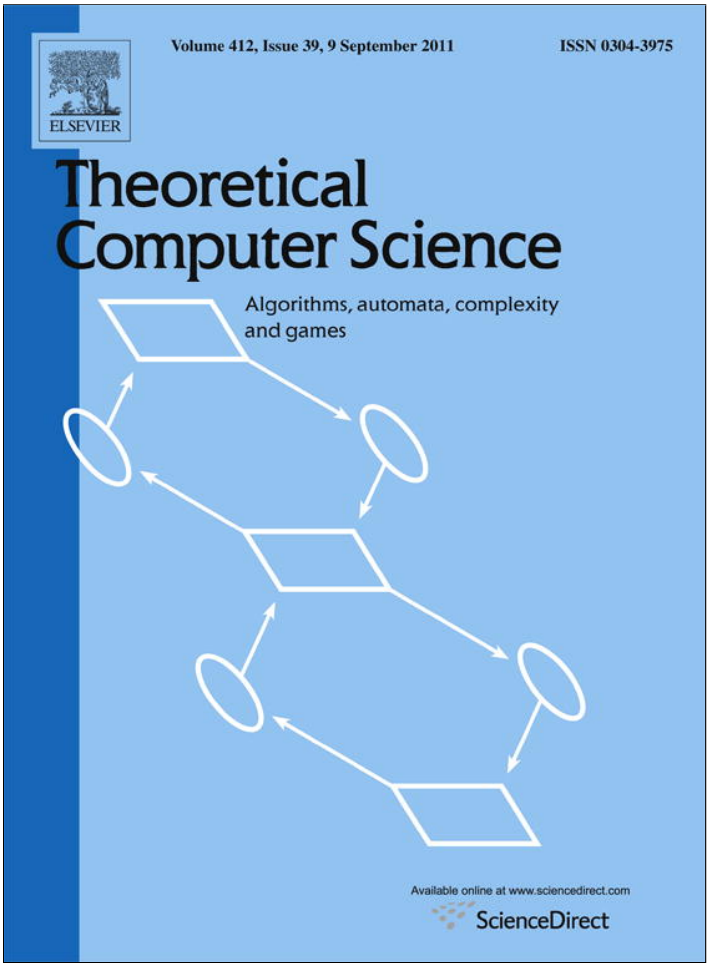
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The convergence classes of Collatz function

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ABSTRACT

The Collatz conjecture, also known as the $3x + 1$ conjecture, can be stated in terms of the reduced Collatz function $R(x) = (3x + 1)/2^h$ (where 2^h is the larger power of 2 that divides $3x + 1$). The conjecture is: *Starting from any odd positive integer and repeating* $R(x)$ *we eventually get to* 1. G_k, the k-th convergence class, is the set of odd positive integers x such that $R^k(x) = 1$.

In this paper an infinite sequence of binary strings s_h of length $2 \cdot 3^{h-1}$ (the *seeds*) are defined and it is shown that the binary representation of all $x \in G_k$ is the concatenation of k periodic strings whose periods are $s_k, \ldots, s_1$. More precisely $x = s^{[n_1)}_{k,d_{k,1}} \ldots s^{[n_k)}_{1,d_{k,k}}$ where $s^{[n_i)}_{k,d_{k,i}}$ is the substring of length n_i that starts in position $d_{k,i}$ in a sufficiently long repetition of the seed s_i.

Finally, starting positions $d_{k,i}$ and lengths n_i for which $s^{[n_1)}_{k,d_{k,1}} \ldots s^{[n_k)}_{1,d_{k,k}} \in G_k$ are defined, thus giving a complete characterization of classes G_k.

1. Introduction

The Collatz function is defined on all positive integers x by:

$$f(x) = \begin{cases} x/2 & x \text{ even} \\ 3x + 1 & x \text{ odd.} \end{cases}$$

Given any odd integer x, let $x' = (3x + 1)/2^h$ where 2^h is the highest power of 2 that divides $3x + 1$. The reduced form of the Collatz function is $R(x) = x'$ and is defined only for odd integers.

The Collatz conjecture says that for all integers $x > 0$ there exists i such that $f^i(x) = 1$ or, equivalently, that there exists k such that $R^k(x) = 1$.

Despite the efforts of many people for about seventy years, the conjecture is still undecided. The efforts are well documented in a very large literature. The problem has been attacked from many viewpoints. The Collatz function has been studied in large domains: Integer, rational, real and even complex numbers (where a beautiful fractal has been obtained) [5,3,4,9]. The Collatz conjecture has been also proved equivalent to many other conjectures in different contexts: Rewriting systems, tag systems, etc. [7,2,6].

Our bibliography contains only a very small and incomplete selection of papers; we refer interested readers to the large annotated bibliography in Lagarias [1]. The paper by Jean Paul Van Bendegem [10] is a philosophical essay on the $3x + 1$ problem.

The paper is organized as follows: Section 2 shows the direct computation of G_k, as sets of binary strings, for the first few values of k. Those computational experiments suggest that binary strings in G_k are the concatenation of k periodic strings whose periods, that we call *seeds*, are of length $2, 6, 18, \ldots, 2 \cdot 3^{k-1}$. In Section 3 some useful (and beautiful) properties of seeds are proved. Section 4 contains the main result: A complete characterization of classes G_k as sets of binary strings.

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2. Computational experiments

Define the inverse $R^{-1}(x)$ of the reduced Collatz function as the set of odd integers such that $y \in R^{-1}(x)$ iff $R(y) = x$. We can easily see that

$$R^{-1}(x) = \begin{cases} \emptyset & \text{if } x \equiv 0 \pmod 3 \\ \left\{ \dfrac{x 2^{2m+2} - 1}{3} \ : \ m \geq 0 \right\} & \text{if } x \equiv 1 \pmod 3 \\ \left\{ \dfrac{x 2^{2m+1} - 1}{3} \ : \ m \geq 0 \right\} & \text{if } x \equiv 2 \pmod 3). \end{cases}$$

Let G_k the class of odd integers x that converge to 1 in k steps, i.e. such that $R^k(x) = 1$.

The class G_k can be defined inductively by

$$G_0 = \{1\}$$
$$G_k = \bigcup_{x \in G_{k-1}} R^{-1}(x).$$

For a binary string s let $[\![s]\!]$ be the non-negative integer whose binary representation is s. In what follows we see classes G_k as sets of binary strings.

Clearly $G_0 = \{1\}$: The singleton set that contains only the binary string 1.

Let us compute first G_1

$$G_1 = \bigcup_{x_0 \in G_0} R^{-1}(x_0) = R^{-1}(1) = \left\{ \frac{4^{m_1+1} - 1}{3} \ : \ m_1 \geq 0 \right\} = \left\{ \sum_{i=0}^{m_1} 4^i \ : \ m_1 \geq 0 \right\}.$$

If we represent $x_1 = \sum_{i=0}^{m_1} 4^i$ as a binary string of length $2m_1 + 2$ we obtain 01^{m_1+1}, i.e. the concatenation of one or more copies of the binary string $s_1 = 01$ of length 2. Thus

$$G_1 = \left\{ \left[\!\left[s_1^{m_1+1} \right]\!\right] \ : \ m_1 \geq 0 \right\}.$$

Now we can compute G_2 from G_1.

$$G_2 = \bigcup_{x_1 \in G_1} R^{-1}(x_1) = \bigcup_{m_1=0}^{\infty} R^{-1}\left(\left[\!\left[s_1^{m_1+1} \right]\!\right] \right).$$

Since $x_1 = \sum_{i=0}^{m_1} 4^i \equiv m_1 + 1 \pmod 3$ we obtain

$$G_2 = \left\{ \frac{\left[\!\left[s_1^{3k_1+1} \right]\!\right] 4^{m_2+1} - 1}{3} \ : \ k_1, m_2 \geq 0 \right\} \cup \left\{ \frac{2 \left[\!\left[s_1^{3k_1+2} \right]\!\right] 4^{m_2} - 1}{3} \ : \ k_1, m_2 \geq 0 \right\}.$$

Compute first

$$\frac{\left[\!\left[s_1^3 \right]\!\right]}{3} = \frac{\sum_{i=0}^{2} 4^i}{3} = \frac{4^3 - 1}{3^2} = 7$$

and let $s_2 = 000111$ be the binary representation of 7 as a string of length 6.

A simple computation shows that $\left[\!\left[s_1^{3k_1+1} \right]\!\right]/3 = \left[\!\left[s_2^{k_1} s_2^{[2)} \right]\!\right]$, where $s_2^{[2)} = 00$ is the prefix of length 2 of s_2 and that $\left[\!\left[s_1^{3k_1+2} \right]\!\right]/3 = \left[\!\left[s_2^{k_1} s_2^{[4)} \right]\!\right]$, where $s_2^{[4)} = 0001$ is the prefix of length 4 of s_2. Moreover, $\left[\!\left[s_1^{3k_1+1} \right]\!\right] \bmod 3 = 1$ and $\left[\!\left[s_1^{3k_1+2} \right]\!\right] \bmod 3 = 2$.

Then

$$G_2 = \left\{ \left[\!\left[s_2^{k_1} s_2^{[2)} \right]\!\right] 4^{m_2+1} + \frac{4^{m_2+1} - 1}{3} \ : \ k_1, m_2 \geq 0 \right\}$$
$$\cup \left\{ 2 \left[\!\left[s_2^{k_1} s_2^{[4)} \right]\!\right] 4^{m_2} + \frac{4^{m_2+1} - 1}{3} \ : \ k_1, m_2 \geq 0 \right\}$$

We can write $(4^{m_2+1} - 1)/3 = \sum_{i=0}^{m_2} 4^i$ in binary both as $\left[\!\left[s_1^{m_2} s_1^{[2)} \right]\!\right]$ and $\left[\!\left[s_{1,1}^{m_2} s_{1,1}^{[1)} \right]\!\right]$, where $s_{1,1} = 10$ is the left rotation of s_1 by 1 position.

Lengths of strings $s_1^{m_2}s_1^{[2]}$ and $s_{1,1}^{m_2}s_{1,1}^{[1]}$ are respectively $2m_2+2$ and $2m_2+1$. Thus we conclude that

$$G_2 = \left\{ \left[\!\left[s_{2,0}^{k_1} s_{2,0}^{[2]} s_{1,0}^{m_2} s_{1,0}^{[2]} \right]\!\right] : k_1, m_2 \ge 0 \right\} \cup \left\{ \left[\!\left[s_{2,0}^{k_1} s_{2,0}^{[4]} s_{1,1}^{m_2} s_{1,1}^{[1]} \right]\!\right] : k_1, m_2 \ge 0 \right\}$$

where, for uniformity, $s_{1,0} = s_1$ and $s_{2,0} = s_2$ (the unrotated seeds).

We can conclude that G_2 is the set of all integers whose binary representation starts with zero or more copies of $s_2 = 000111$ and continues either by the prefix $s_{2,0}^{[2]} = 00$ of s_2 followed by zero or more copies of $s_{1,0} = 01$ followed by the prefix $s_{1,0}^{[2]} = s_1 = 01$ or by the prefix $s_{2,0}^{[4]} = 0001$ followed by zero or more copies of $s_{1,1} = 10$ followed by the prefix $s_{1,1}^{[1]} = 1$. The representation of G_2 as a tree is:

$$\begin{array}{llll} s_{2,0}^* & \rightarrow s_{2,0}^{[2]} & \rightarrow s_{1,0}^* & \rightarrow s_{1,0}^{[2]} \\ & \searrow s_{2,0}^{[4]} & \rightarrow s_{1,1}^* & \rightarrow s_{1,1}^{[1]} \end{array} \quad \text{or} \quad \begin{array}{llll} (000111)^* & \rightarrow 00 & \rightarrow (01)^* & \rightarrow 01 \\ & \searrow 0001 & \rightarrow (10)^* & \rightarrow 1 \end{array}$$

where s^* means concatenation of zero or more copies of s.[1]

We can compute G_3 in the same way. However it is better to use a computer program to build and print the trees for G_3, G_4 and G_5. The tree for G_6 is too big to be computed and printed.

The program inductively computes the tree for G_{k+1} from the tree for G_k by computing $R^{-1}(z)$ for each branch z of the tree; it is based on two mutually recursive procedures: Div3 and Div3Aux.

Div3(x, r) is called with parameters a node of type $x = s_{h,d}^*$ and an integer r which is the remainder of the division by three of the ancestors of node x ($r = 0$ when the procedure is called with the root as input). The companion procedure Div3Aux(z, y, r) is called with parameters a node of type $y = s_{h,d}^{[\ell]}$ and an integer r which is the remainder of the division by three of the ancestors of node y. Moreover, for each node $x = s_{h,d}^*$, the procedure Div3Aux is called three times with, respectively, $z = s_{h,d}^i$ for $i = 0, 1, 2$.

The two procedures can be described as follows in C-like pseudo code:

```
Div3(x, r)    // x = s*_{h,d}
    w = r · s^3_{h,d}
    // w is the concatenation of the binary string for r with three copies of s_{h,d}.
    s_{h+1,d'} = w/3    // Notice that r = w mod 3 since s^3_{h,d} mod 3 = 0.
    "build a new node x' with label s*_{h+1,d'}"
    for "each son y of x"
        for i = 0  to 2
            y' = Div3Aux(s^i_{h,d}, y, r)
            if y' ≠ NIL
                "add y' as a new son of x'"
    return x'

Div3Aux(z, y, r).    // y = s^[ℓ]_{h,d} and z = s^i_{h,d} for 0 ≤ i ≤ 2.
    ℓ' = ℓ+ length of z
    w = r · z · s^[ℓ]_{h,d}
    s^[ℓ']_{h+1,d'} = w/3
    r' = w mod 3
    if y is a leaf
        if r' == 0
            return NIL
        else   // r' == 1 or r' == 2
            "build a new node y' with label s^[ℓ']_{h+1,d'}"
            if r' == 1
                "add to y' a single son s*_{1,0} followed by a leaf s^[2]_{1,0}"
            else   // r' == 2
                "add to y' a single son s*_{1,1} followed by a leaf s^[1]_{1,1}"
    else   // y is not a leaf. Let x be the son of y
        x' = Div3(x, r')
        "put x' as the son of y'"
    return y'
```

[1] The tree representation used for G_2 (and that that will be used for next classes G_k) is just the syntactic tree of a regular expression $(000111)^*[00(01)^*01 + 0001(10)^*1]$. Thus classes G_k are regular sets of strings.

Many different implementations of those procedure have been written and used, starting from a naive one written when no properties of the classes were already known and refining it as soon as more and more properties were discovered.

Here is the tree for G_3 obtained as output of the program:

(000010010111101101)*				
→ 00	→ (000111)*	→ 00	→ (01)*	→ 01
		↘ 0001	→ (10)*	→ 1
↘ 0000	→ (100011)*	→ 100	→ (01)*	→ 01
		↘ 10001	→ (10)*	→ 1
↘ 00001001	→ (011100)*	→ 01	→ (10)*	→ 1
		↘ 011100	→ (01)*	→ 01
↘ 0000100101	→ (111000)*	→ 1	→ (10)*	→ 1
		↘ 11100	→ (01)*	→ 01
↘ 00001001011110	→ (110001)*	→ 1100	→ (01)*	→ 01
		↘ 110001	→ (10)*	→ 1
↘ 0000100101111011	→ (001110)*	→ 0	→ (01)*	→ 01
		↘ 001	→ (10)*	→ 1

Let $s_3^* = (000010010111101101)^*$ be the root. Its sons are the six prefixes $s_{3,0}^{[2]}$, $s_{3,0}^{[4]}$, $s_{3,0}^{[8]}$, $s_{3,0}^{[10]}$, $s_{3,0}^{[14]}$ and $s_{3,0}^{[16]}$, each one followed by the repetition of a different left rotation of s_2: In order $s_{2,0}^*$, $s_{2,5}^*$, $s_{2,2}^*$, $s_{2,3}^*$, $s_{2,4}^*$ and $s_{2,1}^*$. In turn, each left rotation of s_2 is followed by two of its prefixes of different length and then by the repetition of a rotation $s_{1,0}^*$ or $s_{1,1}^*$ of s_1 followed by a prefix of the rotation. By using this notation the tree becomes

$s_{3,0}^*$	→ $s_{3,0}^{[2]}$	→ $s_{2,0}^*$	→ $s_{2,0}^{[2]}$	→ $s_{1,0}^*$	→ $s_{1,0}^{[2]}$
			↘ $s_{2,0}^{[4]}$	→ $s_{1,1}^*$	→ $s_{1,1}^{[1]}$
	↘ $s_{3,0}^{[4]}$	→ $s_{2,5}^*$	→ $s_{2,5}^{[3]}$	→ $s_{1,0}^*$	→ $s_{1,0}^{[2]}$
			↘ $s_{2,5}^{[5]}$	→ $s_{1,1}^*$	→ $s_{1,1}^{[1]}$
	↘ $s_{3,0}^{[8]}$	→ $s_{2,2}^*$	→ $s_{2,2}^{[2]}$	→ $s_{1,1}^*$	→ $s_{1,1}^{[1]}$
			↘ $s_{2,2}^{[6]}$	→ $s_{1,0}^*$	→ $s_{1,0}^{[2]}$
	↘ $s_{3,0}^{[10]}$	→ $s_{2,3}^*$	→ $s_{2,3}^{[1]}$	→ $s_{1,1}^*$	→ $s_{1,1}^{[1]}$
			↘ $s_{2,3}^{[5]}$	→ $s_{1,0}^*$	→ $s_{1,0}^{[2]}$
	↘ $s_{3,0}^{[14]}$	→ $s_{2,4}^*$	→ $s_{2,4}^{[4]}$	→ $s_{1,0}^*$	→ $s_{1,0}^{[2]}$
			↘ $s_{2,4}^{[6]}$	→ $s_{1,1}^*$	→ $s_{1,1}^{[1]}$
	↘ $s_{3,0}^{[16]}$	→ $s_{2,1}^*$	→ $s_{2,1}^{[1]}$	→ $s_{1,0}^*$	→ $s_{1,0}^{[2]}$
			↘ $s_{2,1}^{[3]}$	→ $s_{1,1}^*$	→ $s_{1,1}^{[1]}$

Experimental results suggest that classes G_k can be defined in terms of an infinite sequence of strings s_h of length $2 \cdot 3^{h-1}$. We call s_h *seed* of order h.

Indeed, we will show that for each $x \in G_k$ there exist integers q_h, d_h and ℓ_h such that

$$x = \left[\!\left[s_{k,d_1}^{q_1} s_{k,d_1}^{[\ell_1]} s_{k-1,d_2}^{q_2} s_{k-1,d_2}^{[\ell_2]} \ldots s_{1,d_k}^{q_k} s_{1,d_k}^{[\ell_k]} \right]\!\right]$$

where $q_h \geq 0$, $0 < \ell_h \leq 2 \cdot 3^{h-1}$, $d_1 = 0$ and, for $h > 1$, $0 \leq d_h < 2 \cdot 3^{h-2}$.

We can extend notation $s^{[\ell]}$ (the prefix of length $\ell \leq \lambda$ of a string s of length λ) to all non-negative integers n (even $n > \lambda$) by letting $s^{[n]}$ denote the prefix of length n of a sufficiently long repetition of s, i.e. if $q = \lfloor n/\lambda \rfloor$ and $\ell = n \bmod \lambda$ then $s^{[n]} = s^q s^{[\ell]}$ is the concatenation of q copies of s followed by the prefix $s^{[\ell]}$.

By using this extended notation, we can write the previous equation in a more compact form as

$$x = \left[\!\left[s_{k,d_1}^{[n_1]} s_{k-1,d_2}^{[n_2]} \ldots s_{1,d_k}^{[n_k]} \right]\!\right] \tag{1}$$

where $n_h = 2 \cdot 3^{h-1} q_h + \ell_h$ for $h = 1, \ldots, k$.

In Section 4 the intuition coming from computational experiments is proved, i.e. that all $x \in G_k$ has the binary representation in Eq. (1).

Moreover the sequences of integers n_h, d_h such that

$$\left[\!\left[s_{k,d_1}^{[n_1)} s_{k-1,d_2}^{[n_2)} \ldots s_{1,d_k}^{[n_k)} \right]\!\right] \in G_k$$

are defined, thus giving a complete characterization of classes G_k.

3. Properties of seeds

Experimental results in Section 2 suggest that seeds are binary strings s_h of length $2\lambda_h$, where $\lambda_h = 3^{h-1}$, and that seeds can be defined inductively as $s_1 = 01$ and $[\![s_h]\!] = \left[\!\left[s_{h-1}^3\right]\!\right]/3$ for $h > 1$. The simple computation

$$[\![s_h]\!] = \frac{\left[\!\left[s_{h-1}^3\right]\!\right]}{3} = [\![s_{h-1}]\!]\, 4^{\lambda_{h-1}} \frac{\sum_{i=0}^{2} 4^i}{3} = 7\, [\![s_{h-1}]\!]\, 4^{\lambda_{h-1}}$$

shows that s_h is well defined since $\left[\!\left[s_{h-1}^3\right]\!\right]/3$ is an integer.

Here are some properties of seeds s_h, of rotations $s_{h,d}$ and of extended prefixes $s_h^{[n)}$.

Lemma 1 (*Properties of Seeds*). *For all seed s_h we have*

$$[\![s_h]\!] = \frac{\sum_{i=0}^{\lambda_h-1} 4^i}{\lambda_h} = \frac{4^{\lambda_h}-1}{\lambda_{h+1}} \tag{2}$$

and

$$[\![s_h]\!] \equiv 1 \pmod 3. \tag{3}$$

Proof. The proof is by induction. For the basis $[\![s_1]\!] = 1 = (4^1-1)/3 = (4^{\lambda_1}-1)/\lambda_2$ and $[\![s_1]\!] \bmod 3 = 1$. For the inductive step

$$\begin{aligned}
[\![s_h]\!] &= \frac{\left[\!\left[s_{h-1}^3\right]\!\right]}{3} = \frac{\sum_{i=0}^{2} [\![s_{h-1}]\!]\, 4^{i\lambda_{h-1}}}{3} = \frac{\sum_{i=0}^{2} \left(\frac{\sum_{j=0}^{\lambda_{h-1}-1} 4^j}{\lambda_{h-1}}\right) 4^{i\lambda_{h-1}}}{3} \\
&= \frac{\sum_{i=0}^{2} \left(\sum_{j=0}^{\lambda_{h-1}-1} 4^j\right) 4^{i\lambda_{h-1}}}{\lambda_h} = \frac{\sum_{i=0}^{2} \sum_{j=0}^{\lambda_{h-1}-1} 4^{i\lambda_{h-1}+j}}{\lambda_h} \\
&= \frac{\sum_{i=0}^{\lambda_h-1} 4^i}{\lambda_h} = \frac{4^{\lambda_h}-1}{\lambda_{h+1}}
\end{aligned}$$

and

$$[\![s_h]\!] \equiv \frac{\sum_{i=0}^{2} [\![s_{h-1}]\!]\, 4^{i\lambda_{h-1}}}{3} \equiv [\![s_{h-1}]\!]\, 4^{\lambda_{h-1}} 7 \equiv 1 \pmod 3. \quad \square$$

Lemma 2 (*Properties of Left Rotations of Seeds*). *For $0 \le d < 2\lambda_h$*

$$[\![s_{h,d}]\!] = \left(2^d \bmod \lambda_{h+1}\right) [\![s_h]\!] \tag{4}$$

and

$$[\![s_{h,d}]\!] \equiv 2^d \pmod 3) \tag{5}$$

and, for $0 \le d < \lambda_h$

$$[\![s_{h,d}]\!] + [\![s_{h,d+\lambda_h}]\!] = 4^{\lambda_h} - 1 \tag{6}$$

(i.e. bits of string $s_{h,d+\lambda_h}$ are the complement of corresponding bits of $s_{h,d}$) and, finally

$$[\![s_{h+1,d}]\!] = r\frac{4^{\lambda_{h+1}}-1}{3} + \frac{\left[\!\left[s_{h,d'}^3\right]\!\right]}{3} \tag{7}$$

where $r = \lfloor d/(2\lambda_h) \rfloor$ and $d' = d \bmod 2\lambda_h$.

Proof. The proof for Eq. (4) is

$$\begin{aligned}
[\![s_{h,d}]\!] &= \left([\![s_h]\!] \bmod 2^{2\lambda_h-d}\right) 2^d + \frac{[\![s_h]\!] - [\![s_h]\!] \bmod 2^{2\lambda_h-d}}{2^{2\lambda_h-d}} \\
&= \frac{\left([\![s_h]\!] \bmod 2^{2\lambda_h-d}\right) 4^{\lambda_h} + [\![s_h]\!] - [\![s_h]\!] \bmod 2^{2\lambda_h-d}}{2^{2\lambda_h-d}} \\
&= \frac{\left([\![s_h]\!] \bmod 2^{2\lambda_h-d}\right)(4^{\lambda_h}-1) + [\![s_h]\!]}{2^{2\lambda_h-d}} \\
&= \frac{\left([\![s_h]\!] \bmod 2^{2\lambda_h-d}\right) \lambda_{h+1} [\![s_h]\!] + [\![s_h]\!]}{2^{2\lambda_h-d}} \\
&= \frac{\lambda_{h+1}\left([\![s_h]\!] \bmod 2^{2\lambda_h-d}\right) + 1}{2^{2\lambda_h-d}} [\![s_h]\!] \\
&= \frac{\lambda_{h+1}\left(\frac{4^{\lambda_h}-1}{\lambda_{h+1}} \bmod 2^{2\lambda_h-d}\right) + 1}{2^{2\lambda_h-d}} [\![s_h]\!] \\
&= \frac{\left(4^{\lambda_h}-1\right) \bmod \lambda_{h+1} 2^{2\lambda_h-d} + 1}{2^{2\lambda_h-d}} [\![s_h]\!] \\
&= \frac{2^{2\lambda_h} \bmod \lambda_{h+1} 2^{2\lambda_h-d}}{2^{2\lambda_h-d}} [\![s_h]\!] \\
&= \frac{2^{2\lambda_h-d} 2^d \bmod \lambda_{h+1} 2^{2\lambda_h-d}}{2^{2\lambda_h-d}} [\![s_h]\!] \\
&= \frac{2^{2\lambda_h-d}\left(2^d \bmod \lambda_{h+1}\right)}{2^{2\lambda_h-d}} [\![s_h]\!] \\
&= \left(2^d \bmod \lambda_{h+1}\right) [\![s_h]\!] .
\end{aligned}$$

The proof for Eq. (5) is

$$[\![s_{h,d}]\!] \equiv [\![s_h]\!] \left(2^d \bmod \lambda_{h+1}\right) \equiv 2^d \pmod 3.$$

We can prove Eq. (6) only for $d = 0$: The cases of $1 \le d < \lambda_h$ are a simple consequence since rotations do not change the pairs of bits at a distance λ_h from each other.

$$\begin{aligned}
[\![s_{h,0}]\!] &= [\![s_h]\!] = \frac{4^{\lambda_h}-1}{\lambda_{h+1}} = \frac{2^{\lambda_h}+1}{\lambda_{h+1}}\left(2^{\lambda_h}-1\right) \\
&= \left(\frac{2^{\lambda_h}+1}{\lambda_{h+1}} - 1\right) 2^{\lambda_h} + 2^{\lambda_h} - \frac{2^{\lambda_h}+1}{\lambda_{h+1}} = [\![s]\!]\, 2^{\lambda_h} + [\![s']\!]
\end{aligned}$$

where s, s' are the binary string of length λ_h such that

$$[\![s]\!] = \frac{2^{\lambda_h}+1}{\lambda_{h+1}} - 1 \quad \text{and} \quad [\![s']\!] = 2^{\lambda_h} - \frac{2^{\lambda_h}+1}{\lambda_{h+1}}.$$

Then

$$\begin{aligned}
[\![s_{h,\lambda_h}]\!] &= [\![s']\!]\, 2^{\lambda_h} + [\![s]\!] = \left(2^{\lambda_h} - \frac{2^{\lambda_h}+1}{\lambda_{h+1}}\right) 2^{\lambda_h} + \frac{2^{\lambda_h}+1}{\lambda_{h+1}} - 1 \\
&= 4^{\lambda_h} - 1 - \frac{2^{\lambda_h}+1}{\lambda_{h+1}}\left(2^{\lambda_h}-1\right) = 4^{\lambda_h} - 1 - [\![s_{h,0}]\!] .
\end{aligned}$$

Finally, by Eq. (7), $2^d \equiv 2^{d'} \pmod{\lambda_{h+1}}$ and $\left\lfloor \frac{2^d \bmod \lambda_{h+2}}{\lambda_{h+1}} \right\rfloor = r$ (by the isomorphism of $\mathbb{Z}^+_{2\lambda_h}$ and $\mathbb{Z}^*_{\lambda_{h+1}}$). Thus

$$2^d \bmod \lambda_{h+2} = \left\lfloor \frac{2^d \bmod \lambda_{h+2}}{\lambda_{h+1}} \right\rfloor \lambda_{h+1} + \left(2^d \bmod \lambda_{h+2}\right) \bmod \lambda_{h+1} = r\lambda_{h+1} + 2^{d'} \bmod \lambda_{h+1}$$

and

$$\begin{aligned}
[\![s_{h+1,d}]\!] &= \left(2^d \bmod \lambda_{h+2}\right)[\![s_{h+1}]\!] \quad \text{(by Eq. (4))}\\
&= \left(r\lambda_{h+1} + 2^{d'} \bmod \lambda_{h+1}\right)[\![s_{h+1}]\!]\\
&= \left(r\lambda_{h+1} + 2^{d'} \bmod \lambda_{h+1}\right)\left[\!\!\left[s_h^3\right]\!\!\right]/3\\
&= \left(r\lambda_{h+1} + 2^{d'} \bmod \lambda_{h+1}\right)[\![s_h]\!]\frac{\sum_{i=0}^{2} 4^{i\lambda_h}}{3}\\
&= \left(r(4^{\lambda_h}-1) + [\![s_{h,d'}]\!]\right)\frac{\sum_{i=0}^{2} 4^{i\lambda_h}}{3}\\
&= r\frac{4^{\lambda_{h+1}}-1}{3} + \frac{\left[\!\!\left[s_{h,d'}^3\right]\!\!\right]}{3}. \quad \square
\end{aligned}$$

Lemma 3 (*Properties of Extensions of Seeds*)**.** *For* $n > 0$, $h > 0$ *and* $q = \lfloor n/(2\lambda_h)\rfloor$, $\ell = n \bmod 2\lambda_h$

$$\left[\!\!\left[s_h^{[n)}\right]\!\!\right] = \left\lfloor \frac{2^n}{\lambda_{h+1}} \right\rfloor \tag{8}$$

$$\left[\!\!\left[s_{h+1}^{[n)}\right]\!\!\right] = \left\lfloor \left[\!\!\left[s_h^{[n)}\right]\!\!\right]/3 \right\rfloor \tag{9}$$

$$\left[\!\!\left[s_h^{[n)}\right]\!\!\right] \equiv q2^\ell + \left[\!\!\left[s_h^{[\ell)}\right]\!\!\right] \pmod 3. \tag{10}$$

Moreover, for $0 \le d < \lambda_{h+1}$ *and* $r = \lfloor d/(2\lambda_h)\rfloor$, $d' = d \bmod 2\lambda_h$

$$\left[\!\!\left[s_{h+1,d}^{[n)}\right]\!\!\right] = \left\lfloor \frac{r2^n + \left[\!\!\left[s_{h,d'}^{[n)}\right]\!\!\right]}{3} \right\rfloor. \tag{11}$$

Proof. The proof of Eq. (8) is by induction on q. For the basis $q = 0$ and $n = \ell < 2\lambda_h$

$$\left[\!\!\left[s_h^{[\ell)}\right]\!\!\right] = \left\lfloor \frac{4^{\lambda_h}-1}{\lambda_{h+1}2^{2\lambda_h-\ell}} \right\rfloor = \left\lfloor \frac{2^\ell}{\lambda_{h+1}} - \frac{2^\ell}{\lambda_{h+1}2^{2\lambda_h}} \right\rfloor = \left\lfloor \frac{2^\ell}{\lambda_{h+1}} \right\rfloor$$

where the last equality follows from

$$\frac{2^\ell}{\lambda_{h+1}2^{2\lambda_h}} < \frac{1}{\lambda_{h+1}} \le \frac{2^\ell}{\lambda_{h+1}} - \left\lfloor \frac{2^\ell}{\lambda_{h+1}} \right\rfloor.$$

For the inductive step let $n' = n - 2\lambda_h$. Then

$$\left[\!\!\left[s_h^{[n)}\right]\!\!\right] = [\![s_h]\!]\,2^{n'} + \left[\!\!\left[s_h^{[n')}\right]\!\!\right] = \frac{4^{\lambda_h}-1}{\lambda_{h+1}}2^{n'} + \left\lfloor \frac{2^{n'}}{\lambda_{h+1}} \right\rfloor = \left\lfloor \frac{2^{2\lambda_h}-1}{\lambda_{h+1}}2^{n'} + \frac{2^{n'}}{\lambda_{h+1}} \right\rfloor = \left\lfloor \frac{2^n}{\lambda_{h+1}} \right\rfloor.$$

For Eq. (9) let $k = \lceil n/(2\lambda_{h+1})\rceil$. Then

$$\left[\!\!\left[s_{h+1}^{[n)}\right]\!\!\right] = \left[\!\!\left[\left(s_{h+1}^k\right)^{[n)}\right]\!\!\right] = \left\lfloor \frac{\left[\!\!\left[s_{h+1}^k\right]\!\!\right]}{2^{k2\lambda_{h+1}-n}} \right\rfloor = \left\lfloor \frac{\left[\!\!\left[s_h^{3k}\right]\!\!\right]}{3\cdot 2^{3k2\lambda_h-n}} \right\rfloor = \left\lfloor \left[\!\!\left[s_h^{[n)}\right]\!\!\right]/3 \right\rfloor$$

where the last equality holds because $\left[\!\!\left[s_h^{3k}\right]\!\!\right] \bmod 3 = 0$.

For Eq. (10)

$$\left[\!\!\left[s_h^{[n)}\right]\!\!\right] \equiv \left[\!\!\left[s_h^q\right]\!\!\right]2^\ell + \left[\!\!\left[s_h^{[\ell)}\right]\!\!\right] \equiv q2^\ell + \left[\!\!\left[s_h^{[\ell)}\right]\!\!\right] \pmod 3$$

where the last equality holds because $\left[\!\!\left[s_h^q\right]\!\!\right] \equiv q \pmod 3$.

Finally, for Eq. (11), let $k = \lceil n/(2\lambda_{h+1}) \rceil$ so that $\llbracket s_{h+1,d}^{[n)} \rrbracket = \llbracket (s_{h+1,d}^{k})^{[n)} \rrbracket$. Then

$$\begin{aligned}\llbracket s_{h+1,d}^{k} \rrbracket &= \sum_{i=0}^{k-1} \llbracket s_{h+1,d} \rrbracket \, 2^{2\lambda_{h+1}} \\ &= \sum_{i=0}^{k-1} \left(r\frac{4^{\lambda_{h+1}} - 1}{3} + \frac{\llbracket s_{h,d'}^{3} \rrbracket}{3} \right) 2^{2\lambda_{h+1}} \qquad \text{(by Eq. (7))} \\ &= r\frac{4^{k\lambda_{h+1}} - 1}{3} + \frac{\llbracket s_{h,d'}^{3k} \rrbracket}{3}\end{aligned}$$

and

$$\begin{aligned}\llbracket (s_{h+1,d}^{k})^{[n)} \rrbracket &= \left\lfloor \frac{2^n}{2^{2k\lambda_{h+1}}} \left(r\frac{4^{k\lambda_{h+1}} - 1}{3} + \frac{\llbracket s_{h,d'}^{3k} \rrbracket}{3} \right) \right\rfloor \\ &= \left\lfloor \frac{2^n}{2^{2k\lambda_{h+1}}} \left(r\left\lfloor \frac{4^{k\lambda_{h+1}}}{3} \right\rfloor + \frac{\llbracket s_{h,d'}^{3k} \rrbracket}{3} \right) \right\rfloor \\ &= \left\lfloor \frac{2^n}{2^{2k\lambda_{h+1}}} \left\lfloor \frac{r4^{k\lambda_{h+1}} + \llbracket s_{h,d'}^{3k} \rrbracket}{3} \right\rfloor \right\rfloor \\ &= \left\lfloor \frac{2^n}{2^{2k\lambda_{h+1}}} \frac{r4^{k\lambda_{h+1}} + \llbracket s_{h,d'}^{3k} \rrbracket}{3} \right\rfloor \\ &= \left\lfloor \frac{1}{3} \left(r2^n + \frac{2^n \llbracket s_{h,d'}^{3k} \rrbracket}{2^{2k\lambda_{h+1}}} \right) \right\rfloor \\ &= \left\lfloor \frac{r2^n + \llbracket s_{h,d'}^{3k} \rrbracket}{3} \right\rfloor . \quad \square\end{aligned}$$

4. Convergence classes

The experimental results in Paragraph 1 suggest that each $x \in G_k$ has the binary representation given by Eq. (1), for some integers n_i, d_i $(1 \le i \le k)$.

Here we characterize integers n_i and d_i such that the x given by Eq. (1) is in G_k. We do so by defining a *scheme* $\mathcal{S}_k$ which is a set of lengths $n_i > 0$ and left rotations $d_{h,i}$ (for $1 \le i \le h \le k$).

Definition 1 (*Scheme* $\mathcal{S}_k$ *for* G_k). A scheme $\mathcal{S}_1$ for G_1 is given by the rotation $d_{1,1} = 0$ and an even length $n_1 \equiv 0 \pmod 2$. For $k > 1$ the lengths $n_i > 0$ and left rotations $d_{h,i}$ of a scheme $\mathcal{S}_k$ are defined by mutual induction by

(a) $n_1 \equiv \pm 2 \pmod 6$,
(b) $n_i \equiv r_{i-1,i-1}(5 - 2[r_{i-1,i-1} - r_{i,i-1}]) \pm 1 \pmod 6$, for $2 \le i < k$
(c) $n_k \equiv r_{k-1,k-1} - 1 \pmod 2$

(d) $d_{h,1} = 0$, for $1 \le h \le k$,
(e) $d_{h,i} = d_{h-1,i} + r_{h-1,i-1} 2\lambda_{h-i-1}$, for $2 \le i < h \le k$
(f) $d_{h,h} = r_{h-1,h-1} - 1$, for $2 \le h \le k$

where $r_{h,i} = \llbracket s_{h,d_{h,1}}^{[n_1)} \dots s_{h-i+1,d_{h,i}}^{[n_i)} \rrbracket \bmod 3$.

Lemma 4. *Let* $\mathcal{S}_k$ *a scheme for* G_k*. Then* $r_{h,h} \neq 0$ *for* $1 \le h < k$.

Proof. By induction on h. For the basis $r_{1,1} = \llbracket s_{1,d_{1,1}}^{[n_1)} \rrbracket \bmod 3$ and

$$\llbracket s_{1,d_{1,1}}^{[n_1)} \rrbracket \equiv \llbracket s_1^{[n_1)} \rrbracket \equiv \llbracket s_1^{n_1/2} \rrbracket \equiv n_1/2 \pmod 3.$$

Then $r_{1,1} \neq 0$ since $n_1 \equiv \pm 2 \pmod 6$.

Let $h > 1$ and assume $r_{h-1,h-1} \neq 0$. Then

$$\begin{aligned} r_{h,h} &\equiv \left[\!\left[s^{[n_1)}_{h,d_{h,1}} \dots s^{[n_h)}_{1,d_{h,h}} \right]\!\right] \pmod 3 \\ &\equiv r_{h,h-1} 2^{n_h} + \left[\!\left[s^{[n_h)}_{1,d_{h,h}} \right]\!\right] \pmod 3 \\ &\equiv r_{h,h-1} r_{h-1,h-1} + \left[\!\left[s^{[n_h)}_{1,r_{h-1,h-1}-1} \right]\!\right] \pmod 3. \end{aligned}$$

If $r_{h-1,h-1} = 1$ then $n_h = 2m$ is even and

$$\begin{aligned} r_{h,h} &\equiv r_{h,h-1} + \left[\!\left[s^{[n_h)}_{1} \right]\!\right] \pmod 3 \\ &\equiv r_{h,h-1} + [\![s^{m}_{1}]\!] \pmod 3 \\ &\equiv r_{h,h-1} + m \pmod 3 \end{aligned}$$

and so $r_{h,h} \neq 0$ iff

$$m + r_{h,h-1} \not\equiv 0 \pmod 3$$

$$n_h + 2r_{h,h-1} \not\equiv 0 \pmod 3$$

$$n_h \not\equiv r_{h,h-1} \pmod 3.$$

If $r_{h-1,h-1} = 2$ then $n_h = 2m + 1$ is odd and

$$\begin{aligned} r_{h,h} &\equiv 2r_{h,h-1} + \left[\!\left[s^{[n_h)}_{1,1} \right]\!\right] \pmod 3 \\ &\equiv 2r_{h,h-1} + \left[\!\left[s^{m+1}_{1} \right]\!\right] \pmod 3 \\ &\equiv 2r_{h,h-1} + m + 1 \pmod 3 \end{aligned}$$

and so $r_{h,h} \neq 0$ iff

$$\begin{aligned} &m + 1 + 2r_{h,h-1} \not\equiv 0 \pmod 3 \\ &n_h + 1 + r_{h,h-1} \not\equiv 0 \pmod 3 \\ &n_h \not\equiv 2(r_{h,h-1} + 1) \pmod 3. \end{aligned}$$

In both cases

$$n_h \not\equiv r_{h-1,h-1}(r_{h-1,h-1} + r_{h,h-1} - 1) \pmod 3.$$

Thus n_h satisfy the congruences

$$n_h \equiv r_{h-1,h-1} - 1 \pmod 2$$

and either

$$n_h \equiv 1 + r_{h-1,h-1}(r_{h-1,h-1} + r_{h,h-1} - 1) \pmod 3$$

or

$$n_h \equiv 2 + r_{h-1,h-1}(r_{h-1,h-1} + r_{h,h-1} - 1) \pmod 3.$$

Using the Chinese Remainder Theorem, we can obtain in the former case

$$\begin{aligned} n_h &\equiv 3(r_{h-1,h-1} - 1) - 2[1 + r_{h-1,h-1}(r_{h-1,h-1} + r_{h,h-1} - 1)] \pmod 6 \\ &\equiv r_{h-1,h-1}[5 - 2(r_{h-1,h-1} + r_{h,h-1})] + 1 \pmod 6 \end{aligned}$$

and in the latter case

$$\begin{aligned} n_h &\equiv 3(r_{h-1,h-1} - 1) - 2[2 + r_{h-1,h-1}(r_{h-1,h-1} + r_{h,h-1} - 1)] \pmod 6 \\ &\equiv r_{h-1,h-1}[5 - 2(r_{h-1,h-1} + r_{h,h-1})] - 1 \pmod 6. \end{aligned}$$

Then $r_{h,h} \neq 0$ iff

$$n_h \equiv r_{h-1,h-1}[5 - 2(r_{h-1,h-1} - r_{h,h-1})] \pm 1 \pmod 6$$

and the later is true by definition of $\mathcal{S}_k$. □

Lemma 5. *Let $k > 1$ and $\mathcal{S}_k$ a scheme for G_k. Then for all $h > 1$*

$$\left[\!\left[s^{[n_1)}_{h,d_{h,1}} \dots s^{[n_{h-1})}_{2,d_{h,h-1}} \right]\!\right] = \left\lfloor \left[\!\left[s^{[n_1)}_{h-1,d_{h-1,1}} \dots s^{[n_{h-1})}_{1,d_{h-1,h-1}} \right]\!\right] / 3 \right\rfloor.$$

Proof. We will prove, by induction on $i = 1, \ldots, h-1$, the more general equation

$$\left[\!\left[s_{h,d_{h,1}}^{[n_1)} \ldots s_{h-i+1,d_{h,i}}^{[n_i)} \right]\!\right] = \left\lfloor \left[\!\left[s_{h-1,d_{h-1,1}}^{[n_1)} \ldots s_{h-i,d_{h-1,i}}^{[n_i)} \right]\!\right] /3 \right\rfloor .$$

For the basis $i = 1$, $d_{h,1} = d_{h-1,1} = 0$ and Eq. (9) gives

$$\left[\!\left[s_{h,d_{h,1}}^{[n_1)} \right]\!\right] = \left[\!\left[s_{h}^{[n_1)} \right]\!\right] = \left\lfloor \left[\!\left[s_{h-1}^{[n_1)} \right]\!\right] /3 \right\rfloor = \left\lfloor \left[\!\left[s_{h-1,d_{h-1,1}}^{[n_1)} \right]\!\right] /3 \right\rfloor .$$

For $i > 1$, by applying the inductive hypothesis, we can obtain

$$\begin{aligned} \left[\!\left[s_{h-1,d_{h-1,1}}^{[n_1)} \ldots s_{h-i,d_{h-1,i}}^{[n_i)} \right]\!\right] /3 &= \left\lfloor \left[\!\left[s_{h-1,d_{h-1,1}}^{[n_1)} \ldots s_{h-i-1,d_{h-1,i-1}}^{[n_{i-1})} \right]\!\right] /3 \right\rfloor 2^{n_i} + \left(r_{h-1,i-1} 2^{n_i} + \left[\!\left[s_{h-i,d_{h-1,i}}^{[n_i)} \right]\!\right] \right) /3 \\ &= \left[\!\left[s_{h,d_{h,1}}^{[n_1)} \ldots s_{h-i,d_{h,i-1}}^{[n_{i-1})} \right]\!\right] 2^{n_i} + \left(r_{h-1,i-1} 2^{n_i} + \left[\!\left[s_{h-1,d_{h-i,i}}^{[n_i)} \right]\!\right] \right) /3 \end{aligned}$$

and then, by using Formula (11),

$$\begin{aligned} \left\lfloor \left[\!\left[s_{h-1,d_{h-1,1}}^{[n_1)} \ldots s_{h-i,d_{h-1,i}}^{[n_i)} \right]\!\right] /3 \right\rfloor &= \left[\!\left[s_{h,d_{h,1}}^{[n_1)} \ldots s_{h-i,d_{h,i-1}}^{[n_{i-1})} \right]\!\right] 2^{n_i} + \left\lfloor \left(r_{h-1,i-1} 2^{n_i} + \left[\!\left[s_{h-1,d_{h-i,i}}^{[n_i)} \right]\!\right] \right) /3 \right\rfloor \\ &= \left[\!\left[s_{h,d_{h,1}}^{[n_1)} \ldots s_{h-i,d_{h,i-1}}^{[n_{i-1})} \right]\!\right] 2^{n_i} + \left[\!\left[s_{h,d_{h-i+1,i}}^{[n_i)} \right]\!\right] = \left[\!\left[s_{h,d_{h,1}}^{[n_1)} \ldots s_{h-i+1,d_{h,i}}^{[n_i)} \right]\!\right] . \quad \square \end{aligned}$$

Lemma 6. *Let $\mathcal{S}_k$ be a scheme for G_k and, for $1 \le h \le k$, let $x_h = \left[\!\left[s_{h,d_{h,1}}^{[n_1)} \ldots s_{1,d_{h,h}}^{[n_h)} \right]\!\right]$. Then $x_h \in G_h$.*

Proof. The proof is by induction on h. For the base case $h = 1$, n_1 is even and

$$x_1 = \left[\!\left[s_{1,d_{1,1}}^{[n_1)} \right]\!\right] = \left[\!\left[s_{1,0}^{[2m+2)} \right]\!\right] = \left[\!\left[s_{1}^{m+1} \right]\!\right] \in G_1 .$$

For $h > 1$ we can prove

$$x_h = \frac{x_{h-1} 2^{n_h} - 1}{3} \in R^{-1}(x_{h-1}) \subseteq G_h .$$

Indeed, by applying the inductive hypothesis, we can obtain

$$\begin{aligned} \frac{x_{h-1} 2^{n_h} - 1}{3} &= \frac{(3 \left\lfloor \frac{x_{h-1}}{3} \right\rfloor + r_{h-1,h-1}) 2^{n_h} - 1}{3} \\ &= \left\lfloor \frac{x_{h-1}}{3} \right\rfloor 2^{n_h} + \frac{r_{h-1,h-1} 2^{n_h} - 1}{3} \\ &= \left\lfloor \frac{x_{h-1}}{3} \right\rfloor 2^{n_h} + \left[\!\left[s_{1,d_{h,h}}^{[n_h)} \right]\!\right] \\ &= \left\lfloor \left[\!\left[s_{h-1,d_{h-1,1}}^{[n_1)} \ldots s_{1,d_{h-1,h-1}}^{[n_{h-1})} \right]\!\right] /3 \right\rfloor 2^{n_h} + \left[\!\left[s_{1,d_{h,h}}^{[n_h)} \right]\!\right] \\ &= \left[\!\left[s_{h,d_{h,1}}^{[n_1)} \ldots s_{2,d_{h,h-1}}^{[n_{h-1})} \right]\!\right] 2^{n_h} + \left[\!\left[s_{1,d_{h,h}}^{[n_h)} \right]\!\right] \\ &= x_h . \quad \square \end{aligned}$$

Lemma 7. *For all $x \in G_k$ there is a scheme $\mathcal{S}_k$ such that $x = \left[\!\left[s_{k,d_{k,1}}^{[n_1)} \ldots s_{1,d_{k,k}}^{[n_k)} \right]\!\right]$.*

Proof. The proof is by induction on k. For the base case $k = 1$ the proof is straightforward:

$$G_1 = \left\{ \left[\!\left[s_1^{m+1} \right]\!\right] \ : \ m \ge 0 \right\} = \left\{ \left[\!\left[s_{1,0}^{[2m+2)} \right]\!\right] \ : \ m \ge 0 \right\}$$

and so $x = \left[\!\left[s_{1,0}^{[2m+2)} \right]\!\right]$ for some $m \ge 0$. Then, choosing $\mathcal{S}_k$ with $n_1 = 2m+2$ and $d_{1,1} = 0$, we obtain $x = \left[\!\left[s_{1,d_{1,1}}^{[n_1)} \right]\!\right]$.

Let $k > 1$ and let $y \in G_{k-1}$ such that $x \in R^{-1}(y)$.

By the inductive hypothesis there is a scheme $\mathcal{S}_{k-1}$ such that

$$y = \left[\!\left[s_{k-1,d_{k-1,1}}^{[n_1)} \ldots s_{1,d_{k-1,k-1}}^{[n_{k-1})} \right]\!\right] .$$

Moreover $y \bmod 3 \ne 0$ and

$$x = \left(y 2^n - 1 \right) /3$$

where, for some $m \ge 0$, $n = 2m+2$ if $y \bmod 3 = 1$ and $n = 2m+1$ if $y \bmod 3 = 2$.

We can extend $\mathcal{S}_{k-1}$ to $\mathcal{S}_k$ by setting $n_k = n$, $d_{k,1} = 0$, $d_{k,i} = d_{k-1,i} + r_{k-1,i-1} 2 \lambda_{k-i-1}$, for $2 \le i < k$, and $d_{k,k} = r_{k-1,k-1} - 1$.

Then $x = \left[\!\left[s_{k,d_{k,1}}^{[n_1)} \ldots s_{1,d_{k,k}}^{[n_k)} \right]\!\right]$ and $x \in G_k$ by the previous lemma. $\square$

Theorem 1 (*Structure of Convergence Classes*). *$x \in G_k$ iff there exists a scheme $\mathcal{S}_k$ for G_k such that*

$$x = \left[\!\left[s^{[n_1)}_{k,d_{k,1}} \ldots s^{[n_k)}_{1,d_{k,k}} \right]\!\right]. \tag{12}$$

Proof. Immediate from the last two lemmas. □

Given a scheme $\mathcal{S}_k$ we can compute the corresponding $x \in G_k$ by Formula (12). In the reverse direction, given $x \in G_k$ we can compute the corresponding scheme $\mathcal{S}_k$ as follows. Let $x_k = x$ and, for $h = k-1, \ldots, 1$, compute $x_h = R(x_{h+1})$ and take as $n_k, n_{k-1}, \ldots, n_1$ the exponents of the power of 2 at the denominators in $R(x_{h+1})$. We can easily prove that this sequence of lengths satisfy points (a), (b) and (c) of the definition of a scheme. Then we can use points (d), (e) and (f) to compute rotations $d_{h,i}$.

For example, for $x = 27 \in G_{41}$ we obtain

i	1	2	3	4	5	6	7	8	9	10
n_i	4	5	1	1	3	4	2	2	4	1
$d_{k,i}$	0	11	107	71	47	122	650	866	1154	6155
i	11	12	13	14	15	16	17	18	19	20
n_i	1	1	3	1	1	1	1	1	2	1
$d_{k,i}$	4103	2735	1823	4859	3239	2159	1439	959	638	851
i	21	22	23	24	25	26	27	28	29	30
n_i	2	1	1	3	2	1	1	1	2	1
$d_{k,i}$	755	566	503	335	890	1187	791	527	350	467
i	31	32	33	34	35	36	37	38	39	40
n_i	1	2	1	2	2	1	1	1	1	2
$d_{k,i}$	311	206	275	182	242	323	53	35	5	2
i	41									
n_i	1									
$d_{k,i}$	1									

Notice that $s^{[n_i)}_{k,d_{k,i}} = 0^{n_i}$ but for $s^{[n_{39})}_{k,d_{k,39}} = 00011$, $s^{[n_{40})}_{k,d_{k,40}} = 01$ and $s^{[n_{41})}_{k,d_{k,41}} = 1$.

A final implementation of procedures Div3 and Div3Aux based on the scheme $\mathcal{S}_k$ (with a nice graphical interface) is described in [8] (in Italian).

References

[1] J.C. Lagarias, The $3n+1$ problem: an annotated bibliography, II (2000–2009). In http://arxiv.org/abs/math/0608208v5.
[2] Lisbeth De Mol, Tag Systems and Collatz-like functions, Theoretical Computer Science 390 (2008) 92–101.
[3] Joseph L. Pe, The $3x+1$ fractal, Computers & Graphics 28 (2004) 431–435.
[4] Jeffrey P. Dumont, Clifford A. Reiter, Visualizing generalized $3x+1$ function dynamics, Computers and Graphics 25 (2001) 883–898.
[5] Pavlos B. Konstadinidis, The real $3x+1$ problem, Acta Arithmetica 122 (2006) 35–44.
[6] Pascal Michel, Small Turing machines and generalized busy beaver competition, Theoretical Computer Science 326 (2004) 45–56.
[7] Giuseppe Scollo, ω-rewriting the Collatz problem, Fundamenta Informaticae 64 (2005) 401–412.
[8] Lorenzo Tessari, Visualizzatore Binario delle Classi di Convergenza della funzione di Collatz. Tesi di Laurea Triennale in Informatica, Dipartimento di Matematica Pura e Applicata, Università di Padova.
[9] Toshio Urata, Some holomorphic functions connected with the Collatz problem, Bulletin of Aichi University of Education (Natural Science) 51 (2002) 13–16.
[10] Jean Paul Van Bendegem, The Collatz conjecture: A case study in mathematical problem solving, Logic and Logical Philosophy 14 (1) (2005) 7–23.