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Lagrangian dynamics for classical, Brownian, and quantum mechanical particles

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In the framework of Nelson's stochastic mechanics [E. Nelson, *Dynamical Theories of Brownian Motion* (Princeton University, Princeton, 1967); F. Guerra, Phys. Rep. **77**, 263 (1981); E. Nelson, *Quantum Fluctuations* (Princeton University, Princeton, 1985)] we seek to develop the particle counterpart of the hydrodynamic results of M. Pavon [J. Math. Phys. **36**, 6774 (1995); Phys. Lett. A **209**, 143 (1995)]. In particular, a *first form* of Hamilton's principle is established. We show that this variational principle leads to the correct equations of motion for the classical particle, the Brownian particle in thermodynamical equilibrium, and the quantum particle. In the latter case, the critical process q satisfies a stochastic Newton law. We then introduce the momentum process p, and show that the pair (q,p) satisfies canonical-like equations. © *1996 American Institute of Physics.* [S0022-2488(96)03507-4]

I. INTRODUCTION

In a recent paper,[1] we established the stochastic mechanics counterpart of the second (hydrodynamic) form of Hamilton's principle. The resulting variational picture is much richer and of a different nature with respect to the one previously considered in the literature. This paper deals with the first (particle) form of Hamilton's principle. Our principle may be viewed as a strengthening of Ref. 2 (pp. 73–75) which in turn was a modification of Yasue's original work.[3] Further related work may be found in Ref. 4 (Chap. 5). We adopt kinematical variables and stochastic derivatives different from Refs. 3 and 2. The critical stochastic process is not Markovian, but becomes Markovian if we adjoin certain mean-forward and mean-backward velocities. This picture is consistent with the classical mechanical picture.

For the purpose of later reference and comparison, we outline below one of the main results of Ref. 1. Assume that the motion of a nonrelativistic, spinless particle can be described by a stochastic process $q=\{q(t);t_0\leqslant t\leqslant t_1\}$, taking the values in $\mathbb{R}^3$ and having a stochastic differential of the form

$$dq(t)=\beta(t)dt+\left(\frac{\hbar}{m}\right)^{1/2}dw_+ , \tag{1}$$

where the *forward drift* $\beta(t)$ is a measurable function of $\{q(\tau);t_0\leqslant\tau\leqslant t\}$, w_+ is a *Wiener process* with increments independent at each time of the past of q satisfying $E\{dw_+dw_+^T\}=I_3dt$. If the diffusion has finite kinetic energy

$$E\left\{\int_{t_0}^{t_1}\beta(t)\cdot\beta(t)d\tau\right\}<\infty,$$

then we also have the reverse-time representation[5]

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$$dq(t)=\gamma(t)dt+\left(\frac{\hbar}{m}\right)^{1/2}dw_-, \tag{2}$$

where γ, called the *backward drift*, depends at each time only on the future of the process q and w_- is another Wiener process whose increments are, at each time, independent of the future of the process q and satisfy $E\{dw_-dw_-^T\}=I_3dt$. When $\hbar$ tends to zero, both β and γ tend to the classical velocity. Hence, the *current velocity* $v(t):=\frac{1}{2}[\beta(t)+\gamma(t)]$ corresponds to the classical velocity, and the *osmotic velocity* $u(t):=\frac{1}{2}[\beta(t)-\gamma(t)]$ tends to zero in the semiclassical limit. In order to develop the Lagrangian and Hamiltonian formalism in stochastic mechanics in a way that naturally extends the classical case we are then naturally led[1,6] to introduce the *complex-valued* velocity (quantum velocity) $v_q(t):=v(t)-iu(t)$ that simultaneously captures $v(t)$ and $u(t)$. From Eqs. (1) and (2) we get

$$dq(t)=v(t)dt+\frac{1}{2}\left(\frac{\hbar}{m}\right)^{1/2}[dw_++dw_-], \tag{3}$$

$$0=u(t)dt+\frac{1}{2}\left(\frac{\hbar}{m}\right)^{1/2}[dw_+-dw_-]. \tag{4}$$

Multiplying the second equation by $-i$, and then adding it to the first, we finally get

$$dq(t)=[v(t)-iu(t)]dt+dw, \tag{5}$$

where

$$dw=\frac{1}{2}\left(\frac{\hbar}{m}\right)^{1/2}[(1-i)dw_++(1+i)dw_-].$$

For the properties of the *quantum noise* dw see Ref. 1 (Sec. VII). The differential (5) of q, differently from Eqs. (1) and (2), enjoys the *time reversal invariance* property, see Ref. 6.

Consider the situation where the particle is subject to an external conservative force deriving from the sufficiently regular potential $V(x)$. Let $L(x,y):=\frac{1}{2}my\cdot y-V(x)$ be the Lagrangian defined on $\mathbb{R}^3\times\mathbb{C}^3$, and let $\mathscr{V}$ denote the family of finite-energy, $\mathbb{C}^3$-valued stochastic processes on $[t_0,t_1]$. For ϕ_0 a complex-valued function on $\mathbb{R}^3$ such that $\psi_0(x):=\exp i/\hbar\,\varphi_0(x)$ has L^2 norm 1, consider the variational problem

$$\text{extremize}_{v_q\in\mathscr{V}}E\left\{\int_{t_0}^{t_1}L(x(t),v_q(t))dt+\varphi_0(x(t_0))\right\} \tag{6}$$

subject to the constraint that the finite-energy, possibly non-Markovian diffusion x has quantum velocity v_q and a prescribed probability density ρ_1 at time t_1. We then have the following result (Ref. 1, Sec. VIII).

Theorem 1: *Suppose that the solution* $\{\psi(x,t),t\in[t_0,t_1]\}$ *of the Schrödinger equation*

$$\frac{\partial\psi}{\partial t}=\frac{i\hbar}{2m}\,\Delta\psi-\frac{i}{\hbar}\,V(x)\psi, \tag{7}$$

with initial condition $\psi(x,t_0)=\psi_0(x)$ *never vanishes and satisfies*

$$E\left\{\int_{t_0}^{t_1}|\nabla\ \log\ \psi(x(t),t)|^2 dt\right\}<\infty$$

for each finite-energy diffusion x on $[t_0,t_1]$. *In Problem (6), let* $\rho_1(x)=|\psi(x,t_1)|^2$. *Then, there exists a stochastic process* $\{x^*(t);t_0\leq t\leq t_1\}$, *called the Nelson process, solving together with its quantum drift* $1/m\nabla\hbar/i\ \log\ \psi(x^*(t),t)$ *Problem (6).*

It is worthwhile to observe that the Markov property of the extremal process is a *result* of the variational principle. Also notice that the probability density $\rho(x,t)$ of $q(t)$ satisfies Born's relation $\rho(x,t)=|\psi(x,t)|^2$. The existence of the Nelson process corresponding to a given solution $\psi(x,t)$ of the Schrödinger equation is, in the general case where $\psi(x,t)$ can vanish, a challenging question that has generated considerable interest, see, e.g., Refs. 7, 4 (Chap IV), and 8, and references therein. The existence of the Nelson probability measure is established in Ref. 1 under the present assumptions by means of the Girsanov transformation theory, cf., e.g., Ref. 9 (Chap 6). The quantum Hamilton principle just recalled was also shown in Ref. 1 to be a consequence of two other variational principles of the min–max type. The first one, called the *saddle-point action principle*, contains as special cases both the Guerra–Morato variational principle[10] and Schrödinger original variational derivation of the time-independent equation, see, e.g., Ref. 11 (p. 118). The second, called the *saddle-point entropy production principle*, concerns the production of configurational entropy. The Nelson process appears then as a *saddle-point equilibrium solution* for both stochastic differential games.

In this paper, we develop the *first (particle) form* of Hamilton's principle in stochastic mechanics. We then show that this variational principle can be applied to a variety of conservative systems such as the classical particle, the Brownian particle in thermodynamical equilibrium, and the quantum particle by simply changing the family of trial motions.

The paper is outlined as follows. In Sec. II we collect some basic facts about the kinematics of stochastic processes. In Sec. III, we develop a stochastic calculus of variations. The corresponding Hamilton's principle is then applied in Sec. IV to various conservative systems. In Sec. V, we develop some basic elements of the Hamilton–Jacobi theory in stochastic mechanics.

II. BACKGROUND ON THE KINEMATICS OF STOCHASTIC PROCESSES

Let $(\Omega,\mathcal{E},\mathbf{P})$ be a complete probability space, and let $\mathcal{A}:=(\mathcal{A}_t)$, $t\in[t_0,t_1]$, be a nondecreasing family of sub σ-algebras of $\mathcal{E}$. Let $x:=\{x(t);t\in[t_0,t_1]\}$ be an $\mathbb{R}^n$-valued, second-order, $\mathcal{A}$-adapted stochastic process, namely the components of $x(t)$ are $\mathcal{A}_t$-measurable for all t in $[t_0,t_1]$. Suppose that x is a.s. and mean-square continuous. We say that x is *mean-forward differentiable* with respect to the filtration $\mathcal{A}$ if the limit

$$(D_+^{\mathcal{A}}x)(t)=\lim_{h\searrow 0} E\left\{\frac{x(t+h)-x(t)}{h}\,\middle|\,\mathcal{A}_t\right\}$$

exists for $t\in[t_0,t_1)$, and forms a continuous curve in $L_n^2(\Omega,\mathcal{E},\mathbf{P})$. In this case, it may be shown along the lines of Ref. 12 (Sec. 11) that x is a continuous semimartingale of the form

$$x(t)=x(t_0)+\int_{t_0}^{t}(D_+^{\mathcal{A}}x)(s)ds+m_+^{\mathcal{A}}(t), \tag{8}$$

where the integral is a Riemann integral in $L_n^2(\Omega,\mathcal{E},\mathbf{P})$ (Ref. 13 p. 10), and $m_+^{\mathcal{A}}$ is a square-integrable, continuous $\mathcal{A}_t$-martingale with $m_+^{\mathcal{A}}(t_0)\ =\ 0$ a.s. (for the definitions, see, e.g., Ref. 14, p. 78). Similarly, if $\mathcal{B}:=(\mathcal{B}_t)$, $t\in[t_0,t_1]$, is a nonincreasing family of sub σ-algebras of $\mathcal{E}$ to which x is adapted, we say that x is *mean-backward differentiable* with respect to $\mathcal{B}$ if the limit

$$(D_-^{\mathscr{B}}x)(t)=\lim_{h\searrow 0} E\left\{ \frac{x(t)-x(t-h)}{h} \middle| \mathscr{B}_t \right\}$$

exists for $t\in(t_0,t_1]$, and forms a continuous curve in $L_n^2(\Omega,\mathscr{E},\mathbf{P})$. In that case, x admits the backward semimartingale representation

$$x(t)=x(t_1)+\int_{t_1}^{t}(D_-^{\mathscr{B}}x)(s)ds-m_-^{\mathscr{B}}(t), \tag{9}$$

where $m_-^{\mathscr{B}}$ is a reverse-time, square-integrable, continuous, $\mathscr{B}_t$-martingale with $m_-^{\mathscr{B}}(t_1) = 0$ a.s. Notice that $D_+^{\mathscr{A}}x$ and $D_-^{\mathscr{B}}x$ depend crucially on the filtrations $\mathscr{A}$ and $\mathscr{B}$. Obviously, for x to be mean-square differentiable, $\mathscr{A}_t$ must contain $\mathscr{F}_t:=\sigma\{x(s);t_0\leqslant s\leqslant t\}$ and $\mathscr{B}_t$ must contain $\mathscr{G}_t:=\sigma\{x(s);t\leqslant s\leqslant t_1\}$. If x is mean-forward and mean-backward differentiable with respect to $\mathscr{F}:=(\mathscr{F}_t)$ and $\mathscr{G}:=(\mathscr{G}_t)$, respectively, we call $\beta(t):=(D_+^{\mathscr{F}}x)(t)$ the *forward drift* of x and $\gamma(t):=(D_-^{\mathscr{G}}x)(t)$ the *backward drift* of x. Of course, for mean-square differentiable processes, we have $\beta(t)=\gamma(t)=\dot{x}(t)$.

For stochastic processes that are simultaneously mean-forward and mean-backward differentiable with respect to the filtrations $\mathscr{A}$ and $\mathscr{B}$, we can introduce two more stochastic derivatives (using the notation introduced in Ref. 15) by

$$(D^{\mathscr{A},\mathscr{B}}x)(t):=\frac{(D_+^{\mathscr{A}}x)(t)+(D_-^{\mathscr{B}}x)(t)}{2},$$

$$(\delta D^{\mathscr{A},\mathscr{B}}x)(t):=\frac{(D_+^{\mathscr{A}}x)(t)-(D_-^{\mathscr{B}}x)(t)}{2}.$$

In particular, $v(t):=(D^{\mathscr{F},\mathscr{G}}x)(t)=(\beta(t)+\gamma(t))/2$ and $u(t):=(\delta D^{\mathscr{F},\mathscr{G}}x)(t)=(\beta(t)-\gamma(t))/2$ are the *current drift* and the *osmotic drift* of x, respectively. Representations (8) and (9) now give

$$x(t)-x(s)=\int_s^t(D^{\mathscr{A},\mathscr{B}}x)(\sigma)d\sigma+\frac{1}{2}\left[m_+^{\mathscr{A}}(t)-m_+^{\mathscr{A}}(s)+m_-\mathscr{B}(t)-m_-^{\mathscr{B}}(s)\right], \tag{10}$$

$$0=\int_s^t(\delta D^{\mathscr{A},\mathscr{B}}x)(\sigma)d\sigma+\frac{1}{2}\left[m_+^{\mathscr{A}}(t)-m_+^{\mathscr{A}}(s)-m_-\mathscr{B}(t)+m_-^{\mathscr{B}}(s)\right]. \tag{11}$$

Multiplying Eq. (11) by $-i$, and then adding it to Eq. (10), we finally get a generalization of Eq. (5),

$$x(t)-x(s)=\int_s^t((D-i\,\delta D)^{\mathscr{A},\mathscr{B}}x)(\sigma)d\sigma+m^{\mathscr{A},\mathscr{B}}(t)-m^{\mathscr{A},\mathscr{B}}(s), \tag{12}$$

where

$$m^{\mathscr{A},\mathscr{B}}(t):=\tfrac{1}{2}[(1-i)m_+^{\mathscr{A}}(t)+(1+i)m_-^{\mathscr{B}}(t)].$$

As for the diffusion processes of Sec. I, we call $v_q(t):=((D-i\,\delta D)^{\mathscr{F},\mathscr{G}}x)(t)$ the *quantum drift* of x and $dm^{\mathscr{F},\mathscr{G}}(t)$ the *quantum noise*.

Remark 1: Notice that when $((D-i\,\delta D)^{\mathscr{A},\mathscr{B}}x)=f(x(t),t)$, *$x$ is a Markov process. Indeed, it admits a forward differential given by*

$$dx=[\Re f(x(t),t)-\Im f(x(t),t)]dt+dm_+^{\mathscr{A}},$$

where $\Re$ *and* $\Im$ *denote real and imaginary part, respectively.*

We now state Nelson's product rule.

Lemma 1: Let $x,y:[t_0,t_1]\rightarrow L_n^2(\Omega,\mathscr{E},\mathbf{P})$ *be two* a.s. *and mean-square continuous stochastic processes. Suppose that* x *and* y *are simultaneously mean-forward and mean-backward differentiable with respect to the filtrations* $\mathscr{A}$ *and* $\mathscr{B}$*, respectively. Suppose, moreover, that the processes* $D_+^{\mathscr{A}}x$, $D_-^{\mathscr{B}}x$, $D_+^{\mathscr{A}}y$, *and* $D_-^{\mathscr{B}}y$ *have continuous paths. Then*

$$E\{x(t_1)\cdot y(t_1)-x(t_0)\cdot y(t_0)\}=E\left\{\int_{t_0}^{t_1}[(D_+^{\mathscr{A}}x)(t)\cdot y(t)+x(t)\cdot(D_-^{\mathscr{B}}y)(t)]dt\right\}. \tag{13}$$

Exchanging the roles of x and y in Eq. (13), adding and subtracting, we get two more formulas corresponding to Eqs. (10) and (11).

Corollary 1: Let x *and* y *be as in the previous lemma. Then*

$$E\{x(t_1)\cdot y(t_1)-x(t_0)\cdot y(t_0)\}=E\left\{\int_{t_0}^{t_1}[(D^{\mathscr{A},\mathscr{B}}x)(t)\cdot y(t)+x(t)\cdot(D^{\mathscr{A},\mathscr{B}}y)(t)]dt\right\}, \tag{14}$$

$$0=E\left\{\int_{t_0}^{t_1}[(\delta D^{\mathscr{A},\mathscr{B}}x)(t)\cdot y(t)+x(t)\cdot(\delta D^{\mathscr{A},\mathscr{B}}y)(t)]dt\right\}. \tag{15}$$

Multiplying Eq. (15) by $-i$, and then adding it to the first, we finally get a fundamental integration by parts formula related to representation (12).

Corollary 2: Let x *and* y *be as in the above lemma. Then*

$$\begin{aligned}&E\{x(t_1)\cdot y(t_1)-x(t_0)\cdot y(t_0)\}\\&\quad=E\left\{\int_{t_0}^{t_1}[((D-i\,\delta D)^{\mathscr{A},\mathscr{B}}x)(t)\cdot y(t)+x(t)\cdot((D+i\,\delta D)^{\mathscr{A},\mathscr{B}}y)(t)]dt\right\}.\end{aligned} \tag{16}$$

So far we have dealt with $\mathbb{R}^n$-valued stochastic processes. A moment's thought, however, reveals that everything we have done holds true if the processes are $\mathbb{C}^n$-valued.

We now consider the case where the process x is an n-dimensional, finite-energy Markovian diffusion with constant diffusion coefficient $\sigma^2 I_n$. We denote by $b_+(x(t),t)$ and $b_-(x(t),t)$ its forward and backward drifts, respectively. Moreover, let

$$v(x(t),t)=\frac{b_+(x(t),t)+b_-(x(t),t)}{2}$$

and

$$u(x(t),t)=\frac{b_+(x(t),t)-b_-(x(t),t)}{2}$$

denote the current and osmotic drifts. We then have Nelson's relation

$$u(x,t)=\frac{\sigma^2}{2}\,\nabla\,\log\,\rho(x,t), \tag{17}$$

where $\rho(x,t)$ is the sufficiently smooth probability density of $x(t)$ [set $u(x,t)$ equal to zero whenever $\rho(x,t)=0$]. Moreover, the Fokker–Planck equation governing the evolution of ρ

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(b_+\rho)=\frac{\sigma^2}{2}\,\Delta\rho,$$

may be rewritten as a *continuity equation* of fluid dynamics

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(v\rho)=0. \tag{18}$$

As before, let $\mathcal{F}_t:=\sigma\{x(s);t_0\leqslant s\leqslant t\}$ and $\mathcal{G}_t:=\sigma\{x(s);t\leqslant s\leqslant t_1\}$. Also let $f(x,t):\mathbb{R}^n\times[t_0,t_1]\to\mathbb{R}$ be a function with compact support of class $C^{2,1}$. From the two changes of variables formulas (Ito's rules) corresponding to the forward and backward representations for the increments of x [Eqs. (1) and (2)] we get (Ref. 12, p. 104)

$$(D_+^{\mathcal{F}}f(x(t),t))=\left[\left(\frac{\partial}{\partial t}+b_+\cdot\nabla+\frac{\sigma^2}{2}\,\Delta\right)f\right](x(t),t), \tag{19}$$

$$(D_-^{\mathcal{G}}f(x(t),t))=\left[\left(\frac{\partial}{\partial t}+b_-\cdot\nabla-\frac{\sigma^2}{2}\,\Delta\right)f\right](x(t),t). \tag{20}$$

The semisum and the semidifference of Eqs. (19) and (20) give

$$(D^{\mathcal{F},\mathcal{G}}f(x(t),t))=\left[\frac{\partial}{\partial t}+v(t)\cdot\nabla\right]f(x(t),t), \tag{21}$$

$$(\delta D^{\mathcal{F},\mathcal{G}}f(x(t),t))=\left[u(t)\cdot\nabla+\frac{\sigma^2}{2}\,\Delta\right]f(x(t),t). \tag{22}$$

We can now establish a rather interesting formula.

Lemma 2: Let $x(t)$ be a finite-energy Markovian diffusion whose never vanishing probability density ρ is of class $C^{2,1}$. Then

$$[D_+^{\mathcal{F}}(D_+^{\mathcal{F}}x)](t)-[D_-^{\mathcal{G}}(D_-^{\mathcal{G}}x)](t)=2\{[D^{\mathcal{F},\mathcal{G}}(\delta D^{\mathcal{F},\mathcal{G}}x)](t)+[\delta D^{\mathcal{F},\mathcal{G}}(D^{\mathcal{F},\mathcal{G}}x)](t)\}=0,\quad \text{a.s.} \tag{23}$$

for all $t\in[t_0,t_1]$.

Proof: From Eq. (18), we get

$$\frac{\partial \log \rho}{\partial t}=-\nabla\cdot v-v\cdot\nabla\log\rho.$$

Using Eq. (17), we get

$$\frac{\partial u}{\partial t}=-\frac{\sigma^2}{2}\,\Delta v-u\cdot\nabla v-v\cdot\nabla u. \tag{24}$$

In view of Eqs. (21) and (22), Eq. (24) can now be written as

$$(D^{\mathcal{F},\mathcal{G}}u)(t)+(\delta D^{\mathcal{F},\mathcal{G}}v)(t)=[D^{\mathcal{F},\mathcal{G}}(\delta D^{\mathcal{F},\mathcal{G}}x)](t)+[\delta D^{\mathcal{F},\mathcal{G}}(D^{\mathcal{F},\mathcal{G}}x)](t)=0, \tag{25}$$

which is Eq. (23). □

Equation (24) was derived by Nelson in his early work [Ref. 12, Eq. (5) on p. 106]. Curiously, nobody seems to have noticed the straightforward reformulation of Eq. (24) given by Eqs. (25) and (23). The corresponding hydrodynamical equation (Madelung equation) occurs in the context of the *saddle-point entropy production principle*, cf. Ref. 1, Sec. V.

III. STOCHASTIC CALCULUS OF VARIATIONS

Let $\mathscr{X}(x_0,x_1)$ denote the family of $\mathbb{R}^n$-valued, mean-square continuous processes $x=\{x(t);t\in[t_0,t_1]\}$ such that $x(t_0)=x_0$ a.s., $x(t_1)=x_1$ a.s., and satisfying the following two properties.

(i) x is mean-forward and mean-backward differentiable with respect to its past and future filtrations $\mathscr{F}=(\mathscr{F}_t)$ and $\mathscr{G}=(\mathscr{G}_t)$, respectively; we denote by $X(t)$ the augmented process

$$X(t)=\begin{pmatrix} x(t) \\ \beta(t) \\ \gamma(t) \end{pmatrix}.$$

(ii) The process X is a simultaneously mean-forward and mean-backward differentiable with respect to the filtrations $\mathbf{F}=(\mathbf{F}_t)$ and $\mathbf{G}=(\mathbf{G}_t)$, where $\mathbf{F}_t:=\sigma\{X(s);t_0\leqslant s\leqslant t\}$ and $\mathbf{G}_t:=\sigma\{X(s);t\leqslant s\leqslant t_1\}$.

Obviously, $\mathscr{F}_t\subseteq\mathbf{F}_t$ and $\mathscr{G}_t\subseteq\mathbf{G}_t$. In order to avoid any confusion, we stress the fact that, in general, $(D_+^{\mathbf{F}}x)(t)\neq(D_+^{\mathscr{F}}x)(t)=\beta(t)$ and $(D_-^{\mathbf{G}}x)(t)\neq(D_-^{\mathscr{G}}x)(t)=\gamma(t)$. Consequently, $(D^{\mathbf{F},\mathbf{G}}x)\times(t)\neq v(t)$ and $(\delta D^{\mathbf{F},\mathbf{G}}x)(t)\neq u(t)$. Also notice that $\mathscr{X}(x_0,x_1)$ contains as a proper subset the family of finite-energy, *Markovian* diffusions with the prescribed end-point marginals.

For each $x\in\mathscr{X}(x_0,x_1)$, we define the family of *variations* y of x to be the set $\mathscr{Y}(x)$ of $\mathbb{R}^n$-valued, mean-square continuous processes $y=\{y(t);t\in[t_0,t_1]\}$ satisfying the two properties: (i) $y(t_0)=y(t_1)=0$ a.s.; (ii) y is simultaneously mean-forward and mean-backward differentiable with respect to the filtrations $\mathbf{F}$ and $\mathbf{G}$, respectively.

Remark 2: Let $f:\mathbb{R}^n\times[t_0,t_1]\to\mathbb{R}^n$, *having compact support in* $\mathbb{R}^n\times(t_0,t_1)$, *be of class* $C^{2,1}$. *Then,* $y(t):=f(x(t),t)$ *belongs to* $\mathscr{Y}(x)$. *These are precisely the variations considered by Nelson in Ref. 3.*

Let $L:\mathbb{R}^n\times\mathbb{C}^n\times\mathbb{C}^n\times[t_0,t_1]\to\mathbb{C}^n$ be a sufficiently regular function. Namely, $L(x,z_1,z_2,t)$ is continuously differentiable with respect to x and t, entirely as a function of z_1 and entirely as a function of z_2. Define $I:\mathscr{X}(x_0,x_1)\to\mathbb{C}$ by

$$I(x)=E\left\{\int_{t_0}^{t_1}L(x(t),((D-i\,\delta D)^{\mathbf{F},\mathbf{G}}x)(t),((D+i\,\delta D)^{\mathbf{F},\mathbf{G}}x)(t),t)dt\right\}.$$

Notice that in this action integral appear the conditional derivatives relative to the pair of filtrations $(\mathbf{F},\mathbf{G})$ rather than with respect to $(\mathscr{F},\mathscr{G})$. Also notice that our choice of kinematical variables differs from those previously considered in the literature [Refs. 3, 2 (pp. 73–75), and 4 (Chap. 5). Let dom I denote the subset of x in $\mathscr{X}(x_0,x_1)$ such that $I(x)<\infty$.

Definition 1: The process $x\in\mathrm{dom}\,I$ *is critical for* I *if for all processes* $y\in\mathscr{Y}(x)$ *we have*

$$I(x+y)-I(x)=o(\|y\|),$$

where

$$\|y\|^2:=E\left\{\int_{t_0}^{t_1}[y(t)\cdot y(t)+(D^{\mathbf{F},\mathbf{G}}y)(t)\cdot(D^{\mathbf{F},\mathbf{G}}y)(t)+(\delta D^{\mathbf{F},\mathbf{G}}y)(t)\cdot(\delta D^{\mathbf{F},\mathbf{G}}y)(t)]dt\right\}.$$

We are now ready for the fundamental theorem of stochastic calculus of variations.

Theorem 2: *The stochastic process* $x\in\mathrm{dom}\,I$ *is critical for* I *if and only if it satisfies the Euler–Lagrange equations*

$$\left[(D+i\,\delta D)^{\mathbf{F,G}}\,\frac{\partial L}{\partial z_1}+(D-i\,\delta D)^{\mathbf{F,G}}\,\frac{\partial L}{\partial z_2}-\frac{\partial L}{\partial x}\right]=0\quad\text{a.s.}\tag{26}$$

for almost all $t\in[t_0,t_1]$.

Proof: Let $x\in\operatorname{dom}I$ be critical for I, and let $y\in\mathscr{Y}(x)$ be a variation of x. By Taylor's formula

$$I(x+y)-I(x)=o(\|y\|)+E\left\{\int_{t_0}^{t_1}\left[\frac{\partial L}{\partial x}\cdot y+\frac{\partial L}{\partial z_1}\cdot((D-i\,\delta D)^{\mathbf{F,G}}y)+\frac{\partial L}{\partial z_2}\cdot((D+i\,\delta D)^{\mathbf{F,G}}y)\right]dt\right\}.$$

Applying Eq. (16), and taking into account the fact that y vanishes at the end points, we get

$$I(x+y)-I(x)=o(\|y\|)+E\left\{\int_{t_0}^{t_1}\left[\frac{\partial L}{\partial x}-(D+i\,\delta D)^{\mathbf{F,G}}\left(\frac{\partial L}{\partial z_1}\right)-(D-i\,\delta D)^{\mathbf{F,G}}\left(\frac{\partial L}{\partial z_2}\right)\right]\cdot y\,dt\right\}.\tag{27}$$

Since the expectation in Eq. (27) must vanish for all $y\in\mathscr{Y}(x)$, it follows that both the real and the imaginary parts of

$$\left[\frac{\partial L}{\partial x}-(D+i\,\delta D)^{\mathbf{F,G}}\left(\frac{\partial L}{\partial z_1}\right)-(D-i\,\delta D)^{\mathbf{F,G}}\left(\frac{\partial L}{\partial z_2}\right)\right]$$

must vanish. Conversely, if Eq. (26) holds, x is critical because of Eq. (27). □

Corollary 3: Let $L(x,z_1,z_2,t):=\frac{1}{2}mz_1\cdot z_2-V(x)$, *where* V *is of class* C^1. *Then* $x\in\operatorname{dom}I$ *is critical for* I *if and only if the stochastic Newton law*

$$m[((D^{\mathbf{F,G}}D^{\mathbf{F,G}}-\delta D^{\mathbf{F,G}}\delta D^{\mathbf{F,G}})x)(t)]=-\nabla V(x(t))\quad\text{a.s.}\tag{28}$$

for all $t\in[t_0,t_1]$.

Proof: Notice that in this case

$$\frac{\partial L}{\partial z_1}\,(x(t),((D-i\,\delta D)^{\mathbf{F,G}}x)(t),((D+i\,\delta D)^{\mathbf{F,G}}x)(t),t)=\frac{m}{2}\,((D+i\,\delta D)^{\mathbf{F,G}}x)(t),$$

$$\frac{\partial L}{\partial z_2}\,(x(t),((D-i\,\delta D)^{\mathbf{F,G}}x)(t),((D+i\,\delta D)^{\mathbf{F,G}}x)(t),t)=\frac{m}{2}\,((D-i\,\delta D)^{\mathbf{F,G}}x)(t).$$

From Eq. (26), we get that x is critical for I if and only if

$$\frac{m}{2}\,[((D+i\,\delta D)^{\mathbf{F,G}}(D+i\,\delta D)^{\mathbf{F,G}}+(D-i\,\delta D)^{\mathbf{F,G}}(D-i\,\delta D)^{\mathbf{F,G}})x)](t)=-\nabla V(x(t)),\quad\text{a.s.}\tag{29}$$

and Eq. (28) follows. □

Remark 3: Notice that Eqs. (28) and (29) can also be written in the form

$$\frac{m}{2}\,[((D_+^{\mathbf{F}}D_-^{\mathbf{G}}+D_-^{\mathbf{G}}D_+^{\mathbf{F}})x)(t)]=-\nabla V(x(t)).\tag{30}$$

The comparison between the left-hand sides of Eqs. (29) and (30) gives the long sought probabilistic meaning for the Nelson stochastic acceleration, cf. Ref. 2, Problem 6, p. 133. Nelson's acceleration may also be viewed as the real part of $((D-i\,\delta D)^{\mathbf{F,G}}(D-i\,\delta D)^{\mathbf{F,G}})x)$ *which occurs in the global Newton's law (40) below.*

IV. HAMILTON'S PRINCIPLE

We now require that the conservative motion of a particle of mass m be critical for the action I introduced in Sec. III.

A. Classical particle

Consider a classical particle subject to an external conservative force induced by the potential function V. As trajectories we take deterministic, C^2 functions. Hence, $\mathscr{F}_t = \mathbf{F}_t = \mathscr{G}_t = \mathbf{G}_t = \{\Omega, \varnothing\}$, namely the trivial σ field. In particular, x_0 and x_1 are two points in $\mathbb{R}^3$. Moreover, $(D^{\mathbf{F},\mathbf{G}}x)(t) = \dot{x}(t)$, $(\delta D^{\mathbf{F},\mathbf{G}}x)(t) = 0$, and $(D^{\mathbf{F},\mathbf{G}}\dot{x})(t) = \ddot{x}(t)$. Thus, Corollary 2 gives that x is critical for I if and only if Newton's law

$$m\ddot{x}(t) = -\nabla V(x(t)) \tag{31}$$

is satisfied for all $t \in [t_0, t_1]$.

B. Classical particle with uncertain end points

Suppose we have a classical (C^2 trajectories) particle with uncertain initial and terminal positions. This uncertainty is described through initial and final probability densities ρ_0 and ρ_1, respectively. Let x_0 and x_1 be distributed according to ρ_0 and ρ_1, respectively. Then, for all admissible motions x we have $\mathscr{F}_t = \mathbf{F}_t = \sigma(x_0)$ and $\mathscr{G}_t = \mathbf{G}_t = \sigma(x_1)$. As before, $\mathscr{D}^{\mathbf{F},\mathbf{G}}x(t) = \dot{x}(t)$, and $\delta\mathscr{D}^{\mathbf{F},\mathbf{G}}x(t) = 0$. Thus, by Corollary 2, the stochastic process x satisfying $x(t_0) = x_0$ a.s., $x(t_1) = x_1$ a.s. with C^2 paths is critical for I if and only if Eq. (31) holds for all times with probability one.

C. Brownian particle

Consider a Brownian particle in thermodynamical equilibrium. We assume that its motion may be described by a stochastic process x with differentiable sample paths and that forms a diffusion with constant diffusion coefficient together with its derivative $\dot{x}$. Hence, we have $\dot{x}(t) = \beta(t) = \gamma(t) = \mathscr{D}^{\mathbf{F},\mathbf{G}}x$, and $\delta\mathscr{D}^{\mathbf{F},\mathbf{G}}x = 0$. Moreover, $\mathbf{F}_t = \sigma\{x(s), \dot{x}(s); t_0 \leqslant s \leqslant t\}$ and $\mathbf{G}_t = \sigma\{x(s), \dot{x}(s); t \leqslant s \leqslant t_1\}$. By Corollary 2, x is critical for I if and only if

$$m(D^{\mathbf{F},\mathbf{G}}\dot{x})(t) = -\nabla V(x(t)) \quad \text{a.s.} \tag{32}$$

for all $t \in [t_0, t_1]$. It follows, in particular, that the critical process x is such that $(x, \dot{x})$ is Markovian (see Remark 1).

Theorem 3: *The stochastic process x in the above described class is critical for I if and only if the forward drift of $\dot{x}$ is given by*

$$(D^{\mathbf{F}}\dot{x})(t) = -\lambda\dot{x}(t) - \frac{1}{m}\nabla V(x(t)), \quad \text{a.s.}, \tag{33}$$

where $\lambda = \sigma^2 m/(2kT)$ and σ^2 is the diffusion coefficient of $\dot{x}$.

Proof: By the Gibbsian postulate, the equilibrium distribution is the Maxwell–Boltzmann distribution

$$\rho(x.\dot{x}) = c\ \exp\left\{\frac{-\frac{1}{2}m\dot{x}\cdot\dot{x} - V(x)}{kT}\right\}. \tag{34}$$

Moreover, since

$$\begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}$$

is Markovian, Nelson's relation (17) between the forward and the backward drift of $\dot{x}$ yields

$$(D^{\mathbf{F},\mathbf{G}}\dot{x})(t)=(D_{+}^{\mathbf{F}}\dot{x})(t)-\frac{\sigma^2}{2}\nabla_{\dot{x}}\log\rho(x(t),\dot{x}(t)). \tag{35}$$

Equations (34) and (35) now give

$$(D^{\mathbf{F},\mathbf{G}}\dot{x})(t)=(D_{+}^{\mathbf{F}}\dot{x})(t)+\sigma^2\frac{m}{2kT}\dot{x}(t). \tag{36}$$

If x is critical, then Eqs. (32) and (36) give Eq. (33). Conversely, if the process x has forward drift of $\dot{x}$ given by Eq. (33), and has the invariant density (34), then it satisfies the Newton law (32), see Refs. 1 (p. 102) and 16. □

Remark 4: It follows, in particular, that the Markovianess of $(x,\dot{x})$, *the form of the forward drift of* $\dot{x}$ *in the Ornstein–Uhlenbeck model of physical Brownian motion (Ref. 12, Chaps. 9 and 10), and its relation to the diffusion coefficient* (Einstein's fluctuation–dissipation relation) *are consequences of the Gibbsian postulate and of the Newton law (34). Conversely, given that the forward drift of* $\dot{x}$ *lies in a certain class, necessary and sufficient conditions can be given for the particle to obey the Maxwell–Boltzmann distribution in equilibrium, see Ref. 17, Sec. III.*

D. Quantum particle

Consider a nonrelativistic, spinless quantum mechanical particle moving in a force field. As class of motions we take the subclass of $\mathscr{X}(x_0,x_1)$ of the finite-energy diffusions with constant diffusion coefficient $\sigma^2=\hbar/m$. In this case, the action is given by

$$I(x)=\int_{t_0}^{t_1}\left[\frac{1}{2}m((D-i\delta D)^{\mathbf{F},\mathbf{G}}x)(t)\cdot((D+i\delta D)^{\mathbf{F},\mathbf{G}}x)(t)-V(x(t))\right]dt. \tag{37}$$

Then x satisfies Hamilton's principle if and only if it satisfies the stochastic Newton law (28) or, equivalently, Eq. (29) which may be rewritten as follows

$$m\Re[((D-i\delta D)^{\mathbf{F},\mathbf{G}}(D-i\delta D)^{\mathbf{F},\mathbf{G}})x)](t)=-\nabla V(x(t)),\quad \text{a.s.} \tag{38}$$

Next we postulate

$$[(D^{\mathbf{F},\mathbf{G}}\delta D^{\mathbf{F},\mathbf{G}}+\delta D^{\mathbf{F},\mathbf{G}}D^{\mathbf{F},\mathbf{G}})x](t)\equiv 0. \tag{39}$$

Putting together Eqs. (39) with (38), we get

$$m[((D-i\delta D)^{\mathbf{F},\mathbf{G}}(D-i\delta D)^{\mathbf{F},\mathbf{G}})x)](t)=-\nabla V(x(t)),\quad \text{a.s.} \tag{40}$$

Assumption (39) simply means that the acceleration in the left-hand side of Eq. (40) must be real. Also notice that Eq. (39) is precisely Eq. (23) for the position process x. Finally notice that the extremizing process x is such that the augmented process

$$\begin{pmatrix} x(t) \\ ((D-i\delta D)^{\mathbf{F},\mathbf{G}}x)(t) \end{pmatrix} \tag{41}$$

is Markovian. To see this, recall Remark 1 and observe that

$$\left((D-i\,\delta D)^{\mathbf{F,G}}\begin{pmatrix} x(t) \\ ((D-i\,\delta D)^{\mathbf{F,G}}x) \end{pmatrix}\right)(t)=\begin{pmatrix} ((D-i\,\delta D)^{\mathbf{F,G}}x)(t) \\ -\dfrac{1}{m}\,\nabla V(x(t)) \end{pmatrix}.$$

The process x by itself, however, is in general *non Markovian*.

We are now ready to introduce the *momentum process* corresponding to such a process by

$$p(t):=\frac{\partial L}{\partial z_2}\,(x(t),((D-i\,\delta D)^{\mathbf{F,G}}x)(t),((D+i\,\delta D)^{\mathbf{F,G}}x)(t),t)=m((D-i\,\delta D)^{\mathbf{F,G}}x)(t),$$

$$\overline{p(t)}:=\frac{\partial L}{\partial z_1}\,(x(t),((D-i\,\delta D)^{\mathbf{F,G}}x)(t),((D+i\,\delta D)^{\mathbf{F,G}}x)(t),t)=m((D+i\,\delta D)^{\mathbf{F,G}}x)(t).$$

Then Eq. (40) reads

$$((D-i\,\delta D)^{\mathbf{F,G}}p)(t)=-\nabla V(x(t)),\text{a.s.} \tag{42}$$

or equivalently

$$((D+i\,\delta D)^{\mathbf{F,G}}\overline{p)}(t)=-\nabla V(x(t)),\text{a.s.} \tag{43}$$

Let $H(x,y):=(1/2m)y\cdot y+V(x)$ be the *Hamiltonian function* defined on $\mathbb{R}^3\times\mathbb{C}^3$, and write $q(t)$ instead of $x(t)$ for the position of the quantum particle. We then get the *canonical-like equations*

$$((D-i\,\delta D)^{\mathbf{F,G}}q)(t)=\nabla_y(q(t),p(t)), \tag{44}$$

$$((D-i\,\delta D)^{\mathbf{F,G}}p)(t)=-\nabla_x(q(t),p(t)), \tag{45}$$

or equivalently

$$((D+i\,\delta D)^{\mathbf{F,G}}q)(t)=\nabla_y H(q(t),\overline{p(t)}), \tag{46}$$

$$((D+i\,\delta D)^{\mathbf{F,G}}\overline{p)}(t)=-\nabla_x H(q(t),\overline{p(t)}). \tag{47}$$

The closest in spirit previous attempt to define the momentum process within stochastic mechanics is Ref. 18. See Ref. 2, pp. 95–98 and Ref. 4, pp. 117–119] for further work and discussion on this topic.

We close the section with a comment. In Ref. 19 (p. 110), Bohm and Hiley write concerning Nelson's stochastic acceleration: ''*If it could be made clear that this definition is physically or kinematically plausible then Nelson's approach would evidently have an important advantage*.'' As observed in Remark 3, the Nelson acceleration may be viewed as the real part of the second-order stochastic derivative $((D-i\,\delta D)^{\mathbf{F,G}}(D-i\,\delta D)^{\mathbf{F,G}})x)$ which occurs in Eq. (40).

In Sec. V, we show that indeed the Nelson process associated with a particular solution of the Schrödinger equation satisfies the global Newton's law (40). Hence, we feel that the results of this paper, together with Refs. 12, 15, 2, 1, 6, clearly demonstrate the physical and kinematical plausibility of Nelson's acceleration.

V. ELEMENTS OF HAMILTON–JACOBI THEORY

Following Ref. 6, we now develop the basic elements of a Hamilton–Jacobi theory of stochastic mechanics (see Ref. 15, Sec. 1 for a beautiful account of the classical theory). Suppose $\{\psi(x,t);t_0\leq t\leq t_1\}$ is a never vanishing solution of the *Schrödinger equation*

$$\frac{\partial \psi}{\partial t} = \frac{i\hbar}{2m} \nabla \psi - \frac{i}{\hbar} V(x) \psi.$$

Then $S_q(x,t) := \hbar / i \log \psi(\mathrm{x,t})$ satisfies

$$\frac{\partial S_q}{\partial t} + \frac{1}{2m} \nabla S_q \cdot \nabla S_q + V(x) - \frac{i\hbar}{2m} \Delta S_q = 0. \tag{48}$$

This is the *Hamilton–Jacobi equation* of stochastic mechanics. Indeed, we can now rephrase Theorem 1 as follows. Suppose $\{S_q(x,t); t_0 \leqslant t \leqslant t_1\}$ solves Eq. (48) with the initial condition $S_q(x,t_0) = \phi_0(x)$, and satisfies

$$E\left\{ \int_{t_0}^{t_1} |\nabla S_q(x(t),t)|^2 dt \right\} < \infty$$

for all finite-energy diffusions x on $[t_0, t_1]$. Let $\rho_1(x) = |\exp(\mathrm{i}/\hbar) \mathrm{S_q(x,t_1)}|^2$. Then, there is a stochastic process $\{q(t); t_0 \leqslant t \leqslant t_1\}$, called the Nelson process, solving together with its quantum drift $(1/m)\nabla S_q(q(t),t)$ Problem (6). Corresponding to such an S_q, we define the *momentum field* by $p(x,t) = \nabla S_q(x,t)$, and the momentum process by $p(t) := p(q(t),t) = \nabla S_q(q(t),t)$. In Ref. 6 it was shown that the process $p(t)$ has the same first and second moments as the quantum momentum operator. It was also shown that the uncertainty relations admit a simple stochastic interpretation in terms of the pair $(q(t), p(t))$.

Theorem 4: *The pair* $(q(t), p(t))$ *satisfies the stochastic Hamilton equations (44) and (45).*

Proof: Let us first notice that $(D_+^{\mathbf{F}} q)(t) = (D_+^{\mathscr{F}} q)(t) = b_+(q(t),t)$. Indeed, since $q(t)$ is Markovian, so is

$$X(t) = \begin{pmatrix} q(t) \\ b_+(q(t),t) \\ b_-(q(t),t) \end{pmatrix}.$$

We then have

$$\begin{aligned}
(D_+^{\mathbf{F}} q)(t) &= \lim_{h \searrow 0} E\left\{ \frac{q(t+h) - q(t)}{h} \middle| \mathbf{F}_t \right\} \\
&= \lim_{h \searrow 0} E\left\{ \frac{q(t+h) - q(t)}{h} \middle| \begin{pmatrix} q(t) \\ b_+(q(t),t) \\ b_-(q(t),t) \end{pmatrix} \right\} \\
&= \lim_{h \searrow 0} E\left\{ \frac{q(t+h) - q(t)}{h} \middle| q(t) \right\} \\
&= (D_+^{\mathscr{F}} q)(t) = b_+(q(t),t).
\end{aligned}$$

Similarly, we get $(D_-^G q)(t) = (D_-^C q)(t) = b_-(q(t),t)$. Hence, $(D^{\mathbf{F,G}} q)(t) = (D^{\mathscr{F},\mathscr{G}} q)(t) = v(q(t),t)$ and $(\delta D^{\mathbf{F,G}} q)(t) = (\delta D^{\mathscr{F},\mathscr{G}} q)(t) = u(q(t),t)$. We then have $((D - i\,\delta D)^{\mathbf{F,G}}) q)(t) = ((D - i\,\delta D)^{\mathscr{F},\mathscr{G}}) q)(t) = v(q(t),t) - iu(q(t),t) = v_q(q(t),t) = (1/m)\nabla S_q(q(t),t) = \nabla_y H(q(t), p(q(t),t)) = \nabla_y H(q(t), p(t))$. To prove Eq. (45), recall from Ref. 1, Sec. VII that if $\phi(x,t)$ is a complex-valued function with sufficiently regular real and imaginary parts, then

$$d[\phi(q(t),t)] = \left[\frac{\partial}{\partial t} + v_q(q(t),t) \cdot \nabla - \frac{i\hbar}{2m} \Delta \right] \phi(q(t),t) dt + \nabla \phi(q(t),t) \cdot dw, \tag{49}$$

where $dw = dq - v_q(q(t),t)dt$ is the quantum noise corresponding to q. Applying Eq. (49) to $p(q(t),t) = \nabla S_q(q(t),t)$, we get

$$d[p(q(t),t)]=\left[\frac{\partial}{\partial t}+v_q(q(t),t)\cdot\nabla-\frac{i\hbar}{2m}\,\nabla\right]\nabla S_q(q(t),t)dt+D(q(t),t)dw, \tag{50}$$

where the 3×3 matrix D has ijth entry $d_{ij}(q(t),t)=(\partial^2/\partial x_i\partial x_j)S_q(q(t),t)$. Replacing $v_q(q(t),t)$ in Eq. (50) with $(1/m)\nabla S_q(q(t),t)$, and then employing Eq. (48), we get

$$d[p(q(t),t)]=-\nabla V(q(t))dt+D(q(t),t)dw. \tag{51}$$

Hence, the quantum drift of $p(t)$ is $-\nabla V(q(t))$ and Eq. (45) holds. □

By the same procedure, we can handle more general (sufficiently regular) Hamiltonian functions $H(x,y,t)$ if $S_q(x,t)$ now satisfies

$$\frac{\partial S_q}{\partial t}+H(x,\nabla S_q,t)-\frac{i\hbar}{2m}\,\Delta S_q=0,$$

and if we can construct a Markov process q with quantum drift

$$v_q(q(t),y)=\nabla_y H(q(t),\nabla S_q(q(t),t),t),$$

and prescribed initial condition.

We now isolate a crucial step in the proof of Theorem 4. In view of Eq. (49), define the *quantum acceleration field* by the substantial derivative

$$a_q(x,t):=\left[\frac{\partial}{\partial t}+v_q(x,t)\cdot\nabla-\frac{i\hbar}{2m}\,\Delta\right]v_q(x,t), \tag{52}$$

where $v_q(x,t)=(1/m)\nabla S_q(x,t)$. Using Eq. (48) in Eq. (52), we finally get

$$a_q(x,t)=-\frac{1}{m}\,\nabla V(x). \tag{53}$$

Equation (53) is the local form counterpart of Eq. (40).

Remark 5: Let $\{\psi(x,t);t_0\leqslant t\leqslant t_1\}$ *be a never vanishing solution of the Schrödinger equation satisfying Carlen's finite-energy condition (Ref. 7). Then the corresponding Nelson process satisfies Eq. (40) with end points distributed according to* $\rho_0(x)=|\psi(x,t_0)|^2$ *and* $\rho_1(x)=|\psi(x,t_1)|^2$.

VI. DISCUSSION

In this paper, we have developed a particle form of Hamilton's principle. We have then applied the principle to various conservative systems only changing the class of admissible motions. In the case of a quantum particle, we have seen that the critical process x satisfies the stochastic Newton law (40). This process is not Markovian, but the corresponding augmented process (41) is Markovian.

In Ref. 1, see also the outline in Sec. I, we have developed the second, hydrodynamic version of Hamilton's principle in the context of stochastic mechanics. The critical process q is there Markovian. Indeed, it is the Nelson process. Introducing the momentum field, and then the momentum process p as in Sec. V, we have obtained a pair of stochastic processes satisfying the stochastic Hamilton equations (44) and (45).

If we agree that in a deterministic context Markovian means ''satisfies a first-order differential equation,'' we see that the similarity with classical mechanics is striking. Much remains to be done, however, to develop a satisfactory Lagrangian and Hamiltonian formalism in stochastic mechanics even in the simplest case considered in this paper.

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