

## **$N=2$ supersymmetric Yang-Mills renormalization group scale as modulus for Witten-Dijkgraaf-Verlinde-Verlinde equations**

Gaetano Bertoldi\* and Marco Matone†

*Department of Physics “G. Galilei” and INFN, Padova, Italy*

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We derive a new set of Witten-Dijkgraaf-Verlinde-Verlinde equations for  $N=2$  SYM theory in which the renormalization scale  $\Lambda$  is identified with the distinguished modulus which naturally arises in topological field theories. [S0556-2821(98)01010-8]

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A fascinating aspect of Seiberg-Witten theory [1] is that it relies on deep geometrical aspects such as uniformization theory [2]. On the other hand uniformization theory turns out to be relevant also for topological field theories and nonperturbative 2D quantum gravity [3] to which the Seiberg-Witten theory is related. Furthermore, as a consequence of the Seiberg-Witten results, it has been possible to derive the explicit expression for the beta-function [4–6] which has been recently reconsidered in [7–12]. In [10,11] the analogue of the Zamolodchikov  $c$  theorem [13] in the framework of the 4D  $N=2$  super Yang-Mills (SYM) theory has been considered (see also [7–9]).

In the following we will consider the  $SU(n)$  gauge group and denote by  $\mathcal{F}$  the prepotential and by<sup>1</sup>  $\tau_{ij} = \partial^2 \mathcal{F} / \partial a^i \partial a^j$  the effective couplings where  $a^i = \langle \phi^i \rangle$  are the vacuum expectation values (VEV's) of the scalar component.

Let us consider the beta-function (matrix)

$$\beta_{i\hat{j}} = \Lambda \left. \frac{\partial \tau_{i\hat{j}}}{\partial \Lambda} \right|_{u^2, u^3, \dots}, \quad (1)$$

where  $u^\alpha \equiv u = \langle \text{tr } \phi^\alpha \rangle$ ,  $\alpha = 2, \dots, n$ .

Very recently, it has been shown in [11] that  $\beta_{i\hat{j}}$  plays the role of the metric whose inverse satisfies the Witten-Dijkgraaf-Verlinde-Verlinde- (WDVV-)like equations. In doing this we first considered the WDVV-like equations derived in the framework of the  $SU(3)$  reduced Picard-Fuchs equations [14,15]. Next, we used the WDVV-like equations derived for  $SU(n)$ ,  $n \geq 4$  in [16]. In [11] it has been observed that, by the Euler theorem,

$$\beta_{i\hat{j}} = -\Delta^{\hat{k}} \mathcal{F}_{\hat{k}i\hat{j}}, \quad (2)$$

where

$$\Delta^{\hat{k}} = \Delta_{u^\gamma} a^{\hat{k}} = \sum_{\gamma=2}^n \gamma u^\gamma \frac{\partial a^{\hat{k}}}{\partial u^\gamma}. \quad (3)$$

This strongly indicates, as should be for a topological field theory [17], that there is actually a natural distinguished modulus: the renormalization scale  $\Lambda$ . This strongly suggests

\*Email address: bertoldi@padova.infn.it

†Email address: matone@padova.infn.it

<sup>1</sup>We will denote by  $\hat{i}, \hat{j}, \hat{k}, \dots$  the indices running from 1 to  $n-1$  and by  $i, j, k, \dots$  those running from 0 to  $n-1$ , where  $\mathcal{F}_0 = \partial_\Lambda \mathcal{F}$ .

considering the extension of the WDVV-like equations to fully topological WDVV equations. The idea is to look for WDVV equations where the derivatives of the prepotential include that with respect to the distinguished modulus  $\Lambda$ . We will show that there are remarkable relations which actually lead to such a result.

The result in [11] is that the  $\beta$ -function of  $N=2$  SYM with gauge group  $SU(n)$ , satisfies the WDVV-like equations

$$\mathcal{F}_{\hat{k}\hat{l}} \beta^{\hat{l}\hat{m}} \mathcal{F}_{\hat{m}\hat{n}\hat{j}} = \mathcal{F}_{\hat{j}\hat{k}\hat{l}} \beta^{\hat{l}\hat{m}} \mathcal{F}_{\hat{m}\hat{n}\hat{i}}, \quad (4)$$

where  $\beta^{\hat{l}\hat{m}}$  is the inverse of the matrix  $\beta_{i\hat{m}}$  and

$$\mathcal{F}_{\hat{i}_1 \dots \hat{i}_k} = \frac{\partial^k \mathcal{F}}{\partial a^{\hat{i}_1} \dots \partial a^{\hat{i}_k}},$$

$\hat{i}_1, \dots, \hat{i}_k = 1, \dots, n-1$ .

Let us write down the WDVV-like equations derived in [16]:

$$F_i F_k^{-1} F_j = F_j F_k^{-1} F_i, \quad (5)$$

where  $(F_k)_{ij} = \mathcal{F}_{kij}$ . We want to show that

$$F_i F_k^{-1} F_j = F_j F_k^{-1} F_i, \quad (6)$$

where  $(F_k)_{ij} = \mathcal{F}_{kij}$  and all the indices run from 0 to  $n-1$  and  $a^0 \equiv \Lambda$ . Observe that this is not a trivial extension as we also have extended the range of summation. Writing explicitly the summation indices, Eq. (6) is

$$(\mathcal{F}_i)_{kl} (\mathcal{F}_p)^{lm} (\mathcal{F}_j)_{mn} = (\mathcal{F}_j)_{kl} (\mathcal{F}_p)^{lm} (\mathcal{F}_i)_{mn}, \quad (7)$$

where  $(\mathcal{F}_p)^{lm}$  denotes the inverse of the matrix  $(\mathcal{F}_p)_{lm} = \mathcal{F}_p^{lm}$ .

Observe that we can define the generalized beta function by

$$\beta_{ij} = a^0 \left. \frac{\partial \mathcal{F}_{ij}}{\partial a^0} \right|_{u^2, u^3, \dots}, \quad (8)$$

$i, j = 0, \dots, n-1$ . Generalizing the beta-function  $\beta^{(a)}$  evaluated in [5], we can introduce

$$\beta_{ij}^{(a)} = a^0 \left. \frac{\partial \mathcal{F}_{ij}}{\partial a^0} \right|_{a^1, a^2, \dots}. \quad (9)$$

A key point in the derivation of Eq. (7) is the following identity:

$$\begin{aligned} \mathcal{F}_{ijk} &= \mathcal{F}_{ijk}(1 - \delta_{i0})(1 - \delta_{j0})(1 - \delta_{k0}) - a^{0^{-1}} a^{\hat{l}} [\mathcal{F}_{lij}(1 - \delta_{i0})(1 - \delta_{j0}) \delta_{k0} + \mathcal{F}_{lik}(1 - \delta_{i0})(1 - \delta_{k0}) \delta_{j0} + \mathcal{F}_{ljk}(1 - \delta_{j0}) \\ &\quad \times (1 - \delta_{k0}) \delta_{i0}] + a^{0^{-2}} a^{\hat{l}} a^{\hat{m}} [\mathcal{F}_{\hat{l}\hat{m}i}(1 - \delta_{i0}) \delta_{j0} \delta_{k0} + \mathcal{F}_{\hat{l}\hat{m}j} \delta_{i0} (1 - \delta_{j0}) \delta_{k0} + \mathcal{F}_{\hat{l}\hat{m}k} \delta_{i0} \delta_{j0} (1 - \delta_{k0})] \\ &\quad - a^{0^{-3}} a^{\hat{l}} a^{\hat{m}} a^{\hat{n}} \mathcal{F}_{\hat{l}\hat{m}\hat{n}} \delta_{i0} \delta_{j0} \delta_{k0}, \end{aligned} \tag{10}$$

where the summation on the indices  $\hat{l}, \hat{m}, \hat{n}$  goes from 1 to  $n - 1$ . Observe that on the right hand side of Eq. (10) there never appears  $\mathcal{F}$  derived with respect to  $a^0$  as these possible terms are killed by the delta functions.

The identity (10) is obtained by first observing that, by the Euler theorem,

$$a^i \partial_{a^i} \mathcal{F}_{i_1, \dots, i_k} = (2 - k) \mathcal{F}_{i_1, \dots, i_k}. \tag{11}$$

Therefore, for example,

$$\mathcal{F}_i = \mathcal{F}_i(1 - \delta_{i0}) + a^{0^{-1}} (2\mathcal{F} - a^{\hat{k}} \mathcal{F}_{\hat{k}}) \delta_{i0}. \tag{12}$$

Observe that in this way we get an expression for the derivatives of  $\mathcal{F}$  with respect to any  $a^k$ , including  $a^0$ , in which only the derivatives of  $\mathcal{F}$  with respect to  $a^{\hat{k}}$  appear. In this way we can express the Eq. (7) in terms of  $\mathcal{F}$  and its derivatives with respect to  $a^{\hat{k}}$ .

In the following we will denote by  $\hat{F}_i$  the matrix obtained from  $F_i$  with the matrix indices running from 1 to  $n - 1$ . In other words  $(\hat{F}_i)_{j\hat{k}} = \mathcal{F}_{ij\hat{k}}$ . Similarly, by  $\hat{F}_i^{-1}$  we will denote the matrix obtained from  $F_i^{-1}$  by restricting the range of the matrix indices to  $1, \dots, n - 1$ . We also define  $G_i$  the matrix inverse of  $\hat{F}_i$  (observe that  $G_i \neq \hat{F}_i^{-1}$ ). Let us derive the expression for the inverse of the  $F_i$  matrix. We set

$$\rho_i = (F_i)_{00}, \quad (\sigma_i)_j = (F_i)_{0j}, \tag{13}$$

$$\mu_i = (F_i)_{00}^{-1}, \quad (\nu_i)_j = (F_i)_{0j}^{-1}. \tag{14}$$

We have

$$\mu_i \rho_i + {}^t \sigma_i \nu_i = 1, \quad \hat{F}_i \nu_i + \mu_i \sigma_i = 0,$$

for  $i = 0, \dots, n - 1$ , so that

$$\mu_i = \frac{1}{\rho_i - {}^t \sigma_i G_i \sigma_i}, \tag{15}$$

and

$$\nu_i = -\mu_i G_i \sigma_i. \tag{16}$$

Furthermore, we have

$$\hat{F}_i^{-1} = [G_i(\mathbf{1} - \sigma_i \otimes \nu_i)]_{j\hat{k}}. \tag{17}$$

Therefore, we now have  $F_i^{-1}$  in terms of  $\rho_i$ ,  $\sigma_i$ , and  $G_i$ .

In terms of  $\rho_i$ ,  $\sigma_i$ , etc. Eq. (7) is equivalent to the following identities:

$$\mu_p \rho_j \rho_j + \rho_j {}^t \sigma_i \nu_p + \rho_i {}^t \nu_p \sigma_j + {}^t \sigma_i \hat{F}_p^{-1} \sigma_j = \mu_p \rho_j \rho_i + \rho_i {}^t \sigma_j \nu_p + \rho_j {}^t \nu_p \sigma_i + {}^t \sigma_j \hat{F}_p^{-1} \sigma_i, \tag{18}$$

$$\rho_j \mu_p \sigma_i + \rho_j \hat{F}_i \nu_p + \hat{F}_i \hat{F}_p^{-1} \sigma_j + (\sigma_i \otimes \nu_p) \sigma_j = \rho_i \mu_p \sigma_j + \rho_i \hat{F}_j \nu_p + \hat{F}_j \hat{F}_p^{-1} \sigma_i + (\sigma_j \otimes \nu_p) \sigma_i, \tag{19}$$

$$\hat{F}_i \hat{F}_p^{-1} \hat{F}_j + (\sigma_i \otimes \nu_p) \hat{F}_j + (\mu_p \sigma_i + \hat{F}_i \nu_p) \otimes \sigma_j = \hat{F}_j \hat{F}_p^{-1} \hat{F}_i + (\sigma_j \otimes \nu_p) \hat{F}_i + (\mu_p \sigma_j + \hat{F}_j \nu_p) \otimes \sigma_i. \tag{20}$$

While we immediately see that Eq. (18) holds, the other two equations (19) and (20) are satisfied as a consequence of the identity

$$\hat{F}_i G_j \sigma_k = \hat{F}_k G_j \sigma_i, \tag{21}$$

where  $i, j, k = 0, \dots, n - 1$ , which follows from Eqs. (5), (10), and (13). Actually we have

$$\begin{aligned} (\hat{F}_i G_j \sigma_k)_i &= (\hat{F}_i)_{i\hat{m}} (G_j)_{\hat{m}\hat{p}} [-a^{0^{-1}} a^{\hat{s}} \mathcal{F}_{s\hat{k}\hat{p}} (1 - \delta_{k0}) \\ &\quad + a^{0^{-2}} a^{\hat{s}} a^{\hat{t}} \mathcal{F}_{s\hat{t}\hat{p}} \delta_{k0}]. \end{aligned}$$

Then if both  $i$  and  $k$  are different from 0, the identity follows directly from Eq. (5), whereas if either  $i$  or  $k$  is equal to 0, then we simply use the fact that by Eq. (11)

$$(\hat{F}_0)_{i\hat{k}} = -a^{0^{-1}} a^{\hat{m}} (\hat{F}_m)_{i\hat{k}},$$

to reduce to previous case.

To see how the identity works, we give the example of Eq.(19) that by Eq. (16) and (17) is equivalent to

$$\begin{aligned} \mu_p [\rho_j \sigma_i - \rho_j \hat{F}_i G_p \sigma_p + \hat{F}_i G_p (\sigma_p \otimes G_p \sigma_p) \sigma_j - (\sigma_i \otimes G_p \sigma_p) \sigma_j \\ - \rho_i \sigma_j - \rho_i \hat{F}_j G_p \sigma_p + \hat{F}_j G_p (\sigma_p \otimes G_p \sigma_p) \sigma_i \\ - (\sigma_j \otimes G_p \sigma_p) \sigma_i] + \hat{F}_i G_p \sigma_j - \hat{F}_j G_p \sigma_i = 0, \end{aligned} \tag{22}$$

which is satisfied thanks to Eq. (21).

In conclusion, we have derived a new set of WDVV equations for  $N = 2$  SYM, in which the renormalization scale  $\Lambda$  is identified with the distinguished modulus which naturally

arises in topological field theories [17]. In particular, repeating the construction in [11], we have that Eq. (7) implies

$$(\mathcal{F}_i)_{kl}\beta^{lm}(\mathcal{F}_j)_{mn} = (\mathcal{F}_j)_{kl}\beta^{lm}(\mathcal{F}_i)_{mn}, \quad (23)$$

where  $\beta^{lm}$  is the inverse of the generalized beta matrix  $\beta_{lm}$  defined in Eq. (8). Furthermore, it is easy to see that

$$(\mathcal{F}_i)_{kl}\beta^{(a)lm}(\mathcal{F}_j)_{mn} = (\mathcal{F}_j)_{kl}\beta^{(a)lm}(\mathcal{F}_i)_{mn}, \quad (24)$$

where  $\beta^{(a)lm}$  is the inverse of the beta matrix defined in Eq. (9).

Finally, we stress that it would be desirable to understand the structure of the equations for  $\mathcal{F}$  deriving from the conditions of flatness of the WDVV metrics  $\beta_{lm}$  and  $\beta_{lm}^{(a)}$ .

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