dual formulation of the theory, where $\mathcal{F}(\Psi)$ is replaced by $\mathcal{F}_D(\Psi_D)$, the function $S\circ K(\Psi,\bar\Psi)$ should be constructed with the building block $\mathcal{F}_D(\Psi_D)$ [which replaces $\mathcal{F}(\Psi)$ in the construction of $K(\Psi,\bar\Psi)$].

Let us discuss the reasons why $\mathcal{F}(\Psi)$ should be considered as a building block for $K(\Psi,\bar\Psi)$. First of all, one can observe that the geometry determined by $\mathcal{F}$ is that of the Riemann sphere with three punctures. Then, by S-duality, modular invariance, and general arguments, it is quite natural to believe that K should be a well-defined function on $\mathcal{M}_{\mathrm{SU}(2)}$, that is, a real "function" of $u,\bar u$. On the other hand, the inversion formula (2.28) tells us that we can express u by means of $\mathcal{F}(\Psi)$. Therefore, $\mathcal{F}(\Psi)$ is the building block for $K(\Psi,\bar\Psi)$. This is a useful result since, as we shall see, it implies a differential equation for $K(\Psi,\bar\Psi)$, which is the nonchiral analogue of Eq. (2.28). Furthermore, Eq. (2.28), which is equivalent to a second-order equation, is actually a (anomalous) superconformal Ward identity [13]. Then, the equation we shall get should be interpreted as a nonchiral superconformal Ward identity.

The request of modular invariance indicates that K should be constructed in terms of the geometrical building blocks of the thrice-punctured Riemann sphere $\mathcal{M}_{\mathrm{SU}(2)}$. The comparison between the asymptotics (3.5) and (3.6) suggests that the Poincaré metric should have a rôle in defining K. In particular, we observe that, in order to be well defined on the u-moduli space, the logarithmic terms should come out of a function which has to be globally defined. This would also respect the symmetries of the theory. The above analysis suggested the following proposal [18]:

$$K(\Psi,\bar\Psi)=\alpha\,\frac{e^{-\varphi(\mathcal{G}(\Psi),\overline{\mathcal{G}(\Psi)})}}{|\mathcal{G}^2(\Psi)-\Lambda^4|}, \qquad (3.10)$$

where α is a real constant to be determined via an explicit calculation, $u=\mathcal{G}(\Psi)$ and $e^{\varphi(u,\bar u)}$ is defined in Eq. (3.2). The expression (3.10) can also be written in the form

$$K(\Psi,\bar\Psi)=4\alpha\pi^2 e^{2\varphi_{\mathrm{SW}}}|\mathcal{G}^2(\Psi)-\Lambda^4|, \qquad (3.11)$$

or

[3]It is a trivial exercise to show that Eq. (3.7) is completely equivalent to the superspace-integrated version of the equation (3.7) of [22].

$$K(\Psi,\overline{\Psi})=2\alpha\pi\,\exp\left(\varphi_{\rm SW}(\mathcal{G}(\Psi),\overline{\mathcal{G}(\Psi)})-\frac{\varphi}{2}\,(\mathcal{G}(\Psi),\overline{\mathcal{G}(\Psi)})\right), \tag{3.12}$$

where

$$e^{\varphi_{\rm SW}(u,\overline{u})}=|\partial_u a|^2\ \text{Im}\ \tau, \tag{3.13}$$

is the Seiberg-Witten metric on $\mathcal{M}_{\rm SU(2)}$.

Let us now consider the geometrical meaning of $K(\Psi,\overline{\Psi})$. According to Eq. (3.12) the (1/2,1/2)-differential K is proportional to the Seiberg-Witten metric times the inverse square root of the Poincaré metric. The interesting point is that the structure of Eq. (3.13) does not prevent us from considering for K a suitable modification of the Liouville equation which is satisfied by the Poincaré metric. In particular, looking at the structure of Eq. (3.13), it is easy to see that after a sufficient number of times one acts with the derivative operators, the effect of the Seiberg-Witten metric on the Liouville equation can be eliminated. In particular, setting

$$Y(\Psi,\overline{\Psi})=K(\Psi,\overline{\Psi})\partial_\Psi\partial_{\overline{\Psi}}\ln K(\Psi,\overline{\Psi}), \tag{3.14}$$

we have the "nonchiral superconformal Ward identity" [4]

$$\partial_{\overline{\Psi}}\partial_\Psi \ln Y(\Psi,\overline{\Psi})=0. \tag{3.15}$$

A. $K(\Psi,\overline{\Psi})$ from the β function

In [34] the renormalization group equation (RGE) and the exact β function were derived in the SU(2) case. Similar structures were also considered in the framework of the Witten-Dijkgraaf-Verlinde-Verlinde equation in the SU(3) case [35]. It would be interesting to understand the scaling properties of K. As it is constructed in terms of $\mathcal{F}$, one could imagine that the RGE for $\mathcal{F}$ should play a role. The RGE, derived in [34], is

$$\partial_\Lambda \mathcal{F}(a,\Lambda)=\frac{\Lambda}{\Lambda_0}\ \partial_{\Lambda_0}\mathcal{F}(a_0,\Lambda_0)\exp\left(-2\int_{\tau_0}^{\tau}dx\,\beta^{-1}(x)\right), \tag{3.16}$$

where

$$\beta(\tau)=\Lambda(\partial_\Lambda\tau)_u \tag{3.17}$$

is the β function. Remarkably, the β function admits a geometrical interpretation as the chiral block for the Pointcaré metric: namely [34],

$$ds_P^2=\left|\frac{\beta}{2u\ \text{Im}\tau}\right|^2|du|^2=e^{\varphi}|du|^2. \tag{3.18}$$

On physical grounds, it is clear that the β function should vanish at $u=0$. However, this degeneracy should not appear in the relevant geometrical objects. Remarkably, this is actually the case. To be more precise, K admits the equivalent general representation

$$K(a,\overline{a})=4\alpha\pi\,\frac{|\mathcal{G}(a)|(\text{Im}\tau)^2}{|\beta||\partial_a\mathcal{G}(a)|^2}. \tag{3.19}$$

B. The $1\leqslant N_F\leqslant 4$ case

As the above expression (3.19) for K does not refer to a particular underlying geometry, we can consider it as a general model-independent expression for K. In particular, observe that its asymptotic expansion can be performed by just using the one for the prepotential $\mathcal{F}$. However, there is still another equivalent form for K which is particularly useful in order to perform asymptotic analyses. We have in mind the fact that, in the presence of massless hypermultiplets, only instantons with even k contribute. Then, in order to get a suitable expression for K, we introduce the function [34]

$$\beta^{(a)}(\tau)=\Lambda(\partial_\Lambda\tau)_a, \tag{3.20}$$

whose relation with the β function is [34]

$$\beta(\tau)=2u\,\frac{\partial_u a}{a}\,\beta^{(a)}(\tau). \tag{3.21}$$

By Eqs. (3.19) and (3.21) we have

$$K(a,\overline{a})=2\alpha\pi\,\frac{|a|(\text{Im}\tau)^2}{|\beta^{(a)}||\partial_a\mathcal{G}(a)|}. \tag{3.22}$$

To better illustrate the rôle of the β function in the nonholomorphic contribution, we use a result in [34] where it was shown that

$$u=\Lambda^2\ \exp\left(-2\int_{\tau_0}^{\tau}dx\,\beta^{-1}(x)\right), \tag{3.23}$$

where $u(\tau_0)=\Lambda^2$ (in the $N_F=0$ case, $\tau_0=0$). Then, thanks to Eqs. (3.22) and (3.23), it follows that K has the form

$$K(a,\overline{a})=\alpha\pi\left|\frac{a}{\Lambda}\right|^2|F|^2\ \exp\left(\int_{\tau_0}^{\tau}\beta^{-1}+\overline{\int_{\tau_0}^{\tau}\beta^{-1}}\right)(\text{Im}\tau)^2, \tag{3.24}$$

where

$$F(a,\overline{a})=\frac{\beta^{1/2}}{\beta^{(a)}}. \tag{3.25}$$

As a consequence of Eq. (3.24), K satisfies the RGE:

$$\Lambda(\partial_\Lambda K(a,\overline{a}))_{a,\overline{a}}=2\left[\text{Re}\left(\frac{\beta^{(a)}}{\beta}+\beta^{(a)}\partial_\tau\ln F\right)+\frac{\text{Im}\beta^{(a)}}{\text{Im}\tau}-1\right]K(a,\overline{a}). \tag{3.26}$$

One can check that when only instantons with even k contribute to $\mathcal{F}$, then this would also be the case for the expression (3.22) for K.

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[4]We thank Gaetano Bertoldi for interesting discussions on this equation.

Finally we note that in the $N_F=4$ case the above construction breaks down. Here, in particular, the underlying geometry is trivial. As a consequence, the nontrivial global aspects of moduli spaces, which actually generate nonperturbative corrections, do not arise for $N_F=4$. This is already clear for the chiral part $\mathcal{F}$ which is proportional to a^2. Since in general the function K is built in terms of $\mathcal{F}$, we see that there is no way to get nonholomorphic contributions to K but the one-loop term, whose structure has a global meaning since the underlying geometry is trivial. This is an alternative way to express the nonrenormalization theorem of [19].

IV. NONPERTURBATIVE CONTRIBUTIONS TO $K(\Psi,\overline{\Psi})$

Let us now discuss the series expansion for $K(\Psi,\overline{\Psi})$ in the case of SYM theory. We can rewrite Eq. (3.10) as[5]

$$K(\Psi,\overline{\Psi})=\frac{64\alpha}{\pi^2}\frac{|\mathcal{G}^2(\Psi)-4\Lambda^4|(\mathrm{Im}\,\tau(\Psi))^2}{|\Psi|^4|\hat{\tau}(\Psi)-\tau(\Psi)|^4}, \tag{4.1}$$

where

$$\hat{\tau}(\Psi)=\frac{1}{\Psi}\frac{\partial\mathcal{F}}{\partial\Psi}. \tag{4.2}$$

The constant α can be fixed by using the result in [22]; however, as we shall discuss in the following, this is not enough to get a complete check of the validity of Eqs. (3.19) and (3.22).

Expanding Eq. (4.1) up to the order relevant to 2-instanton calculations, and neglecting purely chiral or antichiral terms, we find

$$\begin{aligned}K(x,\overline{x})\simeq\alpha\Big\{&\ln x\,\ln\overline{x}+x^4(3\ln\overline{x}-2\ln^2\overline{x}-2\ln\overline{x}\,\ln x)\\&+x^8\left(-\frac{21}{2}\ln x\,\ln\overline{x}-\frac{21}{2}\ln^2\overline{x}+\frac{57}{8}\ln\overline{x}\right)\\&+x^4\overline{x}^4\left(\frac{9}{4}-6\ln\overline{x}+2\ln^2\overline{x}+4\ln x\,\ln\overline{x}\right)+\mathrm{H.c.}\Big\},\end{aligned} \tag{4.3}$$

where $x=\Lambda/\Psi$.

Let us briefly comment on the functional dependence of the various terms appearing in the expansion. The first logarithmic term represents the one-loop perturbative contribution to $K(\Psi,\overline{\Psi})$, which was first derived in [15]; it is to be noted that there are no higher-order (higher-loop) logarithmic corrections to $K(\Psi,\overline{\Psi})$. As far as the terms $x^{4k}\ln\overline{x}$ are concerned, they appear explicitly in the k-instanton calculations, while the terms with $x^{4k}\ln\overline{x}\ln x$ and $x^{4k}\ln^2\overline{x}$ are expected to be one-loop corrections around the k-instanton configuration. As a matter of fact, in this case there are no constraints coming from holomorphicity requirements and from the anomalous $\mathrm{U}(1)_R$ symmetry which forbid the existence of loop corrections around instanton configurations [29]. Finally, the terms $x^{4m}\overline{x}^{4n}$ and logarithmic corrections are expected to represent m-instanton/n-antiinstanton contributions and loop corrections around these configurations. In the case of SQCD this situation is modified in the presence of massless hypermultiplets in a simple way, since the expansion contains only nonperturbative contributions from m-instanton/n-antiinstanton, where m,n are even numbers, and one-loop corrections around these configurations. In the sequel we shall perform 1- and 2-instanton calculations which will give contributions to $K(\Psi,\overline{\Psi})$ of the form expected from the conjecture in [18]. Let us now make a remark which will become clear after the instanton computation will be performed. If we differentiate $K(x,\overline{x})$ twice with respect to x and twice with respect to $\overline{x}$ (to obtain $K_{xx\overline{x}\overline{x}}$), the terms containing $\ln x\overline{x}$ and $\ln^2 x\overline{x}$ give contributions which sum. Therefore, for an unambiguous check of the conjectures (3.19), (3.22), one needs not only 1-instanton or 2-instanton but also mixed 1-instanton–1-antiinstanton results and perturbative corrections around all the aforementioned configurations. Anyway, as a first step towards the check of these proposals, we now compute the nonperturbative (1-instanton and 2-instanton) contributions to $K(\Psi,\overline{\Psi})$.

In terms of the $N=1$ superspace the four-derivative term reads [14]

$$\begin{aligned}\frac{1}{16}\int d^2\theta d^2\overline{\theta}[&K_{\phi\overline{\phi}}(\Phi,\overline{\Phi})(D^\alpha D_\alpha\Phi\overline{D}_{\dot\alpha}\overline{D}^{\dot\alpha}\overline{\Phi}+2\overline{D}_{\dot\alpha}D^\alpha\Phi D_\alpha\overline{D}^{\dot\alpha}\overline{\Phi}+4D^\alpha W_\alpha\overline{D}_{\dot\alpha}\overline{W}^{\dot\alpha}-4D^{(\alpha}W^{\beta)}D_{(\alpha}W_{\beta)}-4\overline{D}_{(\dot\alpha}\overline{W}_{\dot\beta)}\overline{D}^{(\dot\alpha}\overline{W}^{\dot\beta)}\\&-2D^\alpha D_\alpha(W^\beta W_\beta)-2\overline{D}_{\dot\alpha}\overline{D}^{\dot\alpha}(\overline{W}_{\dot\beta}\overline{W}^{\dot\beta}))-2K_{\phi\phi\overline{\phi}}(\Phi,\overline{\Phi})W^\alpha W_\alpha D^\beta D_\beta\Phi-2K_{\phi\overline{\phi}\overline{\phi}}(\Phi,\overline{\Phi})\overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}\overline{D}_{\dot\beta}\overline{D}^{\dot\beta}\overline{\Phi}\\&+K_{\phi\phi\overline{\phi}\overline{\phi}}(\Phi,\overline{\Phi})(-8W^\alpha D_\alpha\Phi\overline{W}_{\dot\alpha}\overline{D}^{\dot\alpha}\overline{\Phi}+4W^\alpha W_\alpha\overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha})],\end{aligned} \tag{4.4}$$

where $K_\phi=\partial K/\partial\phi$. When written in the x space this Lagrangian contains a four-field strength vertex which is the one we shall focus our attention on in our calculations:

$$\frac{1}{4}\int d^4x d^2\theta d^2\overline{\theta}K_{\phi\phi\overline{\phi}\overline{\phi}}(\Phi,\overline{\Phi})W^\alpha W_\alpha\overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}=\frac{1}{256}K_{aa\overline{a}\overline{a}}(a,\overline{a})\int d^4x\,\mathrm{Tr}(\sigma^{ab}\sigma^{cd})\mathrm{Tr}(\overline{\sigma}^{ef}\overline{\sigma}^{gh})F_{ab}F_{cd}F_{ef}F_{gh}. \tag{4.5}$$

[5]From now on we shall denote by Λ the Pauli-Villars RGI scale.

Thus, the correlator we intend to study is

$$\langle F_{\mu\nu}(x_1)F_{\rho\sigma}(x_2)F_{\lambda\tau}(x_3)F_{\kappa\theta}(x_4)\rangle. \tag{4.6}$$

A. The $k=1$ semiclassical computation

The relevant configuration which contributes to this Green function is dictated by the sweeping-out procedure at the next to leading order of [9];

$$F_{\mu\nu}=F_{\mu\nu}^{(0)}+i\xi(x)\sigma_{[\nu}D_{\mu]}\overline{\lambda^{(0)}}+2i\overline{\lambda^{(0)}}\overline{\sigma_{\mu\nu}}\overline{\varepsilon}$$
$$+2ig\xi^2(x)\overline{\lambda^{(0)}}\overline{\sigma_{\mu\nu}}\overline{\lambda^{(0)}}, \tag{4.7}$$

where $F_{\mu\nu}^{(0)}$ satisfies the equation

$$D^\mu F_{\mu\nu}^{(0)}=-2ig[\phi_{\rm cl}^\dagger,D_\nu\phi_{\rm cl}], \tag{4.8}$$

with

$$\phi_{\rm cl}=\frac{x^2}{x^2+\rho^2}\,a^c(\sigma^c/2), \tag{4.9}$$

and

$$\xi(x)=\xi+\rho^{-1}x^\mu\sigma_\mu\overline{\varepsilon,} \tag{4.10}$$

$$\overline{\lambda^{(0)}}=-i\sqrt{2}\xi' \not{D}\phi_{\rm cl}^\dagger. \tag{4.11}$$

Here, for simplicity, x stands for $x-x_0$, where x_0, is the center of the 1-instanton configuration, and ρ is its size; finally $\overline{\eta'^a_{\mu\nu}}=R^a{}_b\overline{\eta^b_{\mu\nu}}$, where $R^a{}_b$ is an SU(2) rotation matrix which corresponds to global color rotations.

We start by rederiving the result of [16] for the 1-instanton case in a different way. In the case $k=1$, the $N=2$ SYM measure on the moduli space is simply [28,36]

$$\int d^3\Theta d^4x_0\,\frac{d\rho}{\rho^5}\,\frac{2^7\pi^6}{g^8}\,(\mu\rho)^8\left(\frac{16\pi^2\mu}{g^2}\right)^{-2}d^2\xi d^2\xi'\left(\frac{32\pi^2\rho^2\mu}{g^2}\right)^{-2}d^2\overline{\varepsilon}\overline{d}^2\overline{\varepsilon'}\,\exp(-S_{\rm inst}), \tag{4.12}$$

where $\xi,\xi'(\overline{\varepsilon,\varepsilon^1})$ are the $N=2$ supersymmetric (superconformal) Grassmann collective coordinates, $S_{\rm inst}$ is the sum of the classical action, the Higgs and the Yukawa terms, and Θ^a, $a=1,2,3$ denote the moduli associated with global color rotations. We observe first that $F_{\mu\nu}$ does not contain the superconformal collective coordinate $\overline{\varepsilon'}$ so that the corresponding integration must be completely saturated by the Yukawa action and we can ignore the terms in $F_{\mu\nu}$ which depend on the fermionic coordinate $\overline{\varepsilon}$. Therefore in evaluating the correlator (4.6), only the first, the second, and the fourth term in the rhs of Eq. (4.7) will be of interest. To lowest order in $g^2\rho^2|a|^2$, $F_{\mu\nu}^{(0)}$ becomes

$$F_{\mu\nu}^{\rm cl}=\frac{4\rho^2}{g}\,\frac{1}{x^2(x^2+\rho^2)^2}\,(-x^2\overline{\eta'^a_{\mu\nu}}+2x_\lambda x_\nu\overline{\eta'^a_{\mu\lambda}}$$
$$+2x_\lambda x_\mu\overline{\eta'^a_{\lambda\nu}})\,\frac{\sigma^a}{2}, \tag{4.13}$$

and the term proportional to ξ^2 is negligible. Then, in order to saturate the integration over the supersymmetric collective coordinates ξ,ξ', the product in Eq. (4.6) boils down to

$$F_{\mu\nu}^{\rm cl}(x_1)F_{\rho\sigma}^{\rm cl}(x_2)[i\xi\sigma_{[\nu}D_{\mu]}\overline{\lambda^{(0)}}(x_3)][i\xi\sigma_{[\nu}D_{\mu]}\overline{\lambda^{(0)}}(x_4)]. \tag{4.14}$$

Now we have to extrapolate the relevant long-distance effective U(1) fields (4.13):

$$F_{\mu\nu}^{(3){\rm cl,LD}}(x)=\frac{4\rho^2}{g}\,\frac{1}{x^6}\,(-x^2\overline{\eta'^3_{\mu\nu}}+2x_\lambda x_\nu\overline{\eta'^3_{\mu\lambda}}+2x_\lambda x_\mu\overline{\eta'^3_{\lambda\nu}}) \tag{4.15}$$

and

$$i\xi\sigma_{[\nu}\partial_{\mu]}\overline{\lambda_{\rm LD}^{(0)}}, \tag{4.16}$$

where $\overline{\lambda_{\rm LD}^{(0)}}=-i\sqrt{2}\xi'\not{\partial}\phi_{\rm cl}^{\dagger{\rm LD}}$ and the suffix LD stands for long distance. In this limit the covariant derivative becomes a simple one. In [9] a nice relationship between the scalar Higgs field and the Higgs action in the long-distance limit was derived:

$$\phi_{\rm cl,LD}^\dagger=\overline{a}-a^{-1}S_HG(x,\,x_0), \tag{4.17}$$

where $G(x,x_0)=1/4\pi^2(x-x_0)^2$ is the massless scalar propagator. As a consequence of this observation it is possible to recast Eq. (4.16) into the form

$$\frac{\sqrt{2}}{2}\,\frac{\partial}{\partial a}\,S_H\xi\sigma^{ab}\xi' G_{\mu\nu,ab}(x,x_0), \tag{4.18}$$

where $G_{\mu\nu,ab}(x,x_0)$ is the gauge-invariant propagator of the U(1) field strength:

$$G_{\mu\nu,ab}(x,x_0)=(\delta_{\nu b}\partial_\mu\partial_a-\delta_{\nu a}\partial_\mu\partial_b-\delta_{\mu b}\partial_\nu\partial_a$$
$$+\delta_{\mu a}\partial_\nu\partial_b)G(x,x_0). \tag{4.19}$$

The integration on the superconformal collective coordinates, which are lifted in the background of the constrained instanton, is completely saturated by the Yukawa action S_Y, and one gets [37]

$$\int d^2\overline{\varepsilon}\overline{d}^2\overline{\varepsilon'}\,\exp(-S_Y)=-2^9\pi^4g^{-2}\rho^4\overline{a^2}. \tag{4.20}$$

The key observation is that the only dependence on the coordinates Θ^a is due to the insertion of $F_{\mu\nu}^{(3){\rm cl}}$ and that, in the long-distance limit,

$$\int_{SU(2)/Z_2} d^3\Theta F^{(3)\mathrm{cl}}_{\mu\nu}(x_1)F^{(3)\mathrm{cl}}_{\rho\sigma}(x_2)=\frac{8\pi^2}{3}F^{a\mathrm{cl}}_{\mu\nu}(x_1)F^{a\mathrm{cl}}_{\rho\sigma}(x_2)=-\frac{16\pi^6\rho^4}{3g^2}\mathrm{Tr}(\overline{\sigma^{ef}}\overline{\sigma^{gh}})G_{\mu\nu,ef}(x_1,x_0)G_{\rho\sigma,gh}(x_2,x_0). \tag{4.21}$$

Taking into account the other two insertions which saturate the integration over ξ, ξ', we finally obtain

$$\langle F_{\mu\nu}(x_1)F_{\rho\sigma}(x_2)F_{\lambda\tau}(x_3)F_{\kappa\theta}(x_4)\rangle_{k=1}=-\frac{15}{64\pi^2}\frac{\Lambda^4}{g^4\bar{a}^2a^6}\int d^4x_0\ \mathrm{Tr}(\sigma^{ab}\sigma^{cd})G_{\mu\nu,ab}(x_1,x_0)G_{\rho\sigma,cd}(x_2,x_0)$$
$$\times\mathrm{Tr}(\overline{\sigma^{ef}}\overline{\sigma^{gh}})G_{\lambda\tau,ef}(x_3,x_0)G_{\kappa\theta,gh}(x_4,x_0). \tag{4.22}$$

On the other hand, the computation of the four-field strength vertex performed by making use of the effective Lagrangian yields

$$\langle F_{\mu\nu}(x_1)F_{\rho\sigma}(x_2)F_{\lambda\tau}(x_3)F_{\kappa\theta}(x_4)\rangle_{L\text{-eff}}=\frac{3}{32}K_{aa\bar{a}\bar{a}}(a,\bar{a})\int d^4x\ \mathrm{Tr}(\sigma^{ab}\sigma^{cd})G_{\mu\nu,ab}(x_1,x_0)G_{\rho\sigma,cd}(x_2,x_0)$$
$$\times\mathrm{Tr}(\overline{\sigma^{ef}}\overline{\sigma^{gh}})G_{\lambda\tau,ef}(x_3,x_0)G_{\kappa\theta,gh}(x_4,x_0), \tag{4.23}$$

which finally reproduces the result [16]

$$K(a,\bar{a})=\frac{1}{8\pi^2g^4}\frac{\Lambda^4}{a^4}\ln\bar{a}. \tag{4.24}$$

We can rewrite the 1-instanton correlator in a form which is well suited to the generalization to SQCD with $1\leqslant N_F\leqslant 4$ massive hypermultiplets and gauge group SU(2) (in the case in which at least one hypermultiplet is massless the nonperturbative contributions are expected to come only from m-instanton–n-antiinstanton configurations where m, n are even),

$$\langle F_{\mu\nu}(x_1)F_{\rho\sigma}(x_2)F_{\lambda\tau}(x_3)F_{\kappa\theta}(x_4)\rangle_{k=1}$$
$$=\frac{\pi^4}{2}\left(\int d^4x_0\ \mathrm{Tr}(\sigma^{ab}\sigma^{cd})G_{\mu\nu,ab}(x_1,x_0)G_{\rho\sigma,cd}(x_2,x_0)\mathrm{Tr}(\overline{\sigma^{ef}}\overline{\sigma^{gh}})G_{\lambda\tau,ef}(x_3,x_0)G_{\kappa\theta,gh}(x_4,x_0)\right)\frac{\partial^2}{\partial a^2}\left[\int d\tilde{\mu}_1\rho^4\right], \tag{4.25}$$

where $d\tilde{\mu}_1$ is the ‘‘reduced’’ instanton measure obtained by extracting from the full measure the integration over the bosonic and fermionic translational coordinates [9,12]. This formula generalizes immediately by replacing $d\tilde{\mu}_1$ with $d\tilde{\mu}_1^{N_F}$ [9], where

$$\int d\tilde{\mu}_1^{N_F}=-\frac{1}{16\pi^2g^4}\frac{\Lambda_{N_F}^{4-N_F}}{a^2}\prod_{i=1}^{N_F}m_i, \tag{4.26}$$

and m_i is the mass of the ith hypermultiplet. By doing this we obtain

$$K(a,\bar{a})|_{N_F}=\frac{1}{8\pi^2g^4}\frac{\Lambda_{N_F}^{4-N_F}}{a^4}\ln\bar{a}\prod_{i=1}^{N_F}m_i, \tag{4.27}$$

which is consistent with the relation between the RGI scales Λ for different numbers of flavors,

$$m_{N_F}\Lambda_{N_F}^{4-N_F}=\Lambda_{N_F-1}^{5-N_F}. \tag{4.28}$$

It is to be noted that in the case $N_F=4$ the β function vanishes identically so that the scale $\Lambda_{N_F}^{4-N_F}$ must be replaced by

$$q=\exp(2i\pi\tau_{\mathrm{cl}}), \tag{4.29}$$

where τ_{cl} is defined in Eq. (2.6).

B. The $k=2$ computation

Let us now describe the calculation of the 2-instanton contribution to the real function $K(\Psi,\bar{\Psi})$. Again, the Green function which we are going to study is the simplest one, the four-field strength one. We shall then be able to immediately generalize our calculation to the case of SQCD and to check the validity of the nonrenormalization theorem in the case $N_F=4$ found in [19].

We start by briefly recalling how to determine gauge field configurations for a generic winding number k. The instanton field can be conveniently written in terms of the Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction [38,39]. To find an instanton solution of winding number k, one introduces a $(k+1)\times k$ quaternionic matrix

$$\Delta=a+bx, \tag{4.30}$$

where x denotes a point of the one-dimensional quaternionic space $\mathbb{H}\equiv\mathbb{C}^2\equiv\mathbb{R}^4$, $x=x^\mu\sigma_\mu$.[6] The gauge connection is then written in the form

[6]We use the conventions of [12].

$$A_\mu^{\rm cl}=U^\dagger \partial_\mu U, \tag{4.31}$$

where U is a $(k+1)\times 1$ matrix of quaternions providing an orthonormal frame of $\mathrm{Ker}\Delta^\dagger$: i.e.,

$$\Delta^\dagger U=0, \tag{4.32}$$

$$U^\dagger U=\mathbb{1}_2. \tag{4.33}$$

The constraint (4.33) ensures that $A_\mu^{\rm cl}$ is an element of the Lie algebra of the SU(2) gauge group. The condition of self-duality on the field strength of Eq. (4.31) is imposed by restricting the matrix Δ to obey

$$\Delta^\dagger\Delta=f^{-1}\otimes \mathbb{1}_2, \tag{4.34}$$

with f an invertible Hermitian $k\times k$ matrix (of real numbers). The reparametrization invariances of the ADHM construction [40] can be used to simplify the expressions of a and b. Exploiting this fact, in the following we shall choose the matrix b to be

$$b=-\begin{pmatrix} 0_{1\times k}\\ \mathbb{1}_{k\times k}\end{pmatrix}. \tag{4.35}$$

From Eq. (4.31) one can compute the field strength of the gauge field, which reads

$$F_{\mu\nu}^{\rm cl}=2U^\dagger b\sigma_{\mu\nu} f b^\dagger U. \tag{4.36}$$

In the so-called singular gauge, one has

$$U_0=\sigma_0\left(1-\frac{1}{2} f_{lm}\, \mathrm{Tr} v_l \overline{v_m}\right)^{1/2},$$

$$U_p=-\frac{1}{|U_0|^2}\,\Delta_{pl} f_{lm}\overline{v_m}U_0, \tag{4.37}$$

where $v_p=\Delta_{0p}$ and $l,m,p=1,\dots,k$. In the following we shall need only the long-distance limit of these functions:

$$\Delta_{pl}\sim b_{pl}x, \quad f_{lm}\sim \frac{1}{x^2}\,\delta_{lm},$$

$$U_k\sim -\frac{1}{x^2}\, x\overline{v_k}U_0, \quad U_0\sim\sigma_0,$$

$$\Delta_{0l}\sim 0. \tag{4.38}$$

When $k=2$ the most general instanton configuration can be written starting from the ADHM matrix

$$a=\begin{pmatrix} v_1 & v_2\\ x_0+e & d\\ d & x_0-e\end{pmatrix}. \tag{4.39}$$

Here

$$d=\frac{e}{4|e|^2}\,(\overline{v_2}v_1-\overline{v_1}v_2), \tag{4.40}$$

as a consequence of the ADHM defining equations [41].

The fermionic zero modes $\lambda_{\beta\dot A}^{(0)}$ are easily deduced from the gauge field zero modes [40]

$$Z_\mu=U^\dagger C\overline{\sigma_\mu} f b^\dagger U-U^\dagger b f\sigma_\mu C^\dagger U, \tag{4.41}$$

by recalling that, due to $N=2$ SUSY,

$$\lambda_{\beta\dot A}^{(0)}=\sigma^\mu_{\beta\dot A}Z_\mu, \tag{4.42}$$

($\dot A=1,2$ labels the two SUSY charges and $\beta=1,2$ is a spin index). For Eq. (4.41) to be transverse zero modes, the $(k+1)\times k$ matrix C (for a generic instanton number k) must satisfy

$$\Delta^\dagger C=(\Delta^\dagger C)^T, \tag{4.43}$$

where the superscript T stands for transposition of the quaternionic elements of the matrix (without transposing the quaternions themselves). The number of C's satisfying Eq. (4.43) is $8k$ [40]. In order to describe the zero modes of the $N=2$ gauginos $\lambda_{\beta\dot A}^{(0)}$, we also need the form of the matrix C appearing in Eq. (4.41), which is constrained by Eq. (4.43). To parallel the form of Eq. (4.39), we shall put

$$C_{\dot 1}=\begin{pmatrix} \mu_1 & \mu_2\\ 4\xi+\eta & \delta\\ \delta & 4\xi-\eta\end{pmatrix}, \tag{4.44}$$

$$C_{\dot 2}=\begin{pmatrix} \nu_1 & \nu_2\\ 4\xi'+\eta' & \delta'\\ \delta' & 4\xi'-\eta'\end{pmatrix}, \tag{4.45}$$

where δ,δ' are constrained by Eq. (4.43) to be

$$\delta=\frac{e}{2|e|^2}\,(2\overline{d}\eta+\overline{v_2}\mu_1-\overline{v_1}\mu_2),$$

$$\delta'=\frac{e}{2|e|^2}\,(2\overline{d}\eta'+\overline{v_2}\nu_1-\overline{v_1}\nu_2). \tag{4.46}$$

In the long-distance limit, the 2-instanton field strength factorizes in

$$\begin{aligned}F_{\mu\nu}^{\rm cl,LD}&=\frac{2}{x^6}\,[v_1\overline{x}\overline{\sigma}_{\mu\nu}x\overline{v_1}+(v_1\to v_2)]\\ &=\frac{1}{x^6}\,[v_1(-x^2\overline{\sigma_{\mu\nu}}+2x^\rho x_\mu\overline{\sigma_{\rho\nu}}+2x^\rho x_\nu\overline{\sigma_{\mu\rho}})\overline{v_1}\\ &\quad+(v_1\to v_2)].\end{aligned} \tag{4.47}$$

On the other hand, in [9] it was proved that, thanks to the geometrical properties of the ADHM construction, the relationship between the ξ,ξ' bilinear part in Eq. (4.7) and the Higgs action continues to hold for every winding number.

We start with the $k=2$ $N=2$ supersymmetric measure, which reads

$$\frac{1}{\mathcal{S}_2}\int d^4x_0 d^4e d^4v_1 d^4v_2 d^2\xi d^2\xi' d^2\eta d^2\eta' d^2\mu_1 d^2\mu_2 d^2\nu_1 d^2\nu_2 \exp(-S_{\rm inst})\left(\frac{J_B}{J_F}\right)^{1/2}. \tag{4.48}$$

$\mathcal{S}_2$ is the $k=2$ symmetry factor which eliminates all the redundant copies of each field configuration which appears in the ADHM formalism [40,9], and $J_B(J_F)$ is the Jacobian of the change of variables for the bosonic (fermionic) degrees of freedom. As in the calculation of the 2-instanton contribution to the $N=2$ prepotential [9], we find it convenient to define the four combinations of the bosonic parameters:

$$L=|v_1|^2+|v_2|^2,$$

$$H=L+4|d|^2+4|e|^2,$$

$$\Omega=v_1\overline{v_2}-v_2\overline{v_1},$$

$$\omega=\frac{1}{2}\,{\rm Tr}\Omega A_{00}, \tag{4.49}$$

where $A_{00}=(i/2)a^c\sigma^c$. In terms of these new variables it is possible to write the Higgs action as

$$S_H=16\pi^2\left(L|A_{00}|^2-\frac{|\omega|^2}{H}\right)=4\pi^2|a|^2\left(L-\frac{|\Omega|^2\cos^2\theta}{H}\right), \tag{4.50}$$

and the Yukawa action as

$$S_Y=4\sqrt{2}\pi^2[-\nu_k\overline{A_{00}}\mu_k+(\overline{\omega}/H)(\mu_1\nu_2-\nu_1\mu_2+2\eta\delta' -2\eta'\delta)], \tag{4.51}$$

where $|\omega|=\frac{1}{2}|\Omega||a||\cos\theta|$ defines the polar angle θ. Finally

$$\frac{1}{\mathcal{S}_2}\left(\frac{J_B}{J_F}\right)^{1/2}\exp(-S_{\rm cl})=2^6\pi^{-8}\Lambda^8\,\frac{\| e|^2-|d|^2|}{H}. \tag{4.52}$$

As in the 1-instanton case the integration over the nonsupersymmetric fermionic coordinates is saturated by the Yukawa action, which gives

$$\int d^2\eta d^2\eta' d^2\mu_1 d^2\mu_2 d^2\nu_1 d^2\nu_2 \exp(-S_Y) =-\frac{2^5\pi^6\overline{a^6}\cos^2\theta}{|e|^4H'^2}L^2\left[\left(1+\frac{\cos^2\theta}{H'}\right)^2 + \frac{1-|\Omega'|^2}{H'^2}\sin^2\theta\cos^2\theta\right], \tag{4.53}$$

where we have redefined $\Omega'=\Omega/L$, $H'=H/L$. The integration over the variable e is traded for the integration on H, i.e.,

$$\int d^4e\,\frac{\| e|^2-|d|^2|}{|e|^4}\to\frac{\pi^2}{2}\int_{L+2|\Omega|}^{\infty}dH. \tag{4.54}$$

As far as the two insertions of $F_{\mu\nu}$ bilinear in ξ,ξ' are concerned, it is possible to use a trick already exploited in the 1-instanton case. It consists in writing them as a second derivative of the instanton measure with respect to a [9]; the remaining two insertions, however, will have to be integrated explicitly. First of all, let us write v_2 as a function of v_1,Ω,L;

$$v_2=\left(\frac{\overline{\Omega}}{2}+\sqrt{|v_1|^2(L-|v_1|^2)-\frac{|\Omega|^2}{4}}\right)\frac{v_1}{|v_1|^2}, \tag{4.55}$$

and insert this form in the long-distance limit of the 2-instanton classical configuration. The integration measure over v_1,L,Ω is written as

$$2\int_0^\infty dL\int_{|\Omega|\leq L}d^3\Omega\int_{L_-}^{L_+}d|v_1|^2 \times\frac{1}{32\sqrt{(L_+-|v_1|^2)(|v_1|^2-L_-)}}\int_{S^3}d^3\Theta, \tag{4.56}$$

where $\int_{S^3}d^3\Theta=2\pi^2$ is the integration over the global color rotations of the first center of the instanton and $L_\pm=\frac{1}{2}(L\pm\sqrt{L^2-|\Omega|^2})$. On the other hand,

$$\int d^3\Omega=L^3\int_0^{2\pi}d\varphi\int_{-1}^{1}d(\cos\theta)\int_0^1|\Omega'|^2d|\Omega'|, \tag{4.57}$$

where θ is the angle between Ω and the direction singled out by the vacuum expectation value of the Higgs field. Again, as in the 1-instanton case, we observe that, in the long-distance limit,

$$\int d^3\Theta F^{3{\rm cl}}_{\mu\nu}(x)F^{3{\rm cl}}_{\rho\sigma}(y)=\frac{2\pi^2}{3}F^{a{\rm cl}}_{\mu\nu}(x)F^{a{\rm cl}}_{\rho\sigma}(y)=-\frac{2\pi^6}{3}{\rm Tr}(\overline{\sigma^{ab}}\,\overline{\sigma^{cd}})G_{\mu\nu,ab}(x,x_0)G_{\rho\sigma,cd}(y,x_0)\left(L|v_1|^2-\frac{|\Omega|^2}{2}\sin^2\theta\right). \tag{4.58}$$

Putting everything together one obtains the following integral for the correlator:

$$\int d^4x_0\int_0^1 d|\Omega'||\Omega'|^6\int_{-1}^1 d(\cos\theta)\cos^2\theta\int_{1+2|\Omega'|}^\infty \frac{dH'}{H'^3}\int_0^\infty dLL^7[1-|\Omega'|^2\sin^2\theta](-4\pi^{14})\Lambda^8\overline{a^6}\left[\left(1+\frac{\cos^2\theta}{H'}\right)^2 + \frac{1-|\Omega'|^2}{H'^2}\sin^2\theta\cos^2\theta\right]\mathrm{Tr}(\overline{\sigma^{ab}}\overline{\sigma^{cd}})G_{\mu\nu,ab}(x_1,x_0)G_{\rho\sigma,cd}(x_2,x_0)\mathrm{Tr}(\sigma^{ef}\sigma^{gh})G_{\lambda\tau,ef}(x_3,x_0)G_{\kappa\theta,gh}(x_4,x_0) \times\frac{\partial^2}{\partial a^2}\exp\left[-4\pi^2L|a|^2\left(1-\frac{|\Omega'|^2\cos^2\theta}{H'}\right)\right], \tag{4.59}$$

and, after a trivial integration on L, we get

$$\int d^4x_0\int_0^1 d|\Omega'||\Omega'|^6\int_{-1}^1 d(\cos\theta)\cos^2\theta\int_{1+2|\Omega'|}^\infty \frac{dH'}{H'^3}[1-|\Omega'|^2\sin^2\theta] \times\left(-\frac{5\times3^4\times7}{2^6\pi^2}\right)\frac{\Lambda^8}{\overline{a^2}a^{10}}\frac{(1+\cos^2\theta/H')^2+(1-|\Omega'|^2)\sin^2\theta\cos^2\theta/H'^2}{(1-|\Omega'|^2\cos^2\theta/H')^8} \times\mathrm{Tr}(\overline{\sigma^{ab}}\overline{\sigma^{cd}})G_{\mu\nu,ab}(x_1,x_0)G_{\rho\sigma,cd}(x_2,x_0)\mathrm{Tr}(\sigma^{ef}\sigma^{gh})G_{\lambda\tau,ef}(x_3,x_0)G_{\kappa\theta,gh}(x_4,x_0). \tag{4.60}$$

The remaining integrations over the adimensional variables $|\Omega'|,\cos\theta,H'$ can be easily performed by using a standard algebraic manipulation routine, and give 1/42. The final result is then, restoring the explicit g dependence,

$$\langle F_{\mu\nu}(x_1)F_{\rho\sigma}(x_2)F_{\lambda\tau}(x_3)F_{\kappa\theta}(x_4)\rangle_{k=2}=-\frac{5\times3^3}{2^7\pi^2g^8}\frac{\Lambda^8}{\overline{a^2}a^{10}}\int d^4x_0\,\mathrm{Tr}(\overline{\sigma^{ab}}\overline{\sigma^{cd}})G_{\mu\nu,ab}(x_1,x_0)G_{\rho\sigma,cd}(x_2,x_0) \times\mathrm{Tr}(\sigma^{ef}\sigma^{gh})G_{\lambda\tau,ef}(x_3,x_0)G_{\kappa\theta,gh}(x_4,x_0). \tag{4.61}$$

Comparing this result to that of the effective Lagrangian one gets

$$K(a,\bar{a})|_{k=2}=\frac{5}{32\pi^2g^8}\frac{\Lambda^8}{a^8}\ln\bar{a}, \tag{4.62}$$

which is our prediction for the 2-instanton contribution to the real function $K(\Psi,\bar{\Psi})$. The 2-antiinstanton configuration contribution to K is simply the complex conjugate of Eq. (4.62).

Let us generalize our result to the case of $N_F\leqslant 4$ massless hypermultiplets, which receives the first nonperturbative contribution from the 2-instanton sector and verify the nonrenormalization theorem of [19] for $N_F=4$. As in the 1-instanton case [see Eq. (4.25)] it is possible to rewrite the four-field strength correlator as a double derivative of the ‘‘reduced’’measure with respect to a,

$$\langle F_{\mu\nu}(x_1)F_{\rho\sigma}(x_2)F_{\lambda\tau}(x_3)F_{\kappa\theta}(x_4)\rangle_{k=2}=\frac{\pi^4}{4}\frac{\partial^2}{\partial a^2}\left[\int d\tilde{\mu}_2\left(|v_1|^2L-\frac{|\Omega|^2}{2}\sin^2\theta\right)\right]\int d^4x_0\,\mathrm{Tr}(\overline{\sigma^{ab}}\overline{\sigma^{cd}})G_{\mu\nu,ab}(x_1,x_0) \times G_{\rho\sigma,cd}(x_2,x_0)\mathrm{Tr}(\sigma^{ef}\sigma^{gh})G_{\lambda\tau,gh}(x_3,x_0)G_{\kappa\theta,gh}(x_4,x_0), \tag{4.63}$$

and the extension to the case $N_F>0$ is performed by substituting the ‘‘reduced’’ measure $d\tilde{\mu}_2$ with $d\tilde{\mu}_2^{N_F}$ as defined in [11]:

$$\int d\tilde{\mu}_2^{N_F}=-2^9\pi^7\overline{a^2}\Lambda_{N_F}^{2(4-N_F)}\int_0^1 d|\Omega||\Omega|^2\int_{-1}^1 d(\cos\theta)\int_{1+2|\Omega|}^\infty\frac{dH}{H^3}\int_{S^3}d^3\Theta\int_0^\infty dLL\int_{L_-}^{L_+}\frac{d|v_1|^2}{\sqrt{(L_+-|v_1|^2)(|v_1|^2-L_-)}} \times\exp\left[-4\pi^2L|a|^2\left(1-\frac{|\Omega|^2\cos^2\theta}{H}\right)\right]\sum_{n=0}^{N_F}\frac{M_{N_F-n}^{(N_F)}}{\pi^{4n}}\frac{\partial^{2n}G}{\partial Z^{2n}}\bigg|_{Z=0}. \tag{4.64}$$

We have dropped for simplicity the primes on H,Ω; $G(Z)$ contains the contribution from the integration measure over the hypermultiplets and has the form

$$G(Z)=\left(\overline{\omega L}+\frac{iZ}{8\sqrt{2}}\right)^2\left[\frac{\overline{a^2}}{16}|\Omega|^2L^2+\frac{L}{2H}\bar{a}\overline{\omega L}\left(\overline{\omega L}+\frac{iZ}{8\sqrt{2}}\right)+\frac{1-|\Omega|^2\sin^2\theta}{4H^2}\left(\overline{\omega L}+\frac{iZ}{8\sqrt{2}}\right)^2\right]\exp\left[\frac{i\pi^2Z}{\sqrt{2}H}|\Omega|aL\cos\theta\right]. \tag{4.65}$$

The $M^{(N_F)}_{N_F-n}$ are a set of SO$(2N_F)$-invariant polynomials in the masses m_n of the hypermultiplets:

$$M_0^{(N_F)}=1,$$

$$M_1^{(N_F)}=\sum_{n=1}^{N_F} m_n^2,$$

$$M_2^{(N_F)}=\sum_{n<p}^{N_F} m_n^2 m_p^2, \tag{4.66}$$

$$\vdots \quad \vdots$$

$$M_{N_F}^{(N_F)}=\prod_{n=1}^{N_F} m_n^2.$$

In the case of massless hypermultiplets, the only contribution to the correlator will come from the term with the $2N_F$-th derivative of $G(Z)$ and, writing the generic contribution to $K(\Psi,\bar{\Psi})$ as

$$K(\Psi,\bar{\Psi})|_{N_F<4}=K_2^{(N_F)}\frac{1}{\pi^2 g^8}\left(\frac{\Lambda_{N_F}}{\Psi}\right)^{2(4-N_F)}\ln\bar{\Psi}, \tag{4.67}$$

we find

$$K_2^{(0)}=\frac{5}{32}, \quad K_2^{(1)}=-\frac{3^3}{2^{10}},$$

$$K_2^{(2)}=\frac{3}{2^{10}}, \quad K_2^{(3)}=-\frac{1}{2^{12}}. \tag{4.68}$$

For the case $N_F=4$ we get

$$K(\Psi,\bar{\Psi})|_{N_F=4}=\frac{q^2}{3^3 2^{11}\pi^2 g^8}\ln\bar{\Psi}, \tag{4.69}$$

which is a purely antichiral term. When integrated over the whole superspace it does not contribute to the effective action; this confirms thus the nonrenormalization theorem of [19]. In the case in which there are massive hypermultiplets, Eqs. (4.67) and (4.69) generalize immediately to the formula

$$K(\Psi,\bar{\Psi})|_{N_F}=\sum_{n=0}^{N_F} M_{N_F-n}^{(N_F)}K_2^{(N_F)}\frac{1}{\pi^2 g^8}\left(\frac{\Lambda_{N_F}}{\Psi}\right)^{2(4-N_F)}\ln\bar{\Psi}, \tag{4.70}$$

provided that one replaces $\Lambda_{N_F}^{2(4-N_F)}$ with q^2 [q is defined in Eq. (4.29)] when $N_F=4$. In this case the nonrenormalization theorem of [19], as already noted in [21], is spoilt by the presence of other energy scales represented by the masses of the hypermultiplets.

V. CONCLUSIONS AND OUTLOOK

In this paper we investigated the next-to-leading corrections to the low-energy Wilsonian effective actions which describe the dynamics of the light degrees of freedom of the Coulomb phase for $N=2$ SYM and SQCD theories. These terms are determined by the full superspace integral of a real, adimensional, modular invariant function $K(\Psi,\bar{\Psi})$. In particular, extending [18] to the case of SQCD with $N_F<4$ hypermultiplets, we proposed a solution for K which satisfies all the physical requirements of the model. We found its behavior under the renormalization group action, and a differential equation which we interpreted as a fully nonperturbative ''nonchiral Ward identity.'' To support our proposals we performed multi-instanton calculations around configurations of winding number $k=1,2$. This way we checked a nonrenormalization theorem in the scale-invariant $N_F=4$ SQCD.

We observe that our investigation is strictly connected to the ''nonchiral'' analogue of the Picard-Fuchs equations [7,42] and to the related integrable structure [43]. This approach deserves being generalized to the case of higher-rank groups [44,45] and to the study of the strong coupling region [46]. As a final remark, we would like to point out that many aspects of the theory seem to be related to the Duistermaat-Heckman theorem [47]. In this context, we observe that in a recent paper a ''Gaussian approach'' to compute supersymmetric effective actions has also been worked out [48].

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