Università degli Studi di Padova

DIPARTIMENTO DI MATEMATICA "TULLIO LEVI-CIVITA"
Corso di Dottorato di Ricerca in Scienze Matematiche
Curriculum Mathematics
XXXV Ciclo

Tesi di dottorato di ricerca

Gap Phenomena in Optimal Control with State Constraints

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Acknowledgements

Mi arrogo la libertà di scrivere i ringraziamenti nella mia lingua madre, così da potermi esprimere al meglio. Infatti, una cosa di cui mi sono convinto in questi anni di approccio alla Matematica, è che essa è Scienza Libera. Libera da istituzioni politiche e religiose, così come da costrizioni economiche, etiche, sociali e culturali. Le uniche cose che vincolano la Matematica sono il formalismo e il ragionamento logico, ma è proprio in ciò che consiste la perfetta Libertà. Perché la Libertà non viene esercitata bene quando si dice o si fa ciò che si vuole, ma dicendo ciò che è Vero, e facendo ciò che è Giusto. Semmai, si può discutere sul Giusto, mentre del Vero si occupa la Matematica, che diventa così piena espressione di Libertà.

Ringrazio innanzitutto la mia supervisora per il sostegno che mi ha dato in questo percorso, nonché per i numerosi stimoli e problemi di ricerca a cui mi ha sottoposto, contribuendo in larga parte alla mia crescita come matematico e ad un rinvigorito amore per questa materia. Infatti, ho potuto sperimentare che, al fine di raggiungere la gioia della scoperta matematica, sono indispensabili la determinazione di uno sportivo, la pazienza di un asceta, la precisione di un chirurgo, la passione di un innamorato.

Voglio rivolgere un pensiero a tutti gli amici che hanno contribuito a rallegrare le mie giornate, non aggiungendo ulteriori problemi alla mia vita oltre a quelli matematici che ho dovuto affrontare nel mio lavoro quotidiano.

Infine, un grazie particolare va alla mia famiglia, sia quella attuale che quella di origine. Mia moglie, i miei genitori, mio fratello, le nonne, gli zii e i cugini tutti, sono le persone che mi sono affianco da sempre e su cui so di poter contare, soprattutto nelle difficoltà. Infatti, in questi tre anni sono cambiate moltissime cose: sono dapprima diventato marito e, successivamente, padre di un bambino che ha portato gioia a tutti noi. Tutto ciò non sarebbe stato possibile senza di te, Arianna, mia spalla di vita, insostituibilmente mio primo pensiero e mia prima intenzione.

Abstract

When existence of minimizers of an optimal control problem is not guaranteed, it is a common practice in Control Theory to extend the set of admissible solutions, so that to construct an auxiliary optimization problem that admits minimizers. The first fundamental requirement of such an auxiliary problem for it to be well posed is the density (e.g. in the L^∞-norm) of the set of trajectories of the original system into that of the auxiliary one. Nevertheless, due to the presence of constraints, it might happen that the minimum of the auxiliary problem is strictly smaller than the infimum of the original one. We refer to this phenomenon as *infimum gap*.

In the literature, sufficient conditions for no gap are sometimes expressed in terms of normality of the sets of multipliers of the Maximum Principle. However, in the common situation of active state constraints at the initial point, there always exist degenerate – consequently, abnormal – sets of multipliers.

In this thesis, we establish a gap-abnormality relation for a general auxiliary problem that comprehends as special cases both the compactification of the control set and the convexification of the dynamics in a novel unified framework. Furthermore, we provide refined no infimum gap conditions in order to deal with the presence of state constraints. In particular, under a suitable constraint qualification condition, we prove that if the minimizer of the auxiliary problem is a nondegenerate normal extremal, i.e. it is normal in the subset of nondegenerate multipliers only, then there is no infimum gap.

We highlight the relevance and novelties of our results with several examples, and we analyze in detail the special case of control-polynomial impulsive optimization problems.

Contents

Chapter 1

Introduction

How can I go from Earth to Moon with the lowest fuel consumption? How can I water and expose to artificial light an indoor plant cultivation with a view to maximizing its growth? What is the ideal trajectory a Formula 1 driver should follow in order to carry out a fast lap? What is the most efficient advertising investment campaign for a firm aiming to launch a cutting edge product?

All these questions might appear very different from each other. However, they share a common denominator, that is, the possibility for an external agent – commonly named *controller* – to influence an underlying dynamical system by means of his own action – usually called *control* – in order to optimize a given performance criterion. Indeed, all the situations portrayed above can be modeled in terms of the minimization of an assigned cost functional over control-trajectory pairs satisfying certain constraints. For instance, in the example of the Formula 1 driver we have a minimum time problem, where the control is the action the driver implements by means of the accelerator, the gear and the steering wheel – so that the dynamics is completely determined by the laws of physics –, the initial and the final point of the trajectory must belong to the starting line of the circuit, and the trajectories are constrained to remain inside track limits.

This is precisely the paradigm behind Optimal Control Theory, a branch of Mathematical Analysis born in the middle of the previous century (see [15, 63]) with an attempt to generalize problems in Calculus of Variations. Initially, Optimal Control Theory was utilized to address problems arising in scheduling and the control of engineering devices. Now, seventy years on, there are several applications in very different areas such as epidemiology, medicine, agriculture, robotics, environmental sciences, logistics and marketing.

A general example of (fixed end-time) optimal control problem is given by the following

$$
(P)\begin{cases}
\qquad\qquad \text{Minimize } \Phi(x(0),x(T)) \\
\text{over measurable functions } \alpha:[0,T]\to\mathbb{R}^m \text{ and absolutely continuous} \\
\text{arcs } x:[0,T]\to\mathbb{R}^n \text{ satisfying} \\
\dot{x}(t)=F(t,x(t),\alpha(t)) \qquad \text{a.e. } t\in[0,T], & (1.1)\\
\alpha(t)\in A \qquad \text{a.e. } t\in[0,T], & (1.2)\\
(x(0),x(T))\in\mathcal{C}, & (1.3)\\
\psi(t,x(t))\le 0 \qquad \text{for all } t\in[0,T], & (1.4)
\end{cases}
$$

where the data comprise the *cost function* $\Phi:\mathbb{R}^{n+n}\to\mathbb{R}$, the *dynamics function* $F:\mathbb{R}^{1+n+m}\to\mathbb{R}^n$, the *state constraint function* $\psi:\mathbb{R}^{1+n}\to\mathbb{R}$, the *control set* $A\subset\mathbb{R}^m$ and the *target* $\mathcal{C}\subset\mathbb{R}^{n+n}$.

We refer to an element (α,x) as *process* if it satisfies the controlled differential equation (1.1) and the control constraint (1.2). In this case, we refer to α as *control*, and to x as *trajectory*. A process (α,x) is said to be *feasible* if in addition it satisfies the endpoint constraint (1.3) and the state constraint (1.4). A feasible process $(\bar{\alpha},\bar{x})$ is called *minimizer* if

$$\Phi(\bar{x}(0),\bar{x}(T))\le\Phi(x(0),x(T))$$

for any feasible process (α,x).

Once introduced this formal apparatus, a natural question arises: is there a method that allows us to identify minimizers? A partial answer is provided by the celebrated Pontryagin's Maximum Principle (a seminal version of it can be found in [63]), namely, the set of necessary conditions that have to be satisfied by every minimizer and that can be seen as the generalization of the Lagrange multipliers rule to infinite dimension.

Referring the reader to Theorem 2.5.7 below for the exact statement in the general free end-time setting, the Maximum Principle can be roughly declared as follows: assuming smooth data, every minimizer $(\bar{\alpha},\bar{x})$ for problem (P) turns out to be an *extremal* for (P), i.e. there exist an absolutely continuous path $p:[0,T]\to\mathbb{R}^n$, a real number $\lambda \geq 0$, and a real-valued measure μ defined on the Borel subsets of $[0,T]$, fulfilling the following conditions

$$\|p\|_{L^\infty} + \lambda + \mu([0,T]) \neq 0 \tag{1.5}$$

$$-\dot{p}(t) = \Big(p(t) + \int_{[0,t[} \nabla_x \psi(t', \bar{x}(t'))\mu(dt')\Big) \cdot \nabla_x F(t, \bar{x}(t), \bar{\alpha}(t)) \quad \text{a.e. } t \in [0,T] \tag{1.6}$$

$$\Big(p(0), -p(T) - \int_{[0,T]} \nabla_x \psi(t, \bar{x}(t))\mu(dt)\Big) \in \lambda \nabla \Phi(\bar{x}(0), \bar{x}(T)) + N_{\mathcal{C}}(\bar{x}(0), \bar{x}(T)) \tag{1.7}$$

$$\begin{aligned}&\Big(p(t) + \int_{[0,t[} \nabla_x \psi(t', \bar{x}(t'))\mu(dt')\Big) \cdot F(t, \bar{x}(t), \bar{\alpha}(t)) \\ &\quad = \max_{a \in A} \Big\{\Big(p(t) + \int_{[0,t[} \nabla_x \psi(t', \bar{x}(t'))\mu(dt')\Big) \cdot F(t, \bar{x}(t), a)\Big\} \quad \text{a.e. } t \in [0,T]\end{aligned} \tag{1.8}$$

$$\operatorname{spt}(\mu) \subset \{t \in [0,t] : \psi(t, \bar{x}(t)) = 0\}, \tag{1.9}$$

where $N_{\mathcal{C}}$ denotes the Clarke's limiting normal cone (see Subsection 2.4.1 below for the precise definition) and $\operatorname{spt}(\mu)$ designates the support of the measure μ.

The Maximum Principle represents a powerful tool in order to isolate candidates of optimality. Despite this, it might become of no use if the optimal control problem under consideration does not admit minimizers. Consider, for instance, the following classic optimization problem without right endpoint and state constraints

$$\begin{cases} \qquad\qquad \text{Minimize} \quad (x^2(1))^2 \\ \text{over measurable } \alpha : [0,1] \to \mathbb{R}, \text{ absolutely continuous } x : [0,1] \to \mathbb{R}^2 \text{ satisfying} \\ (\dot{x}^1(t), \dot{x}^2(t)) = (\alpha(t), (x^1(t))^2) \qquad \text{a.e. } t \in [0,1] \\ \alpha(t) \in \{-1, 1\} \quad \text{a.e. } t \in [0,1] \\ (x^1(0), x^2(0)) = (0,0). \end{cases}$$

It is very easy to see that there are no feasible processes (α, x) with final cost $(x^2(1))^2$ equal to 0. Indeed, if such a feasible process existed, one immediately would get

$$0 = x^2(1) - x^2(0) = \int_0^1 (x^1(t))^2 dt \ \Rightarrow\ x^1 \equiv 0 \ \Rightarrow \dot{x}^1 = \alpha \equiv 0.$$

A contradiction, since the control set is $\{-1, 1\}$. In spite of that, let us consider the sequence of highly oscillatory controls (α_i) defined by

$$\alpha_i(t) := \begin{cases} 1 & \text{if } \sin(it) \geq 0 \\ -1 & \text{otherwise.} \end{cases}$$

It is straightforward that the sequence (x_i^1) of absolutely continuous functions satisfying $\dot{x}_i^1 = \alpha_i$ and $x_i^1(0) = 0$ converges uniformly to 0. As a consequence of the Dominated Convergence Theorem (see Theorem 2.2.4), the sequence (x_i^2) of absolutely continuous functions enjoying $\dot{x}_i^2 = (x_i^1)^2$ and $x_i^2(0) = 0$ satisfies

$$x_i^2(1) = \int_0^1 (x_i^1(t))^2 dt \to 0,$$

Therefore, the optimization problem under consideration has no feasible processes with final cost equal to 0, but we managed to construct a minimizing sequence to 0.

In Optimal Control Theory, when an optimization problem (P) does not admit minimizers, it is a common practice to enlarge the set of admissible solutions, so that to construct an *auxiliary* optimal control problem (P_a) for which the existence of minimizers is guaranteed. The first fundamental characteristic that is required of such an extension for it to be well posed is some kind of density of the set of processes of the original problem into the set of processes of the auxiliary one, as for instance the L^∞-density between the sets of trajectories. This leads us to introduce two well-known auxiliary problems present in the literature.

On the one hand, we have the *extended* problem, where a bounded but possibly not closed control set A is replaced by its closure. This auxiliary problem is legitimized by a reparameterization technique utilized to compactify the control set of control-affine systems with unbounded controls, in view of the graph completion approach [6, 21, 22, 23, 24, 50, 51, 56]. In such a way, the set of solutions is enlarged by admitting discontinuous trajectories with jumps, usually called *impulses*, hence the name of impulsive systems. The motivation behind this is provided by aerospace applications, in which the dynamics is nonlinear in the state variable and affine in the control one, and where an impulse control is an idealization of high intensity control action over a time interval of short duration. In this perspective, discontinuous trajectories can be identified as punctual limits of sequences of ordinary state trajectories. We refer the reader to [9, 10, 41, 42, 43] for an equivalent approach to impulsive systems based on a discontinuous time change.

On the other hand we have the *relaxed* problem, where the possibly non-convex dynamical system (1.1)-(1.2) is replaced by its convexification, namely, the following control system

$$\begin{cases} \dot{x}(t) = \sum_{j=0}^{n} \gamma^j(t) F(t, x(t), \alpha^j(t)) & \text{a.e. } t \in [0,T], \\ (\alpha^0(t), \dots, \alpha^n(t)) \in A^{n+1} & \text{a.e. } t \in [0,T], \\ (\gamma^0(t), \dots, \gamma^n(t)) \in \Delta_n & \text{a.e. } t \in [0,T], \end{cases} \tag{1.10}$$

where Δ_n denotes the n-dimensional symplex (see Section 2.1 below). Indeed, the well-known Relaxation Theorem (see Theorem 2.3.5) ensures that the set of trajectories of (1.1)-(1.2) is L^∞-dense into that of (1.10). Moreover, the convexity of the dynamics is one of the crucial assumption in order to guarantee existence of minimizers (see Theorem 2.5.4). Not less important is the interpretation we can give to the relaxed system: it is in fact acclaimed that it is equivalent to an "extension in measure" of the original system, where the controller plays probability measures on the control set instead of ordinary control functions (see [16, 77]).

However, even though the density requisite is satisfied, it might happen that the minimum of the auxiliary problem is strictly smaller than the infimum of the original one, namely

$$\min_{(\tilde{\alpha},\tilde{x}) \text{ feasible process for } (\mathrm{P}_a)} \Phi(\tilde{x}(0), \tilde{x}(T)) \; < \; \inf_{(\alpha,x) \text{ feasible process for } (\mathrm{P})} \Phi(x(0), x(T)).$$

This phenomenon, which we refer to as *infimum gap*, occurs due to the presence of constraints, since it takes place when an L^∞-neighborhood of the minimizing trajectory of (P_a) contains no feasible trajectories of (P). Thus, the absence of an infimum gap is closely related to the *controllability* of the constrained original system to a feasible trajectory of the auxiliary one, by which we mean the possibility to approximate in the L^∞-norm a feasible trajectory of (P_a) with feasible trajectories of (P) (see [45, 79] and references therein).

For numerical purposes, it is of obvious interest to have sufficient conditions that make certain the absence of an infimum gap. Indeed, in case of no infimum gap one first looks for minimizers of the auxiliary problem and then, by means of approximation techniques, finds ε-minimizers for the original problem, that is, feasible processes $(\alpha_\varepsilon, x_\varepsilon)$ for (P) such that

$$\Phi(x_\varepsilon(0), x_\varepsilon(T)) \leq \inf\{\Phi(x(0), x(T)) : (\alpha, x) \text{ feasible process for (P)}\} + \varepsilon.$$

The presence of an infimum gap renders also the method of dynamic programming inapplicable, since the solution of the corresponding Hamilton-Jacobi equation typically coincides with the value function of the relaxed (extended) problem.

In his seminal works [76, 77, 79], Warga pioneered research on this topic, establishing that when a minimizer for (P_a) is a *normal* extremal for (P_a), by which we mean that $\lambda > 0$ for any set of multipliers (p, λ, μ) satisfying conditions (1.5)–(1.9) for problem (P_a) [1], then there is no infimum gap. These results were subsequently refined by Palladino and Vinter [60, 61] for the relaxation of a state constrained fixed end-time optimization problem (see also [72] for the relaxation of free end-time problems without state constraints), and by other authors [58, 37] for the impulsive extension of control-affine optimization problems (see also [52] for higher order normality conditions). It is worth to mention also [59] where an abstract auxiliary problem is considered, but only for smooth data and no state constraints.

Apart from the different auxiliary problem under consideration, all the above mentioned results vary for the techniques employed in the proofs, hence for the set of necessary conditions they derive. Indeed, on the one hand, we notice a variational approach based on the Ekeland's Variational Principle, that leads to the Maximum Principle expressed in terms of Clarke's subdifferentials, on the other hand, we observe a "geometric control" approach that leans on the construction of approximating cones and reachable sets and on set separation arguments, that provides necessary conditions that make use of suitable cones. To the former group belong [60, 61, 58, 37], to the latter one [59, 52], where the authors utilize "Quasi Differential Quotient" approx-

[1] When we write that '(p, λ, μ) satisfies (1.5)–(1.9) for (P_a)' we mean that conditions (1.6) and (1.8) strongly depend on the dynamics function and on the control set, so that they have to be adapted to the particular form of the dynamics and control set of (P_a). For instance, if (P_a) coincide with the relaxed problem (see (1.10)), then in conditions (1.6), (1.8) we have to replace $F(t, x, \alpha)$ with $\sum_{j=0}^{n} \gamma^j F(t, x, \alpha^j)$ and the control set A in (1.8) has to be substituted by $A^{n+1} \times \Delta_n$.

imating cones, a generalization of the classical Boltyanski Cone that can be seen as special case of Sussmann's Approximate Generalized Differential Quotient cone (see [68]). In the middle between these two approaches we find [79]. Indeed, Warga adopts a smooth perturbation technique that depends on the construction of "Derivative containers" defined in [78].

With respect to the literature, in this thesis we analyze an auxiliary optimization problem that comprehends both a general extension and the relaxation in a novel unified framework. On the one hand, we include all the previous results in the context of the variational approach and, on the other hand, we generalize them, in a way that allows us to deal with optimal control problems of great relevance in many applications, as the control-polynomial impulsive ones, analyzed in Section 5.2.

The key idea – first introduced in [38] and then utilized in [39, 40] – is to consider a pair of control functions (ω, u) in place of α, where initially only ω is extended, while subsequently both are relaxed. The different role played by these controls is emphasized by the assumptions made on the dynamics function, as we need to demand an additional uniformly continuity of F and its generalized gradient with respect to the w-variable. For instance, this requirement is automatically met by dynamics with a control-polynomial dependence on w and coefficients that are Lipschitz continuous with respect to both the time and the state variables (see the proof of Theorem 5.2.5 below).

Momentarily ignoring precise hypotheses on data, that in this thesis are even more minimal than in [38, 40], the core of our first main outcome can be stated as follows (see Theorem 3.1.7).

First main result. *Take a minimizer for the auxiliary optimization problem* (P_a) *associated with* (P)*. If that minimizer is a normal extremal for* (P_a)*, then there is no infimum gap.*

However, in presence of state constraints, this theorem can not be regarded fully satisfactory, because there are situations in which abnormal sets of multipliers always exist. For instance, this is the case of initial active state constraint. Indeed, if $\mathcal{C} = \{\check{z}_0\} \times \tilde{\mathcal{C}}$ for some $\check{z}_0 \in \mathbb{R}^n$ and $\tilde{\mathcal{C}} \subset \mathbb{R}^n$, and in addition it holds $\psi(0, \check{z}_0) = 0$, then any feasible process turns out to be an extremal for (P) with set of *degenerate* multipliers

(p, λ, μ) defined by

$$p \equiv -\nabla_x \psi(0, \check{z}_0)\mu(\{0\}), \qquad \lambda = 0, \qquad \mu \equiv \mu(\{0\}) \neq 0. \tag{1.11}$$

In this case, the Maximum Principle becomes of no use not only to identify gap phenomena, but also to select minimizers, precisely because it is not possible to distinguish between minimizers and feasible processes only, as both are extremals. For this reason, in the literature there has been an extensive effort aimed to come up with sufficient conditions to avoid presence of sets of degenerate multipliers. In particular, several authors [2, 3, 7, 8, 32, 33, 34, 35, 37, 47, 57, 62] managed to prove that, under suitable nondegeneracy hypotheses expressed in terms of inward pointing conditions at the initial point or constraint qualifications involving the optimal process, every minimizer $(\bar{\alpha}, \bar{x})$ for (P) turns out to be a *nondegenerate* extremal for (P), by which we mean that there exists a set of multipliers (p, λ, μ) fulfilling relations (1.5)–(1.9) and, in addition, the following strengthened nontriviality condition

$$\|q\|_{L^\infty} + \lambda + \mu(]0, T]) \neq 0, \tag{1.12}$$

where $q(t) = p(t) + \int_{[0,t[} \nabla_x \psi(t', x(t'))\mu(dt')$ for a.e. $t \in [0, T]$. Indeed, it is immediate to see that sets of multipliers as in (1.11) do not satisfy (1.12).

The second novelty of our research has been to come up with sufficient conditions for no gap in the very common situation of initial active state constraint (see [38, 40]). In order to do so, it is not enough to add to our previous assumptions one of the standard nondegeneracy hypotheses in the literature cited above, since we need stronger nondegeneracy relations that remains so when we pass to the limit. We refer the reader to Section 3.2 for a detailed discussion of this point and for the precise statement of our appropriate nondegeneracy assumption (see Hypothesis 3.2.6 below). In particular, our second main result (see Theorem 3.2.9) can be roughly stated as follows.

Second main result. *Take a minimizer for the auxiliary optimization problem* (P_a) *associated with* (P) *and assume that it satisfies a suitable nondegeneracy hypothesis. If that minimizer is a nondegenerate normal extremal for* (P_a)*, by wich we mean that* $\lambda > 0$ *for all sets of multipliers* (p, λ, μ) *satisfying conditions* (1.5)–(1.9) *and* (1.12) *for* (P_a)*, then there is no infimum gap.*

The improvement with respect to our first main result is clear, since no infimum gap can now be deduced checking normality not among all sets of multipliers, but only among the nondegenerate ones. In fact, as illustrated in several examples, we will show how this second main result can be applied in order to conclude absence of gap in situations in which normality alone is not met. Furthermore, we accompany our main results with corollaries dealing with the controllability issue of the original constrained control system (1.1)–(1.4) to feasible trajectories of the auxiliary problem.

Although the above stated results might appear theoretical rather than practical, as they subsume that one is capable to calculate all sets of multipliers, in Section 5.4 we supply some sufficient conditions, expressed in terms of state constraint and target qualification conditions, that ensure the nondegenerate normality of multipliers of the impulsive Maximum Principle.

We briefly explain how the contents of the thesis are divided. In Chapter 2 we present the fundamental theoretical preliminaries at the basis of the subsequent discussion. In particular, our overview covers elements of Set-valued and Nonsmooth Analysis and topics of Optimal Control Theory, with special emphasis to the necessary conditions of optimality.

Chapter 3 is the heart of the thesis, in it we state our main results concerning sufficient conditions for no gap and controllability with reference to free end-time optimization problems with measurable time dependence.

In Chapter 4 we adapt the above mentioned results to the case of fixed end-time optimal control problems. Afterwards, we apply them to address the stability issue regarding when a minimizer for the original problem is still a minimizer for the auxiliary one. We point out that a separate analysis of fixed end-time optimization problems is crucial not only because many applications can be formulate in this setting, but also due to the fact that from this case we are able to deduce enhanced gap-abnormality relations for free end-time optimal control problems with Lipschitz continuous time dependence, as explained more in detail at the beginning of Chapter 5.

Indeed, in Chapter 5 we first introduce a time rescaling procedure that allows us to deal with these latter problems, so that to characterize the Hamiltonian function almost everywhere. Then, we devote ample space to control-polynomial impulsive optimization problems, supplying simplified hypotheses for nondegeneracy and easily verifiable conditions guaranteeing nondegenerate normality of multipliers.

Finally, in Chapter 6 we summarize the outcomes obtained, highlighting future directions of investigation and drawing conclusions.

Chapter 2

Preliminaries

The goal of this chapter is to assemble the fundamental theoretical apparatus on which the thesis is based. This chapter is not intended to be exhaustive and many of the results that we will state in the next sections actually hold under less restrictive assumptions. If the readers are interested in the minimal assumptions or in a more detailed explanation of these topics, we suggest them to look at the bibliographical references that we will supply throughout the chapter.

2.1 Notation

Given $a, b \in \mathbb{R}$, an integer $k \geq 1$, a real number $r > 0$, and vectors z, $w \in \mathbb{R}^k$ we set:

- $\mathbb{R}^k_{\geq 0} := [0, +\infty[^k$;
- $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$;
- $r\mathbb{B}_k$ for the closed unit ball of radius r in $\mathbb{R}^k$. We do not specify the dimension when the meaning is clear.
- $\Delta_k := \left\{\gamma = (\gamma^0, \dots, \gamma^k) : \ \ \gamma^j \geq 0 \ \text{ for any } \ j = 0, \dots, k \ \text{ and } \ \sum_{j=0}^k \gamma^j = 1\right\}$ is the k-dimensional symplex.
- $|z|$ the Euclidean norm of z.
- $\langle z, w\rangle$ or equivalently $z \cdot w$, the scalar product between z and w.

Given a nonempty subset $X \subseteq \mathbb{R}^k$, we set:

- $d_X : \mathbb{R}^k \to \mathbb{R}_{\geq 0}$ the distance function from X;
- $\ell(X)$ the Lebesgue measure of X;
- $\mathrm{co}(X)$ the convex hull of X, i.e. the smallest convex set in $\mathbb{R}^k$ that contains X;
- $\mathrm{Int}(X)$ the interior of X;
- $\overline{X}$ the closure of X;
- ∂X the boundary of X;
- χ_X the characteristic function of X, namely $\chi_X(z) = 1$ if $z \in X$ and $\chi_X(z) = 0$ if $z \in \mathbb{R}^k \setminus X$;
- for any subset $Y \subset \mathbb{R}^k$, $\mathrm{proj}_Y X$ will denote the projection of X on Y. In particular, if $X \subseteq \mathbb{R}^{k_1+k_2}$ for some natural numbers k_1, k_2, then $\mathrm{proj}_{x_j} X$ will denote the projection of X on $\mathbb{R}^{k_j}$, for $j = 1, 2$.

Given two nonempty subsets X_1, X_2 of $\mathbb{R}^k$, we set:

- $X_1 + X_2 := \{z_1 + z_2 \,|\, z_1 \in X_1,\ z_2 \in X_2\}$;
- $d_H(X_1, X_2)$ the Hausdorff distance between X_1 and X_2, that is

$$d_H(X_1, X_2) := \max\left\{ \sup_{z_1 \in X_1} d_{X_2}(z_1),\ \sup_{z_2 \in X_2} d_{X_1}(z_2) \right\}.$$

Given an interval $I \subseteq \mathbb{R}$ and a set $X \subseteq \mathbb{R}^k$, we set

- $I \cdot X := \{r\, z \,:\, r \in I,\ z \in X\}$;
- $C(I, X)$ the space of continuous functions from I to X;
- $W^{1,1}(I, X)$ the space of absolutely continuous functions from I to X;
- $\mathcal{M}(I, X)$ the space of Lebesgue measurable functions from I to X;
- $L^1(I, X)$ the space of Lebesgue integrable functions from I to X;

- $L^\infty(I, X)$ the space of essentially bounded functions from I to X;
- $C^*(I, X)$ the topological dual space of $C(I, X)$. In view of the Riesz's representation Theorem (see [26, Thm. 4.31]), this coincides with the space of signed and regular measures $\mu : \mathscr{B}(I) \to X$ (from now on we will refer to such μ simply as measures), where $\mathscr{B}(I)$ denotes the set of Borel measurable subsets of I. In particular, we set $C^\oplus(I) := C^*(I, \mathbb{R}_{\geq 0})$.

We will not specify domain and codomain when the meaning is clear and we will use $\|\cdot\|_{L^1(I)}$, $\|\cdot\|_{L^\infty(I)}$, or also $\|\cdot\|_{L^1}$, $\|\cdot\|_{L^\infty}$ to denote the L^1 and the ess-sup norm, respectively. We recall that the induced norm $\|\cdot\|_{C^\oplus(I)}$ on $C^\oplus(I)$ coincides with the *total variation*, i.e. $\|\mu\|_{C^\oplus(I)} = \mu(I)$ for any $\mu \in C^\oplus(I)$. Finally, given $\mu \in C^*$ we denote by $\mathrm{spt}(\mu)$ the support of the measure μ, and we write μ-*a.e.* in place of "almost everywhere with respect to μ" (when we do not specify μ we implicitly refer to the Lebesgue measure).

2.2 Some fundamental theorems

In this section we state basic results of Mathematical Analysis that we will need in the following. A comprehensive list of references for these results is given by [13, 26, 49, 69].

We start with a characterization of the convex hull of a subset of $\mathbb{R}^k$.

Theorem 2.2.1 (Caratheodory's Theorem). *Let $\mathcal{K}$ be a subset of $\mathbb{R}^k$. Then it holds*

$$\mathrm{co}(\mathcal{K}) = \left\{ \sum_{j=0}^{k} \gamma^j z_j \,:\, (\gamma^0, \dots, \gamma^n) \in \Delta_k, \quad z_j \in \mathcal{K} \right\}.$$

In addition, if $\mathcal{K}$ is compact then $\mathrm{co}(\mathcal{K})$ *is compact as well.*

The last assertion of the theorem corresponds to [13, Prop. 0.5.6]. Notice that a direct consequence of the Caratheodory's Theorem is that for any $\mathcal{K}_1$, $\mathcal{K}_2 \subset \mathbb{R}^k$ it holds

$$\mathrm{co}(\mathcal{K}_1 + \mathcal{K}_2) \subset \mathrm{co}(\mathcal{K}_1) + \mathrm{co}(\mathcal{K}_2). \tag{2.2.1}$$

A similar representation formula is valid for the convex hull of the union of a collection of convex subsets of $\mathbb{R}^k$ (see [13, Prop. 0.5.3]).

Proposition 2.2.2. *Let $\mathcal{K}_j$ be a convex subset of $\mathbb{R}^k$ for any $j = 0, \ldots, n$. Then one has*

$$\mathrm{co}\left(\bigcup_{j=0}^{n} \mathcal{K}_j\right) = \left\{\sum_{j=0}^{n} \gamma^j z_j \,:\, (\gamma^0, \ldots, \gamma^n) \in \Delta_n, \quad z_j \in \mathcal{K}_j\right\}.$$

Let us now recall a version of the crucial Gronwall's estimation that can be found in [49, pp. 356-357] and the celebrated Dominated Convergence Theorem.

Theorem 2.2.3 (Gronwall's Lemma)**.** *Take $v \in L^\infty([t_1, t_2], \mathbb{R}_{\geq 0})$ and assume that there exist a nondecreasing function $c_0 \in L^\infty([t_1, t_2], \mathbb{R}_{\geq 0})$ and $c_1 \in L^1([t_1, t_2], \mathbb{R}_{\geq 0})$ for which it holds*

$$v(t) \leq c_0(t) + \int_{t_1}^{t_2} c_1(s) v(s)\, ds \qquad \text{a.e. } t \in [t_1, t_2].$$

Then, one has

$$v(t) \leq c_0(t)\, e^{\int_{t_1}^{t_2} c_1(s)\, ds} \qquad \text{a.e. } t \in [t_1, t_2].$$

Theorem 2.2.4 (Dominated convergence Theorem)**.** *Let $(v_i) \subset L^1([t_1, t_2], \mathbb{R}^k)$, let $v : [t_1, t_2] \to \mathbb{R}^k$ and let $c \in L^1([t_1, t_2], \mathbb{R}_{\geq 0})$ be such that:*

(i) for any i one has $|v_i(t)| \leq c(t)$ for a.e. $t \in [t_1, t_2]$;

(ii) $v_i(t) \to v(t)$ for a.e. $t \in [t_1, t_2]$.

Then $v \in L^1([t_1, t_2], \mathbb{R}^k)$ and $v_i \to v$ in L^1.

The next theorem constitutes the basis of the variational approach considered in this thesis (see Section 3.3 below).

Theorem 2.2.5 (Ekeland's Theorem)**.** *Let $(Z, \mathbf{d})$ be a complete metric space and let $\Psi : Z \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. Let $r > 0$ be a real number and let $z_0 \in Z$ be an r^2-minimizer for Ψ, i.e. $\Psi(z_0) \leq \inf_{z \in Z} \Psi(z) + r^2$. Then there exists $\bar{z} \in Z$ that satisfies conditions (i)–(iii) below:*

(i) $\Psi(\bar{z}) \leq \Psi(z_0)$;

(ii) $\mathbf{d}(z_0, \bar{z}) \leq r$;

(iii) $\Psi(\bar{z}) \leq \Psi(z) + r\, \mathbf{d}(z, \bar{z})$ for all $z \in Z$.

2.2.1 Convergence of measures

In this subsection we list some preliminary results about measures. We start recalling the notion of convergence in $C^*([t_1,t_2],\mathbb{R}^k)$ where, from now on, t_1, $t_2 \in \mathbb{R}$ with $t_1 < t_2$ are fixed.

Definition 2.2.6 (Weak* convergence in C^*)**.** Given a sequence of measures $(\mu_i) \subset C^*([t_1,t_2],\mathbb{R}^k)$, and $\mu \in C^*([t_1,t_2],\mathbb{R}^k)$, we write $\mu_i \rightharpoonup^* \mu$ if it holds

$$\lim_i \int_{[t_1,t_2]} \varphi(t)\mu_i^j(dt) = \int_{[t_1,t_2]} \varphi(t)\mu^j(dt), \qquad \forall \varphi \in C([t_1,t_2],\mathbb{R}),\ \forall j = 1,\dots,k,$$

where μ_i^j denotes the j-th component of μ_i and μ^j the j-th component of μ. In this case, we say that (μ_i) *weakly* converges to* μ.

In view of the Banach-Alaoglu's Theorem (see [26, Thm. 3.16]) and since $C([t_1,t_2],\mathbb{R})$ is a separable space, we have the following result (see [26, Cor. 3.30]).

Theorem 2.2.7. *Let $M > 0$ be a real number, and let $(\mu_i) \subset C^\oplus([t_1,t_2])$ be such that $\mu_i([t_1,t_2]) \leq M$ for any i. Then there exist $\mu \in C^\oplus([t_1,t_2])$ and a subsequence (μ_{i_j}) of (μ_i) such that $\mu_{i_j} \rightharpoonup^* \mu$ as $j \to \infty$.*

Let us conclude this subsection recalling another convergence result for measures (see [69, Prop. 9.2.1] and [4, Prop. 1.62]).

Theorem 2.2.8. *Let $(\mu_i) \subset C^\oplus([t_1,t_2])$ and $\mu \in C^\oplus([t_1,t_2])$ be such that $\mu_i \rightharpoonup^* \mu$. Let $M > 0$ be a real number, let (Ω_i) be a sequence of closed product subsets*[2] *of $[t_1,t_2] \times \mathbb{R}^k$ such that $\Omega_i \subset M\mathbb{B}_{1+n}$ for any i, and let Ω be a closed product subset of $[t_1,t_2] \times \mathbb{R}^k$. Assume that $\Omega(t) = \{z \in \mathbb{R}^k : (t,z) \in \Omega\}$ is convex for any $t \in \mathrm{dom}(\Omega)$ and suppose that*[3]

$$\limsup_{i\to\infty} \Omega_i \subset \Omega \subset M\mathbb{B}_{1+n}.$$

Finally, let (m_i) be a sequence of Borel measurable functions from $[t_1,t_2]$ to $\mathbb{R}^k$ such that, for any i, one has

$$m_i(t) \in \Omega_i(t) \qquad \text{for } \mu_i\text{-a.e. } t \in [t_1,t_2].$$

[2] A is a product subset of $[t_1,t_2] \times \mathbb{R}^k$ if $A = X_1 \times X_2$ with $X_1 \subset [t_1,t_2]$ and $X_2 \subset \mathbb{R}^k$. In this case, we set $\mathrm{dom}(A) := X_1$ and for any $t \in \mathrm{dom}(A)$ we set $A(t) := \{z \in X_2 : (t,z) \in A\}$.

[3] We recall that $\limsup_{i\to\infty} \Omega_i := \big\{(t,z) \in [t_1,t_2] \times \mathbb{R}^k : \liminf_{i\to\infty} d_{\Omega_i}(t,z) = 0\big\}$.

Then, there exist a subset $\mathcal{D}$ of $[t_1, t_2]$ with $\ell(\mathcal{D}) = t_2 - t_1$ and $t_2 \in \mathcal{D}$, a Borel measurable function $m : [t_1, t_2] \to \mathbb{R}^k$ satisfying

$$m(t) \in \Omega(t) \qquad \textit{for } \mu\textit{-a.e. } t \in [t_1, t_2],$$

and a subsequence (m_{i_j}, μ_{i_j}) of $(m_i, \mu_i) \subset C^*([t_1, t_2], \mathbb{R}^k)$ such that

$$\begin{aligned} m_{i_j}(t)\mu_{i_j}(dt) &\rightharpoonup^* m(t)\mu(dt) \in C^*([t_1, t_2], \mathbb{R}^k), \\ \int_{[0,t]} m_{i_j}(t')\mu_{i_j}(dt') &\to \int_{[0,t]} m(t')\mu(dt') \qquad \textit{for all } t \in \mathcal{D}. \end{aligned} \tag{2.2.2}$$

2.3 Set-valued maps

In this section we present a brief introduction to the theory of set-valued maps. Our main sources on this subject are [13, 14, 27, 69].

Given $Z \subset \mathbb{R}^l$, a *set-valued map* $\mathfrak{F} : Z \rightsquigarrow \mathbb{R}^k$ (or, equivalently, a *multifunction*) is a map that associates to any element of Z a subset of $\mathbb{R}^k$. Of course, if $\mathfrak{F}(z)$ is a singleton for any $z \in Z$, then $\mathfrak{F}$ turns out to be a classic function from Z to $\mathbb{R}^k$. Therefore, a set-valued map can be seen as a generalization of the usual notion of function. A multifunction $\mathfrak{F}$ is said to be *closed* [resp. *convex*, *compact*, *nonempty*] if $\mathfrak{F}(z)$ is a closed [resp. convex, compact, nonempty] subset of $\mathbb{R}^k$ for any $z \in Z$. If $\mathcal{O}$ is a σ-algebra on Z, $\mathfrak{F}$ is called $\mathcal{O}$-*measurable* if for any open set $X \subset \mathbb{R}^k$ one has

$$\{z \in Z : \mathfrak{F}(z) \cap X \neq \emptyset\} \in \mathcal{O}.$$

When we do not specify $\mathcal{O}$, we implicitly mean Lebesgue measurability. We recall that if $\mathfrak{F}$ is $\mathcal{O}$-measurable then it is $\overline{\mathfrak{F}}$ (i.e. its closure).

We say that $\mathfrak{F}$ is *upper semicontinuous at* $\bar{z} \in Z$ if for any open set $V \supset \mathfrak{F}(\bar{z})$ there exists a neighborhood U of $\bar{z}$ such that $\mathfrak{F}(z) \subset V$ for any $z \in U$; we say that $\mathfrak{F}$ is upper semicontinuous if it is so at every $z \in Z$.

Finally, we denote by $\mathrm{Gr}(\mathfrak{F})$ the *graph* of the multifunction $\mathfrak{F}$, namely the subset of $Z \times \mathbb{R}^k$ defined by

$$\mathrm{Gr}(\mathfrak{F}) := \{(z, w) \in Z \times \mathbb{R}^k : w \in \mathfrak{F}(z)\}.$$

One might think that if $\mathfrak{F}$ has closed graph then it is upper semicontinuous, but this actually holds only if the codomain of $\mathfrak{F}$ is compact (see [13, Cor. 1.1.1]).

Now we state some results about multifunctions that we will recall in the following. We start with a theorem that ensures that the projection of a measurable function on a closed measurable multifunction is still measurable and admits a measurable selection (see [14, Cor. 8.2.13]). Afterwards we establish a criterion for the measurability of a multifunction defined on a real interval (see [69, Thm. 2.3.7]).

Theorem 2.3.1. *Let $Z \subset \mathbb{R}^l$, let $\mathcal{O}$ be a σ-algebra on Z, let $\mathfrak{F} : Z \rightsquigarrow \mathbb{R}^k$ be a closed and nonempty $\mathcal{O}$-measurable set-valued map, and let $f : Z \to \mathbb{R}^k$ be a (classic) $\mathcal{O}$-measurable function. Then the projection set-valued map defined by*

$$\Pi_{\mathfrak{F}(z)} f(z) := \left\{ w \in \mathfrak{F}(z) : \ |w - f(z)| = d_{\mathfrak{F}(z)}(f(z)) \right\}$$

is $\mathcal{O}$-measurable. In particular, there exists a $\mathcal{O}$-measurable selection $h(z) \in \mathfrak{F}(z)$ such that

$$|h(z) - f(z)| = d_{\mathfrak{F}(z)}(f(z)) \qquad \forall z \in Z.$$

Proposition 2.3.2. *Let $I \subset \mathbb{R}$ be an interval and let $\mathfrak{F} : I \rightsquigarrow \mathbb{R}^k$ be a closed set-valued map. Then $\mathfrak{F}$ is Lebesgue-measurable if and only if $\mathrm{Gr}(\mathfrak{F})$ is an $\mathscr{L} \times \mathscr{B}^k$-measurable set.*

Fix $T > 0$ and let $\mathfrak{F} : [t_1, t_2] \times \mathbb{R}^k \rightsquigarrow \mathbb{R}^k$ be a set-valued map. A function $x \in W^{1,1}([t_1, t_2], \mathbb{R}^k)$ is called $\mathfrak{F}$-*trajectory* if it satisfies the following *differential inclusion*

$$\dot{x}(t) \in \mathfrak{F}(t, x(t)) \qquad \text{a.e. } t \in [t_1, t_2].$$

Theorem 2.3.3 (Filippov's existence Theorem)**.** *Let $\mathfrak{F} : [t_1, t_2] \times \mathbb{R}^k \rightsquigarrow \mathbb{R}^k$ be a closed and nonempty set-valued map which is $\mathscr{L} \times \mathscr{B}^k$-measurable*[4]*, let $\tilde{x} \in W^{1,1}([t_1, t_2], \mathbb{R}^k)$ and let $\check{z} \in \mathbb{R}^k$. Assume that there exists $c \in L^1([t_1, t_2], \mathbb{R}_{\geq 0})$ for which the following Lipschitz continuity hypothesis holds*

$$\mathfrak{F}(t, z_1) \subset \mathfrak{F}(t, z_2) + c(t)|z_1 - z_2|\mathbb{B}_n \qquad \forall z_1,\ z_2 \in \mathbb{R}^k, \quad a.e.\ t \in [t_1, t_2]. \tag{2.3.1}$$

[4] $\mathscr{L} \times \mathscr{B}^k$ denotes the smallest σ-algebra of $[t_1, t_2] \times \mathbb{R}^k$ that contains all the sets $Z_1 \times Z_2$ where $Z_1 \subset [t_1, t_2]$ is Lebesgue measurable and $Z_2 \subset \mathbb{R}^k$ is Borel measurable.

Finally, let us suppose that

$$\beta := \int_{t_1}^{t_2} \inf\{|w - \dot{\tilde{x}}(t)| \,:\, w \in \mathfrak{F}(t, \tilde{x}(t))\}\, dt < +\infty. \tag{2.3.2}$$

Then, there exists an $\mathfrak{F}$*-trajectory* x *fulfilling* $x(t_1) = \check{z}$ *and such that*

$$\|x - \tilde{x}\|_{L^\infty(t_1,t_2)} \le |x(t_1) - \tilde{x}(t_1)| + \int_{t_1}^{t_2} |\dot{x}(t) - \dot{\tilde{x}}(t)|\, dt \le (|\check{z} - \tilde{x}(t_1)| + \beta) e^{\int_{t_1}^{t_2} c(t)\, dt}.$$

The following result (see [69, Thm. 2.5.3]) will be very useful in the convergence analysis of the proof of our main result in Section 3.3.

Theorem 2.3.4 (Compactness of trajectories Theorem)**.** *Let* $\mathfrak{F} : [t_1, t_2] \times \mathbb{R}^k \rightsquigarrow \mathbb{R}^k$ *be a closed, convex and nonempty set-valued map that is* $\mathscr{L} \times \mathscr{B}^k$*-measurable and such that* $\mathrm{Gr}(\mathfrak{F}_t)$ *is a closed set for any* $t \in [t_1, t_2]$*, where* $\mathfrak{F}_t : \mathbb{R}^k \rightsquigarrow \mathbb{R}^k$ *is the multifunction defined by* $\mathfrak{F}_t(z) := \mathfrak{F}(t, z)$ *for any* $z \in \mathbb{R}^k$*. Let* (Ω_i) *be a sequence of Lebesgue measurable subsets of* $[t_1, t_2]$ *such that* $\ell(\Omega_i) \to t_2 - t_1$*, and let* $(\rho_i) \subset L^1([t_1, t_2], \mathbb{R}_{\ge 0})$ *be such that* $\rho_i \to 0$ *in* L^1*. Finally, let* $(x_i) \subset W^{1,1}([t_1, t_2], \mathbb{R}^k)$ *and let* $c \in L^1([t_1, t_2], \mathbb{R}_{\ge 0})$ *be such that:*

(i) the sequence $(x_i(t_1))$ *is bounded;*

(ii) For all i *one has* $|\dot{x}_i(t)| \le c(t)$ *for a.e.* $t \in [t_1, t_2]$ *(namely, the sequence* $(\dot{x}_i)$ *is uniformly integrably bounded);*

(iii) one has

$$\mathfrak{F}(t, x_i(t)) \subset c(t)\mathbb{B}_k \qquad \text{for a.e. } t \in \Omega_i \text{ and for all } i;$$

(iv) one has

$$\dot{x}_i(t) \in \mathfrak{F}(t, x_i(t)) + \rho_i(t)\mathbb{B}_k \qquad \text{for a.e. } t \in \Omega_i \text{ and for all } i.$$

Then, there exist an $\mathfrak{F}$*-trajectory* $x \in W^{1,1}([t_1, t_2], \mathbb{R}^k)$ *and a subsequence* (x_{i_j}) *of* (x_i)

such that, as $j \to \infty$, one has[5]

$$x_{i_j} \to x \quad \text{in } L^\infty, \qquad \dot{x}_{i_j} \rightharpoonup \dot{x} \quad \text{weakly in } L^1.$$

We have just seen that the convexity of a set-valued map $\mathfrak{F}$ is a crucial requirement for the compactness of the set of the $\mathfrak{F}$-trajectories, so that to play a fundamental role in the existence of minimizers for an optimization problem as well (see Theorem 2.5.4 below). At this point, given a non-convex multifunction $\mathfrak{F} : [t_1, t_2] \times \mathbb{R}^k \rightsquigarrow \mathbb{R}^k$ one might ask what relation occurs between the set of $\mathfrak{F}$-trajectories and the set of co($\mathfrak{F}$)-trajectories, where this latter set is commonly called *set of relaxed $\mathfrak{F}$-trajectories*. The following result answers this question establishing that $\mathfrak{F}$-trajectories are L^∞-dense into the set of relaxed $\mathfrak{F}$-trajectories.

Theorem 2.3.5 (Relaxation Theorem). *Let $\mathfrak{F} : [t_1, t_2] \times \mathbb{R}^k \rightsquigarrow \mathbb{R}^k$ be a closed and nonempty set-valued map which is $\mathscr{L} \times \mathscr{B}^k$-measurable. Assume that there exists $c \in L^1([t_1, t_2], \mathbb{R}_{\geq 0})$ for which condition* (2.3.1) *holds and such that*

$$\mathfrak{F}(t, z) \subset c(t)\mathbb{B}_k \qquad \text{for any } (t, z) \in [t_1, t_2] \times \mathbb{R}^k.$$

Let $\tilde{x} \in W^{1,1}([t_1, t_2], \mathbb{R}^k)$ be a relaxed $\mathfrak{F}$-trajectory (i.e., $\dot{\tilde{x}}(t) \in \operatorname{co}\mathfrak{F}(t, \tilde{x}(t))$ for a.e. $t \in [t_1, t_2]$) and let $\varepsilon > 0$ be a real number. Then there exists an $\mathfrak{F}$-trajectory $x \in W^{1,1}([t_1, t_2], \mathbb{R}^k)$ with $x(t_1) = \tilde{x}(t_1)$ that satisfies

$$\|x - \tilde{x}\|_{L^\infty([t_1,t_2])} \leq \varepsilon.$$

2.4 Nonsmooth Analysis

In this section we summarize some basic constructs and fundamental results of Nonsmooth Analysis, a branch of Mathematical Analysis that aims to extend to a nonsmooth setting classic notions that are usually reserved to smooth objects. For instance, the concept of derivative usually involves C^1 functions, but in this section we provide a suitable generalized definition that applies to lower semicontinuous func-

[5]We recall that a sequence of functions $(h_i) \subset L^1([t_1, t_2], \mathbb{R}^k)$ converges weakly in L^1 to $h \in L^1([t_1, t_2], \mathbb{R}^k)$ if $\int_{t_1}^{t_2} \langle h_i(t), \varphi(t)\rangle dt \to \int_{t_1}^{t_2} \langle h(t), \varphi(t)\rangle dt$ for any $\varphi \in L^\infty([t_1, t_2], \mathbb{R}^k)$.

tions in such a way that standard results (e.g. necessary conditions for minimizers or the Lagrange multipliers rule) still hold with minor changes.

Our main references for these topics are [27, 30, 69].

2.4.1 Normal cones

First of all, we recall that a subset $\mathcal{K} \subseteq \mathbb{R}^k$ is said to be a *cone* if $c\,w \in \mathcal{K}$ for any $c > 0$, whenever $w \in \mathcal{K}$. A cone $\mathcal{K} \subseteq \mathbb{R}^k$ is said to be *pointed* if it contains no straight line, namely, if w, $-w \in \mathcal{K}$ implies that $w = 0$.

Definition 2.4.1. Take a closed set $\mathcal{C} \subseteq \mathbb{R}^k$ and a point $\bar{z} \in \mathcal{C}$. The *proximal normal cone* $N^P_{\mathcal{C}}(\bar{z})$ of $\mathcal{C}$ at $\bar{z}$ is defined as

$$N^P_{\mathcal{C}}(\bar{z}) := \left\{ \xi \in \mathbb{R}^k : \exists M > 0 \ \text{ such that } \ \langle \xi, z - \bar{z} \rangle \leq M|z - \bar{z}|^2 \quad \forall z \in \mathcal{C} \right\}.$$

The *limiting normal cone* $N_{\mathcal{C}}(\bar{z})$ of $\mathcal{C}$ at $\bar{z}$ is given by

$$N_{\mathcal{C}}(\bar{z}) := \left\{ \xi \in \mathbb{R}^k : \exists z_i \xrightarrow{\mathcal{C}} \bar{z},\, \xi_i \to \xi \ \text{ such that } \ \xi_i \in N^P_{\mathcal{C}}(z_i) \ \ \forall i \right\},$$

in which the notation $z_i \xrightarrow{\mathcal{C}} \bar{z}$ is used to indicate that all points in the converging sequence (z_i) lay in $\mathcal{C}$.

In general, $N^P_{\mathcal{C}}(\bar{z}) \subseteq N_{\mathcal{C}}(\bar{z})$ and $N^P_{\mathcal{C}}(\bar{z})$ is a convex set. It is possible to prove that if $\bar{z} \in \text{Int}(\mathcal{C})$ then $N^P_{\mathcal{C}}(\bar{z}) = N_{\mathcal{C}}(\bar{z}) = \{0\}$, while if $\bar{z} \in \partial\mathcal{C}$ then $N_{\mathcal{C}}(\bar{z})$ contains nonzero vectors. Similarly, $N_{\{\bar{z}\}}(\bar{z}) = \mathbb{R}^k$ and in the case there exist closed subset $\mathcal{C}_1$ and $\mathcal{C}_2$ such that $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$ and $\bar{z} = (\bar{x}, \bar{y}) \in \mathcal{C}_1 \times \mathcal{C}_2$, then

$$N_{\mathcal{C}_1 \times \mathcal{C}_2}(\bar{x}, \bar{y}) = N_{\mathcal{C}_1}(\bar{x}) \times N_{\mathcal{C}_2}(\bar{y}).$$

Now we state two important properties of the nonsmooth normal cones defined above. The former gives a useful geometrical interpretation of $N^P_{\mathcal{C}}(\bar{z})$, relating it with the set of points in $\mathbb{R}^k \setminus \mathcal{C}$ such that $\bar{z}$ is their closest point in $\mathcal{C}$ (with respect to the usual Euclidean norm). The latter underlines the robustness of $N_{\mathcal{C}}(\bar{z})$ in the passage to the limit.

Proposition 2.4.2. *Take a closed set $\mathcal{C} \subset \mathbb{R}^k$, a point $\bar{z} \in \mathcal{C}$ and a vector $\xi \in \mathbb{R}^k$. Then one has:*

(i) $\xi \in N^P_{\mathcal{C}}(\bar{z})$ *if and only if there exist a point $z \in \mathbb{R}^k$ and a real number $r > 0$ such that*

$$|z - \bar{z}| = \min\{|z - y| \,:\, y \in \mathcal{C}\} \qquad \textit{and} \qquad \xi = r(z - \bar{z}).$$

(ii) If $z_i \xrightarrow{\mathcal{C}} \bar{z}$, $\xi_i \to \xi$ and $\xi_i \in N_{\mathcal{C}}(z_i)$ for any i, then $\xi \in N_{\mathcal{C}}(\bar{z})$. Namely the set-valued map that associates to every point in $\mathcal{C}$ its limiting normal cone has closed graph.

2.4.2 Subdifferentials

In this subsection we collect all definitions and properties of the generalized gradients of lower semicontinuous functions that we will consider from now on.

Definition 2.4.3. Take a lower semicontinuous function $g : \mathbb{R}^k \to \mathbb{R}$ and a point $\bar{z} \in \mathbb{R}^k$, the *proximal subdifferential* $\partial^P g(\bar{z})$ of g at $\bar{z}$ is defined as

$$\partial^P g(\bar{z}) := \Big\{\xi \in \mathbb{R}^k \,:\, \exists M,\ \varepsilon > 0 \text{ such that } \langle \xi, z - \bar{z}\rangle \leq g(z) - g(\bar{z}) + M|z - \bar{z}|^2 \quad \forall z \text{ satisfying } |z - \bar{z}| \leq \varepsilon \Big\}.$$

The *limiting subdifferential* of g at $\bar{z}$ is given by

$$\partial g(\bar{z}) := \left\{\xi \,:\, \exists \xi_i \to \xi,\ z_i \to \bar{z} \text{ such that } \xi_i \in \partial^P g(z_i) \ \ \forall i\right\}.$$

If g is lower semicontinuous then the set of points with nonempty proximal subdifferential is dense in $\mathbb{R}^k$ (see [30, Thm. 3.1]), so that the definition of limiting subdifferential is well posed. One might think that if $g \in C^1$, then $\partial^P g = \partial g$ and they coincide with the classic gradient operator. Actually, this is granted only for C^2 functions (see [30, Cor. 1.2.6]).

We can give a geometrical interpretation to $\partial^P g(\bar{z})$: it is the set of slopes of paraboloids with vertex in $\bar{z}$ and laying below the graph of g. Incidentally, there is an equivalent formulation of the subdifferentials defined above in terms of the normal cones of the previous subsection. Indeed, it holds

$$\partial^P g(\bar{z}) = \{\xi \in \mathbb{R}^k : (\xi, -1) \in N^P_{\mathrm{epi}(g)}(\bar{z}, g(\bar{z}))\},$$
$$\partial g(\bar{z}) = \{\xi \in \mathbb{R}^k : (\xi, -1) \in N_{\mathrm{epi}(g)}(\bar{z}, g(\bar{z}))\},$$

where $\mathrm{epi}(g) := \{(z, w) \in \mathbb{R}^k \times \mathbb{R} : w \geq g(z)\}$ is the *epigraph* of g. In view of the above representation, and thanks to Proposition 2.4.2 it is straightforward that the set-valued map that associates to each point in $\mathbb{R}^k$ the limiting subdifferential of g at that point has closed graph. We point out that, in the case $k = k_1 + k_2$ and $\bar{z} = (\bar{x}, \bar{y}) \in \mathbb{R}^{k_1} \times \mathbb{R}^{k_2}$, we write $\partial_x g(\bar{x}, \bar{y})$, $\partial_y g(\bar{x}, \bar{y})$ to denote the *partial limiting subdifferential* of g at $(\bar{x}, \bar{y})$ with respect to x, y, respectively.

The next proposition provides the main tools of subdifferential calculus.

Proposition 2.4.4. *Let $g_j : \mathbb{R}^k \to \mathbb{R}$ be a lower semicontinuous function for any $j = 0, \ldots, l$ and let $\bar{z} \in \mathbb{R}^k$. Then one has:*

(i) $\partial \left(\sum_{j=0}^{l} g_j \right)(\bar{z}) \subset \sum_{j=0}^{l} \partial g_j(\bar{z})$;

(ii) $\partial \left(\max_{j=0,\ldots,l} g_j \right)(\bar{z}) \subset \left\{ \sum_{j=0}^{l} \varsigma^j \partial g_j(\bar{z}) : (\varsigma^0, \ldots, \varsigma^l) \in \Delta_l \right\}$.

2.4.3 Lipschitz continuous functions

When $g : \mathbb{R}^k \to \mathbb{R}$ is a Lipschitz continuous function, it is possible to define new generalized gradients that enjoy further powerful properties, useful for the next chapters.

Definition 2.4.5. Given a locally Lipschitz continuous function $g : \mathbb{R}^k \to \mathbb{R}$ and $\bar{z} \in \mathbb{R}^k$, the *reachable hybrid subdifferential* of g at $\bar{z}$ is defined as

$$\partial^{*>} g(\bar{z}) := \{\xi : \exists (z_i) \subset \mathrm{diff}(g) \setminus \{\bar{z}\} \text{ such that } z_i \to \bar{z},\ g(z_i) > 0\ \forall i,\ \nabla g(z_i) \to \xi\},$$

while the *reachable gradient* of g at $\bar{z}$ is given by

$$\partial^* g(\bar{z}) := \{\xi : \exists (z_i) \subset \mathrm{diff}(g) \setminus \{\bar{z}\} \text{ such that } z_i \to \bar{z} \text{ and } \nabla g(z_i) \to \xi\}$$

where $\mathrm{diff}(g)$ denotes the set of differentiability points of g and ∇g denotes the usual gradient operator. Finally, we refer to $\partial^{>} g(\bar{x}) := \mathrm{co}\,\partial^{*>} g(\bar{x})$ as the *hybrid subdifferential.*

Notice that the previous subdifferentials are well defined, since $\mathrm{diff}(g)$ is a dense set in $\mathbb{R}^k$, in view of the Lipschitz continuity of g. It is clear that $\partial^{*>} g(\bar{z}) \subset \partial^* g(\bar{z})$ and, in the case $k = k_1 + k_2$ and $\bar{z} = (\bar{x}, \bar{y}) \in \mathbb{R}^{k_1} \times \mathbb{R}^{k_2}$, we write $\partial_x^{>} g(\bar{x}, \bar{y})$, $\partial_y^{>} g(\bar{x}, \bar{y})$ [resp. $\partial_x^* g(\bar{x}, \bar{y})$, $\partial_y^* g(\bar{x}, \bar{y})$] to denote the *partial hybrid subdifferential* [resp. the *partial reachable gradient*] of g at $(\bar{x}, \bar{y})$ with respect to x, y, respectively.

The following proposition establishes a link between the reachable gradient and the limiting subdifferential of a Lipschitz continuous function. In particular, the second part of the statement is a consequence of [69, Prop. 4.7.1, Thm. 5.3.1].

Theorem 2.4.6. *If $g : \mathbb{R}^k \to \mathbb{R}$ is locally Lipschitz continuous, then the set-valued map that associates to each point in $\mathbb{R}^k$ the reachable gradient of g at that point is nonempty, closed and upper semicontinuous. Moreover, it holds*

$$\mathrm{co}\,\partial^* g(\bar{z}) = \mathrm{co}\,\partial g(\bar{z}) \subset L\mathbb{B}_k \qquad \forall \bar{z} \in \mathbb{R}^k, \tag{2.4.1}$$

where $L > 0$ is the Lipschitz constant of g in a neighborhood of $\bar{z}$. Furthermore, if $k = k_1 + k_2$ and $\bar{z} = (\bar{x}, \bar{y}) \in \mathbb{R}^{k_1} \times \mathbb{R}^{k_2}$, then one has

$$\partial_x g(\bar{x}, \bar{y}) = \{\xi_1 \in \mathbb{R}^{k_1} \,:\, \exists \xi_2 \in \mathbb{R}^{k_2} \text{ such that } (\xi_1, \xi_2) \in \partial g(\bar{x}, \bar{y})\}. \tag{2.4.2}$$

Let us now recall a crucial mean value inequality and some useful properties of the distance function.

Theorem 2.4.7 (Lebourg's Theorem)**.** *Let $g : \mathbb{R}^k \to \mathbb{R}$ be locally Lipschitz continuous and let z_1, $z_2 \in \mathbb{R}^k$. Then, there exists $\hat{z} \in \{sz_1 + (1-s)z_2 \,:\, s \in [0,1]\}$ and $\xi \in \mathrm{co}\,\partial g(\hat{z})$ such that*

$$g(z_2) - g(z_1) = \langle \xi, z_2 - z_1 \rangle.$$

Proposition 2.4.8. *Let $\mathcal{C} \subset \mathbb{R}^k$ be closed, then the distance function $d_{\mathcal{C}} : \mathbb{R}^k \to \mathbb{R}_{\geq 0}$ is Lipschitz continuous with Lipschitz constant equal to 1. Moreover, it holds*

$$\partial d_{\mathcal{C}}(z) = N_{\mathcal{C}}(z) \cap \mathbb{B}_k \qquad \forall z \in \mathcal{C},$$

and for any $z \in \mathbb{R}^k \setminus \mathcal{C}$ *one has*

$$|\xi| = 1 \qquad \forall \xi \in \partial d_{\mathcal{C}}(z).$$

We conclude this subsection extending some of the notions presented so far to multivariate functions. Given a locally Lipschitz continuous function $G : \mathbb{R}^k \to \mathbb{R}^l$ and $\bar{z} \in \mathbb{R}^k$, we write $DG(\bar{z})$ to denote the *Clarke generalized Jacobian*, defined as

$$DG(\bar{z}) := \text{co } \{\xi : \exists (z_i) \subset \text{diff}(G) \setminus \{\bar{z}\} \text{ s.t. } z_i \to \bar{z} \text{ and } \nabla G(z_i) \to \xi\},$$

where now ∇G denotes the classical Jacobian matrix of G.

We notice that when $l = 1$ one has $DG = \text{co}\, \partial G$ and that it holds (see [30, Ex. 3.3.10])

$$p \cdot DG(x) = \text{co}\, \partial (p \cdot G)(x) \qquad \text{for all } (p, x) \in \mathbb{R}^l \times \mathbb{R}^k. \tag{2.4.3}$$

Once again, if $k = k_1 + k_2$ and $\bar{z} = (\bar{x}, \bar{y}) \in \mathbb{R}^{k_1} \times \mathbb{R}^{k_2}$, then $D_x G(\bar{x}, \bar{y})$, $D_y G(\bar{x}, \bar{y})$ denote the *partial Clarke generalized Jacobian* of G at $(\bar{x}, \bar{y})$ with respect to x, y, respectively.

2.5 Optimal Control Theory

Taking account of the theory developed in the previous sections, we are now able to offer a short, but comprehensive, introduction to Optimal Control Theory. We start dealing with control systems, addressing in particular the question of existence of solutions and that of the continuity of the input-output map. The second subsection is devoted to the problem of existence of minimizers for optimal control problems. Finally, we conclude announcing the most celebrated set of necessary conditions for minimizers of optimization problems, namely the Pontryagin's Maximum Principle. A general list of references for these topics is given by [16, 20, 27, 69, 77].

2.5.1 Control systems

Let $\mathscr{A} : [t_1, t_2] \rightsquigarrow \mathbb{R}^l$ be a nonempty set-valued map, and let $\mathcal{A}([t_1, t_2])$ be the set of measurable selections of $\mathscr{A}$, namely

$$\mathcal{A}([t_1, t_2]) := \{\alpha \in \mathcal{M}([t_1, t_2], \mathbb{R}^l) : \alpha(t) \in \mathscr{A}(t) \text{ for a.e. } t \in [t_1, t_2]\}.$$

Given a vector-valued function $F : \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^k$, we consider the following dynamical system

$$\begin{cases} \dot{x}(t) = F(t, x(t), \alpha(t)) & \text{a.e. } t \in [t_1, t_2], \\ \alpha \in \mathcal{A}([t_1, t_2]). \end{cases} \tag{2.5.1}$$

We refer to $\mathcal{A}([t_1, t_2])$ as *control set*, and to elements α of $\mathcal{A}([t_1, t_2])$ as *control functions*, or simply *controls*. Accordingly, a dynamical system as in (2.5.1) is called *control system* and the differential equation governing a control system is called *controlled differential equation*.

We will refer to a pair $(\alpha, x) \in \mathcal{M}([t_1, t_2], \mathbb{R}^l) \times W^{1,1}([t_1, t_2], \mathbb{R}^k)$ as *process* if it satisfies the constraints in (2.5.1). In this case, we will refer to x as *trajectory*. Let $\mathrm{Vel}_F : [t_1, t_2] \times \mathbb{R}^k \rightsquigarrow \mathbb{R}^k$ be the *set-valued map of the velocities* defined by

$$\mathrm{Vel}_F(t, z) := \{F(t, z, a) : a \in \mathscr{A}(t)\} \qquad \forall (t, z) \in [t_1, t_2] \times \mathbb{R}^k.$$

Under mild assumptions, the set of trajectories to (2.5.1) coincides with the set of Vel_F-trajectories (see [69, Thm. 2.3.13]).

Theorem 2.5.1 (Filippov's selection Theorem). *Assume that* $\mathrm{Gr}(\mathscr{A})$ *is an* $\mathscr{L} \times \mathscr{B}^k$*-measurable set. Moreover, suppose that for any* $z \in \mathbb{R}^k$ *the function* $[t_1, t_2] \times \mathbb{R}^l \ni (t, a) \mapsto F(t, z, a)$ *is* $\mathscr{L} \times \mathscr{B}^m$*-measurable and that for any* $(t, a) \in [t_1, t_2] \times \mathbb{R}^l$ *the function* $\mathbb{R}^k \ni z \mapsto F(t, z, a)$ *is continuous. Then there is a one-to-one correspondence between the set of* Vel_F*-trajectories and the set of trajectories for* (2.5.1)*. Namely, one has:*

(i) if x *is an* Vel_F*-trajectory, then there exists* $\alpha \in \mathcal{A}([t_1, t_2])$ *such that* (α, x) *is a process for* (2.5.1)*;*

(ii) viceversa, if (α, x) *is a process for* (2.5.1)*, then* x *is an* Vel_F*-trajectory.*

Now we state a theorem of existence of solutions to (2.5.1), together with the continuity of the input-output map property.

Theorem 2.5.2. *Assume that* $\mathscr{A}$ *and* F *satisfy the same assumptions as in Theorem 2.5.1. Moreover, suppose that there exists* $c \in L^1([t_1, t_2], \mathbb{R}_{\geq 0})$ *such that*

$$\begin{aligned} &|F(t, z, a)| \leq c(t) \qquad \forall\, t \in [t_1, t_2],\ z \in \mathbb{R}^k,\ a \in \mathscr{A}(t), \\ &|F(t, z_1, a) - F(t, z_2, a)| \leq c(t)|z_1 - z_2| \qquad \forall\, t \in [t_1, t_2],\ z_1,\ z_2 \in \mathbb{R}^k,\ a \in \mathscr{A}(t). \end{aligned} \tag{2.5.2}$$

Then, for any $\check{z} \in \mathbb{R}^k$ *and any* $\alpha \in \mathcal{A}([t_1, t_2])$ *there exists exactly one function* $x \in W^{1,1}([t_1, t_2], \mathbb{R}^k)$ *with* $x(t_1) = \check{z}$ *such that* (α, x) *is a process for* (2.5.1)*. We will denote by* $x[\check{z}, \alpha]$ *such function. In addition, assume that there exist a compact subset* $A \subset \mathbb{R}^l$ *such that* $\mathscr{A}(t) \subset A$ *for any* $t \in [t_1, t_2]$ *and a continuous increasing function* $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *with* $\rho(0) = 0$ *that fulfills*

$$|F(t, z, a_1) - F(t, z, a_2)| \leq \rho(|a_1 - a_2|) \qquad \forall (t, z, a_1),\ (t, z, a_2) \in [t_1, t_2] \times \mathbb{R}^k \times A. \tag{2.5.3}$$

Then for any sequence $(\alpha_i) \subset \mathcal{A}([t_1, t_2])$ *such that* $\alpha_i \to \alpha$ *in* L^1 *and for any sequence* $(z_i) \subset \mathbb{R}^k$ *such that* $z_i \to \check{z}$*, one has that* $x[z_i, \alpha_i] \to x[\check{z}, \alpha]$ *in* L^∞*. Namely, the input-output map* $(\check{z}, \alpha) \mapsto x[\check{z}, \alpha]$ *is continuous if the space of initial states is endowed with the Euclidean norm, the space of controls with the* L^1*-norm and the space of trajectories with the* L^∞*-norm.*

Proof. The first assertion follows by Theorem 2.3.3 applied to the function $\mathfrak{F}(t, z) := \{F(t, z, \alpha(t))\}$, since the first condition in (2.5.2) implies (2.3.2), while the second one implies (2.3.1) (this latter, for any $\tilde{x} \in W^{1,1}$).

The second statement is not completely trivial, hence we provide some hints. Consider a subsequence (α_{i_j}) of (α_i), then there exists a subsequence $(\alpha_{i_{j_l}})$ of (α_{i_j}) such that $\alpha_{i_{j_l}}(t) \to \alpha(t)$ for a.e. $t \in [t_1, t_2]$, as $l \to \infty$. Set $x_{i_{j_l}} := x[z_{i_{j_l}}, \alpha_{i_{j_l}}]$ and

$x := x[\check{z}, \alpha]$, so that one has

$$\begin{aligned}|x(t) - x_{i_{j_l}}(t)| &\le |z_{i_{j_l}} - \check{z}| + \int_{t_1}^{t} |F(s, x(s), \alpha(s)) - F(s, x_{i_{j_l}}(s), \alpha_{i_{j_l}}(s))| \, ds \\ &\le |z_{i_{j_l}} - \check{z}| + \int_{t_1}^{t} c(s)|x(s) - x_{i_{j_l}}(s)| \, ds + \int_{t_1}^{t} \rho(|\alpha(s) - \alpha_{i_{j_l}}(s)|) \, ds.\end{aligned}$$

where in the last step we have added and subtracted $F(s, x_{i_{j_l}}(s), \alpha(s))$. Since the right hand side of the above inequality is a nondecreasing function of t, it follows

$$\|x - x_{i_{j_l}}\|_{L^\infty(t_1,t)} \le |z_{i_{j_l}} - \check{z}| + \int_{t_1}^{t} c(s)\|x - x_{i_{j_l}}\|_{L^\infty(t_1,s)} \, ds + \int_{t_1}^{t} \rho(|\alpha(s) - \alpha_{i_{j_l}}(s)|) \, ds.$$

By applying the Gronwall's Lemma 2.2.3 to the function $t \mapsto \|x - x_{i_{j_l}}\|_{L^\infty(t_1,t)}$ one obtains

$$\|x - x_{i_{j_l}}\|_{L^\infty(t_1,t_2)} \le \Big(|z_{i_{j_l}} - \check{z}| + \int_{t_1}^{t_2} \rho(|\alpha(t) - \alpha_{i_{j_l}}(t)|) \, dt\Big) e^{\int_{t_1}^{t_2} c(t)\, dt}.$$

Now $\displaystyle\int_{t_1}^{t_2} \rho(|\alpha(t) - \alpha_{i_{j_l}}(t)|) \, dt \to 0$ by the Dominated convergence Theorem 2.2.4, so that the conclusion follows by the arbitrariness of the first subsequence (α_{i_j}). □

Remark 2.5.3. The input-output map can be made Lipschitz continuous with respect to the controls without requiring (2.5.3), but endowing the set of controls with the so called "Ekeland distance" $d_{\mathcal{A}}$ in place of the L^1-norm (see for instance [20, Sec. 3.2]), where

$$d_{\mathcal{A}}(\alpha, \alpha') := \ell(\{t \in [t_1, t_2] : \alpha(t) \ne \alpha'(t)\}) \qquad \forall \alpha, \alpha' \in \mathcal{A}([t_1, t_2]).$$

We conclude this subsection introducing the following relaxed control system.

$$\begin{cases} \dot{x}(t) = \sum_{j=0}^{k} \gamma^j(t) F(t, x(t), \alpha^j(t)) & \text{a.e. } t \in [t_1, t_2], \\ (\gamma^0(t), \dots, \gamma^k(t)) \in \Delta_k & \text{a.e. } t \in [t_1, t_2], \\ (\alpha^0, \dots, \alpha^k) \in \mathcal{A}^{k+1}([t_1, t_2]). \end{cases} \tag{2.5.4}$$

In view of the Caratheodory's Theorem 2.2.1, the Relaxation Theorem 2.3.5 and the Filippov's selection Theorem 2.5.1, it is straightforward that the set of trajectories to control system (2.5.1) is dense in the set of trajectories to the relaxed control system (2.5.4).

We recall that the relaxation via convexification is equivalent to an extension in measure of the control system, where the new set of control is composed by the probability measures on the ordinary control set (see [16, 77]).

2.5.2 Existence of minimizers

We have previously introduced Control Theory emphasizing that control functions are chosen by an external agent in order to reach some preassigned goal. Classic purposes for a controller might be to steer the system from one state to another one, or to optimize a given performance criterion. In the first case one is interested in Stabilizability Theory (see for instance the monographs [18, 30] and the articles [29, 36]), in the second case in Optimal Control Theory, subject of this thesis.

The goal of this subsection is to provide a sufficient condition for the existence of minimizers of an optimal control problem, which is very important in order to motivate the particular extensions of optimization problems that we will consider in the next chapters. A classic example of optimal control problem is given by

$$(P)\begin{cases} \text{Minimize } \Phi(t_1, x(t_1), t_2, x(t_2)) + \int_{t_1}^{t_2} \mathcal{L}(t, x(t), \alpha(t))\, dt \\ \text{over } (t_1, t_2, \alpha, x) \in \mathbb{R} \times \mathbb{R} \times \mathcal{M}([t_1, t_2], \mathbb{R}^l) \times W^{1,1}([t_1, t_2], \mathbb{R}^k) \text{ satisfying} \\ \dot{x}(t) = F(t, x(t), \alpha(t)) \qquad \text{a.e. } t \in [t_1, t_2], \\ \alpha(t) \in \mathscr{A}(t) \qquad \text{a.e. } t \in [t_1, t_2], \\ (t_1, x(t_1), t_2, x(t_2)) \in \mathcal{C}, \\ \psi(t, x(t)) \le 0 \qquad \forall t \in [t_1, t_2]. \end{cases}$$

The data comprise functions $\Phi : \mathbb{R}^{1+k+1+k} \to \mathbb{R}$, $\mathcal{L} : \mathbb{R}^{1+k+m} \to \mathbb{R}$, $F : \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^k$ and $\psi : \mathbb{R}^{1+k} \to \mathbb{R}$, a set-valued map $\mathscr{A} : \mathbb{R} \rightsquigarrow \mathbb{R}^l$ and a set $\mathcal{C} \subset \mathbb{R}^{1+k+1+k}$. We say that $(\bar{t}_1, \bar{t}_2, \bar{\alpha}, \bar{x})$ is a *feasible process* if it satisfies all the constraints in (P).

Moreover, a feasible process is said to be a *minimizer* if

$$\Phi(\bar{t}_1, \bar{x}(\bar{t}_1), \bar{t}_2, \bar{x}(\bar{t}_2)) + \int_{\bar{t}_1}^{\bar{t}_2} \mathcal{L}(t, \bar{x}(t), \bar{\alpha}(t))\, dt$$
$$\leq \Phi(t_1, x(t_1), t_2, x(t_2)) + \int_{t_1}^{t_2} \mathcal{L}(t, x(t), \alpha(t))\, dt$$

for any feasible process (t_1, t_2, α, x).

We are now ready for our existence theorem in the simpler case $\mathcal{L} \equiv 0$. If the reader is interested to understand how the result modifies for nonzero lagrangian, we suggest him to browse [20, Sec. 5.2].

Theorem 2.5.4. *Let $\mathscr{A}$ be a compact multifunction, let $\mathrm{Gr}(\mathscr{A})$ be an $\mathscr{L} \times \mathscr{B}^l$-measurable set, let $\mathcal{L} \equiv 0$, let $\Phi : \mathbb{R}^{1+k+1+k} \to \mathbb{R}$ be lower semicontinuous, let $\mathcal{C}$ be a closed subset of $\mathbb{R}^{1+k+1+k}$ such that $\mathrm{proj}_{(t_1,x_1,t_2)}\mathcal{C}$ is bounded, and let ψ be Lipschitz continuous in the state variable, uniformly with respect to the time variable. Assume that for any $(z, a) \in \mathbb{R}^k \times \mathbb{R}^l$ the function $\mathbb{R} \ni t \mapsto F(t, z, a)$ is Lebesgue measurable and that for any $t \in \mathbb{R}$ the function $\mathbb{R}^k \times \mathbb{R}^l \ni (z, a) \mapsto F(t, z, a)$ is continuous. Moreover, suppose that the set-valued map of the velocities Vel_F is convex and that there exists $C \in L^1(\mathbb{R}, \mathbb{R}_{\geq 0})$ satisfying*

$$|F(t, z, a)| \leq C(t)(1 + |z|) \qquad \text{for any } (t, z, a) \in \mathbb{R}^{1+k+m}. \tag{2.5.5}$$

Then, if there exists at least one feasible process, the optimization problem (P) admits a minimizer.

The above result can be easily proven with a standard extraction of subsequence argument that relies on Theorem 2.3.4. Indeed, the continuity of F with respect to the control variable a together with the compactness of $\mathscr{A}$ implies that Vel_F is closed, while the particular bound (2.5.5) allows the application of the Gronwall's Lemma 2.2.3 in order to deduce that any minimizing sequence fulfills conditions (ii)-(iii) in the statement of Theorem 2.3.4. Finally the boundedness of $\mathrm{proj}_{x_1}\mathcal{C}$ implies that also condition (i) is met, while the boundedness of $\mathrm{proj}_{(t_1,t_2)}\mathcal{C}$ implies that all trajectories of the minimizing sequence can be defined by constant extrapolation in a common compact interval $[\bar{t}_1, \bar{t}_2]$. The other hypotheses permit to deduce that the extraction argument produces a minimizer.

2.5.3 A nonsmooth Pontryagin's Maximum Principle

In this subsection we announce the necessary conditions of optimality for a state constrained free end-time optimal control problem with measurable time dependence. This result has been stated for the first time in [46] in a differential inclusions setting and assuming that the state constraint is inactive at the optimal end-times (see also [46, 74]), and it was eventually improved in [67] with the removal of that demanding assumption. To this aim, we first need to introduce a set-valued map that can be seen as the generalization of the point evaluation of a continuous function.

Definition 2.5.5. Given an open interval $I \subset \mathbb{R}$, an essentially bounded function $g : I \to \mathbb{R}$ and a point $\bar{t} \in I$, the *essential value* of g at $\bar{t}$ is the set

$$\operatorname*{ess}_{t\to\bar{t}} g(t) := \left[\lim_{\delta\downarrow 0}\left(\operatorname*{ess\,inf}_{t\in[\bar{t}-\delta,\bar{t}+\bar{\delta}]} g(t)\right),\ \lim_{\delta\downarrow 0}\left(\operatorname*{ess\,sup}_{t\in[\bar{t}-\delta,\bar{t}+\bar{\delta}]} g(t)\right)\right],$$

where "ess inf" and "ess sup" denote the essential infimum and the essential supremum, respectively.

It is immediate to see that if g has left and right limits $g(\bar{t}^-)$ and $g(\bar{t}^+)$, then $\operatorname*{ess}_{t\to\bar{t}} g(t) = [g(\bar{t}^-) \wedge g(\bar{t}^+),\ g(\bar{t}^-) \vee g(\bar{t}^+)]$. As a consequence, if g is continuous at $\bar{t}$, then $\operatorname*{ess}_{t\to\bar{t}} g(t) = \{g(\bar{t})\}$. In the following proposition we provide a stability property of the essential values (see [69, Prop. 8.3.2]).

Proposition 2.5.6. *Let $I \subset \mathbb{R}$ be an open interval, let $Z \subset \mathbb{R}^k$, let $g : I \times Z \to \mathbb{R}$ and let $(\bar{t}, \bar{z}, \bar{\xi}) \in I \times Z \times \mathbb{R}$. Assume that for any $z \in Z$ the function $I \ni t \mapsto g(t, z)$ is essentially bounded and that the function $Z \ni z \mapsto g(t, z)$ is continuous, uniformly with respect to $t \in I$. Then, if $(z_i) \subset Z$, $(t_i) \subset I$ and $(\xi_i) \subset \mathbb{R}$ are such that $z_i \to \bar{z}$, $t_i \to \bar{t}$, $\xi_i \to \bar{\xi}$ and $\xi_i \in \operatorname*{ess}_{t\to t_i} g(t, z_i)$ for any i, then one has $\bar{\xi} \in \operatorname*{ess}_{t\to\bar{t}} g(t, \bar{z})$.*

Before stating the Pontryagin's Maximum Principle, we introduce the *unmaximized Hamiltonian* defined by

$$H_\lambda(t, z, p, a) := p \cdot F(t, z, a) - \lambda\mathcal{L}(t, z, a) \qquad \forall (t, z, p, a) \in \mathbb{R}^{1+k+k+l} \text{ and } \forall\lambda \geq 0.$$

Theorem 2.5.7 (Nonsmooth Maximum Principle)**.** *Suppose that $\mathcal{C}$ is a closed set, Φ is locally Lipschitz continuous and* $\mathrm{Gr}(\mathscr{A})$ *is an $\mathscr{L} \times \mathscr{B}^l$-measurable set. Assume that ψ is continuous and that there exists $L_\psi > 0$ such that*

$$|\psi(t, z_1) - \psi(t, z_2)| \leq L_\psi |z_1 - z_2| \qquad \forall (t, z_1), (t, z_2) \in \mathbb{R} \times \mathbb{R}^k.$$

Moreover, suppose that for any $z \in \mathbb{R}^k$ the function $\mathbb{R} \times \mathbb{R}^l \ni (t, a) \mapsto (\mathcal{L}, F)(t, x, a)$ is $\mathscr{L} \times \mathbb{B}^l$-measurable, that for any $(t, a) \in \mathbb{R} \times \mathbb{R}^l$ the function $\mathbb{R}^k \ni x \mapsto (\mathcal{L}, F)(t, x, a)$ is continuous and that there exists $c \in L^1(\mathbb{R}, \mathbb{R}_{\geq 0})$ such that

$$\begin{aligned} &|(\mathcal{L}, F)(t, z, a)| \leq c(t) \qquad \forall (t, z, a) \in \mathbb{R}^{1+k+l}, \\ &|(\mathcal{L}, F)(t, z_1, a) - (\mathcal{L}, F)(t, z_2, a)| \leq c(t) \qquad \forall (t, z_1, a), (t, z_2, a) \in \mathbb{R}^{1+k+l}. \end{aligned}$$

Let $(\bar{t}_1, \bar{t}_2, \bar{\alpha}, \bar{x})$ be a minimizer for (P). Then there exist a path $p \in W^{1,1}([\bar{t}_1, \bar{t}_2], \mathbb{R}^k)$, numbers h_1, $h_2 \in \mathbb{R}$, $\lambda \geq 0$, $\beta_1 \geq 0$, $\beta_2 \geq 0$, a measure $\mu \in C^{\oplus}([\bar{t}_1, \bar{t}_2])$, and a Borel measurable and μ-integrable function $m : [\bar{t}_1, \bar{t}_2] \to \mathbb{R}^k$, such that:

$$\|p\|_{L^\infty([\bar{t}_1, \bar{t}_2])} + \mu([\bar{t}_1, \bar{t}_2]) + \beta_1 + \beta_2 + \lambda \neq 0;$$
$$-\dot{p}(t) \in \operatorname{co} \partial_x H_\lambda(t, \bar{x}(t), q(t), \alpha(t)) \qquad a.e.\ t \in [\bar{t}_1, \bar{t}_2];$$

$$\begin{aligned} (-h_1, p(\bar{t}_1), h_2, -q(\bar{t}_2)) \in\ & \lambda \partial \Phi(\bar{t}_1, \bar{x}(\bar{t}_1), \bar{t}_2, \bar{x}(\bar{t}_2)) \\ & + N_{\mathcal{C}}(\bar{t}_1, \bar{x}(\bar{t}_1), \bar{t}_2, \bar{x}(\bar{t}_2)) + \beta_1 \partial \psi(\bar{t}_1, \bar{x}(\bar{t}_1)) \times \beta_2 \partial \psi(\bar{t}_2, \bar{x}(\bar{t}_2)); \end{aligned} \tag{2.5.6}$$

$$h_j \in \operatorname*{ess}_{t \to \bar{t}_j} \Big(\max_{a \in \mathscr{A}(t)} H_\lambda(t, \bar{x}(\bar{t}_j), a, q(\bar{t}_j)) \Big) \qquad \textit{for } j = 1, 2; \tag{2.5.7}$$

$$H_\lambda(t, \bar{x}(t), q(t), \bar{\alpha}(t)) = \max_{a \in \mathscr{A}(t)} H_\lambda(t, \bar{x}(t), q(t), a) \qquad a.e.\ t \in [\bar{t}_1, \bar{t}_2];$$

$$m(t) \in \partial_x^{>} \psi\,(t, \bar{x}(t)) \qquad \mu\text{-}a.e.\ t \in [\bar{t}_1, \bar{t}_2];$$

$$\operatorname{spt}(\mu) \subseteq \{t \in [\bar{t}_1, \bar{t}_2] \,:\, \psi\,(t, \bar{x}(t)) = 0\},$$

where $q : [\bar{t}_1, \bar{t}_2] \to \mathbb{R}^k$ is defined by

$$q(t) := \begin{cases} p(t) + \int_{[\bar{t}_1, t[} m(t') \mu(dt') & t \in [\bar{t}_1, \bar{t}_2[, \\ p(\bar{t}_2) + \int_{[\bar{t}_1, \bar{t}_2]} m(t') \mu(dt') & t = \bar{t}_2. \end{cases}$$

Furthermore, for $j \in \{1, 2\}$, $\beta_j = 0$ *if either* $\psi(\bar{t}_j, \bar{x}(\bar{t}_j)) < 0$ *or the* t_j*-component of the endpoint constraint set* $\mathcal{C}$ *is the single point* $\{\bar{t}_j\}$.

Theorem 2.5.7 holds even for local notions of minimizer, in particular for that we will introduce in the next chapter (see Definition 3.0.3).

Moreover, in view of the last statement, Theorem 2.5.7 applies also to fixed end-time problems, i.e. when $\mathrm{proj}_{(t_1,t_2)}\mathcal{C} = \{\bar{t}_1\} \times \{\bar{t}_2\}$ for some $\bar{t}_1$, $\bar{t}_2 \in \mathbb{R}$, $\bar{t}_1 < \bar{t}_2$. In this case, condition (2.5.7) are of no longer use, since in the transversality condition (2.5.6) one has $N_{\{\bar{t}_j\}}(\bar{t}_j) = \mathbb{R}$ for $j = 1, 2$.

Chapter 3

Free end-time problems with measurable time dependence

For any pair t_1, $t_2 \in \mathbb{R}$, $t_1 < t_2$, consider the *original control system*

$$\begin{cases} \dot{x}(t) = \mathcal{F}(t, x(t), \omega(t), u(t)), \\ \omega(t) \in \mathscr{V}(t), \quad u(t) \in \mathscr{U}(t), \end{cases} \qquad \text{a.e. } t \in [t_1, t_2] \tag{3.0.1}$$

and the state and endpoint constraints

$$\psi(t, x(t)) \leq 0 \quad \forall t \in [t_1, t_2], \qquad (t_1, x(t_1), t_2, x(t_2)) \in \mathcal{C}. \tag{3.0.2}$$

The data comprise the functions $\mathcal{F} : \mathbb{R}^{1+n+m+q} \to \mathbb{R}^n$, $\psi : \mathbb{R}^{1+n} \to \mathbb{R}$, the closed set $\mathcal{C} \subset \mathbb{R}^{1+n+1+n}$, and the set-valued maps $\mathscr{U} : \mathbb{R} \rightsquigarrow \mathbb{R}^q$, $\mathscr{V} : \mathbb{R} \rightsquigarrow \mathbb{R}^m$, where $\mathscr{U}$ takes as values compact sets while the values of $\mathscr{V}$ are bounded but not necessarily closed sets. Generic elements of the domain of $\mathcal{F}$ will be denoted by (t, z, w, v).

In order to introduce the precise concepts of strict sense, extended and relaxed extended process, for any pair t_1, $t_2 \in \mathbb{R}$, $t_1 < t_2$, we set

$$\begin{aligned} &\mathcal{U}([t_1, t_2]) := \{u \in \mathcal{M}([t_1, t_2]; \mathbb{R}^q) : \ u(t) \in \mathscr{U}(t) \text{ a.e. } t \in [t_1, t_2]\}, \\ &\mathcal{V}([t_1, t_2]) := \{\omega \in \mathcal{M}([t_1, t_2]; \mathbb{R}^m) : \ \omega(t) \in \mathscr{V}(t) \text{ a.e. } t \in [t_1, t_2]\}, \\ &\mathcal{W}([t_1, t_2]) := \{\omega \in \mathcal{M}([t_1, t_2]; \mathbb{R}^m) : \ \omega(t) \in \overline{\mathscr{V}(t)} \text{ a.e. } t \in [t_1, t_2]\}, \\ &\Gamma([t_1, t_2]) := \mathcal{M}([t_1, t_2]; \Delta_n). \end{aligned}$$

Definition 3.0.1. We refer to any element (t_1, t_2, ω, u, x) with $t_1 < t_2$, controls $\omega \in \mathcal{W}([t_1, t_2])$, $u \in \mathcal{U}([t_1, t_2])$, and trajectory $x \in W^{1,1}([t_1, t_2]; \mathbb{R}^n)$ that satisfies

$$\dot{x}(t) = \mathcal{F}(t, x(t), \omega(t), u(t)) \quad \text{a.e. } t \in [t_1, t_2],$$

as *extended process*. An extended process (t_1, t_2, ω, u, x) is called a *strict sense process* if $\omega \in \mathcal{V}([t_1, t_2])$. A strict sense or extended process is *feasible* when it fulfills (3.0.2), namely, if $\psi(t, x(t)) \leq 0$ for all $t \in [t_1, t_2]$ and $(t_1, x(t_1), t_2, x(t_2)) \in \mathcal{C}$.

We define *relaxed (extended) process* any element $(t_1, t_2, \underline{\omega}, \underline{u}, \gamma, x)$, where $t_1 < t_2$, $\underline{\omega} \in \mathcal{W}^{1+n}([t_1, t_2])$, $\underline{u} \in \mathcal{U}^{1+n}([t_1, t_2])$, $\gamma \in \Gamma([t_1, t_2])$, and $x \in W^{1,1}([t_1, t_2]; \mathbb{R}^n)$ satisfies

$$\dot{x}(t) = \sum_{j=0}^{n} \gamma^j(t) \mathcal{F}(t, x(t), \omega^j(t), u^j(t)) \quad \text{a.e. } t \in [t_1, t_2]. \tag{3.0.3}$$

A relaxed process is *feasible* when it satisfies (3.0.2). We will use Σ, Σ_e, and Σ_r to denote the subsets of feasible strict sense, feasible extended, and feasible relaxed extended processes, respectively.

Remark 3.0.2. We do observe that we can identify any extended process (t_1, t_2, ω, u, x) with the relaxed process $(t_1, t_2, \underline{\omega}, \underline{u}, \gamma, x)$ having $\underline{\omega} = (\omega, \dots, \omega)$, $\underline{u} = (u, \dots, u)$ and $\gamma = (1+n)^{-1}(1, \dots, 1)$. As a consequence, we get $\Sigma \subseteq \Sigma_e \subseteq \Sigma_r$.

Since we are interested in local properties, we introduce a concept of distance between trajectories, including left and right endpoints. Precisely, for all (t_1, t_2, x), (t_1', t_2', x') with $t_1 < t_2$, $t_1' < t_2'$, and $x : [t_1, t_2] \to \mathbb{R}^n$, $x' : [t_1', t_2'] \to \mathbb{R}^n$ continuous functions, we define the distance

$$d_\infty\big((t_1, t_2, x), (t_1', t_2', x')\big) := |t_1 - t_1'| + |t_2 - t_2'| + \|\tilde{x} - \tilde{x}'\|_{L^\infty(\mathbb{R})}, \tag{3.0.4}$$

where $\tilde{x} : \mathbb{R} \to \mathbb{R}^n$ denotes the extension of the function x obtained by setting $\tilde{x}(t) := x(t_1)$ for all $t < t_1$ and $\tilde{x}(t) := x(t_2)$ for all $t > t_2$.

We now define *local* notions of *minimum, infimum gap*, and *controllability.*

Definition 3.0.3. Let $\tilde{\Sigma} \in \{\Sigma, \Sigma_e, \Sigma_r\}$. Given a continuous function $\Phi : \mathbb{R}^{1+n+1+n} \to \mathbb{R}$, a process $\mathfrak{Z} := (\bar{t}_1, \bar{t}_2, \underline{\bar{\omega}}, \underline{\bar{u}}, \bar{\gamma}, \bar{x}) \in \tilde{\Sigma}$ is called a *local Φ-minimizer for problem* $(P_{\tilde{\Sigma}})$

if, for some $\delta > 0$, one has

$$\Phi(\bar{t}_1, \bar{x}(\bar{t}_1), \bar{t}_2, \bar{x}(\bar{t}_2)) = \min \Big\{ \Phi(t_1, x(t_1), t_2, x(t_2)) : \quad (t_1, t_2, \underline{\omega}, \underline{u}, \gamma, x) \in \tilde{\Sigma}, \\ d_\infty((t_1, t_2, x), (\bar{t}_1, \bar{t}_2, \bar{x})) < \delta \Big\}.$$

The process $\mathfrak{Z} \in \tilde{\Sigma}$ is a (*global*) Φ-*minimizer for problem* $(P_{\tilde{\Sigma}})$ if

$$\Phi(\bar{t}_1, \bar{x}(\bar{t}_1), \bar{t}_2, \bar{x}(\bar{t}_2)) = \min_{\tilde{\Sigma}} \Phi(t_1, x(t_1), t_2, x(t_2)).$$

Definition 3.0.4. Let $\Phi : \mathbb{R}^{1+n+1+n} \to \mathbb{R}$ be a continuous function. Fix $\mathfrak{Z} := (\bar{t}_1, \bar{t}_2, \bar{\underline{\omega}}, \bar{\underline{u}}, \bar{\gamma}, \bar{x}) \in \Sigma_r$. We say that *at $\mathfrak{Z}$ there is a local Φ-infimum gap* if, for some $\delta > 0$,

$$\Phi(\bar{t}_1, \bar{x}(\bar{t}_1), \bar{t}_2, \bar{x}(\bar{t}_2)) < \inf \Big\{ \Phi(t_1, x(t_1), t_2, x(t_2)) : (t_1, t_2, \omega, u, x) \in \Sigma, \\ d_\infty\big((t_1, t_2, x), (\bar{t}_1, \bar{t}_2, \bar{x})\big) < \delta \Big\}.^{6} \tag{3.0.5}$$

Definition 3.0.5. Let us fix a process $\mathfrak{Z} := (\bar{t}_1, \bar{t}_2, \bar{\underline{\omega}}, \bar{\underline{u}}, \bar{\gamma}, \bar{x}) \in \Sigma_r$. We call $\mathfrak{Z}$ *isolated* if, for some $\delta > 0$,

$$\Big\{ (t_1, t_2, \omega, u, x) \in \Sigma : \quad d_\infty\big((t_1, t_2, x), (\bar{t}_1, \bar{t}_2, \bar{x})\big) < \delta \Big\} = \emptyset.$$

We say that *the constrained control system* (3.0.1)-(3.0.2) *is controllable to* $\mathfrak{Z}$ if $\mathfrak{Z}$ is not isolated, that is, for any $\varepsilon > 0$ there is some $(t_1, t_2, \omega, u, x) \in \Sigma$ such that $d_\infty\big((t_1, t_2, x), (\bar{t}_1, \bar{t}_2, \bar{x})\big) < \varepsilon$[7].

The notion of local infimum gap is a purely dynamical property, independent of the cost function Φ. Precisely, it turns out to be equivalent to the topological property of isolation.

Proposition 3.0.6. *Given a feasible relaxed process $\mathfrak{Z} := (\bar{t}_1, \bar{t}_2, \bar{\underline{\omega}}, \bar{\underline{u}}, \bar{\gamma}, \bar{x})$, the following properties are equivalent:*

[6] As customary, when the set is empty we set the infimum equal to $+\infty$.

[7] We point out that the definitions of isolation and controllability could be given for arcs $y : [t_1, t_2] \to \mathbb{R}^n$ that are not necessarily feasible relaxed processes.

(i) $\mathfrak{Z}$ *is isolated;*

(ii) *for every continuous function* Φ*, at* $\mathfrak{Z}$ *there is a local* Φ*-infimum gap;*

(iii) *given a continuous function* Φ*, at* $\mathfrak{Z}$ *there is a local* Φ*-infimum gap.*

Proof. The implication (i)⇒(ii) is immediate, since if $\mathfrak{Z}$ is isolated, then the right-hand-side in (3.0.5) is equal to $+\infty$. Also the fact that (ii)⇒(iii) is obvious. It remains only to show that (iii)⇒(i). Assume by contradiction that (iii) holds true but $\mathfrak{Z}$ is not isolated. Then, for some $\delta > 0$ as in Definition 3.0.4 and any sequence $(\varepsilon_i)_i \subset]0, \delta[$, $\varepsilon_i \downarrow 0$, there exists a sequence of feasible strict sense processes $(t_{1_i}, t_{2_i}, \omega_i, u_i, x_i) \in \Sigma$ such that $d_\infty\big((t_{1_i}, t_{2_i}, x_i), (\bar{t}_1, \bar{t}_2, \bar{x})\big) < \varepsilon_i < \delta$, so that

$$\Phi(\bar{t}_1, \bar{x}(\bar{t}_1), \bar{t}_2, \bar{x}(\bar{t}_2)) \leq \inf \Big\{ \Phi(t_1, x(t_1), t_2, x(t_2)) : \quad (t_1, t_2, \omega, u, x) \in \Sigma, \\ d_\infty\big((t_1, t_2, x), (\bar{t}_1, \bar{t}_2, \bar{x})\big) < \delta \Big\} - \eta \leq \Phi(t_{1_i}, x_i(t_{1_i}), t_{2_i}, x_i(t_{2_i})) - \eta,$$

for some $\eta > 0$. As $i \to +\infty$, we get the desired contradiction and the proof is complete. □

From Proposition 3.0.6 it follows that having a local Φ-infimum gap at $\mathfrak{Z}$ is independent of the choice of Φ. For this reason, in the following we simply say that *at* $\mathfrak{Z}$ *there is a local infimum gap.*

From now on, we consider two related problems: (i) find necessary conditions to have at a feasible relaxed process $\mathfrak{Z}$ a local infimum gap; (ii) determine sufficient controllability conditions for the original constrained control system to a feasible relaxed process $\mathfrak{Z}$. Note that, even when the set of strict sense processes is d_∞-dense in the set of relaxed processes, a local infimum gap may occur. Indeed, the presence of constraints might imply that there does not exist any sequence of feasible approximating strict sense processes.

As a first main result, we prove that, if a local infimum gap occurs at a feasible relaxed process $\mathfrak{Z}$, then the free end-time, constrained, nonsmooth version of the Maximum Principle is valid in abnormal form – i.e., with zero cost multiplier – at $\mathfrak{Z}$. We derive as corollaries that: *normality* of multipliers – i.e., all sets of multipliers with cost multiplier $\neq 0$ – guarantees the absence of gap, and *non-existence of non trivial* – i.e., not identically zero – *abnormal multipliers* implies controllability.

However, when the state constraint is active at the initial point, a situation which is difficult to exclude a priori, it is well known that there can always be degenerate multipliers, with zero cost multiplier. In this case, a normality test for gap avoidance becomes useless, unless only nondegenerate multipliers can be considered. This is the question we address in Section 3.2. Here, under some additional constraint qualification conditions, we prove that, if there is a local infimum gap at a feasible relaxed process, then $\mathfrak{Z}$ is *nondegenerate abnormal*, that is, abnormal for a nondegenerate version of the Maximum Principle considered above.

Controllability of a control system to a reference trajectory, which might not solve the original system, and occurrence of infimum gaps, when the original class of processes is extended in order to achieve existence of minimizers, are largely investigated issues. In particular, links between these properties and normality of multipliers in the Maximum Principle have been established since the early works [76, 78, 79, 45], up to the more recent results [60, 61, 58, 59, 37, 38, 39].

The novelties of our results lie, on the one hand, into the generality of the extension, which includes as particular cases both the convex relaxation investigated in [60, 61] and the impulsive extension treated in [58, 37], allowing for measurable time dependence of the data and (active) state constraints. On the other hand, we relate nondegeneracy with the conditions for no gap occurence.

Apart from the recent paper [72], which, however, only deals with the relaxed problem without state constraints, all previous works have addressed, exclusively, either fixed end-time optimal control problems (see e.g., [76, 79, 45, 59, 60, 61]) or free end-time problems with Lipschitz continuous time dependence and control sets independent of time ([58, 37, 38, 39]). We point out that the Lipschitz case differs substantially from the case with measurable time dependence of the data, in that the former can be reduced to a fixed end-time problem by a change of independent variable. Free end-time problems with measurable time dependence and state constraints have received considerable attention since the late '80s, especially in relation to the study of optimality conditions (see e.g. [46, 74] and references therein). In particular, a motivation to investigate situations with active state constraint at the optimal free end-times came from the observation that a minimizing trajectory evolving on the boundary of the constraint set and terminating at a discontinuity point of the dynamics was a frequently encountered phenomenon in a variety of threshold problems (associated, for instance, with abrupt changes in a tariff or rate of return on investment at prespecified times, as described in [31, 67] and references therein).

The question of determining sufficient conditions to avoid the gap in the form of nondegenerate normality conditions, has been addressed for the first time only recently, in [37, 38, 40]. In particular, in [37] we introduced, just for the impulsive extension, sufficient conditions for each set of multipliers to be nondegenerate. These conditions, however, did not cover the case of fixed initial point, for which we provided sufficient nondegeneracy conditions in [38, 40].

In the present chapter, we unify and extend all the previous results to the general free end-time problem with measurable time dependence and with time-dependent control constraint sets. The results we state are based on those in [40], but with milder assumptions on the (time-varying) control sets and on the dynamics function.

It is worth mentioning that, although our conditions are partially inspired by well-known conditions for the nondegeneracy of the Maximum Principle (see for instance [7, 33, 34, 47, 57] and references therein), the techniques of the proofs utilized in Section 3.3 below are original. In particular, by means of perturbation and penalization techniques and by Ekeland's variational principle, we construct a sequence of approximating problems with strict sense optimal processes, whose multipliers are shown to converge to an abnormal nondegenerate multiplier for the given relaxed extended process $\mathfrak{Z}$.

3.1 Sufficient conditions for no gap

Now we explain the assumptions we make on the data and we give the definition of extremal. Afterwards, we establish a link between the occurrence of gap phenomena and the presence of abnormal extremals.

3.1.1 Main assumptions

The hypotheses we invoke are of local nature: they relate to a reference feasible relaxed process $(\bar{t}_1, \bar{t}_2, \underline{\bar{\omega}}, \underline{\bar{u}}, \bar{\gamma}, \bar{x})$ and a parameter $\eta > 0$. In particular, we define the *η-tube* of the process $(\bar{t}_1, \bar{t}_2, \underline{\bar{\omega}}, \underline{\bar{u}}, \bar{\gamma}, \bar{x})$ as

$$\mathcal{S}_\eta := \{(t,z) \in \mathbb{R} \times \mathbb{R}^n : \quad t \in [\bar{t}_1 - \eta, \bar{t}_2 + \eta], \ |z - \bar{x}(t)| \leq \eta\},$$

where $\bar{x}$ is extended by constant extrapolation.

Hypothesis 3.1.1. The set-valued map $\mathscr{U} : [\bar{t}_1 - \eta, \bar{t}_2 + \eta] \rightsquigarrow \mathbb{R}^q$ has $\mathscr{L} \times \mathscr{B}^q$ measurable graph. The set-valued map $\mathscr{V} : [\bar{t}_1 - \eta, \bar{t}_2 + \eta] \rightsquigarrow W$ is measurable and $W \subseteq \mathbb{R}^m$ is a compact set. Moreover, for every $i \in \mathbb{N}$ there exists a closed set-valued map with $\mathscr{L} \times \mathscr{B}^m$ measurable graph $\mathscr{V}_i : [\bar{t}_1 - \eta, \bar{t}_2 + \eta] \rightsquigarrow W$ such that $\mathscr{V}_i(t) \subseteq \mathscr{V}(t)$ for a.e. $t \in [\bar{t}_1 - \eta, \bar{t}_2 + \eta]$, and

$$d_H(\mathscr{V}_i(t), \overline{\mathscr{V}(t)}) \leq \sigma_i(t) \qquad \text{a.e. } t \in [\bar{t}_1 - \eta, \bar{t}_2 + \eta],$$

where $(\sigma_i) \subset L^1([\bar{t}_1 - \eta, \bar{t}_2 + \eta], \mathbb{R}_{\geq 0}) \cap L^\infty([\bar{t}_1 - \eta, \bar{t}_1 + \eta] \cup [\bar{t}_2 - \eta, \bar{t}_2 + \eta], \mathbb{R}_{\geq 0})$ is such that

$$\|\sigma_i\|_{L^1([\bar{t}_1 - \eta, \bar{t}_2 + \eta])} \to 0, \tag{3.1.1}$$

$$\|\sigma_i\|_{L^\infty([\bar{t}_1 - \eta, \bar{t}_1 + \eta] \cup [\bar{t}_2 - \eta, \bar{t}_2 + \eta])} \to 0. \tag{3.1.2}$$

Hypothesis 3.1.2. The target $\mathcal{C} \subset \mathbb{R}^{1+n+1+n}$ is closed. The constraint function ψ is Lipschitz continuous on $\mathcal{S}_\eta$, i.e. there is some constant $L_\psi > 0$ such that

$$|\psi(t,z) - \psi(t',z')| \leq L_\psi |(t,z) - (t',z')| \qquad \forall (t,z),\ (t',z') \in \mathcal{S}_\eta.$$

Hypothesis 3.1.3. (i) For all $(z,w) \in \mathrm{proj}_{\mathbb{R}^n} \mathcal{S}_\eta \times W$ the function $[\bar{t}_1 - \eta, \bar{t}_2 + \eta] \times \mathbb{R}^q \ni (t,v) \mapsto \mathcal{F}(t,z,w,v)$ is $\mathscr{L} \times \mathscr{B}^q$-measurable[8]. Moreover, there exists $c \in L^1([\bar{t}_1 - \eta, \bar{t}_2 + \eta]; \mathbb{R}_{\geq 0}) \cap L^\infty([\bar{t}_1 - \eta, \bar{t}_1 + \eta] \cup [\bar{t}_2 - \eta, \bar{t}_2 + \eta], \mathbb{R}_{\geq 0})$ such that

$$|\mathcal{F}(t,z,w,v)| \leq c(t), \qquad |\mathcal{F}(t,z',w,v) - \mathcal{F}(t,z,w,v)| \leq c(t)|z' - z|,$$

for all (t,z,w,v), $(t,z',w,v) \in \mathcal{S}_\eta \times W \times \mathscr{U}(t)$. In particular, we assume that there exists $L_{\mathcal{F}} > 0$ such that $c(t) \leq L_{\mathcal{F}}$ for a.e. $t \in [\bar{t}_1 - \eta, \bar{t}_1 + \eta] \cup [\bar{t}_2 - \eta, \bar{t}_2 + \eta]$.
(ii) There exists some continuous increasing function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\rho(0) = 0$

[8] According to Subsection 2.1, $\mathrm{proj}_{\mathbb{R}^n} \mathcal{S}_\eta = \{z \in \mathbb{R}^n : (t,z) \in \mathcal{S}_\eta \text{ for some } t \in [\bar{t}_1 - \eta, \bar{t}_2 + \eta]\}$.

such that, for any $(t, z, v) \in \mathcal{S}_\eta \times \mathscr{U}(t)$, we have

$$\begin{aligned} &|\mathcal{F}(t, z, w', v) - \mathcal{F}(t, z, w, v)| \leq \rho(|w' - w|) \qquad \forall w',\, w \in W, \\ &D_x\mathcal{F}(t, z, w', v) \subseteq D_x\mathcal{F}(t, z, w, v) + \rho(|w' - w|)\,\mathbb{B} \qquad \forall w',\, w \in W. \end{aligned}$$

Remark 3.1.4. Some comments on the hypotheses are in order.

(i) By Hypothesis 3.1.1, for any $\delta > 0$ there exists some $\iota_\delta \in \mathbb{N}$ such that, for every $i \geq \iota_\delta$, one has

$$\int_{\bar{t}_1-\eta}^{\bar{t}_2+\eta} \sigma_i(t')\,dt' \leq \delta, \qquad \|\sigma_i\|_{L^\infty([\bar{t}_1-\eta,\bar{t}_1+\eta]\cup[\bar{t}_2-\eta,\bar{t}_2+\eta])} \leq \delta.$$

Thus, given an arbitrary measurable function $\omega(t) \in \overline{\mathscr{V}(t)}$ for a.e. $t \in [\bar{t}_1 - \eta, \bar{t}_2 + \eta]$, from Theorem 2.3.1 (and Proposition 2.3.2) it follows that there is some measurable selection $\omega_\delta(t) \in \Pi_{\mathscr{V}_{\iota_\delta}(t)}(\omega(t))$ for a.e. $t \in [\bar{t}_1 - \eta, \bar{t}_2 + \eta]$ such that

$$\begin{cases} \|\omega_\delta - \omega\|_{L^1([\bar{t}_1-\eta,\bar{t}_2+\eta])} \leq \displaystyle\int_{\bar{t}_1-\eta}^{\bar{t}_2+\eta} \sigma_{\iota_\delta}(t')\,dt' \leq \delta, \\ \|\omega_\delta - \omega\|_{L^\infty([\bar{t}_1-\eta,\bar{t}_1+\eta]\cup[\bar{t}_2-\eta,\bar{t}_2+\eta])} \leq \delta. \end{cases} \tag{3.1.3}$$

As a consequence, Hypothesis 3.1.1 implies in particular the density of the control set $\mathcal{V}([t_1, t_2])$ in $\mathcal{W}([t_1, t_2])$ in the L^1-norm, for every $t_1,\, t_2 \in \mathbb{R}$ such that $\bar{t}_1 - \eta \leq t_1 < t_2 \leq \bar{t}_2 + \eta$.

(ii) Hypotheses 3.1.2 and 3.1.3(i) are quite standard assumptions, while Hypothesis 3.1.3(ii), which prescribes additional regularity properties of the dynamics $\mathcal{F}$ in the w-variable, reflects the different roles played by the controls u and ω, as only the set of w-control values is extended by replacing $\mathscr{V}(t)$ with $\overline{\mathscr{V}(t)}$ for a.e. t. Hypothesis 3.1.3(ii) is fulfilled, for instance, when $\mathcal{F}(t, z, w, v) = \mathcal{F}_1(t, z, v) + \mathcal{F}_2(t, z, w, v)$, where $\mathcal{F}_1$, $\mathcal{F}_2$ satisfy Hypothesis 3.1.3(i), $\mathcal{F}_2(t, \cdot, w, v)$ is C^1, and $\nabla_x\mathcal{F}_2$ is continuous on the compact set $\mathcal{S}_\eta \times W \times \mathscr{U}(t)$. It is also verified when the dynamics function has a polynomial dependence on the control variable w, with coefficients satisfying Hypothesis 3.1.3(i) in the remaining variables, as proved in Theorem 5.2.5 below (see also Examples 3.2.11, 3.2.12 and 5.3.4).

3.1.2 Abnormality and local infimum gap

In this subsection we state a theorem relating the existence of a gap and the validity of a constrained Maximum Principle in abnormal form for a free-time optimal control problem, in which both end-times are choice variables. From this result we deduce that normality of multipliers is a sufficient condition for gap-avoidance and a controllability.

Definition 3.1.5. Let $\mathfrak{Z} := (\bar{t}_1, \bar{t}_2, \underline{\bar{\omega}}, \underline{\bar{u}}, \bar{\gamma}, \bar{x})$ be a feasible relaxed process for which Hypotheses 3.1.1–3.1.3 are verified. Given a function $\Phi : \mathbb{R}^{1+n+1+n} \to \mathbb{R}$ which is Lipschitz continuous on a neighborhood of $(\bar{t}_1, \bar{x}(\bar{t}_1), \bar{t}_2, \bar{x}(\bar{t}_2))$, we say that $\mathfrak{Z}$ is a Φ-*extremal* if there exist a path $p \in W^{1,1}([\bar{t}_1, \bar{t}_2], \mathbb{R}^n)$, numbers h_1, $h_2 \in \mathbb{R}$, $\lambda \geq 0$, $\beta_1 \geq 0$, $\beta_2 \geq 0$, a measure $\mu \in C^{\oplus}([\bar{t}_1, \bar{t}_2])$, and a Borel measurable and μ-integrable function $m : [\bar{t}_1, \bar{t}_2] \to \mathbb{R}^n$, such that:

$$\|p\|_{L^\infty([\bar{t}_1,\bar{t}_2])} + \mu([\bar{t}_1, \bar{t}_2]) + \beta_1 + \beta_2 + \lambda \neq 0; \tag{3.1.4}$$

$$-\dot{p}(t) \in \sum_{j=0}^{n} \bar{\gamma}^j(t) \operatorname{co} \partial_x \left(q(t) \cdot \mathcal{F}(t, (\bar{x}, \bar{\omega}^j, \bar{u}^j)(t))\right) \qquad \text{a.e. } t \in [\bar{t}_1, \bar{t}_2]; \tag{3.1.5}$$

$$\begin{aligned}(-h_1, p(\bar{t}_1), h_2, -q(\bar{t}_2)) \in{}& \lambda \partial \Phi(\bar{t}_1, \bar{x}(\bar{t}_1), \bar{t}_2, \bar{x}(\bar{t}_2)) \\ &+ N_{\mathcal{C}}(\bar{t}_1, \bar{x}(\bar{t}_1), \bar{t}_2, \bar{x}(\bar{t}_2)) + \beta_1 \partial \psi(\bar{t}_1, \bar{x}(\bar{t}_1)) \times \beta_2 \partial \psi(\bar{t}_2, \bar{x}(\bar{t}_2));\end{aligned} \tag{3.1.6}$$

$$h_j \in \operatorname*{ess}_{t \to \bar{t}_j} \Big(\max_{(w,v) \in \overline{\mathscr{V}(t)} \times \mathscr{U}(t)} q(\bar{t}_j) \cdot \mathcal{F}\big(t, \bar{x}(\bar{t}_j), w, v\big)\Big) \qquad \text{for } j = 1, 2; \tag{3.1.7}$$

for every $j = 0, \dots, n$, for a.e. $t \in [\bar{t}_1, \bar{t}_2]$, one has

$$q(t) \cdot \mathcal{F}\big(t, \bar{x}(t), \bar{\omega}^j(t), \bar{u}^j(t)\big) = \max_{(w,v) \in \overline{\mathscr{V}(t)} \times \mathscr{U}(t)} q(t) \cdot \mathcal{F}\big(t, \bar{x}(t), w, v\big); \tag{3.1.8}$$

$$m(t) \in \partial_x^{>} \psi(t, \bar{x}(t)) \qquad \mu\text{-a.e.}; \tag{3.1.9}$$

$$\operatorname{spt}(\mu) \subseteq \{t \in [\bar{t}_1, \bar{t}_2] : \psi(t, \bar{x}(t)) = 0\}, \tag{3.1.10}$$

where $q : [\bar{t}_1, \bar{t}_2] \to \mathbb{R}^n$ is given by

$$q(t) := \begin{cases} p(t) + \int_{[\bar{t}_1, t[} m(t')\mu(dt') & t \in [\bar{t}_1, \bar{t}_2[, \\ p(\bar{t}_2) + \int_{[\bar{t}_1, \bar{t}_2]} m(t')\mu(dt') & t = \bar{t}_2. \end{cases}$$

Furthermore, for $j \in \{1,2\}$, $\beta_j = 0$ if either $\psi(\bar{t}_j, \bar{x}(\bar{t}_j)) < 0$ or the t_j-component of the endpoint constraint set $\mathcal{C}$ is the single point $\{\bar{t}_j\}$.

We will call a Φ-extremal *normal* if all multipliers $(p, h_1, h_2, \beta_1, \beta_2, \lambda, \mu, m)$ as above have $\lambda > 0$, and *abnormal* when it is not normal. Clearly, an abnormal Φ-extremal is abnormal for every Φ, thus in the following it will be simply called an *abnormal extremal.*

Theorem 3.1.6. *Let* $\mathfrak{Z} := (\bar{t}_1, \bar{t}_2, \underline{\bar{\omega}}, \underline{\bar{u}}, \bar{\gamma}, \bar{x})$ *be a feasible relaxed process for which Hypotheses* 3.1.1–3.1.3 *are verified. If at* $\mathfrak{Z}$ *there is a local infimum gap, then* $\mathfrak{Z}$ *is an abnormal extremal.*

We postpone the proof of this result to Section 3.4 below. As corollaries of Theorem 3.1.6, we get the following results.

Theorem 3.1.7. *Let* $\mathfrak{Z} := (\bar{t}_1, \bar{t}_2, \underline{\bar{\omega}}, \underline{\bar{u}}, \bar{\gamma}, \bar{x})$ *be a feasible relaxed process for which Hypotheses* 3.1.1–3.1.3 *are verified. Let* $\Phi : \mathbb{R}^{1+n+1+n} \to \mathbb{R}$ *be a Lipschitz continuous function on a neighborhood of* $(\bar{t}_1, \bar{x}(\bar{t}_1), \bar{t}_2, \bar{x}(\bar{t}_2))$*. When* $\mathfrak{Z}$ *is a local* Φ*-minimizer for* (P_{Σ_e}) *or* (P_{Σ_r}) *which is a normal* Φ*-extremal, then*

$$\Phi(\bar{t}_1, \bar{x}(\bar{t}_1), \bar{t}_2, \bar{x}(\bar{t}_2)) = \inf \Phi(t_1, x(t_1), t_2, x(t_2)),$$

over all processes $(t_1, t_2, \omega, u, x) \in \Sigma$ *with* $d_\infty((t_1, t_2, x), (\bar{t}_1, \bar{t}_2, \bar{x})) < \delta$.
Similarly, if $\mathfrak{Z}$ *is* Φ*-minimizer for* (P_{Σ_e}) *or* (P_{Σ_r}) *which is a* Φ*-normal extremal, then the above equality holds for the infimum over the whole set* Σ*.*

Proof. Since $\Sigma \subseteq \Sigma_e \subseteq \Sigma_r$, when $\mathfrak{Z}$ is a local Φ-minimizer for (P_{Σ_e}) or (P_{Σ_r}) there exists some $\delta > 0$ such that

$$\begin{aligned}\Phi(\bar{t}_1, \bar{x}(\bar{t}_1), \bar{t}_2, \bar{x}(\bar{t}_2)) \leq \inf \Big\{ &\Phi(t_1, x(t_1), t_2, x(t_2)) : \\ &(t_1, t_2, \omega, u, x) \in \Sigma, \quad d_\infty\big((t_1, t_2, x), (\bar{t}_1, \bar{t}_2, \bar{x})\big) < \delta \Big\}.\end{aligned} \tag{3.1.11}$$

At this point, the proof of the first statement is trivial: indeed, if $\mathfrak{Z}$ satisfies (3.1.11) as a strict inequality, then at $\mathfrak{Z}$ there is a local Φ-infimum gap. But in this case $\mathfrak{Z}$ could not be a normal Φ-extremal, in view of Theorem 3.1.6. Hence, the inequality in (3.1.11) is in fact an equality. Let now $\mathfrak{Z}$ be a Φ-minimizer for (P_{Σ_e}) or (P_{Σ_r}). Then, it satisfies the relation

$$\Phi(\bar{t}_1, \bar{x}(\bar{t}_1), \bar{t}_2, \bar{x}(\bar{t}_2)) \le \inf_{(t_1,t_2,\omega,u,x)\in\Sigma} \Phi(t_1, x(t_1), t_2, x(t_2))$$

and, if we suppose that the inequality is strict, this implies again that at $\mathfrak{z}$ there is a local infimum gap. Thus, arguing as above we still get a contradiction. $\square$

Remark 3.1.8. By the free end-time constrained Maximum Principle (see Theorem 2.5.7), local Φ-minimizers of (P_{Σ_r}) are Φ-extremals in a stronger form than in Definition 3.1.5, in which the costate differential inclusion (3.1.5) is replaced by

$$-\dot{p}(t) \in \operatorname{co}\partial_x \left(\sum_{j=0}^{n} \bar{\gamma}^j(t)\, q(t) \cdot \mathcal{F}(t, (\bar{x}, \bar{\omega}^j, \bar{u}^j)(t)) \right) \text{ a.e. } t \in [\bar{t}_1, \bar{t}_2]. \tag{3.1.12}$$

The need to consider (3.1.5) derives from the perturbation technique used in the proof of Theorem 3.1.6 (see also [61, 38]). In fact, (3.1.5) may differ from (3.1.12) only in case of nonsmooth dynamics. Precisely, if $\mathcal{F}(t, \cdot, \bar{\omega}^j(t), \bar{u}^j(t))$ is continuously differentiable at $\bar{x}(t)$, for all $j = 0, \dots, n$ and a.e. $t \in [\bar{t}_1, \bar{t}_2]$, then both differential inclusions reduce to the adjoint equation

$$-\dot{p}(t) = \sum_{j=0}^{n} \bar{\gamma}^j(t)\, q(t) \cdot \nabla_x \mathcal{F}(t, (\bar{x}, \bar{\omega}^j, \bar{u}^j)(t)) \ \text{ a.e. } t \in [\bar{t}_1, \bar{t}_2].$$

In order to establish sufficient controllability conditions, given a reference process $\mathfrak{z} := (\bar{t}_1, \bar{t}_2, \underline{\bar{\omega}}, \underline{\bar{u}}, \bar{\gamma}, \bar{x}) \in \Sigma_r$ for which Hypotheses 3.1.1–3.1.3 are verified, we introduce the set $\mathscr{M}(\mathfrak{z})$ of multipliers $(p, h_1, h_2, \beta_1, \beta_2, \mu, m)$, where $p \in W^{1,1}([\bar{t}_1, \bar{t}_2], \mathbb{R}^n)$, h_1, $h_2 \in \mathbb{R}$, $\beta_1, \beta_2 \ge 0$, $\mu \in C^{\oplus}([\bar{t}_1, \bar{t}_2])$, $m : [\bar{t}_1, \bar{t}_2] \to \mathbb{R}^n$ is a Borel measurable and μ-integrable function, that meet conditions (3.1.5), (3.1.7)–(3.1.10) (for q as in Definition 3.1.5), and such that

$$\|p\|_{L^\infty([\bar{t}_1,\bar{t}_2])} + \mu([\bar{t}_1, \bar{t}_2]) + \beta_1 + \beta_2 \neq 0,$$
$$(-h_1, p(\bar{t}_1), h_2, -q(\bar{t}_2)) \in N_{\mathcal{C}}(\bar{t}_1, \bar{x}(\bar{t}_1), \bar{t}_2, \bar{x}(\bar{t}_2)) + \beta_1 \partial\psi(\bar{t}_1, \bar{x}(\bar{t}_1)) \times \beta_2 \partial\psi(\bar{t}_2, \bar{x}(\bar{t}_2)).$$

Theorem 3.1.9. *Let $\mathfrak{z} := (\bar{t}_1, \bar{t}_2, \underline{\bar{\omega}}, \underline{\bar{u}}, \bar{\gamma}, \bar{x})$ be a feasible relaxed process and assume that Hypotheses 3.1.1–3.1.3 are verified. If $\mathscr{M}(\mathfrak{z}) = \emptyset$, then the constrained control* *system* (3.0.1)-(3.0.2) *is controllable to* $\mathfrak{Z}$.

Proof. Theorem 3.1.9 is simply the contrapositive statement of Theorem 3.1.6. Indeed, if the constrained control system (3.0.1)-(3.0.2) is not controllable to $\mathfrak{Z}$, then $\mathfrak{Z}$ is an isolated process, which means that at $\mathfrak{Z}$ there is a local infimum gap, in view of Proposition 3.0.6. Now, Theorem 3.1.6 implies that $\mathfrak{Z}$ is an abnormal extremal, and this guarantees that $\mathscr{M}(\mathfrak{Z}) \neq \emptyset$. □

Remark 3.1.10. In order to simplify the exposition, we considered a Mayer problem with a single state constraint inequality. Actually, from quite standard arguments (see e.g. [37, 69]) all the results of this thesis could be extended: (i) to a Bolza problem, with cost of the form

$$J(t_1, t_2, \omega, u, x) := \Phi(t_1, x(t_1), t_2, x(t_2)) + \int_{t_1}^{t_2} \mathcal{L}(t', x(t'), \omega(t'), u(t'))\, dt',$$

with $\mathcal{L} : \mathbb{R}^{1+n+m+q} \to \mathbb{R}$ which satisfies the same regularity assumptions as the dynamics $\mathcal{F}$; (ii) to $N \geq 1$ inequality state constraints $\psi_j(t, x(t)) \leq 0$ for all $t \in [t_1, t_2]$ ($j = 1, \ldots, N$), where each ψ_j satisfies Hypothesis 3.1.2; (iii) to implicit time-dependent state constraints of the form $x(t) \in X(t)$ for all $t \in [t_1, t_2]$, where $x : \mathbb{R} \rightsquigarrow \mathbb{R}^n$ is a Lipschitz continuous set-valued map.

3.2 The case of initial active state constraint

The normality test to avoid a local infimum gap at some process $\mathfrak{Z} := (\bar{t}_1, \bar{t}_2, \underline{\bar{\omega}}, \underline{\bar{u}}, \bar{\gamma}, \bar{x}) \in \Sigma_r$ established in Theorem 3.1.7 might be useless when the initial point $(\bar{t}_1, \bar{x}(\bar{t}_1)) \in \partial\mathcal{Q}$, where $\mathcal{Q}$ is the state constraint set, defined by

$$\mathcal{Q} := \{(t, z) \in \mathbb{R}^{1+n} : \psi(t, z) \leq 0\}. \tag{3.2.1}$$

Indeed, in this case $\mathfrak{Z}$ is very often an abnormal extremal, since, at least disregarding the endpoint constraints, there may be *degenerate* sets $(p, h_1, h_2, \beta_1, \beta_2, \lambda, \mu, m)$ of multipliers that meet all the conditions of Definition 3.1.5 with

$$\mu \equiv \mu(\{\bar{t}_1\}) \neq 0, \qquad p \equiv -m(\bar{t}_1)\mu(\{\bar{t}_1\}), \qquad \lambda = \beta_1 = \beta_2 = 0. \tag{3.2.2}$$

This section is devoted to provide some sufficient conditions to refine the results of Section 3.1, in order to exclude degenerate multipliers. We will conclude with some examples.

Let us point out that we cannot simply consider any of the conditions of nondegeneracy known in the literature to prove that a process $\mathfrak{Z}$ at which there is a local infimum gap is abnormal *and nondegenerate*. In particular, our strategy to prove that $\mathfrak{Z}$ is an abnormal extremal is to apply the Ekeland Principle to a sequence of optimization problems over strict sense processes, so that the sequence of Ekeland minimizers approximate the reference relaxed process $\mathfrak{Z}$. By applying the Maximum Principle to these minimizers we derive, in the limit, a maximum principle in abnormal form for $\mathfrak{Z}$. Hence, on the one hand, we would need a condition of nondegeneracy for each of these minimizers, which remains so by passing to the limit. On the other hand, for the approximating problems we cannot invoke, for instance, controllability conditions of the kind introduced in [3, 2] (see also [7], [69, Sec. 10.6]), since they require Hamiltonians which are Lipschitz continuous in time, while the Hamiltonians of our Ekeland optimization problems are at most measurable in time (see problems (P_i) in the proof of Theorem 3.2.8 below). Let us recall also [62], where this kind of nondegeneracy conditions are extended to differential inclusions with bounded variation in time.

3.2.1 Hypotheses for nondegeneracy for general endpoint constraints

In the case of general endpoint constraints, we consider the following condition, which ensures that a multiplier as in (3.2.2) cannot exist.

Hypothesis 3.2.1. A process $(\bar t_1, \bar t_2, \underline{\bar\omega}, \underline{\bar u}, \bar\gamma, \bar x) \in \Sigma_r$ is said to satisfy the condition for nondegeneracy if

$$\partial_x^{>}\psi(\bar t_1, \bar x(\bar t_1)) \cap \big(-\mathrm{proj}_{x_1} N_{\mathfrak{C}}(\bar t_1, \bar x(\bar t_1), \bar t_2, \bar x(\bar t_2))\big) = \emptyset. \tag{3.2.3}$$

Hypothesis 3.2.1 extends a condition first introduced in [37], for the impulsive extension with Lipschitz continuous data in the time variable. It is *a posteriori* requirement, that ensures the nondegeneracy of *every* (Φ-)extremal, similarly to the strengthened nontriviality conditions derived in [34, Cor. 3.1].

Proposition 3.2.2. *Let $(\bar{t}_1, \bar{t}_2, \underline{\bar{\omega}}, \underline{\bar{u}}, \bar{\gamma}, \bar{x})$ be a Φ-extremal for which Hypothesis 3.2.1 is satisfied, for some $\Phi : \mathbb{R}^{1+n+1+n} \to \mathbb{R}$ which is Lipschitz continuous on a neighborhood of $(\bar{t}_1, \bar{x}(\bar{t}_1), \bar{t}_2, \bar{x}(\bar{t}_2))$. Then, any multiplier $(p, h_1, h_2, \beta_1, \beta_2, \lambda, \mu, m)$ that meets all the condition of Definition 3.1.5, is a* nondegenerate multiplier*, that is, it satisfies the following additional strengthened nontriviality condition*

$$\|q\|_{L^\infty([\bar{t}_1, \bar{t}_2])} + \lambda + \mu(]\bar{t}_1, \bar{t}_2]) + \beta_1 + \beta_2 \neq 0, \tag{3.2.4}$$

where q is as in Definition 3.1.5.

Proof. Let $(p, h_1, h_2, \beta_1, \beta_2, \lambda, \mu, m)$ be a multiplier associated to the relaxed process $(\bar{t}_1, \bar{t}_2, \underline{\bar{\omega}}, \underline{\bar{u}}, \bar{\gamma}, \bar{x})$, as in Definition 3.1.5. Assume by contradiction that (3.2.4) is not satisfied. Then by conditions (3.1.4)–(3.1.10), $(p, h_1, h_2, \beta_1, \beta_2, \lambda, \mu, m)$ satisfies (3.2.2) and one has

$$m(\bar{t}_1) \in \partial_x^> \psi(\bar{t}_1, \bar{x}(\bar{t}_1)), \qquad p(\bar{t}_1) \in \mathrm{proj}_{x_1} N_{\mathfrak{C}}(\bar{t}_1, \bar{x}(\bar{t}_1), \bar{t}_2, \bar{x}(\bar{t}_2)).$$

Since $N_{\mathfrak{C}}(\bar{t}_1, \bar{x}(\bar{t}_1), \bar{t}_2, \bar{x}(\bar{t}_2))$ is a cone, this implies

$$m(\bar{t}_1) \in \partial_x^> \psi(\bar{t}_1, \bar{x}(\bar{t}_1)) \cap \left(-\mathrm{proj}_{x_1} N_{\mathfrak{C}}(\bar{t}_1, \bar{x}(\bar{t}_1), \bar{t}_2, \bar{x}(\bar{t}_2))\right),$$

in contradiction with (3.2.3). □

As a consequence of Proposition 3.2.2, when Hypothesis 3.2.1 is valid, in Theorem 3.1.6 we can equivalently consider nondegenerate multipliers only. Precisely, we get the following result.

Theorem 3.2.3. *Let $\mathfrak{Z} := (\bar{t}_1, \bar{t}_2, \underline{\bar{\omega}}, \underline{\bar{u}}, \bar{\gamma}, \bar{x})$ be a feasible relaxed process for which Hypotheses* 3.1.1–3.1.3 *and 3.2.1 are verified. If at $\mathfrak{Z}$ there is a local infimum gap, then there exists a set of multipliers $(p, h_1, h_2, \beta_1, \beta_2, \lambda, \mu, m)$ satisfying conditions* (3.1.4)–(3.1.10) *and* (3.2.2) *with $\lambda = 0$.*

Accordingly, from Theorem 3.2.3 we can deduce strengthened sufficient conditions for no gap and for controllability of the original constrained control system to a feasible relaxed process, in the case Hypothesis 3.2.1 is fulfilled.

Hypothesis 3.2.1 is trivially satisfied when, for instance, $(\bar{t}_1, \bar{x}(\bar{t}_1)) \in \mathrm{Int}(\mathcal{Q})$, as in this case $\partial_x^{>}\psi(\bar{t}_1, \bar{x}(\bar{t}_1)) = \emptyset$. For less trivial situations in which Hypothesis 3.2.1 is met, we refer the readers to Subsection 5.4, and in particular to Remark 5.4.2.

3.2.2 Hypotheses for nondegeneracy for fixed initial point

We now analyze the case with fixed initial point, for which it is immediate to see that Hypothesis 3.2.1 is never verified if the point lies on $\partial\mathcal{Q}$. Given some value $\check{z}_0 \in \mathbb{R}^n$ and a closed set $\tilde{\mathcal{C}} \subseteq \mathbb{R}^{1+n}$, the endpoint constraint set $\mathcal{C}$ takes now the form

$$\mathcal{C} = \{(0, \check{z}_0)\} \times \tilde{\mathcal{C}}. \tag{3.2.5}$$

Since the initial time is always zero, in this subsection for any $T > 0$ we simply write (T, ω, u, x), $(T, \underline{\omega}, \underline{u}, \gamma, x)$, $\mathcal{V}(T)$, $\mathcal{W}(T)$, $\mathcal{U}(T)$, $\Gamma(T)$ in place of $(0, T, \omega, u, x)$, $(0, T, \underline{\omega}, \underline{u}, \gamma, x)$, $\mathcal{V}([0,T])$, $\mathcal{W}([0,T])$, $\mathcal{U}([0,T])$, $\Gamma([0,T])$, respectively. Furthermore, we imply that all processes satisfy $x(0) = \check{z}_0$.

Definition 3.2.4. Let $\mathfrak{Z} := (\bar{T}, \underline{\bar{\omega}}, \underline{\bar{u}}, \bar{\gamma}, \bar{x})$ be a feasible relaxed process for which Hypotheses 3.1.1–3.1.3 are verified. Given a function $\Phi : \mathbb{R}^{1+n} \to \mathbb{R}$ which is Lipschitz continuous on a neighborhood of $(\bar{T}, \bar{x}(\bar{T}))$, we call *nondegenerate multiplier* any element $(p, h, \beta, \lambda, \mu, m)$ that meets conditions (3.1.5), (3.1.8), (3.1.9) and (3.1.10) of Definition 3.1.5, obviously with $[0, \bar{T}]$ replacing $[\bar{t}_1, \bar{t}_2]$, and satisfies the strengthened nontriviality condition

$$\mu(]0, \bar{T}]) + \|q\|_{L^\infty([0,\bar{T}])} + \lambda + \beta \neq 0, \tag{3.2.6}$$

the transversality condition

$$(h, -q(\bar{T})) \in \lambda \partial\Phi(\bar{T}, \bar{x}(\bar{T})) + N_{\tilde{\mathcal{C}}}(\bar{T}, \bar{x}(\bar{T})) + \beta\, \partial\psi(\bar{T}, \bar{x}(\bar{T})), \tag{3.2.7}$$

and

$$h \in \operatorname*{ess}_{t \to \bar{T}} \Big(\max_{(w,v) \in \overline{\mathscr{V}(t)} \times \mathscr{U}(t)} q(\bar{T}) \cdot \mathcal{F}\big(t, \bar{x}(\bar{T}), w, v\big) \Big), \tag{3.2.8}$$

where

$$q(t) := \begin{cases} p(t) + \int_{[0,t[} m(t')\mu(dt') & \text{if } t \in [0, \bar{T}[, \\ p(\bar{T}) + \int_{[0,\bar{T}]} m(t')\mu(dt') & \text{if } t = \bar{T}. \end{cases} \tag{3.2.9}$$

Moreover, $\beta = 0$ if either $\psi(\bar{T}, \bar{x}(\bar{T})) < 0$ or $\tilde{\mathfrak{C}} \subseteq \{\bar{T}\} \times \mathbb{R}^n$. We call $\mathfrak{Z}$ a *nondegenerate* Φ-*extremal* if nondegenerate multipliers exist. Then, we say that $\mathfrak{Z}$ is *nondegenerate normal* if all possible choices of nondegenerate multipliers have $\lambda > 0$, and *nondegenerate abnormal* if there is at least one nondegenerate multiplier with $\lambda = 0$.

A nondegenerate abnormal extremal is an abnormal extremal and any normal Φ-extremal is nondegenerate normal. However, we have examples of nondegenerate normal Φ-extremals that are abnormal (see Example 3.2.12 and Example 5.3.4 below). In these situations, the nondegenerate normality test established in Theorem 3.2.9 below detects the absence of gap, while Theorem 3.1.7 gives no information.

To introduce sufficient nondegeneracy conditions, we first extend the relaxed control system by introducing a new variable, ζ. Precisely, with a small abuse of notation, in the following we call relaxed process any element $(T, \underline{\omega}, \underline{u}, \gamma, \zeta, x)$ with $T > 0$ and $(\underline{\omega}, \underline{u}, \gamma, \zeta, x) \in \mathcal{W}^{1+n}(T) \times \mathcal{U}^{1+n}(T) \times \Gamma(T) \times W^{1,1}([0,T], \mathbb{R}^{1+n} \times \mathbb{R}^n)$, which satisfies the Cauchy problem

$$\begin{cases} (\dot{\zeta}, \dot{x})(t) = \Big(\gamma(t), \sum_{j=0}^{n} \gamma^j(t)\mathcal{F}(t, (y, \omega^j, u^j)(t))\Big) \text{ a.e. } t \in [0,T], \\ (\zeta, x)(0) = (0, \check{z}_0). \end{cases} \tag{3.2.10}$$

Remark 3.2.5. Define the subset $\Gamma^1(T) := \mathcal{M}([0,T], \Delta_n^1) \subset \Gamma(T)$, where

$$\Delta_n^1 := \bigcup_{j=0}^{n} \{e_j\} \quad (e_0, \dots, e_n \text{ canonical basis of } \mathbb{R}^{1+n}). \tag{3.2.11}$$

One may observe that a relaxed process $(T, \underline{\omega}, \underline{u}, \gamma, \zeta, x)$ with $\gamma \in \Gamma^1(T)$ corresponds to the extended process (T, ω, u, ζ, x), where [9]

$$(\omega, u) := \sum_{j=0}^{n} (\omega^j, u^j)\chi_{\{t\in[0,T]:\ \gamma(t)=e_j\}}.$$

[9] According to the above convention, we include in the string also the new variable ζ.

Let $(\bar{T}, \underline{\bar{\omega}}, \underline{\bar{u}}, \bar{\gamma}, \bar{\zeta}, \bar{x})$ be the reference feasible relaxed process. We consider the following hypothesis.

Hypothesis 3.2.6. If $(0, \check{z}_0) \in \partial\mathcal{Q}$ (for $\mathcal{Q}$ as in (3.2.1)), there exist $\tilde{\delta} > 0$, $C > 1$, $(\varepsilon_i) \subseteq]0, \bar{T}]$ with $\varepsilon_i \downarrow 0$, $(\tilde{\varphi}_i) \subseteq L^1([0, \bar{T}], \mathbb{R}_{\geq 0})$ with $\lim_{i \to +\infty} \|\tilde{\varphi}_i\|_{L^1([0,\bar{T}])} \to 0$, a sequence $(\tilde{\Omega}_i)$ of Lebesgue measurable subsets of $[0, \bar{T}]$ with $\lim_{i \to +\infty} \ell(\tilde{\Omega}_i) = \bar{T}$, a sequence of extended processes $(\bar{T}, \tilde{\omega}_i, \tilde{u}_i, \tilde{\gamma}_i, \tilde{\zeta}_i, \tilde{x}_i)$ with $(\tilde{\omega}_i, \tilde{u}_i, \tilde{\gamma}_i) \in (\mathcal{W}(\bar{T}) \cap \mathcal{V}(C\sqrt[4]{\varepsilon_i})) \times \mathcal{U}(\bar{T}) \times \Gamma^1(\bar{T})$ for every i, and a sequence of extended controls $(\hat{\omega}_i, \hat{u}_i) \in \mathcal{W}(C\sqrt[4]{\varepsilon_i}) \times \mathcal{U}(C\sqrt[4]{\varepsilon_i})$, enjoying for any i the following properties:

(i) one has
$$\|(\tilde{\zeta}_i, \tilde{x}_i) - (\bar{\zeta}, \bar{x})\|_{L^\infty([0,\bar{T}])} \leq \varepsilon_i; \tag{3.2.12}$$

(ii) one has
$$\psi(t, \tilde{x}_i(t)) \leq 0 \qquad \forall t \in [0, C\sqrt[4]{\varepsilon_i}]\,; \tag{3.2.13}$$

(iii) for a.e. $t \in \tilde{\Omega}_i$, one has
$$(\tilde{\omega}_i, \tilde{u}_i, \tilde{\gamma}_i)(t) \in \bigcup_{j=0}^{n} \{(\bar{\omega}^j(t), \bar{u}^j(t), e^j)\} + (\tilde{\varphi}_i(t)\mathbb{B}_m) \times \{0\} \times \{0\}; \tag{3.2.14}$$

(iv) for all $(\xi_0, \xi) \in \partial^*\psi(0, \check{z}_0)$, for a.e. $t \in [0, C\sqrt[4]{\varepsilon_i}]$ one has
$$\xi \cdot \big[\mathcal{F}(t, \check{z}_0, (\hat{\omega}_i, \hat{u}_i)(t)) - \mathcal{F}(t, \check{z}_0, (\tilde{\omega}_i, \tilde{u}_i)(t))\big] \leq -\tilde{\delta}. \tag{3.2.15}$$

Remark 3.2.7. Some comments on Hypothesis 3.2.6 are in order.

(1) It prescribes additional conditions to Hypotheses 3.1.1–3.1.3 only when the initial point $(0, \check{z}_0)$ lies on the boundary of the constraint set $\mathcal{Q}$. Incidentally, this is not equivalent to having $\psi(0, \check{z}_0) = 0$, as it may clearly happen that $\psi(0, \check{z}_0) = 0$ but $(0, \check{z}_0) \in \mathrm{Int}(\mathcal{Q})$.

(2) When $(0, \check{z}_0) \in \partial\mathcal{Q}$, the first part of Hypothesis 3.2.6 substantially requires the existence of strict sense processes that approximate the reference process and satisfy the state constraint on some (small) interval, with controls which are close to controls $(\bar{\omega}_i, \bar{u}_i, \bar{\gamma}_i)$ belonging to $\bigcup_{j=0}^{n}\{(\bar{\omega}^j(t), \bar{u}^j(t), e^j)\}$ for a.e. $t \in [0, \bar{T}]$. Let

us point out that, disregarding the state constraint (3.2.13), the existence of approximating controls that satisfy the remaining conditions (3.2.12), (3.2.14) follows by the Relaxation Theorem 2.3.5 together with Hypothesis 3.1.1, as we will see in the proof of Theorem 3.1.6 below, in Section 3.4. Relation (3.2.15), on the other hand, is an adaptation of known constraint qualification conditions (see e.g. [32, 33]), that will be crucial in order to show that the reference process is a nondegenerate extremal, as well as abnormal.

(3) Hypothesis 3.2.6 is trivially satisfied when the reference process is in fact a strict sense process on some interval $[0, \bar{t}\,]$ and satisfies a classical constraint qualification condition introduced in [33], namely, if there exist $\bar{t}$, $\tilde{\delta} > 0$, $(\tilde{\omega}, \tilde{u})$, $(\hat{\omega}, \hat{u}) \in \mathcal{W}(\bar{t}) \times \mathcal{U}(\bar{t})$ such that for a.e. $t \in [0, \bar{t}\,]$, $\dot{\bar{x}} = \mathcal{F}(t, \bar{x}, \tilde{\omega}, \tilde{u})$ and

$$\sup_{(\xi_0,\xi)\in\partial^*\psi(0,\check{z}_0)} \xi \cdot \left[\mathcal{F}(t, \check{z}_0, (\hat{\omega}, \hat{u})(t)) - \mathcal{F}(t, \check{z}_0, (\tilde{\omega}, \tilde{u})(t))\right] \leq -\tilde{\delta}.$$

See Section 5.3 (in particular, Lemma 5.3.3) for easy verifiable conditions implying Hypothesis 3.2.6 in the case of control-polynomial impulsive optimization problems.

(4) If Hypotheses 3.1.1–3.1.3 with reference to $\mathfrak{Z}$ are verified, then in Hypothesis 3.2.6 one can assume that the control sequence $(\hat{\omega}_i, \hat{u}_i)$ belongs to the strict sense control set $\mathcal{V}(C\sqrt[4]{\varepsilon_i}) \times \mathcal{U}(C\sqrt[4]{\varepsilon_i})$ rather than $\mathcal{W}(C\sqrt[4]{\varepsilon_i}) \times \mathcal{U}(C\sqrt[4]{\varepsilon_i})$. Indeed, using the notation of Hypotheses 3.1.1–3.1.3, let us choose some $r > 0$ such that $L_\psi L_{\mathcal{F}} \rho(r) \leq \frac{\tilde{\delta}}{2}$, and let $j \in \mathbb{N}$ verify $\|\sigma_j\|_{L^\infty([0,\eta])} \leq r$. Hence, for every $i \in \mathbb{N}$ such that $C\sqrt[4]{\varepsilon_i} \leq \eta$ there exists a measurable selection $\hat{\omega}_i^*(t) \in \mathrm{proj}_{\mathscr{V}_j(t)}(\hat{\omega}_i(t))$ for a.e. $t \in [0, T]$, such that $\|\hat{\omega}_i^* - \hat{\omega}_i\|_{L^\infty([0,C\sqrt[4]{\varepsilon_i}])} \leq r$ (see also Remark 3.1.4), and, by adding and subtracting '$\xi \cdot \mathcal{F}(t, \check{z}_0, (\hat{\omega}_i^*, \hat{u}_i)(t))$' for all $(\xi_0, \xi) \in \partial^*\psi(0, \check{z}_0)$ one has

$$\xi \cdot \left[\mathcal{F}(t, \check{z}_0, (\hat{\omega}_i^*, \hat{u}_i)(t)) - \mathcal{F}(t, \check{z}_0, (\tilde{\omega}_i, \tilde{u}_i)(t))\right] \leq -\frac{\tilde{\delta}}{2}, \quad \text{a.e. } t \in [0, C\sqrt[4]{\varepsilon_i}],$$

as soon as $(\hat{\omega}_i, \hat{u}_i)$ satisfies (3.2.15).

(5) When Hypothesis 3.1.3 is verified, then the upper semicontinuity of the set-valued map $\partial^*\psi(\cdot, \cdot)$ and (3.2.15) imply that there exist $\hat{\delta}$ and $\varepsilon > 0$ such that,

for any $(\xi_0, \xi) \in \partial^* \psi(\tau, z)$ with $\tau \in [0, \varepsilon]$ and $z \in \{\check{z}_0\} + \varepsilon\mathbb{B}$, for any $t \leq C\sqrt[4]{\varepsilon_i}$, for any continuous path $x : [0, t] \to \{\check{z}_0\} + \varepsilon\mathbb{B}$ and for any measurable map $\vartheta : [0, t] \to \{0, 1\}$, the following integral condition holds:

$$\int_0^t \vartheta(t')\, \xi \cdot \big[\mathcal{F}(t', x, \hat{\omega}_i, \hat{u}_i)(t') - \mathcal{F}(t', x, \tilde{\omega}_i, \tilde{u}_i)(t')\big]\, dt' \leq -\hat{\delta}\, \ell[\vartheta](t), \qquad (3.2.16)$$

where

$$\ell[\vartheta](t) := \ell(\{\tau \in [0, t] \,:\, \vartheta(\tau) = 1\}). \qquad (3.2.17)$$

In particular, relation (3.2.16) holds for any $(\xi_0, \xi) \in \operatorname{co} \partial^* \psi(\tau, z)$, as the scalar product is bilinear. Accordingly, (3.2.16) still holds for all $\xi \in \partial_x^{>} \psi(\tau, z)$ since, by Theorem 2.4.6, for any $(\tau, z) \in \mathbb{R}^{1+n}$ one has

$$\begin{aligned} \partial_x^{>} \psi(\tau, z) &\subset \operatorname{co} \partial_x^* \psi(\tau, z) = \operatorname{co} \partial_x \psi(\tau, z) \subset \operatorname{co} \{\xi \,:\, \exists \xi_0 \text{ s.t. } (\xi_0, \xi) \in \partial \psi(\tau, z)\} \\ &\subset \{\xi \,:\, \exists \xi_0 \text{ s.t. } (\xi_0, \xi) \in \operatorname{co} \partial \psi(\tau, z)\} = \{\xi \,:\, \exists \xi_0 \text{ s.t. } (\xi_0, \xi) \in \operatorname{co} \partial^* \psi(\tau, z)\}. \end{aligned}$$

Relation (3.2.16) is in fact the condition used in the proof of Theorem 3.2.8 below, in Section 3.3.

In the case with fixed initial point, Theorem 3.1.6 can be refined as follows:

Theorem 3.2.8. *Let $\mathcal{C}$ be as in* (3.2.5)*. Let $\mathfrak{Z} := (\bar{T}, \underline{\bar{\omega}}, \underline{\bar{u}}, \bar{\gamma}, \bar{\zeta}, \bar{x})$ be a feasible relaxed process for which Hypotheses 3.1.1–3.1.3 and 3.2.6 are verified. Then, if at $\mathfrak{Z}$ there is a local infimum gap, $\mathfrak{Z}$ is a* nondegenerate *abnormal extremal.*

The proof of this result will be given in Section 3.3. Arguing as in the previous section, from Theorem 3.2.8 we can derive the following results.

Theorem 3.2.9. *Let $\mathcal{C}$ be as in* (3.2.5)*. Let $\mathfrak{Z} := (\bar{T}, \underline{\bar{\omega}}, \underline{\bar{u}}, \bar{\gamma}, \bar{\zeta}, \bar{x})$ be a feasible relaxed process for which Hypotheses 3.1.1–3.1.3 and 3.2.6 are verified. Let $\Phi : \mathbb{R}^{1+n} \to \mathbb{R}$ be a Lipschitz continuous function on a neighborhood of $(\bar{T}, \bar{x}(\bar{T}))$. When $\mathfrak{Z}$ is a local Φ-minimizer for (P_{Σ_e}) or (P_{Σ_r}) which is a* nondegenerate *normal Φ-extremal, then*

$$\Phi(\bar{T}, \bar{x}(\bar{T})) = \inf\, \Phi(T, x(T))$$

over all processes $(0, T, \omega, u, x) \in \Sigma$ with $d_\infty\big((0, T, x), (0, \bar{T}, \bar{x})\big) < \delta$.

Similarly, if $\mathfrak{Z}$ *is a* Φ*-minimizer for* (P_{Σ_e}) *or* (P_{Σ_r}) *which is a* nondegenerate *normal* Φ*-extremal, then* $\mathfrak{Z}$ *realizes the infimum of* Φ *over* Σ*.*

Let $\mathfrak{Z} := (\bar{T}, \underline{\bar{\omega}}, \underline{\bar{u}}, \bar{\gamma}, \bar{\zeta}, \bar{x})$ be a feasible relaxed process for which Hypotheses 3.1.1–3.1.3 and 3.2.6 are verified. Define the set $\mathscr{M}_0(\mathfrak{Z})$ of multipliers (p, h, β, μ, m), where $p \in W^{1,1}([0,\bar{T}], \mathbb{R}^n)$, $h \in \mathbb{R}$, $\beta \geq 0$, $\mu \in C^{\oplus}([0,\bar{T}])$, $m : [0,\bar{T}] \to \mathbb{R}^n$ is a Borel measurable and μ-integrable function, that meet conditions (3.1.5), (3.1.8), (3.1.9) and (3.1.10) of Definition 3.1.5 on $[0,\bar{T}]$, and (3.2.8) (for q as in Definition 3.1.5), and such that

$$\|q\|_{L^\infty([0,\bar{T}])} + \mu(]0,\bar{T}]) + \beta \neq 0, \quad (h, -q(\bar{T})) \in N_{\tilde{\mathcal{C}}}(\bar{T}, \bar{x}(\bar{T})) + \beta \partial \psi(\bar{T}, \bar{x}(\bar{T})).$$

Theorem 3.2.10. *Let* $\mathcal{C}$ *be as in* (3.2.5)*. Let* $\mathfrak{Z} := (\bar{T}, \underline{\bar{\omega}}, \underline{\bar{u}}, \bar{\gamma}, \bar{\zeta}, \bar{x})$ *be a feasible relaxed process for which Hypotheses 3.1.1–3.1.3 and 3.2.6 are verified. If* $\mathscr{M}_0(\mathfrak{Z}) = \emptyset$*, then the constrained control system* (3.0.1)-(3.0.2) *is controllable to* $\mathfrak{Z}$*.*

3.2.3 Some examples

The following example shows that the minimum of a constrained optimal control problem, of its extension, and of the relaxed extended problem can all be different from each other. Accordingly with the results in the previous subsections, the extended and the relaxed extended minimizer are abnormal extremals (actually, nondegenerate abnormal extremals, as Hypothesis 3.2.1 is verified).

Example 3.2.11. Consider the optimal control problem

$$\begin{cases} \qquad\qquad \text{Minimize} \quad -x^1(T) \\ \text{over } T > 0, \ (\omega, u, x) \in \mathcal{V}(T) \times \mathcal{U}(T) \times W^{1,1}([0,T], \mathbb{R}^3), \text{ satisfying} \\ \dot{x}(t) = \mathcal{F}(t, x(t), \omega(t), u(t)) \quad \text{a.e. } t \in [0,T], \\ \psi(x(t)) = x^1(t) - 1 \leq 0 \ \ \forall t \in [0,T], \\ x(0) \in \mathbb{R} \times \{0\} \times \{0\}, \quad (T, x(T)) \in \left[\tfrac{3}{4}, +\infty\right[\times \mathbb{R} \times \mathbb{R} \times \mathbb{R}_{\leq 0}, \end{cases} \tag{3.2.18}$$

where $\mathcal{V}(T) := \mathcal{M}([0,T],]0,1])$, $\mathcal{U}(T) := \mathcal{M}([0,T], \{-1,1\})$, and the function $\mathcal{F} : \mathbb{R}^6 \to$

$\mathbb{R}^3$, $\mathcal{F}(t,z,w,v) = \mathcal{F}(t,z^1,z^2,z^3,w,v)$, is given by

$$\mathcal{F}(t,z,w,v) := \begin{cases} \left((z^2)^2, z^1 v, (z^2)^2\right) & t \in \left[0, \frac{1}{2}\right], \\ \left(0, z^1 w - z^3 v, w\right) & t \in \left]\frac{1}{2}, +\infty\right[. \end{cases}$$

First of all, we note that, adopting the terminology of the previous sections, there are no feasible strict sense processes for problem (3.2.18), so the infimum cost for (3.2.18) is $+\infty$. Indeed, if (T,ω,u,x) were a feasible strict sense process, we should have $\omega(t) > 0$ for a.e. $t \in [0,T]$ and $T \geq 3/4$, from which the contradiction follows:

$$0 \geq x^3(T) - x^3(0) = \int_0^{1/2} (x^2(t))^2\, dt + \int_{1/2}^T \omega(t)\, dt > 0. \qquad (3.2.19)$$

Let us now consider the corresponding extended problem, where for any $T > 0$ the extended controls ω belong to the set $\mathcal{M}([0,T],[0,1])$. In this case, the feasible extended process $\mathfrak{Z} = (\bar{T},\bar{\omega},\bar{u},\bar{x})$ given by

$$\bar{T} = 1, \quad \bar{\omega} \equiv 0, \quad \bar{u} \equiv 1, \quad \bar{x} \equiv (0,0,0),$$

is a minimizer of the extended problem, with cost equal to 0. In fact, for any feasible extended process (T,ω,u,x) arguing similarly to (3.2.19), now we have $x^2 \equiv 0$ on $[0,1/2]$ and $\omega = 0$ a.e. on $[1/2,T]$, so that $x^1(t) = x^1(0)$ for every $t \in [0,T]$. Recalling that $u(t) \in \{-1,1\}$, the equalities $0 = \dot{x}^2(t) = u(t)x^1(0)$ for a.e. $t \in [0,1/2]$ imply that $x^1 \equiv 0$.

Finally, we consider the relaxed extended problem which, given the linearity of the dynamics in the control variables, is equivalent to considering problem (3.2.18), with $\mathcal{M}([0,T],[0,1])$ and $\mathcal{M}([0,T],[-1,1])$ that replace the control sets $\mathcal{V}(T)$, $\mathcal{U}(T)$, respectively. As it is easy to see, a feasible relaxed minimizer is now given by the process $\tilde{\mathfrak{Z}} = (\tilde{T},\tilde{\omega},\tilde{u},\tilde{x})$, where

$$\tilde{T} = 1, \quad \tilde{\omega} \equiv 0, \quad \tilde{u} \equiv 0, \quad \tilde{x} \equiv (1,0,0).$$

Observe that, because of the state constraint, any feasible relaxed trajectory must satisfy $x^1(t) \leq 1$ for every $t \in [0,T]$. Thus, the minimum cost of the relaxed problem is equal to -1. In conclusion, the minimum cost is $+\infty$ on feasible strict sense processes, 0 on feasible extended processes, and -1 on feasible relaxed processes.

Note that both the minimizing processes $\mathfrak{Z}$ and $\tilde{\mathfrak{Z}}$ are abnormal extremals, actually, nondegenerate abnormal extremals: just choose in Definition 3.1.5 the set of nondegenerate multipliers $(p, h_1, h_2, \beta_1, \beta_2, \lambda, \mu, m)$, where $p = (p_1, p_2, p_3) \equiv (0, 0, -1)$, $h_1 = h_2 = \beta_1 = \beta_2 = 0$, $\lambda = 0$, and μ, $m \equiv 0$. Furthermore, the nondegeneracy Hypothesis 3.2.1 is trivially satisfied for $\mathfrak{Z}$, as $\psi(\bar{x}(0)) = -1 < 0$, so that $\partial_x^> \psi(\bar{x}(0)) = \emptyset$, but also for $\tilde{\mathfrak{Z}}$, since $\partial_x^> \psi(\tilde{x}(0)) = (1, 0, 0)$ and the normal cone $N_{\mathbb{R}\times\{0\}\times\{0\}}(\tilde{x}(0)) = \{0\} \times \mathbb{R} \times \mathbb{R}$, so that $\partial_x^> \psi(\tilde{x}(0)) \cap N_{\mathbb{R}\times\{0\}\times\{0\}}(\tilde{x}(0)) = \emptyset$.

In the following example there is no infimum gap but this fact cannot be deduced from the normality criterion in Theorem 3.1.7, since the extended minimizer is abnormal. Instead, the absence of gap is detected by Theorem 3.2.9, as Hypothesis 3.2.6 is satisfied and the minimizer is nondegenerate normal.

Example 3.2.12. Let us consider the constrained optimal control problem

$$\begin{cases} \qquad\qquad \text{Minimize} \quad -x^2(T) \\ \text{over } T > 0,\ (\omega, u, x) \in \mathcal{V}(T) \times \mathcal{U}(T) \times W^{1,1}([0,T];\mathbb{R}^4),\ \text{satisfying} \\ \dot{x}(t) = \mathcal{F}(t, x(t), \omega(t), u(t)) \quad \text{a.e. } t \in [0,T], \\ x(0) = (0,1,0,0) \\ x(t) \in \mathcal{Q} \ \ \forall t \in [0,T], \quad x(T) \in \mathcal{C}, \end{cases} \tag{3.2.20}$$

where the function $\mathcal{F} : \mathbb{R}^9 \to \mathbb{R}^4$, $\mathcal{F}(t, z, w, v) = \mathcal{F}(t, z^1, \dots, z^4, w^1, w^2, w^3, v)$, is given by

$$\mathcal{F}(t, z, w, v) := \begin{cases} \big(w^1, w^2, (z^3 + z^4)w^3, -w^3\big) & t \in [0, 1[, \\ \big(w^1, w^2, z^3 z^4 w^1 - w^3, -z^2 w^3\big) & t \in [1, 3[, \\ \big(0, 0, z^2 v, (z^3)^2\big) & t \in [3, +\infty[, \end{cases}$$

and $\mathcal{Q} := \mathbb{R} \times [-1,1]^3$, $\mathcal{C} := \{1\} \times [-1,0] \times [0,1]^2$, $\mathcal{V}(T) := \mathcal{M}([0,T], V)$ for $V := \{w = (w^1, w^2, w^3) \in \mathbb{R}_{>0} \times \mathbb{R}^2 : \ \ |w| = 1\}$, and $\mathcal{U}(T) := \mathcal{M}([0,T], U)$ for $U := \{-1, 1\}$.

Hence, for any $T > 0$ the set of strict sense controls is $\mathcal{V}(T) \times \mathcal{U}(T)$, while the set of extended controls is $\mathcal{W}(T) \times \mathcal{U}(T)$, where $\mathcal{W}(T) := \mathcal{M}([0,T], \overline{V})$.

Since for any $z \in \mathcal{C}$ one has $z^2 \in [-1, 0]$ and the cost function is $\Phi(z) = -z^2$, for the relaxed extended problem associated to problem (3.2.20) every feasible process such that $x^2(T) = 0$ is a minimizer. In particular, the following process $\mathfrak{Z} := (\bar{T}, \bar{\omega}, \bar{u}, \bar{x})$

given by

$$\begin{aligned} \bar{T} = 2, \qquad & \bar{\omega}(t) = (\bar{\omega}^1, \bar{\omega}^2, \bar{\omega}^3)(t) = (1,0,0)\chi_{[0,1]}(t) + (0,-1,0)\chi_{]1,2]}(t), \\ \bar{u} \equiv 0, \qquad & \bar{x}(t) = (\bar{x}^1, \bar{x}^2, \bar{x}^3, \bar{x}^4)(t) = (t,1,0,0)\chi_{[0,1]}(t) + (1, 2-t, 0, 0)\chi_{[1,2]}(t)\,, \end{aligned}$$

is a feasible extended process which is a minimizer of the (relaxed) extended problem. Notice that $\mathfrak{Z}$ is not a strict sense process, since $\bar{\omega}^1 \equiv 0$ on $]1,2]$.

From the free end-time constrained Maximum Principle 2.5.7, $\mathfrak{Z}$ is a Φ-extremal. Hence, there exists a set of multipliers $(p, h, \beta, \lambda, \mu, m)$, that meets the conditions of Definition 3.1.5 on $[0, \bar{T}]$. In particular, $p = (p_1, \dots, p_4) \in W^{1,1}([0,2]; \mathbb{R}^4)$ solves the adjoint system, so that $p \equiv (\bar{p}_1, \dots, \bar{p}_4)$ is constant on $[0,2]$. Furthermore, $\beta = 0$ since $\bar{x}(2) \in \mathrm{Int}(\mathfrak{Q})$, and $\mu([0,t]) = \mu([0,1])$ for every $t \in [1,2]$ as $\bar{x}(t) \in \mathrm{Int}(\mathfrak{Q})$ for every $t \in]1,2]$. Notice that, for $t \in [0,1]$, $\bar{x}(t) \in \mathfrak{Q}$ is equivalent to $\psi(\bar{x}(t)) \leq 0$ for $\psi(z) := z^2 - 1$. Thus, $m(t) \in \partial_x^{>}\psi(\bar{x}(t)) = (0,1,0,0)$ μ-a.e. in $[0,2]$ and the function $q = (q_1, \dots, q_4)$ (as in Definition 3.1.5) is given by

$$q_2(t) = \begin{cases} \bar{p}_2 + \mu([0,t[) & \text{if } t \in [0,1] \\ \bar{p}_2 + \mu([0,1]) & \text{if } t \in]1,2] \end{cases}, \quad (q_1, q_3, q_4) \equiv (\bar{p}_1, \bar{p}_3, \bar{p}_4).$$

From the transversality condition (3.2.7) it follows that $h = 0$, $q_1 = \bar{p}_1 \in \mathbb{R}$, $q_2(2) = \bar{p}_2 + \mu([0,1]) = \lambda - \alpha$ for some $\alpha \geq 0$, $q_3 = \bar{p}_3 \geq 0$, and $q_4 = \bar{p}_4 \geq 0$. The second transversality condition (3.2.8) implies that

$$\max_{(w^1,w^2,w^3)\in\overline{V}} \left\{ \bar{p}_1 w^1 + q_2(2) w^2 - \bar{p}_3 w^3 \right\} = h = 0,$$

from which, considering the controls $(1,0,0)$, $(0,\pm1,0)$, and $(0,0,\pm1)$, it follows that $\bar{p}_1 \leq 0$, $q_2(2) = 0$, and $\bar{p}_3 = 0$. Notice that $q_2(t) = \bar{p}_2 + \mu([0,t[) \leq \bar{p}_2 + \mu([0,1]) = 0$ for every $t \in [0,1[$. Therefore, the maximality condition in $[0,1[$, that reads

$$\max_{(w^1,w^2,w^3)\in\overline{V}} \left\{ \bar{p}_1 w^1 + q_2(t) w^2 - \bar{p}_4 w^3 \right\} = \bar{p}_1 \leq 0,$$

implies that $\bar{p}_1 = 0$, $q_2(t) = \bar{p}_2 + \mu([0,t[) = 0$ for a.e. $t \in [0,1[$, and $\bar{p}_4 = 0$. In particular, $q(t) = 0$ for a.e. $t \in [0,2]$, $\mu([0,t[) = -\bar{p}_2$ for a.e. $t \in [0,1]$, so that $\mu([0,t[) = \mu(\{0\}) = -\bar{p}_2$ for all $t \in [0,2]$, and $\lambda = \alpha$.

At this point, by choosing $\lambda = \alpha = 0$ and $\mu = \delta_{\{0\}}$ we obtain a set of degenerate multipliers that meets all the conditions of the maximum principle, so proving that $\mathfrak{Z}$ is an abnormal extremal.

However, from the above analysis we can deduce that $\mathfrak{Z}$ is nondegenerate normal. Indeed, for each choice of admissible multipliers one has $\beta = h = 0$, $\|q\|_{L^\infty([0,2])} = 0$, and $\mu(]0,2]) = 0$, so that $\lambda \neq 0$ as soon as they verify the strengthened nontriviality condition (3.2.6). Furthermore, as it is easy to check, Hypothesis 3.2.6 is verified if we set $\tilde{\omega}_i := \bar{\omega}$ and $\hat{\omega}_i :\equiv (0,-1,0)$ for every i (see also Remark 3.2.7,(3)). Consequently, Theorem 3.2.9 guarantees that at $\mathfrak{Z}$ there is no infimum gap.

3.3 Proof of Theorem 3.2.8

Since the proofs involve only processes with trajectories close to the reference trajectory, using standard cut-off techniques we can assume without loss of generality that Hypotheses 3.1.2-3.1.3 are satisfied in the whole space $\mathbb{R}^{1+n}$ in place of $\mathcal{S}_\eta$.

In view of the above remark, for any $T > 0$ and any $(\underline{\omega}, \underline{u}, \gamma) \in \mathcal{W}^{1+n}(T) \times \mathcal{U}^{1+n}(T) \times \Gamma(T)$ there exists a unique solution (ζ, x) to the Cauchy problem (3.2.10), and this solution is defined on the whole interval $[0,T]$ (see Theorem 2.5.2). In the following, such solution will be denoted by $(\zeta, x)[\underline{\omega}, \underline{u}, \gamma]$. Similarly, for any $(\omega, u) \in \mathcal{W}(T) \times \mathcal{U}(T)$, we will write $x[\omega, u]$ to denote the corresponding solution to (3.0.1) on $[0,T]$, with initial condition $x(0) = \check{z}_0$.

Step 1. Define the function $\Psi : \mathbb{R}^{1+n+1} \to \mathbb{R}$, given by $\Psi(t,z,k) := d_{\tilde{\mathcal{C}}}(t,z) \vee k$, and for any $T > 0$ and $x \in W^{1,1}([0,T];\mathbb{R}^n)$, introduce the payoff

$$\mathcal{J}(T,x) := \Psi\Big(T, x(T), \max_{t\in[0,T]} \frac{\psi(t,x(t))}{1 \vee L_\psi}\Big).$$

Let (ε_i), $C > 1$, and $(\tilde{\omega}_i, \tilde{u}_i, \tilde{\gamma}_i, \tilde{\zeta}_i, \tilde{x}_i)$ be as in Hypothesis 3.2.6, so that (3.2.12) holds. For every i, let $r_i \geq 0$ satisfy

$$r_i^4 = \sup\left\{\mathcal{J}(T,x) : \ (T,\omega,u,\zeta,x) \in \Sigma, \ d_\infty((0,T,(\zeta,x)),(0,\bar{T},(\bar{\zeta},\bar{x}))) \leq C^4\varepsilon_i\right\},$$

By the Lipschitz continuity of Ψ and since $\mathfrak{Z}$ is an isolated process by Proposition

3.0.6, for $\delta > 0$ as in Definition 3.0.5, for i large enough, we have

$$0 < r_i \leq C\sqrt[4]{\varepsilon_i} < \sqrt[4]{\delta}, \qquad \lim_{i\to+\infty} r_i = 0. \tag{3.3.1}$$

Since Theorem 2.5.2 implies that for any $u \in \mathcal{U}(T)$ the input-output map $\omega \mapsto x[\omega, u]$ is continuous, for every ε_i there exists $\delta_i > 0$ such that for any $\omega \in \mathcal{M}([0,\bar{T}];W)$ with $\|\omega - \tilde{\omega}_i\|_{L^1([0,\bar{T}])} \leq \delta_i$, one has $\|x[\omega,\tilde{u}_i] - \tilde{x}_i\|_{L^\infty([0,\bar{T}])} \leq (C^4-1)\varepsilon_i$. According to Remark 3.1.4,(i), for any i let us choose a measurable control $\mathring{\omega}_i(t) \in \mathscr{V}_{\iota_{\delta_i}}(t)$ for a.e. $t \in [0,\bar{T}]$, such that $\|\mathring{\omega}_i - \tilde{\omega}_i\|_{L^1([0,\bar{T}])} \leq \delta_i$.

For every i, set

$$\mathcal{V}_{\delta_i}(\bar{T}) := \{\omega \in \mathcal{M}([0,\bar{T}];W): \ \omega(t) \in \mathscr{V}_{\iota_{\delta_i}}(t) \text{ a.e. } t \in [0,\bar{T}]\}, \tag{3.3.2}$$

and consider the optimal control problem

$$(\hat{P}_i)\begin{cases} \qquad\qquad \text{Minimize} \quad \mathcal{J}(T,x) \\ \text{over } T>0,\ (\omega,u,\gamma,\vartheta) \in \mathcal{V}_{\delta_i}(T)\times\mathcal{U}(T)\times\Gamma^1(T)\times\mathcal{M}([0,T];\{0,1\}), \\ \text{and trajectories } \ (\zeta,x) \in W^{1,1}([0,T];\mathbb{R}^{1+n}\times\mathbb{R}^n), \text{ satisfying} \\ \dot{\zeta}(t) = \gamma(t) \qquad \text{a.e. } t \in [0,T] \\ \dot{x}(t) = \mathcal{F}(t,y,\tilde{\omega}_i,\tilde{u}_i) + \vartheta(t)[\mathcal{F}(t,y,\hat{\omega}_i,\hat{u}_i) - \mathcal{F}(t,y,\tilde{\omega}_i,\tilde{u}_i)] \text{ a.e. } t \in [0,r_i] \\ \dot{x}(t) = \mathcal{F}(t,x(t),\omega(t),u(t)) \quad \text{ a.e. } t \in]r_i,T] \\ (\zeta,x)(0) = (0,\check{z}_0), \qquad d_\infty((0,T,x),(0,\bar{T},\bar{x})) \leq \delta, \end{cases}$$

where $(\hat{\omega}_i,\hat{u}_i)$ is as in Hypothesis 3.2.6 and is assumed to belong to $\mathcal{V}(C\sqrt[4]{\varepsilon_i})\times\mathcal{U}(C\sqrt[4]{\varepsilon_i})$ in view of Remark 3.2.7,(4). We call an element $(T,\omega,u,\gamma,\vartheta,\zeta,x)$ verifying the constraints in $(\hat{P}_i)$ a *process for problem* $(\hat{P}_i)$ and use Λ_i to denote the set of such processes. By introducing, for every $(T',\omega',u',\gamma',\vartheta',\zeta',y')$, $(T,\omega,u,\gamma,\vartheta,\zeta,x) \in \Lambda_i$, the distance

$$\begin{aligned}\mathbf{d}((T',\omega',u',\gamma',\vartheta',\zeta',y'),(T,\omega,u,\gamma,\vartheta,\zeta,x)) &:= |T-T'| + \|\omega'-\omega\|_{L^1([0,T\wedge T'])} \\ &+\ell\left(\{t \in [0,T\wedge T']:\ (u',\gamma',\vartheta')(t) \neq (u,\gamma,\vartheta)(t)\}\right),\end{aligned} \tag{3.3.3}$$

we can make $(\Lambda_i,\mathbf{d})$ a complete metric space. Notice that, by the very definition of r_i, the process $\check{\mathfrak{Z}}_i := (\bar{T},\check{\omega}_i,\check{u}_i,\check{\gamma}_i,\check{\vartheta}_i,\check{\zeta}_i,\check{x}_i)$ with $\check{\omega}_i := \mathring{\omega}_i$, $\check{u}_i := \tilde{u}_i$, $\check{\gamma}_i := \tilde{\gamma}_i$, $\check{\vartheta}_i \equiv 0$, and

$(\check{\zeta}_i, \check{x}_i)$ the corresponding trajectory in $(\hat{P}_i)$ (belongs to Λ_i and), is an r_i^4-minimizer for problem $(\hat{P}_i)$.[10] In particular, $\check{\zeta}_i = \tilde{\zeta}_i$ and one has $\|(\check{\zeta}_i, \check{x}_i) - (\tilde{\zeta}_i, \tilde{x}_i)\|_{L^\infty([0,\bar{T}])} \leq (C^4 - 1)\varepsilon_i$. Hence, from (3.2.12) it follows that

$$d_\infty\left((0, \bar{T}, (\check{\zeta}_i, \check{x}_i)), (0, \bar{T}, (\bar{\zeta}, \bar{x}))\right) = \|(\check{\zeta}_i, \check{x}_i) - (\bar{\zeta}, \bar{x})\|_{L^\infty([0,\bar{T}])} \leq C^4\varepsilon_i. \tag{3.3.4}$$

Then, from Ekeland's Principle one can deduce that there exists a process $\mathfrak{z}_i := (T_i, \omega_i, u_i, \gamma_i, \vartheta_i, \zeta_i, x_i) \in \Lambda_i$ which is a minimizer of the optimization problem

$$(P_i) \begin{cases} \text{Minimize } \mathcal{J}(T, x) + r_i^2 \left[|T - T_i| + \int_0^T \varrho_i(t, \omega(t), u(t), \gamma(t), \vartheta(t))\, dt\right] \\ \text{over } \ (T, \omega, u, \gamma, \vartheta, \zeta, x) \in \Lambda_i, \end{cases}$$

where the function $\varrho_i : [0, \bar{T} + \delta] \times W \times \mathbb{R}^q \times \Delta_n^1 \times \{0, 1\} \to \mathbb{R}$ is defined as

$$\varrho_i(t, w, v, \gamma, \vartheta) := \begin{cases} |w - \omega_i(t)| + \chi_{\{(a,\gamma,\vartheta) \neq (u_i(t), \gamma_i(t), \vartheta_i(t))\}}(t, v, \gamma, \vartheta), & t \in [0, T_i], \\ 0 & t \in]T_i, \bar{T} + \delta]. \end{cases}$$

Furthermore, $\mathfrak{z}_i$ satisfies

$$\mathbf{d}\big((T_i, \omega_i, u_i, \gamma_i, \vartheta_i, \zeta_i, x_i), (\bar{T}, \check{\omega}_i, \check{u}_i, \check{\gamma}_i, \check{\vartheta}_i, \check{\zeta}_i, \check{x}_i)\big) \leq r_i^2. \tag{3.3.5}$$

In order to apply Ekeland's variational principle, the domain of minimization must be a complete metric space. For this reason, unlike usual, we apply Ekeland's principle to the sequence of problems $(\hat{P}_i)$ on different domains, in each of which strict sense controls ω must belong to a closed subset $\mathcal{V}_{\delta_i}(T)$ of the set of strict sense controls $\mathcal{V}(T)$, which is generally not closed, and consequently not complete, in the L^1-norm: it is in fact dense in the set of extended controls, $\mathcal{W}(T)$. By (3.3.4), (3.3.5), and the continuity of the input-output map associated to (3.2.10), it follows that, possibly for a subsequence, on $[0, \bar{T} + \delta]$ one has

$$\left\|(\zeta_i, x_i) - \left(\bar{\zeta}, \bar{x}\right)\right\|_{L^\infty} \to 0, \quad \left(\dot{\zeta}_i, \dot{x}_i\right) \rightharpoonup \left(\dot{\bar{\zeta}}, \dot{\bar{x}}\right) \quad \text{weakly in } L^1. \tag{3.3.6}$$

[10]Notice that, for any $T > 0$ and any control, the Cauchy problem in $(\hat{P}_i)$ is a special case of (3.2.10), hence it admits a unique solution, which is defined on $[0, T]$. In particular, all processes in Λ_i are strict sense processes.

Here (we do not rename) the functions ζ_i, x_i, $\bar{\zeta}$, $\bar{x}$ are extended to $[0, \bar{T} + \delta]$ by constant extrapolation and the derivatives are set equal to 0 accordingly. Furthermore, Hypothesis 3.2.6 and (3.3.5) imply that there exist a sequence of measurable subsets $\Omega_i \subseteq [0, \bar{T}] \cap [r_i, T_i]$ and $(\varphi_i) \subset L^1([0, \bar{T}]; \mathbb{R}_{\geq 0})$ such that $\ell(\Omega_i) \to \bar{T}$, $\|\varphi_i\|_{L^1([0,\bar{T}])} \to 0$ as $i \to +\infty$, and, for every i and for a.e. $t \in \Omega_i$:

$$(\omega_i, u_i, \gamma_i)(t) \in \bigcup_{j=0}^{n} \{(\bar{\omega}^j(t), \bar{u}^j(t), e^j)\} + (\varphi_i(t)\mathbb{B}_m) \times \{0\} \times \{0\}. \tag{3.3.7}$$

Step 2. For each $i \in \mathbb{N}$, set

$$k_i := \max_{t \in [0, T_i]} \psi(t, x_i(t)), \qquad \tilde{\psi}(t, z, k) := \psi(t, z) - k \quad \forall (t, z, k) \in \mathbb{R}^{1+n+1}.$$

As it is easy to verify, the process $(\mathfrak{z}_i, k_i) = (T_i, \omega_i, u_i, \gamma_i, \vartheta_i, \zeta_i, x_i, k_i)$ turns out to be a minimizer for the optimization problem

$$(Q_i) \begin{cases} \text{Minimize}\, \Psi\left(T, x(T), \frac{k(T)}{1 \vee L_\psi}\right) + r_i^2 \left[|T - T_i| + \int_0^T \varrho_i(t, \omega(t), u(t), \gamma(t), \vartheta(t))\, dt\right] \\ \text{over} \;\; (T, \omega, u, \gamma, \vartheta, \zeta, x) \in \Lambda_i,\;\; k \in W^{1,1}([0, T]; \mathbb{R}),\;\; \text{verifying} \\ \dot{k}(t) = 0, \qquad \tilde{\psi}(t, x(t), k(t)) \leq 0 \quad \forall t \in [0, T]. \end{cases}$$

Since $\mathfrak{z}$ is isolated, from (3.3.6) it follows that $\Psi\left(T_i, x_i(T_i), \frac{k_i}{1 \vee L_\psi}\right) > 0$ for all i, namely, at least one of the following inequalities holds true:

$$d_{\tilde{\mathfrak{C}}}(T_i, x_i(T_i)) > 0, \qquad\qquad k_i > 0. \tag{3.3.8}$$

Possibly passing to a subsequence, we may suppose $k_i > 0$ for every i. Indeed, if this is not the case, condition (3.3.8) implies $d_{\mathfrak{C}}(T_i, x_i(T_i)) > 0$. Thus, the process $(T_i, \omega_i, u_i, \gamma_i, \zeta_i, x_i, k_i)$ can be replaced by $(T_i, \omega_i, u_i, \gamma_i, \zeta_i, x_i, \hat{k}_i)$, which is still a minimizer of problem (Q_i) provided $\hat{k}_i := d_{\tilde{\mathfrak{C}}}(T_i, x_i(T_i))/2 > 0$.

Now we claim that

$$\psi(t, x_i(t)) \leq 0 < k_i \qquad \forall t \in [0, r_i], \tag{3.3.9}$$

namely, the constraint is inactive on $[0, r_i]$. Indeed, for i large enough $c(t) \leq L_{\mathcal{F}}$ a.e.

$t \in [0, r_i]$, so that we can compute

$$\begin{aligned}|x_i(t) - \tilde{x}_i(t)| \leq & \int_0^t |\mathcal{F}(t', x_i, \tilde{\omega}_i, \tilde{u}_i) - \mathcal{F}(t', \tilde{x}_i, \tilde{\omega}_i, \tilde{u}_i)|dt' \\ & + \int_0^t \vartheta_i |\mathcal{F}(t', x_i, \hat{\omega}_i, \hat{u}_i) - \mathcal{F}(t', x_i, \tilde{\omega}_i, \tilde{u}_i)|dt' \\ \leq & \int_0^t L_{\mathcal{F}} |x_i(t') - \tilde{x}_i(t')|dt' + 2L_{\mathcal{F}} \ell[\vartheta_i](t),\end{aligned}$$

where the nondecreasing map $\ell[\cdot](\cdot)$ is as in (3.2.17). By the Gronwall's Lemma 2.2.3 one can deduce that there is $\bar{C} > 0$ such that, for every i, one has

$$|x_i(t) - \tilde{x}_i(t)| \leq \bar{C}\, \ell[\vartheta_i](t) \qquad \forall t \in]0, r_i], \tag{3.3.10}$$

Fix now $i \in \mathbb{N}$. By the Lebourg Mean Value Theorem 2.4.7 (see also Theorem 2.4.6), for every $t \in [0, r_i]$ there exists $(\xi_{0_i}^t, \xi_i^t) \in \mathrm{co}\partial^* \psi(t, y_i(t))$ for some $y_i(t)$ belonging to the segment $\{s x_i(t) + (1-s)\tilde{x}_i(t) \,:\, s \in [0,1]\}$ such that [11]

$$\begin{aligned}\psi(t, x_i(t)) - \psi(t, \tilde{x}_i(t)) &= \xi_i^t \cdot (x_i(t) - \tilde{x}_i(t)) \\ &= \int_0^t \xi_i^t \cdot [\mathcal{F}(t', x_i, \tilde{\omega}_i, \tilde{u}_i) - \mathcal{F}(t', \tilde{x}_i, \tilde{\omega}_i, \tilde{u}_i)]\, dt' \\ &\quad + \int_0^t \vartheta_i(t') \xi_i^t \cdot [\mathcal{F}(t', x_i, \hat{\omega}_i, \hat{u}_i) - \mathcal{F}(t', x_i, \tilde{\omega}_i, \tilde{u}_i)]\, dt' \\ &\leq \int_0^t \bar{C} L_{\psi} L_{\mathcal{F}}\, \ell[\vartheta_i](t') dt' - \hat{\delta}\, \ell[\vartheta_i](t) \leq \ell[\vartheta_i](t) \left(-\hat{\delta} + \bar{C} L_{\psi} L_{\mathcal{F}}\, t\right) \leq 0,\end{aligned}$$

where the last relations follow from (3.2.16), (3.3.10), and the fact that $t \leq r_i \downarrow 0$. Finally, condition (3.2.13) confirms our claim.

Our aim is now to apply a free end-time constrained Pontryagin Maximum Principle to problem (Q_i) with reference to the minimizer $(\mathfrak{z}_i, k_i)$. By the properties of subdifferentials (see Subsection 2.4.2) and the conditions in (3.3.8), we deduce that $\partial^{>}_{t,x,c} \tilde{\psi}(t, x_i(t), k_i) = \partial^{>}_{t,x} \psi(t, x_i(t)) \times \{-1\}$ and that $(\xi_i, \xi_{k_i}) \in \partial \Psi \left(T_i, x_i(T_i), \frac{k_i}{1 \vee L_{\psi}}\right)$

[11] Notice that by the boundedness of the dynamics near 0, both $\tilde{x}_i(t)$ and $x_i(t)$ lay on $\check{z}_0 + tL_{\mathcal{F}}\mathbb{B}$. Hence, for i sufficiently large, $t \in [0, \varepsilon]$ and $y_i(t) \in \check{z}_0 + \varepsilon\mathbb{B}$, where $\varepsilon > 0$ is as in Remark 3.2.7,(4).

implies the existence of some $\varsigma_i^1, \varsigma_i^2 \geq 0$ with $\varsigma_i^1 + \varsigma_i^2 = 1$, verifying

$$\xi_i = (\xi_{t_i}, \xi_{x_i}) \in \varsigma_i^1 \left(\partial d_{\tilde{\mathcal{C}}}(T_i, x_i(T_i)) \cap \partial \mathbb{B}_{1+n}\right), \qquad \xi_{k_i} = \frac{\varsigma_i^2}{1 \vee L_{\psi}}.$$

Furthermore, $\varsigma_i^j = 0$ for $j \in \{1,2\}$, when the maximum in $d_{\tilde{\mathcal{C}}}(T_i, x_i(T_i)) \vee \frac{k_i}{1 \vee L_\psi}$ is strictly greater than the j-th term in the maximization. Thus, the Maximum Principle 2.5.7 yield the existence of a path $(p_i, \pi_i) \in W^{1,1}([0,T_i]; \mathbb{R}^{n+1})$, numbers $h_i \in \mathbb{R}$, $\lambda_i \geq 0$, $\beta_i \geq 0$, $\varsigma_i^1 \geq 0$, $\varsigma_i^2 \geq 0$ with $\sum_{j=1}^2 \varsigma_i^j = 1$ (see Proposition 2.4.4), a measure $\mu_i \in C^{\oplus}([0,T_i])$, and a Borel measurable and μ_i-integrable function $m_i : [0,T_i] \to \mathbb{R}^n$, verifying the following conditions[12]:

(i)′ $\lambda_i + \|p_i\|_{L^\infty([0,T_i])} + \|\pi_i\|_{L^\infty([0,T_i])} + \mu_i([0,T_i]) + \beta_i = 1$;

(ii)′ $-\dot{p}_i(t) \in \text{co}\, \partial_x\Big(q_i(t) \cdot \mathcal{F}(t, (x_i, \omega_i, u_i)(t))\Big)$ for a.e. $t \in [0,T_i]$, and $\dot{\pi}_i(t) = 0$ for a.e. $t \in [0,T_i]$;

(iii)′ $(h_i, -q_i(T_i)) \in (\lambda_i r_i^2\, \mathbb{B}_1 \times \{0_n\}) + \beta_i \partial\psi(T_i, x_i(T_i)) + \lambda_i \varsigma_i^1 \left(\partial d_{\tilde{\mathcal{C}}}(T_i, x_i(T_i)) \cap \partial\mathbb{B}\right)$,
$\pi_i(0) = 0,\ -\pi_i(T_i) + \displaystyle\int_{[0,T_i]} \mu_i(dt') = \lambda_i \frac{\varsigma_i^2}{1 \vee L_\psi} - \beta_i$;

(iv)′ $h_i \in \underset{t \to T_i}{\text{ess}} \Big(\max_{(w,v) \in \mathscr{V}_{\iota_{\delta_i}}(t) \times \mathscr{U}(t)} q_i(T_i) \cdot \mathcal{F}\big(t, x_i(T_i), w, v\big)\Big) + M\lambda_i r_i^2\, \mathbb{B}_1$;

(v)′ $m_i(t) \in \partial_x^{>} \psi\,(t, x_i(t))$, $\qquad \mu_i$-a.e. $t \in [0,T_i]$,

(vi)′ $\text{spt}(\mu_i) \subseteq \{t \in [0,T_i] : \psi\,(t, x_i(t)) - k_i = 0\} \subset [r_i, T_i]$[13],

(vii)′$_1$ $\int_0^{r_i} \vartheta_i\, p_i \cdot [\mathcal{F}(t, x_i, \hat{\omega}_i, \hat{u}_i) - \mathcal{F}(t, x_i, \tilde{\omega}_i, \tilde{u}_i)]\, ds$
$\geq \int_0^{r_i} \Big\{(1 - \vartheta_i)\, p_i \cdot [\mathcal{F}(t, x_i, \hat{\omega}_i, \hat{u}_i) - \mathcal{F}(t, x_i, \tilde{\omega}_i, \tilde{u}_i)] - M\lambda_i r_i^2 \Big\} dt$;[14]

(vii)′$_2$ $\int_{r_i}^{T_i} q_i \cdot \mathcal{F}(t, x_i, \omega_i, u_i) dt \geq \int_{r_i}^{T_i} [q_i \cdot \mathcal{F}(t, x_i, w, u) - M\lambda_i r_i^2]\, dt$
for all $(\omega, u, \gamma) \in \mathcal{V}_{\iota_{\delta_i}}(T_i) \times \mathcal{U}(T_i) \times \Gamma^1(T_i)$,

[12]The costate path associated to the state component ζ_i is the zero function, hence it does not appear in the Maximum Principle's conditions.

[13]The last inclusion follows by (3.3.9).

[14]By (vi)′ it follows that $q_i \equiv p_i$ on $[0, r_i]$. Notice also that (vii)′$_1$ holds in a more general form, in fact we can replace $1 - \vartheta_i$ in the right hand side with any measurable function $\vartheta : [0, r_i] \to \{0, 1\}$.

for some $M > 1$ depending on the diameter of the bounded set W, where

$$q_i(t) := \begin{cases} p_i(t) + \int_{[0,t[} m_i(t')\mu_i(dt') & t \in [0, T_i[, \\ p_i(T_i) + \int_{[0,T_i]} m_i(t')\mu_i(dt') & t = T_i \end{cases}$$

and $\beta_i = 0$ if $\psi(T_i, x_i(T_i)) < k_i$. Notice that, in view of (i)′ and Hypothesis 3.1.2

$$\|q_i\|_{L^\infty(0,T_i)} \leq 1 + L_{\psi}. \tag{3.3.11}$$

Observe that, for each i, by (ii)′ and (iii)′ we derive

$$\mu_i([0, T_i]) = \int_{[0,T_i]} \mu_i(dt) = \lambda_i \frac{\varsigma_i^2}{1 \vee L_{\psi}} - \beta_i \qquad \text{and} \qquad \pi_i \equiv 0. \tag{3.3.12}$$

Furthermore, by (iv)′ we deduce that

$$h_i \leq L_{\mathcal{F}} \|q_i\|_{L^\infty([0,T_i])} + M\lambda_i r_i^2.$$

Accordingly, since $\|m_i\|_{L^\infty([0,T_i])} \leq L_{\psi}$ and $\partial\psi(\cdot,\cdot) \subseteq L_{\psi}\mathbb{B}_{1+n}$, by (iii)′,

$$\lambda_i\varsigma_i^1 - L_{\psi}\beta_i - \lambda_i r_i^2 \leq |(h_i, -q_i(T_i))| \leq (L_{\mathcal{F}}+1)\|p_i\|_{L^\infty([0,T_i])} + L_{\psi}(L_{\mathcal{F}}+1)\mu_i([0, T_i]) + M\lambda_i r_i^2. \tag{3.3.13}$$

By adding up the non-triviality condition (i)′, (3.3.12) and (3.3.13), for i sufficiently large we get

$$\begin{aligned} &(L_{\mathcal{F}} + 2)\|p_i\|_{L^\infty([0,T_i])} + (1 + 1 \vee L_{\psi} + L_{\psi}(1 + L_{\mathcal{F}}))\mu_i([0, T_i]) \\ &\qquad + (1 + L_{\psi} + 1 \vee L_{\psi})\beta_i \geq 1 - \lambda_i + \lambda_i(\varsigma_i^1 + \varsigma_i^2) - (M + 1)\lambda_i r_i^2 \geq \frac{1}{2}, \end{aligned}$$

since $r_i \downarrow 0$ and $\varsigma_i^1 + \varsigma_i^2 = 1$. Hence, scaling the multipliers, we obtain

$$\|p_i\|_{L^\infty([0,T_i])} + \mu_i([0, T_i]) + \beta_i = 1, \qquad \lambda_i \leq \tilde{L}, \tag{3.3.14}$$

where $\tilde{L} := 2[(L_{\mathcal{F}} + 2) \vee (1 + 1 \vee L_{\psi} + L_{\psi}(1 + L_{\mathcal{F}}))]$.

Step 3. Now, we pass to the limit in the relations obtained in *Step 2.* As for the trajectories, consider the functions p_i extended to $[0, \bar{T} + \delta]$ by constant extrapolation to the left and to the right. Extend also the measures μ_i and the functions m_i to $[0, \bar{T} + \delta]$ by setting them identically zero outside $[0, T_i]$. Then, there exist a subsequence of (μ_i), $\mu \in C^{\oplus}([0, \bar{T} + \delta])$, $m : [0, \bar{T} + \delta] \to \mathbb{R}^n$ Borel measurable and μ-integrable, such that (we do not relabel) $\mu_i \rightharpoonup^* \mu$ weakly* and $m_i(t)\mu_i(dt) \rightharpoonup^* m(t)\mu(dt)$ (see Theorem 2.2.8). Furthermore, for any i, $|\dot{p}_i(t)| \leq (1 + L_{\psi})c(t)$ by (3.3.11) and Hypothesis 3.1.3, and $|p_i(\bar{T} + \delta)| \leq 2L_{\psi} + 1$ by (iii)′ and (i)′, so that Theorem 2.3.4 implies that there is some path $p \in W^{1,1}([0, \bar{T} + \delta]; \mathbb{R}^n)$ such that, along a suitable subsequence, on $[0, \bar{T} + \delta]$ one has

$$p_i \to p \text{ in } L^\infty, \qquad \dot{p}_i \rightharpoonup \dot{p} \text{ weakly in } L^1, \qquad q_i \to q \text{ in } L^1, \qquad q_i(T_i) \to q(\bar{T}), \tag{3.3.15}$$

where

$$q(t) := \begin{cases} p(t) + \int_{[0,t[} m(t')\mu(dt') & t \in [0, \bar{T} + \delta[, \\ p(\bar{T} + \delta) + \int_{[0,\bar{T}+\delta]} m(t')\mu(dt') & t = \bar{T} + \delta. \end{cases} \tag{3.3.16}$$

In particular, $q_i \to q$ a.e. in view of (2.2.2) (that implies also $q_i(T_i) \to q(\bar{T})$), so that (3.3.11) allows the application of the Dominated Convergence Theorem 2.2.4, that implies $q_i \to q$ in L^1.

By (i)′ and (iii)′, the real sequences (h_i), (β_i) are bounded. Hence, possibly for a further subsequence, there exist $h \in \mathbb{R}$ and $\beta \geq 0$ such that $h_i \to h$ and $\beta_i \to \beta$, as $i \to +\infty$. In the limit, condition (3.3.14) yields

$$\|p\|_{L^\infty([0,\bar{T}])} + \mu([0, \bar{T}]) + \beta = 1, \tag{3.3.17}$$

while (iii)′, (v)′, and (vi)′ (see also Proposition 2.4.8) imply the transversality conditions (3.2.7), and the properties (3.1.9), (3.1.10) of m and μ, respectively. We point out that, since $T_i \to \bar{T}$, $p_i \to p$ in L^∞, and in view of (3.1.10), we get that (3.3.16) is the extension by constant extrapolation to $[0, \bar{T} + \delta]$ of q given by (3.2.9).

Notice that if $\psi(\bar{T}, \bar{x}(\bar{T})) < 0$, then $\psi(T_i, x_i(T_i)) < 0 < k_i$ for i sufficiently large by (3.3.5) and (3.3.6), hence $\beta_i = 0$ for any i large, so that $\beta = 0$. Similarly, if $\tilde{\mathfrak{C}} \subset \{\bar{T}\} \times \mathbb{R}^n$, then $T_i \equiv \bar{T}$, hence $\beta_i \equiv 0$ and consequently $\beta = 0$.

Condition (3.2.8) on h follows from Proposition 2.5.6 (together with (3.3.6) and the last condition in (3.3.15)), once we observe that Hypothesis 3.1.1 (see Remark 3.1.4,(i)) and Hypothesis 3.1.3,(ii) imply that, possibly reducing δ and for i large enough, for a.e. $t \in [T_i - \delta, T_i + \delta]$ one has

$$
\begin{aligned}
0 &\leq \max_{(w,v)\in\overline{\mathscr{V}(t)}\times\mathscr{U}(t)} q_i(T_i)\cdot\mathcal{F}\big(t, x_i(T_i), w, v\big) - \max_{(w,v)\in\mathscr{V}_{\iota_{\delta_i}}(t)\times\mathscr{U}(t)} q_i(T_i)\cdot\mathcal{F}\big(t, x_i(T_i), w, v\big) \\
&\leq q_i(T_i)\cdot\mathcal{F}(t, x_i(T_i), \bar{w}(t), \bar{v}(t)) - q_i(T_i)\cdot\mathcal{F}\Big(t, x_i(T_i), \Pi_{\mathscr{V}_{\iota_{\delta_i}}(t)}(\bar{w}(t)), \bar{v}(t)\Big) \\
&\leq (1 + L_{\psi})\rho(\delta_i),
\end{aligned}
\tag{3.3.18}
$$

where $\Pi_{\mathscr{V}_{\iota_{\delta_i}}(\cdot)}(\cdot)$ is as in Theorem 2.3.1 and, for a.e. $t \in [T_i - \delta, T_i + \delta]$, $(\bar{w}(t), \bar{v}(t))$ is such that

$$
q_i(t)\cdot\mathcal{F}(t, x_i(T_i), \bar{w}(t), \bar{v}(t)) = \max_{(w,v)\in\overline{\mathscr{V}(t)}\times\mathscr{U}(t)} q_i(T_i)\cdot\mathcal{F}\big(t, x_i(T_i), w, v\big).
$$

Now we obtain the adjoint equation (3.1.5) from (ii)$'$. By adding and subtracting '$q(t)$', using Proposition 2.4.4, (2.2.1) and (2.4.3), for a.e. $t \in [0, T_i]$ one has

$$
\begin{aligned}
\dot{p}_i(t) &\in \operatorname{co}\partial_x\big(q(t)\cdot\mathcal{F}(t, (x_i, \omega_i, u_i)(t))\big) + \operatorname{co}\partial_x\big((q_i(t) - q(t))\cdot\mathcal{F}(t, (x_i, \omega_i, u_i)(t))\big) \\
&\subset \operatorname{co}\partial_x\big(q(t)\cdot\mathcal{F}(t, (x_i, \omega_i, u_i)(t))\big) + (q_i(t) - q(t))\cdot D_x\mathcal{F}(t, (x_i, \omega_i, u_i)(t)) \\
&\subset \operatorname{co}\partial_x\big(q(t)\cdot\mathcal{F}(t, (x_i, \omega_i, u_i)(t))\big) + L_{\mathcal{F}}|q_i(t) - q(t)|\mathbb{B}_n.
\end{aligned}
$$

Now, using 3.1.3,(ii) and (3.3.7) we deduce that, for a.e. $t \in \Omega_i$ one has

$$
\begin{aligned}
\big(-\dot{p}_i, \dot{\zeta}_i, \dot{x}_i\big)(t) \in \bigcup_{j=0}^{n} &\Big(\operatorname{co}\partial_x\big(q(t)\cdot\mathcal{F}(t, (x_i, \bar{\omega}^j, \bar{u}^j)(t))\big), e^j, \mathcal{F}(t, (x_i, \bar{\omega}^j, \bar{u}^j)(t))\Big) \\
&+ \big([(1 + L_{\psi})\rho(\varphi_i(t)) + L_{\mathcal{F}}|q_i(t) - q(t)|]\mathbb{B}_n\big) \times \{0\} \times \big(\rho(\varphi_i(t))\mathbb{B}_n\big).
\end{aligned}
$$

Taking account of (3.3.6) and (3.3.15), we can appeal to the Compactness of Trajectories Theorem 2.3.4 [15] to obtain

$$
\Big(-\dot{p}, \dot{\bar{\zeta}}, \dot{\bar{x}}\Big)(t) \in \operatorname{co}\left(\bigcup_{j=0}^{n}\Big(\operatorname{co}\,\partial_x\left(q(t)\cdot\mathcal{F}(t, (\bar{x}, \bar{\omega}^j, \bar{u}^j)(t))\right), e^j, \mathcal{F}(t, (\bar{x}, \bar{\omega}^j, \bar{u}^j)(t))\Big)\right)
$$

[15] In particular, $\operatorname{co}(q\cdot\mathcal{F})$ is a closed multifunction in view of Theorem 2.4.6 and Theorem 2.2.1; $\rho(\varphi_i) \to 0$ in L^1 by the Dominated Convergence Theorem 2.2.4 since, up to subsequence, $\varphi_i \to 0$ a.e. and (φ_i) is uniformly bounded by a constant that depends on the diameter of W.

for a.e. $t \in [0, \bar{T}]$. Thanks to Proposition 2.2.2 we get

$$\Big(-\dot{p}, \dot{\bar{\zeta}}, \dot{\bar{x}} \Big)(t) \in \sum_{j=0}^{n} \gamma^j(t) \Big(\text{co } \partial_x \left(q(t) \cdot \mathcal{F}(t, (\bar{x}, \bar{\omega}^j, \bar{u}^j)(t)) \right), e^j, \mathcal{F}(t, (\bar{x}, \bar{\omega}^j, \bar{u}^j)(t)) \Big),$$

for a.e. $t \in [0, \bar{T}]$ and for some function $\gamma \in \Gamma(\bar{T})$. This implies that p satisfies (3.1.5), since $\gamma = \dot{\bar{\zeta}} = \bar{\gamma}$ almost everywhere.

To obtain the maximality condition (3.1.8), take an arbitrary $(\omega, u) \in \mathcal{W}(\bar{T}+\delta) \times \mathcal{U}(\bar{T}+\delta)$. In view of Remark 3.1.4,(i), Hypothesis 3.1.1 implies that, for any i, there exists some $\mathrm{v}_i \in \mathcal{V}_{\delta_i}(\bar{T}+\delta)$ such that $\|\omega - \mathrm{v}_i\|_{L^1} \le \delta_i \downarrow 0$. By (vii)$'$, we deduce that, for any i, one has

$$\int_0^{\bar{T}+\delta} q_i \cdot \dot{x}_i \chi_{[r_i, T_i]} \, dt \ge \int_0^{\bar{T}+\delta} \left\{ q_i \cdot \mathcal{F}(t, x_i, \mathrm{v}_i, u) - M \lambda_i r_i^2 \right\} \chi_{[r_i, T_i]} \, dt.$$

In the left hand side we add and subtract the quantities '$q \cdot \dot{x}_i$' and '$q \cdot \dot{\bar{x}}$' and, recalling that $|\dot{x}_i(t)| \le c(t)$ and $|q_i| \le 1 + L_\psi$ for any i, we use (3.3.6) and (3.3.15) in order to pass to the limit (in particular, $\int (q_i - q) \dot{\bar{x}}_i \to 0$ by the Dominated Convergence Theorem 2.2.4). Observing that, up to a subsequence, $\mathrm{v}_i \to \omega$ a.e., in the right hand side we utilize the Dominated Convergence Theorem 2.2.4, so that to obtain

$$\int_0^{\bar{T}} q(t) \cdot \dot{\bar{x}}(t) \, dt \ge \int_0^{\bar{T}} q(t) \cdot \mathcal{F}(t, (\bar{x}, \omega, u)(t)) \, dt.$$

Since this is true for any control (ω, u) as above, we conclude that

$$q(t) \cdot \dot{\bar{x}}(t) = \max_{(w,v) \in \overline{\mathscr{V}(t)} \times \mathscr{U}(t)} q(t) \cdot \mathcal{F}(t, \bar{x}(t), w, v) \qquad \text{a.e. } t \in [0, \bar{T}],$$

which implies (3.1.8). Thus $\mathfrak{Z}$ is an abnormal extremal. To prove that it is in fact a nondegenerate abnormal extremal, it remains to show that the above multipliers fulfill the strengthened non-triviality condition

$$\|q\|_{L^\infty([0,\bar{T}])} + \mu(]0, \bar{T}]) + \beta \ne 0. \tag{3.3.19}$$

Indeed, assume by contradiction that $\|q\|_{L^\infty([0,\bar{T}])} + \mu(]0, \bar{T}]) + \beta = 0$. Then, the non-triviality condition (3.3.17) yields that $\mu(\{0\}) \neq 0$ and $p \equiv -\mu(\{0\})\xi$ for some $\xi \in \partial_x^{>}\psi(0, \check{z}_0)$. For every i, by the maximality condition (vii)$'_1$ and relation (3.2.16) (recalling that $r_i \leq C\sqrt[4]{\varepsilon_i}$ by (3.3.1)) it follows that

$$\begin{aligned}
0 \geq & \int_0^{r_i} (1-2\vartheta_i)\, p \cdot [\mathcal{F}(t, x_i, \hat{\omega}_i, \hat{u}_i) - \mathcal{F}(t, x_i, \tilde{\omega}_i, \tilde{u}_i)]\, dt \\
& + \int_0^{r_i} \left\{ (1-2\vartheta_i)\,(p_i - p) \cdot [\mathcal{F}(t, x_i, \hat{\omega}_i, \hat{u}_i) - \mathcal{F}(t, x_i, \tilde{\omega}_i, \tilde{u}_i)] - M\lambda_i r_i^2 \right\} dt \\
\geq & \int_0^{r_i} p \cdot [\mathcal{F}(t, x_i, \hat{\omega}_i, \hat{u}_i) - \mathcal{F}(t, x_i, \tilde{\omega}_i, \tilde{u}_i)]\, \chi_{\{t' :\, \vartheta_i(t') = 0\}}(t)\, dt \\
& - \int_0^{r_i} p \cdot [\mathcal{F}(t, x_i, \hat{\omega}_i, \hat{u}_i) - \mathcal{F}(t, x_i, \tilde{\omega}_i, \tilde{u}_i)]\, \chi_{\{t' :\, \vartheta_i(t') = 1\}}(t)\, dt \\
& - r_i (2L_{\mathcal{F}} \|p_i - p\|_{L^\infty} + M\tilde{L} r_i^2) \\
\geq & \, \mu(\{0\})\, \hat{\delta}\, \ell[1-\vartheta_i](r_i) - 2L_{\mathcal{F}} L_{\psi}\, \ell[\vartheta_i](r_i) - r_i(2L_{\mathcal{F}} \|p_i - p\|_{L^\infty} + M\tilde{L} r_i^2) \\
\geq & \, r_i \left[\mu(\{0\})\, \hat{\delta} - \mu(\{0\})\, \hat{\delta}\, r_i - 2L_{\mathcal{F}} L_{\psi} r_i - 2L_{\mathcal{F}} \|p_i - p\|_{L^\infty} - M\tilde{L} r_i^2 \right] > 0,
\end{aligned}$$

where we use the facts that $\ell[\vartheta_i](r_i) \leq r_i^2$ and, as a straightforward consequence, that $\ell[1-\vartheta_i](r_i) \geq r_i - r_i^2$, which follow from (3.3.5). Thus, we get a contradiction and the proof is complete.

3.4 Proof of Theorem 3.1.6

For any $t_1 < t_2$, set $\Gamma^1([t_1, t_2]) := \mathcal{M}([t_1, t_2]; \Delta_n^1)$, where Δ_n^1 is as in (3.2.11). Preliminarily, notice that Theorem 2.5.2 implies that for any initial condition $\check{z}_0 \in \mathbb{R}^n$ the input-output map $\mathcal{W}([t_1, t_2]) \times \mathcal{U}([t_1, t_2]) \times \Gamma^1([t_1, t_2]) \ni (\omega, u, \gamma) \mapsto (\zeta, x)$, where $(\zeta, x) = (\zeta, x)[t_1, t_2, \check{z}_0, \omega, u, \gamma]$ denotes the unique solution to

$$(\dot{\zeta}, \dot{x})(t) = (\gamma(t), \mathcal{F}(t, x(t), \omega(t), u(t))) \quad \text{for a.e. } t \in [t_1, t_2] \tag{3.4.1}$$

with initial condition $(\zeta, x)(t_1) = (0, \check{z}_0)$, is well defined and has a continuous dependence with respect to ω. Let $\mathfrak{Z} := (\bar{t}_1, \bar{t}_2, \underline{\bar{\omega}}, \underline{\bar{u}}, \bar{\gamma}, \bar{x})$ be as in the theorem's statement, set $\bar{\zeta}(t) := \int_{\bar{t}_1}^{t} \bar{\gamma}(t')\, dt'$ for all $t \in [\bar{t}_1, \bar{t}_2]$ and observe that $(\bar{\zeta}, \bar{x})$ solves the differential inclusion

$$\big(\dot\zeta,\dot x\big)(t)\in \mathrm{co}\,\bigcup_{j=0}^{n}\{\big(e^j,\mathcal{F}(t,x(t),\bar\omega^j(t),\bar u^j(t))\big)\}\qquad \text{a.e. } t\in[\bar t_1,\bar t_2].$$

Since $\mathfrak{Z}$ is isolated in view of Proposition 3.0.6, there is some $\delta>0$ as in Definition 3.0.5. Fixed a sequence $\varepsilon_i\downarrow 0$, $\varepsilon_i<\delta/2$, by the Relaxation Theorem 2.3.5 for every i there is a measurable control $(\bar\omega_i,\bar u_i,\bar\gamma_i)(t)\in\bigcup_{j=0}^{n}\{(\bar\omega^j(t),\bar u^j(t),e^j)\}$ for a.e. $t\in[\bar t_1,\bar t_2]$, such that the pair $(\bar\zeta_i,\bar x_i)$, where $(\bar\zeta_i,\bar x_i):=(\zeta,x)[\bar t_1,\bar t_2,\bar x(\bar t_1),\bar\omega_i,\bar u_i,\bar\gamma_i]$, satisfies

$$\|(\bar\zeta_i,\bar x_i)-(\bar\zeta,\bar x)\|_{L^\infty([\bar t_1,\bar t_2])}\le\varepsilon_i. \tag{3.4.2}$$

Choose $\delta_i\in]0,\varepsilon_i[$ such that, for any $\omega\in\mathcal{W}([\bar t_1,\bar t_2])$, $\|\omega-\bar\omega_i\|_{L^1([\bar t_1,\bar t_2])}\le\delta_i$, one has

$$\|(\zeta,x)[\bar t_1,\bar t_2,\bar x(\bar t_1),\omega,\bar u_i,\bar\gamma_i]-(\bar\zeta_i,\bar x_i)\|_{L^\infty([\bar t_1,\bar t_2])}\le\varepsilon_i.$$

Let $\check\omega_i(t)\in\mathscr{V}_{\iota\delta_i}(t)$ for a.e. $t\in[\bar t_1,\bar t_2]$ ($\mathscr{V}_{\iota\delta_i}(t)$ as in Remark 3.1.4,(i)) be a strict sense control satisfying $\|\check\omega_i-\bar\omega_i\|_{L^1([\bar t_1,\bar t_2])}\le\delta_i$, which exists owing to Hypothesis 3.1.1. Hence, setting $\check u_i:=\bar u_i$, $\check\gamma_i:=\bar\gamma_i$, and $(\check\zeta_i,\check x_i):=(\zeta,x)[\bar t_1,\bar t_2,\bar x(\bar t_1),\check\omega_i,\check u_i,\check\gamma_i]$, we get a strict sense process $(\bar t_1,\bar t_2,\check\omega_i,\check u_i,\check\gamma_i,\check\zeta_i,\check x_i)$[16] enjoying the properties

$$\|(\check\zeta_i,\check x_i)-(\bar\zeta_i,\bar x_i)\|_{L^\infty([\bar t_1,\bar t_2])}\le\varepsilon_i, \tag{3.4.3}$$

and, for some sequence $(\check\varphi_i)\subset L^1([\bar t_1,\bar t_2];\mathbb{R}_{\ge0})$ converging to 0 in L^1,

$$(\check\omega_i,\check u_i,\check\gamma_i)(t)\in\bigcup_{j=0}^{n}\{(\bar\omega^j(t),\bar u^j(t),e^j)\}+(\check\varphi_i(t)\mathbb{B}_m)\times\{0\}\times\{0\}\quad\text{a.e. } t\in[\bar t_1,\bar t_2].$$

Now, set $\Psi\,(t_1,z_1,t_2,z_2,k):=d_{\mathfrak{C}}(t_1,z_1,t_2,z_2)\vee k$ for all (t_1,z_1,t_2,z_2,k) in $\mathbb{R}^{1+n+1+n+1}$ and, for any $t_1<t_2$, $x\in W^{1,1}([t_1,t_2];\mathbb{R}^n)$, consider the payoff

$$\mathfrak{J}(t_1,t_2,x):=\Psi\Big(t_1,x(t_1),t_2,x(t_2),\ \max_{t\in[t_1,t_2]}\psi(t,x(t))\Big).$$

[16] As in the proof of Theorem 3.2.8, with a small abuse of notation we call *process* also an originally defined process, where the variable ζ is added.

For every i, let $r_i \geq 0$ satisfy

$$r_i^4 = \sup\big\{\mathcal{J}(t_1,t_2,x):\ (t_1,t_2,w,u,x)\in\Sigma,\ d_\infty\big((t_1,t_2,x),(\bar t_1,\bar t_2,\bar x)\big)\leq 2\varepsilon_i\big\}$$

and introduce the set Λ_i of processes $(t_1,t_2,\omega,u,\gamma,\zeta,x)$, where $t_1 < t_2$, the control $(\omega,u,\gamma)\in\mathcal{V}_{\delta_i}([t_1,t_2])\times\mathcal{U}([t_1,t_2])\times\Gamma^1([t_1,t_2])$ with $\mathcal{V}_{\delta_i}([t_1,t_2]) := \{\omega\in\mathcal{M}([t_1,t_2];W):\ \omega(t)\in\mathscr{V}_{\iota\delta_i}$ a.e. $t\in[t_1,t_2]\}$, and (ζ,x) satisfies (3.4.1) and has

$$d_\infty\big((t_1,t_2,x),(\bar t_1,\bar t_2,\bar x)\big)\leq\delta.$$

This set is a complete metric space if endowed with the distance

$$\begin{aligned}\mathbf{d}((t_1',t_2',\omega',u',\gamma',\zeta',y'),(t_1,t_2,\omega,u,\gamma,\zeta,x)) &:= |t_1'-t_1|+|t_2'-t_2|\\ &+|y'(t_1')-x(t_1)|+\|\omega'-\omega\|_{L^1(I)}+\ell\{t\in I:\ (u',\gamma')(t)\neq(u,\gamma)(t)\},\end{aligned}\tag{3.4.4}$$

where $I := [t_1'\vee t_1, t_2'\wedge t_2]$. Notice that by (3.4.2), (3.4.3) it follows that

$$d_\infty((\bar t_1,\bar t_2,\check x_i),(\bar t_1,\bar t_2,\bar x))\leq 2\varepsilon_i,$$

so that the process $(\bar t_1,\bar t_2,\check\omega_i,\check u_i,\check\gamma_i,\check\zeta_i,\check x_i)$ is an r_i^4-minimizer for the optimal control problem

$$\begin{cases}\text{Minimize}\quad \mathcal{J}(t_1,t_2,x)\\ \text{over processes } (t_1,t_2,\omega,u,\gamma,\zeta,x)\in\Lambda_i.\end{cases}$$

From now on, except for minor obvious changes, the proof proceeds similarly to the proof of Theorem 3.2.8 and is actually simpler, since we disregard the nondegeneracy issue. Hence, we omit it. □

Chapter 4

Fixed end-time problems

In this chapter we deal with fixed end-time optimization problems with prescribed initial position and constant control sets. Despite this might appear as a mere special case of what we investigated in Chapter 3, the reasons for giving space to this subject is twofold. On the one hand, fixed end-time optimization problems are the most studied in the literature, both in the derivation of necessary conditions of optimality and in the deduction of sufficient conditions for no gap [59, 60, 61, 63, 76, 79], so that to deserve separate attention. On the other hand, a detailed analysis of the fixed end-time case is preparatory in order to look for sufficient conditions for no gap for free end-time problems with Lipschitz continuous time dependence. Indeed, as we will better explain in Chapter 5, for these latter problems one can provide additional conditions satisfied by the extremals – as, for instance, the constancy of the Hamiltonian in the autonomous case –, that can not be deduced from those of Definition 3.1.5 in any way.

In particular, in Section 4.1 we adapt the definitions and the results of Chapter 3 to the setting of fixed end-time problems with fixed initial position and constant control sets. Afterwards, in Section 4.2, we study the converse problem of wether a minimizer for the original problem is still a minimizer for the extended or the relaxed auxiliary problem. The present chapter is based on [38, 39].

4.1 Gap and controllability theorems for fixed end-time

First of all we have to appropriately modify the notions of (feasible) strict sense, extended and relaxed processes.

Definition 4.1.1. Let $U \subset \mathbb{R}^q$, $V \subset \mathbb{R}^m$, and let $W = \overline{V}$. We refer to (ω, u, x) as *extended process* if $\omega \in \mathcal{W}(T) := \mathcal{M}([0,T], W)$, $u \in \mathcal{U}(T) := \mathcal{M}([0,T], U)$, and $x \in W^{1,1}([0,T], \mathbb{R}^n)$ satisfies

$$\dot{x}(t) = \mathcal{F}(t, x(t), \omega(t), u(t)) \qquad \text{a.e. } t \in [0,T]. \tag{4.1.1}$$

An extended process is called a *strict sense process* if $\omega \in \mathcal{V}(T) := \mathcal{M}([0,T], V)$. A strict sense or extended process is *feasible* if it satisfies the following endpoint and state constraints

$$\psi(t, x(t)) \leq 0 \quad \forall t \in [0,T], \qquad (x(0), x(T)) \in \{\check{z}_0\} \times \check{\mathcal{C}}. \tag{4.1.2}$$

With a small abuse of notation, we refer to $(\underline{\omega}, \underline{u}, \gamma, \zeta, x)$ as *relaxed process* if $\omega \in \mathcal{W}^{1+n}(T)$, $u \in \mathcal{U}^{1+n}(T)$, $\gamma \in \Gamma(T) = \mathcal{M}([0,T], \Delta_n)$ and $(\zeta, x) \in W^{1,1}([0,T], \mathbb{R}^{1+n} \times \mathbb{R}^n)$ satisfies

$$(\dot{\zeta}(t), \dot{x}(t)) = \left(\gamma(t), \sum_{j=0}^{n} \gamma^j(t) \mathcal{F}(t, x(t), \omega^j(t), u^j(t))\right) \qquad \text{a.e. } t \in [0,T]. \tag{4.1.3}$$

A relaxed process is *feasible* when it satisfies (4.1.2). We write $\check{\Sigma}$, $\check{\Sigma}_e$, $\check{\Sigma}_r$, to denote the sets of strict sense, extended, and relaxed processes which are feasible, respectively.

As observed in Remark 3.0.2, we have $\check{\Sigma} \subset \check{\Sigma}_e \subset \check{\Sigma}_r$. We can now adapt the concepts of *local minimizer*, *local infimum gap*, and *isolated process* to fixed end-time problems, by noticing that the distance d_∞ defined in (3.0.4) turns into the L^∞-norm over the interval $[0,T]$, namely $\|\cdot\|_{L^\infty([0,T])}$.

For instance, if $\mathfrak{Z} := (\underline{\bar{\omega}}, \underline{\bar{u}}, \bar{\gamma}, \bar{\zeta}, \bar{x}) \in \check{\Sigma}_r$, *at* $\mathfrak{Z}$ *there is a local infimum* gap if for any continuous function $\Phi : \mathbb{R}^n \to \mathbb{R}$ there is some $\delta > 0$ such that

$$\Phi(\bar{x}(T)) < \inf \left\{ \Phi(x(T)) : \quad (\omega, u, x) \in \check{\Sigma}, \quad \|\bar{x} - x\|_{L^\infty([0,T])} < \delta \right\}.$$

We say that $\mathfrak{Z}$ is an *isolated process* if, for some $\delta > 0$, it holds

$$\left\{(\omega, u, x) \in \check{\Sigma} : \ \|\bar{x} - x\|_{L^\infty([0,T])} < \delta\right\} = \emptyset,$$

while we say that *the constrained control system* (4.1.1)-(4.1.2) *is controllable to* $\mathfrak{Z}$ if $\mathfrak{Z}$ is not isolated. As in previous chapter, it is true that at $\mathfrak{Z}$ there is a local infimum gap if and only if $\mathfrak{Z}$ is isolated. Finally, given $\tilde{\Sigma} \in \{\check{\Sigma}, \check{\Sigma}_e, \check{\Sigma}_r\}$ and a continuous function $\Phi : \mathbb{R}^n \to \mathbb{R}$ we say that $\mathfrak{Z} := (\underline{\bar{\omega}}, \underline{\bar{u}}, \bar{\gamma}, \bar{\zeta}, \bar{x}) \in \tilde{\Sigma}$ is a *local* Φ*-minimizer for problem* $(P_{\tilde{\Sigma}})$ if, for some $\delta > 0$, one has [17]

$$\Phi(\bar{x}(T)) = \min\left\{\Phi(x(T)) : \quad (\underline{\omega}, \underline{u}, \gamma, \zeta, x) \in \tilde{\Sigma} \ \text{ s.t. } \ \|x - \bar{x}\|_{L^\infty([0,T])} < \delta\right\}.$$

The process $\mathfrak{Z}$ is a (*global*) Φ*-minimizer for problem* $(P_{\tilde{\Sigma}})$ if $\Phi(\bar{x}(T)) = \min_{\tilde{\Sigma}} \Phi(x(T))$.

We shall consider the following hypotheses, in which the relaxed reference process $(\underline{\bar{\omega}}, \underline{\bar{u}}, \bar{\gamma}, \bar{\zeta}, \bar{x}) \in \check{\Sigma}_r$ and $\eta > 0$ are given. In particular we defint the η-tube of $(\underline{\bar{\omega}}, \underline{\bar{u}}, \bar{\gamma}, \bar{\zeta}, \bar{x})$ by

$$\check{\mathfrak{S}}_\eta := \{(t, z) \in \mathbb{R} \times \mathbb{R}^n : \quad t \in [0, T], \ z \in \bar{x}(t) + \eta \, \mathbb{B}\}.$$

Hypothesis 4.1.2. The Borel set $U \subset \mathbb{R}^q$ is compact, and the Borel set $V \subset \mathbb{R}^m$ is bounded. Moreover, there exists a sequence (V_i) of closed subsets of V such that $V_i \subseteq V_{i+1}$ for every i and $\bigcup_{i=1}^{+\infty} V_i = V$.

Hypothesis 4.1.3. The target $\check{\mathfrak{C}} \subset \mathbb{R}^n$ is closed. The constraint function ψ is upper semicontinuous and there exists $L_\psi > 0$ such that $|\psi(t, z) - \psi(t, z')| \le L_\psi |z - z'|$ for any (t, z), $(t, z') \in \check{\mathfrak{S}}_\eta$.

Hypothesis 4.1.4. (i) For all $(z, w) \in \mathrm{proj}_{\mathbb{R}^n} \check{\mathfrak{S}}_\eta \times W$, the function $[0, T] \times U \ni (t, v) \mapsto \mathcal{F}(t, z, w, v)$ is $\mathscr{L} \times \mathscr{B}^q$-measurable. Moreover, there exists $c \in L^1([0, T], \mathbb{R}_{\ge 0})$ such that, for all (t, z, w, v), $(t, z', w, v) \in \check{\mathfrak{S}}_\eta \times W \times U$, we have

$$|\mathcal{F}(t, z, w, v)| \le c(t), \qquad |\mathcal{F}(t, z', w, v) - \mathcal{F}(t, z, w, v)| \le c(t) \, |z' - z|.$$

(ii) There exists some continuous increasing function $\rho : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ with $\rho(0) = 0$

[17] If $\tilde{\Sigma} \in \{\check{\Sigma}, \check{\Sigma}_e\}$, the presence of the state arc ζ has to be interpreted in view of Remark 3.2.5.

such that for any $(t,z,v) \in \check{\mathcal{S}}_\eta \times U$, we have

$$|\mathcal{F}(t,z,w',v) - \mathcal{F}(t,z,w,v)| \le \rho(|w'-w|) \qquad \forall w',\, w \in W,$$
$$D_x\mathcal{F}(t,z,w',v) \subseteq D_x\mathcal{F}(t,z,w,v) + \rho(|w'-w|)\,\mathbb{B} \qquad \forall w',\, w \in W.$$

In order to deal with the nondegeneracy issue, we shall consider the following additional assumption, that will be discussed in Remark 4.1.10.

Hypothesis 4.1.5. Let $c \in L^1$ as in Hypothesis 4.1.4. We assume that there exist $\hat{\eta}$, $L_{\mathcal{F}} > 0$ such that $c(t) \le L_{\mathcal{F}}$ for a.e. $t \in [0,\hat{\eta}]$.

We now introduce a notion of normal and abnormal extremal for the relaxed optimization problem with fixed endtime, which follows straightforward by Definition 3.1.5.

Definition 4.1.6. Let $\mathfrak{Z} := (\underline{\bar{\omega}}, \underline{\bar{u}}, \bar{\gamma}, \bar{\zeta}, \bar{x}) \in \check{\Sigma}_r$. Given a function $\Phi : \mathbb{R}^n \to \mathbb{R}$ which is Lipschitz continuous on a neighborhood of $\bar{x}(T)$, we say that $\mathfrak{Z}$ is a Φ-*extremal* if there exist a path $p \in W^{1,1}([0,T],\mathbb{R}^n)$, $\lambda \ge 0$, $\mu \in C^\oplus([0,T])$, $m : [0,T] \to \mathbb{R}^n$ Borel measurable and μ-integrable function, verifying the following conditions:

$$\|p\|_{L^\infty} + \mu([0,T]) + \lambda \neq 0; \tag{4.1.4}$$

$$-\dot{p}(t) \in \sum_{j=0}^{n} \bar{\gamma}^j(t)\, \mathrm{co}\, \partial_x \left(q(t)\cdot \mathcal{F}(t,\bar{x}(t),\bar{\omega}^j(t),\bar{u}^j(t))\right) \qquad \text{a.e. } t \in [0,T]; \tag{4.1.5}$$

$$-q(T) \in \lambda\partial\Phi\left(\bar{x}(T)\right) + N_{\check{\mathcal{C}}}(\bar{x}(T)); \tag{4.1.6}$$

$$\begin{aligned}&\text{for every } j = 0,\dots,n, \text{ for a.e. } t \in [0,T], \text{ one has}\\ &q(t)\cdot\mathcal{F}\big(t,\bar{x}(t),\bar{\omega}^j(t),\bar{u}^j(t)\big) = \max_{(w,v)\in W\times U} q(t)\cdot\mathcal{F}\big(t,\bar{x}(t),w,v\big);\end{aligned} \tag{4.1.7}$$

$$m(t) \in \partial_x^{>}\,\psi\left(t,\bar{x}(t)\right) \qquad \mu\text{-a.e.}; \tag{4.1.8}$$

$$\mathrm{spt}(\mu) \subseteq \{t \in [0,T] :\, \psi\left(t,\bar{x}(t)\right) = 0\}, \tag{4.1.9}$$

where $q : [0,T] \to \mathbb{R}^n$ is defined as

$$q(t) := \begin{cases} p(t) + \int_{[0,t[} m(t')\mu(dt') & t \in [0,T[,\\ p(T) + \int_{[0,T]} m(t')\mu(dt') & t = T.\end{cases} \tag{4.1.10}$$

We will call a Φ-extremal *normal* if all possible choices of (p, λ, μ, m) as above have $\lambda > 0$, and *abnormal* when it is not normal. We will call *nondegenerate multiplier* any set of multipliers (p, λ, μ, m) that meets conditions (4.1.4)–(4.1.10) above, and satisfies the strenghtened nontriviality condition

$$\mu(]0, T]) + \|q\|_{L^\infty} + \lambda \neq 0. \tag{4.1.11}$$

A Φ-extremal is *nondegenerate normal* if all choices of nondegenerate multipliers have $\lambda > 0$, and it is *nondegenerate abnormal* when there exists a nondegenerate multiplier with $\lambda = 0$. In the following, abnormal [nondegenerate abnormal] Φ-extremals will be simply called *abnormal [nondegenerate abnormal] extremals.*

We are now ready to state the gap results to fixed end-time optimization problems. We omit the proofs of the theorems because they are very similar to those you may find in Sections 3.3, 3.4. For more details, see [38].

Theorem 4.1.7. *Let $\mathfrak{z} := (\underline{\bar{\omega}}, \underline{\bar{u}}, \bar{\gamma}, \bar{\zeta}, \bar{x}) \in \check{\Sigma}_r$ and suppose that at $\mathfrak{z}$ there is a local infimum gap.*

(i) *If Hypotheses 4.1.2–4.1.4 are verified, then $\mathfrak{z}$ is an abnormal extremal.*

(ii) *If, in addition, also Hypothesis 3.2.6 (for $\bar{T} = T$) and Hypothesis 4.1.5 are satisfied, then $\mathfrak{z}$ is a nondegenerate abnormal extremal.*

Theorem 4.1.7 implies sufficient conditions for the absence of infimum gap.

Theorem 4.1.8. *Let $\mathfrak{z} := (\underline{\bar{\omega}}, \underline{\bar{u}}, \bar{\gamma}, \bar{\zeta}, \bar{x}) \in \check{\Sigma}_r$ be such that Hypotheses 4.1.2–4.1.4 are satisifed and let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz continuous in a neighborhood of $\bar{x}(T)$.*

(i) *If $\mathfrak{z}$ is a local Φ-minimizer for $(P_{\check{\Sigma}_r})$ or $(P_{\check{\Sigma}_e})$ which is a normal Φ-extremal, then at $\mathfrak{z}$ there is no local Φ-infimum gap. Similarly, if $\mathfrak{z}$ is a Φ-minimizer for $(P_{\check{\Sigma}_r})$ or $(P_{\check{\Sigma}_e})$ which is a normal Φ-extremal, then it realizes the infimum of Φ over $\check{\Sigma}$.*

(ii) *Assume in addition that also Hypothesis 3.2.6 (for $\bar{T} = T$) and Hypothesis 4.1.5 are fulfilled. If $\mathfrak{z}$ is a local Φ-minimizer for $(P_{\check{\Sigma}_r})$ or $(P_{\check{\Sigma}_e})$ which is a nondegenerate normal Φ-extremal, then at $\mathfrak{z}$ there is no local Φ-infimum gap. Similarly, if $\mathfrak{z}$ is a Φ-minimizer for $(P_{\check{\Sigma}_r})$ or $(P_{\check{\Sigma}_e})$ which is a nondegenerate normal Φ-extremal, then it realizes the infimum of Φ over $\check{\Sigma}$.*

In order to state the controllability result, given a relaxed reference process $\mathfrak{Z} := (\underline{\bar{\omega}}, \underline{\bar{u}}, \bar{\gamma}, \bar{x}) \in \check{\Sigma}_r$ for which Hypotheses 4.1.2–4.1.4 are verified, we define $\check{\mathscr{M}}(\mathfrak{Z})$ to be the set of multipliers (p, μ, m) where $p \in W^{1,1}([0,T], \mathbb{R}^n)$, $\mu \in C^{\oplus}([0,T])$, $m : [0,T] \to \mathbb{R}^n$ is a Borel measurable and μ-integrable map, that meet conditions (4.1.5), (4.1.7)–(4.1.10) and such that

$$\|p\|_{L^\infty} + \mu([0,T]) \neq 0,$$
$$-\, q(T) \in N_{\check{\mathfrak{C}}}(\bar{x}(T)).$$

Moreover, we denote by $\check{\mathscr{M}}_0(\mathfrak{Z})$ the subset of $\check{\mathscr{M}}(\mathfrak{Z})$ containing multipliers (p, μ, m) that satisfy the additional strengthened nontriviality condition

$$\|q\|_{L^\infty} + \mu(]0,T]) \neq 0.$$

Theorem 4.1.9. *Let $\mathfrak{Z} := (\underline{\bar{\omega}}, \underline{\bar{u}}, \bar{\gamma}, \bar{\zeta}, \bar{x}) \in \check{\Sigma}_r$ and assume that Hypotheses 4.1.2–4.1.4 are fulfilled.*

(i) *If $\check{\mathscr{M}}(\mathfrak{Z}) = \emptyset$, then the constrained control system* (4.1.1)-(4.1.2) *is controllable to $\mathfrak{Z}$.*

(ii) *If, in addition, also Hypothesis 3.2.6 (for $\bar{T} = T$) and Hypothesis 4.1.5 are satisfied and $\check{\mathscr{M}}_0(\mathfrak{Z}) = \emptyset$, then the constrained control system* (4.1.1)-(4.1.2) *is controllable to $\mathfrak{Z}$.*

Remark 4.1.10. We collect some remarks about the assumptions made and the results obtained.

(1) In the case $\mathscr{V}(\cdot) \equiv V$ and V is bounded, if Hypothesis 4.1.2 is satisfied, then also Hypothesis 3.1.1 is fulfilled. In particular, when $V \subset W$ is such that $\mathrm{Int}(W) \subseteq V \subseteq W$ and $W = \overline{\mathrm{Int}(W)}$, the validity of Hypothesis 4.1.2 follows by elementary properties of closed and open subsets of $\mathbb{R}^n$.

(2) As one can easily deduce from the proof in Section 3.3, Hypothesis 4.1.2 could be replaced by the hypothesis that there exists a subset $\mathcal{V} \subset \mathcal{W} := L^1([0,T]; W)$ which is closed by finite concatenation and verifies:

(i) *there exists an increasing sequence of closed subsets* $(\mathcal{V}_i) \subseteq \mathcal{V}$ *such that* $\cup_i \mathcal{V}_i = \mathcal{V}$ *and, for any* $\omega \in \mathcal{W}$ *and* $\delta > 0$*, there are* i_δ *and* $\omega_\delta \in \mathcal{V}_{i_\delta}$*, such that* $\|\omega_\delta - \omega\|_{L^1} \leq \delta$*;*

(ii) *for every* i*, for the optimization problem obtained from* $(P_{\tilde{\Sigma}})$ *by replacing* $\mathcal{V}$ *with* $\mathcal{V}_i$*, a nonsmooth constrained maximum principle is valid.*

For example, from [44] a condition sufficient for (ii) to hold true is the C^0-closure of the set of the solutions to (4.1.1) as $(\omega, u) \in \mathcal{V}_i \times \mathcal{U}$, for every i.

(3) Surveying the proof of Theorem 3.2.8 in Section 3.3 it can be noticed that condition (3.1.2) of Hypothesis 3.1.1 is needed only to prove relation (3.2.8) (see in particular the calculations in (3.3.18)) and to justify the claim of Remark 3.2.7, (4). Since condition (3.2.8) does not appear in the Maximum Principle of fixed end-time optimization problems, we deduce that, in case of set-valued control sets as in Chapter 3, we could assume the only relation (3.1.1) in Hypothesis 3.1.1, together with the requirement that the sequence $(\hat{\omega}_i)$ of Hypothesis 3.2.6 is strict sense.

(4) We do observe that in Hypothesis 4.1.3 the constraint function ψ is required to be Lipschitz continuous only in the state variable, while Hypothesis 3.1.2 of Section 3.1 the constraint function ψ is required to be Lipschitz continuous in all its variables. This difference is due to the additional multipliers β_1, $\beta_2 \geq 0$ in the definition of Φ-extremal when the dynamics function has a measurable dependence in the time variable (cfr. Definition 3.1.5 and Definition 4.1.6, in particular (3.1.6) and (4.1.6)). Therefore, this leads to estimations that involve the limiting subdifferential of ψ in both its variables (see (3.3.13) in the proof of Theorem 3.2.8). Conversely, these estimations involve the only partial hybrid subdifferential of ψ in the state variable in the case of fixed endtime (see [38, Sec. 6]), and this justifies Hypothesis 4.1.3. For the same reasons in Hypothesis 4.1.4 we do not require the function c to be essentially bounded in a neighborhood of $\bar{T}$ (cfr. Hypothesis 3.1.3). We only need a bound for c near 0 when we deal with the nondegeneracy issue, and this is precisely the reason why we have introduced the additional Hypothesis 4.1.5.

Remark 4.1.11. Since an extended process is a special case of a relaxed process, implicit in the definition of relaxed extremal is the definition of extended extremal. In particular, if $\mathfrak{Z} = (\bar{\omega}, \bar{u}, \bar{x}) \in \check{\Sigma}_e$, clearly the costate differential inclusion (3.1.5) and the maximality condition (3.1.8) in the above definition read, for a.e. $t \in [0, T]$,

$$-\dot{p}(t) \in \operatorname{co} \partial_x \Big(q(t) \cdot \mathcal{F}(t, \bar{x}(t), \bar{\omega}(t), \bar{u}(t)) \Big), \tag{4.1.12}$$

$$q(t) \cdot \mathcal{F}\big(t, \bar{x}(t), \bar{\omega}(t), \bar{u}(t)\big) = \max_{(w,v) \in W \times U} q(t) \cdot \mathcal{F}\big(t, \bar{x}(t), w, v\big), \tag{4.1.13}$$

respectively.

4.2 Stability of minimizers

In the literature, two kinds of relationship between occurrence of an infimum gap and abnormality of associated necessary conditions of optimality have been investigated:

Type S relation: a strict sense local minimizer satisfies the Pontryagin Maximum Principle in abnormal form if it is not also a relaxed local minimizer;

Type R relation: a relaxed local minimizer satisfies the relaxed Pontryagin Maximum Principle in abnormal form if its cost is strictly less than the infimum of the costs of d_∞-neighboring feasible strict sense trajectories.[18]

Differently from Type R relations, which have been widely investigated since the seminal works by Warga [78, 79] until the more recent papers [60, 61, 58, 37, 38, 40, 59], Type S relations have been announced by Warga for state constraint-free optimal control problems with smooth data in [76] and then developed, for the first time, by Palladino and Vinter in [60, 61] (see also [72]), only in the case of the classical extension by convex relaxation.

On the one hand, we aim to extend the properties established in [60, 61] to the more general framework introduced in this thesis, which includes extensions very different from the convex relaxation. On the other hand, this section is complementary to the results of the previous one, where Type R relations have been obtained. The present section is based on [39].

[18] In [62], these two relations are referred to as 'Type A' and 'Type B' relation, respectively.

Theorem 4.2.1. *Assume that Hypotheses 4.1.2–4.1.4 are verified, let $\mathfrak{z} = (\bar{\omega}, \bar{u}, \bar{x}) \in \check{\Sigma}$ and let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz continuous function on a neighborhood of $\bar{x}(T)$. Suppose that $\mathfrak{z}$ is a local Φ-minimizer for problem $(P_{\check{\Sigma}})$, then the following properties are valid.*

(i) *The process $\mathfrak{z}$ is an extremal. In particular, there exists a set of multipliers (p, λ, μ, m) that meets conditions* (4.1.4), (4.1.6), (4.1.8), (4.1.9), (4.1.10), (4.1.12), *and* (4.1.13);

(ii) *if $\mathfrak{z}$ is not a Φ-local minimizer for problem $(P_{\check{\Sigma}_r})$, namely, for every $\varepsilon > 0$ there exists $(\underline{\omega}, \underline{u}, \gamma, \zeta, x) \in \check{\Sigma}_r$ such that*

$$\Phi(x(T)) < \Phi(\bar{x}(T)), \qquad \|x - \bar{x}\|_{L^\infty} < \varepsilon,$$

there exists a choice of multipliers (p, λ, μ, m) as in part (i), *except that* (4.1.12) *is replaced by*

$$-\dot{p}(t) \in \operatorname{co}\left(\bigcup_{(w,v)\in W\times U} \partial_x \Big(q(t) \cdot \mathcal{F}(t, \bar{x}(t), w, v) \Big) \right) \qquad a.e.\ t \in [0, T], \quad (4.2.1)$$

for which $\lambda = 0$.

Proof. We first prove (i). From Hypothesis 4.1.2 it follows that V is $\mathscr{B}^m$-measurable. Hence, part (i) is a well-known version of the nonsmooth constrained Pontryagin Maximum Principle for the original problem (see Theorem 2.5.7).

Now we prove (ii). Assume that $\mathfrak{z}$ is a Φ-local minimizer for $(P_{\check{\Sigma}})$ which is not a Φ-local minimizer for $(P_{\check{\Sigma}_r})$. Let $\delta > 0$ satisfy

$$\Phi(\bar{x}(T)) \leq \Phi(x(T)), \quad \forall (\omega, u, x) \in \check{\Sigma} \quad \text{such that} \quad \|x - \bar{x}\|_{L^\infty} \leq \delta. \qquad (4.2.2)$$

By definition, given any sequence $(\varepsilon_i) \subset]0, \frac{\delta}{2}[$, $\varepsilon_i \downarrow 0$, for every i there exists a feasible relaxed process $\mathfrak{z}_i := (\underline{\omega}_i, \underline{u}_i, \gamma_i, \zeta_i, x_i) \in \check{\Sigma}_r$, such that

$$\Phi(x_i(T)) < \Phi(\bar{x}(T)), \qquad \|x_i - \bar{x}\|_{L^\infty} < \varepsilon_i. \qquad (4.2.3)$$

For any feasible strict sense process $(\omega, u, x) \in \check{\Sigma}$ such that $\|x - x_i\|_{L^\infty} \leq \frac{\delta}{2}$, by (4.2.3) one has

$$\|x-\bar{x}\|_{L^\infty} \leq \|x-x_i\|_{L^\infty} + \|x_i-\bar{x}\|_{L^\infty} \leq \frac{\delta}{2} + \varepsilon_i \leq \delta.$$

From the last relations, we derive that, for every i one has

$$\Phi(x_i(T)) < \inf\left\{\Phi(x(T)) :\ (w,u,x)\in\check{\Sigma},\ \|x-x_i\|_{L^\infty}\leq\frac{\delta}{2}\right\},$$

namely, at $\mathfrak{Z}_i$ there is a Φ-local infimum gap. Thus, from Theorem 4.1.7, (i) applied to any feasible relaxed process $\mathfrak{Z}_i$, we derive that $\mathfrak{Z}_i$ is an abnormal extremal. In particular, for every i there exist multipliers $p_i \in W^{1,1}([0,T];\mathbb{R}^n)$, $\mu_i \in C^\oplus([0,T])$, $m_i : [0,T] \to \mathbb{R}^n$, with m_i Borel measurable and μ_i-integrable function (and $\lambda_i = 0$), verifying:

$$\|p_i\|_{L^\infty} + \mu_i([0,T]) = 1\,^{19};$$

$$-\dot{p}_i(t) \in \sum_{j=0}^{n} \gamma_i^j(t)\,\mathrm{co}\,\partial_x\Big(q_i(t)\cdot\mathcal{F}(t,x_i(t),\omega_i^j(t),u_i^j(t))\Big) \qquad \text{a.e. } t\in[0,T]; \qquad (4.2.4)$$

$$-q_i(T) \in N_{\mathcal{C}}(x_i(T));$$

for every $j = 0,\dots,n$ and a.e. $t \in [0,T]$,

$$q_i(t)\cdot\mathcal{F}\big(t,x_i(t),\omega_i^j(t),u_i^j(t)\big) = \max_{(w,v)\in W\times U} q_i(t)\cdot\mathcal{F}\big(t,x_i(t),w,v\big);$$

$$m_i(t) \in \partial_x^{>}\psi\,(t,x_i(t))\ \ \mu_i\text{-a.e.};$$
$$\mathrm{spt}(\mu_i) \subseteq \{t\in[0,T] :\ \psi\,(t,x_i(t)) = 0\},$$

where

$$q_i(t) := \begin{cases} p_i(t) + \int_{[0,t[} m_i(t')\mu_i(dt') & t\in[0,T[,\\ p_i(T) + \int_{[0,T]} m_i(t')\mu_i(dt') & t = T. \end{cases}$$

Notice that condition (4.2.4) implies that, for a.e. $t \in [0,T]$,

$$-\dot{p}_i(t) \in \sum_{j=0}^{n} \gamma_i^j(t)\,\mathrm{co}\,\partial_x\Big(q_i(t)\cdot\mathcal{F}(t,x_i(t),\omega_i^j(t),u_i^j(t))\Big)$$

[19] As it is always possible, we normalize the multipliers in the nontriviality condition.

$$\subseteq \sum_{j=0}^{n} \gamma_i^j(t) \bigcup_{(w,v)\in W\times U} \operatorname{co} \partial_x \Big(q_i(t) \cdot \mathcal{F}(t, x_i(t), w, v) \Big)$$
$$= \bigcup_{(w,v)\in W\times U} \operatorname{co} \partial_x \Big(q_i(t) \cdot \mathcal{F}(t, x_i(t), w, v) \Big)$$
$$\subseteq \operatorname{co} \left(\bigcup_{(w,v)\in W\times U} \partial_x \Big(q_i(t) \cdot \mathcal{F}(t, x_i(t), w, v) \Big) \right).$$

By Theorem 2.2.8, there exist some subsequence of (μ_i), a measure $\mu \in C^{\oplus}([0,T])$, a function $m : [0,T] \to \mathbb{R}^n$ Borel measurable and μ-integrable, such that $\mu_i \overset{*}{\rightharpoonup} \mu$ weakly* in $C^*([0,T])$ and $m_i(t)\mu_i(dt) \overset{*}{\rightharpoonup} m(t)\mu(dt)$. Moreover, reasoning as in Section 3.3, one can deuce that there is some $p \in W^{1,1}([0,T];\mathbb{R}^n)$ such that, possibly for a further subsequence, $p_i \to p$ in L^∞, and $\frac{dp_i}{dt} \rightharpoonup \frac{dp}{dt}$ weakly in L^1. Similarly, $x_i \to \bar{x}$ in L^∞, $\frac{dx_i}{dt} \rightharpoonup \frac{d\bar{x}}{dt}$ weakly in L^1, $q_i \to q$ in L^1 for q as in (4.1.10) and $q_i(T) \to q(T)$. At this point, passing to the limit in the previous relations as in the proof of Theorem 3.2.8, one finally obtains that (p, μ, m) fulfills conditions (4.2.1), (4.1.8), (4.1.13), (4.1.9) together with

$$\|p\|_{L^\infty} + \mu([0,T]) = 1, \qquad -q(T) \in N_{\mathcal{C}}(\bar{x}(T)).$$

Surveying these conditions we see that the proof of part (b) is complete. □

Remark 4.2.2. We point out that condition (4.2.1) is weaker than the expected costate differential inclusion (4.1.12). This *averaged* version of the adjoint equation for Type S relations does not occur in the case the dynamics constraint takes the form of a differential inclusion and, consequently, the set of necessary condition involved is Clarke's Hamiltonian inclusion (see [61]). Moreover, in [62] it has been shown that the two adjoint relations are the same when the dynamics are affine with respect to the control variable. Nevertheless, it remains an open question whether in general a sharper Type S relation is valid, involving the original costate differential inclusion (4.1.12) instead of (4.2.1).

From Theorem 4.2.1, (ii), one immediately derives the following result as corollary, that can be seen as a stability result for minimizers of problem $(P_{\check{\Sigma}})$ when small perturbations of the dynamics occur.

Theorem 4.2.3. *Assume that Hypotheses 4.1.2–4.1.4 are verified, let* $\mathfrak{Z} = (\bar{\omega}, \bar{u}, \bar{x}) \in \check{\Sigma}$ *and let* $\Phi : \mathbb{R}^n \to \mathbb{R}$ *be a Lipschitz continuous function in a neighborhood of* $\bar{x}(T)$.

Suppose that $\mathfrak{z}$ *is a local* Φ*-minimizer for problem* $(P_{\check{\Sigma}})$*. If, given any set of multipliers* (p, λ, μ, m) *satisfying conditions* (4.1.4), (4.1.6), (4.1.8), (4.1.9), (4.1.10), (4.1.13), *and* (4.2.1) *we have* $\lambda \neq 0$*, then* $\mathfrak{z}$ *is also a* Φ*-local minimizer for problem* $(P_{\check{\Sigma}_r})$.

Chapter 5

Control-polynomial impulsive optimization problems

This chapter is devoted to the analysis of nonlinear optimization problems with control-polynomial dynamics and unbounded control set. We refer to this kind of problems as *impulsive*, as the lack of growth conditions allows minimizing sequences to have larger and larger velocities, so that to converge to discontinuous paths. However, by means of a standard 'change of independent variable' technique, it is possible to reduce an impulsive problem to an *extended* conventional one, where the right end-time is free, the dynamics function is autonomous and the control set is bounded.

At this point, one might hastily apply the results of Chapter 3 to the extended problem, so that to establish sufficient conditions for no gap between the original impulsive problem and its extension. However, it is well known that, in the case the dynamics function does not depend on time, minimizers satisfy an additional 'constancy of the Hamiltonian' condition that can not be deduced by means of the approach utilized to prove Theorem 2.5.7, as explained very well in [69, Ch. 8].

Instead, this crucial additional relation can be obtained by the analysis of general free end-time optimization problems with Lipschitz continuous time dependence, that have a completely different nature with respect to those with measurable time dependence (see also [46, 67, 74]). In fact, adopting a classic reparameterization procedure employed to derive necessary conditions of optimality for this kind of problems, we convert the free end-time problem under consideration into a fixed end-time one, so that to deduce enhanced gap-abnormality relations, thanks to the results of Chapter

4. In particular, we treat the time variable as a further state variable to which we associate an additional costate arc, that turns out to be almost everywhere equal to the Hamiltonian function, and equal to a constant in case of autonomous dynamics.

In Section 5.1 we study the gap issue for free end-time optimal control problems with Lipschitz continuous time dependence. Then, in Section 5.2, we introduce the impulsive (relaxed) extension of a control-polynomial system with unbounded controls and we prove sufficient conditions for no gap for this type of problems. Afterwards, in Section 5.3, we show that, for impulsive optimization problems, it is possible to replace Hypothesis 3.2.6 with simpler nondegeneracy assumptions. Finally, in Section 5.4, we establish easily verifiable conditions for the normality of extremals of linear impulsive problems.

The present chapter is based on [37, 38].

5.1 Free end-time problems with Lipschitz continuous time dependence

First of all, we have to adjust the definitions of (feasible) strict sense, extended and relaxed processes.

Definition 5.1.1. Let $U \subset \mathbb{R}^q$, $V \subset \mathbb{R}^m$, and let $W = \overline{V}$. We refer to (T, ω, u, x) as *extended process* if $T > 0$, $\omega \in \mathcal{W}(T) := \mathcal{M}([0,T], W)$, $u \in \mathcal{U}(T) := \mathcal{M}([0,T], U)$, and $x \in W^{1,1}([0,T], \mathbb{R}^n)$ satisfies

$$\dot{x}(t) = \mathcal{F}(t, x(t), \omega(t), u(t)) \qquad \text{a.e. } t \in [0,T]. \tag{5.1.1}$$

An extended process is called a *strict sense process* if $\omega \in \mathcal{V}(T) := \mathcal{M}([0,T], V)$. A strict sense or extended process is *feasible* if it satisfies the following endpoint and state constraints

$$\psi(t, x(t)) \leq 0 \quad \forall t \in [0,T], \qquad (x(0), T, x(T)) \in \{\check{z}_0\} \times \mathcal{C}^*. \tag{5.1.2}$$

With a small abuse of notation, we refer to $(T, \underline{\omega}, \underline{u}, \gamma, \zeta, x)$ as *relaxed process* if $T > 0$, $\omega \in \mathcal{W}^{1+n}(T)$, $u \in \mathcal{U}^{1+n}(T)$, $\gamma \in \Gamma(T) = \mathcal{M}([0,T], \Delta_n)$ and $(\zeta, x) \in$

$W^{1,1}([0,T],\mathbb{R}^{1+n}\times\mathbb{R}^n)$ satisfies

$$(\dot{\zeta}(t),\dot{x}(t)) = \left(\gamma(t), \sum_{j=0}^{n}\gamma^j(t)\mathcal{F}(t,x(t),\omega^j(t),u^j(t))\right) \qquad \text{a.e. } t\in[0,T]. \qquad (5.1.3)$$

A relaxed process is *feasible* when it satisfies (5.1.2). We write Σ^*, Σ^*_e, Σ^*_r, to denote the sets of strict sense, extended, and relaxed processes which are feasible, respectively.

Identify a continuous function $x:[0,\tau]\to\mathbb{R}^k$ with its extension to $\tilde{x}:\mathbb{R}\to\mathbb{R}^k$ by constant extrapolation of the left and right endpoint values. Then, for all τ_1, $\tau_2>0$, and $(\tilde{x}_1,\tilde{x}_2)\in C^0([0,\tau_1],\mathbb{R}^k)\times C^0([0,\tau_2],\mathbb{R}^k)$, we notice that the distance d_∞ defined in (3.0.4) can be write as

$$d_\infty\big((\tau_1,\tilde{x}_1),(\tau_2,\tilde{x}_2)\big) := |\tau_2-\tau_1| + \|\tilde{x}_2-\tilde{x}_1\|_{L^\infty(\mathbb{R})}. \qquad (5.1.4)$$

Therefore, we can modify the concepts of *local infimum gap* and *isolated process* to free end-time problems with Lipschitz continuous time dependence.

Definition 5.1.2. Let $\mathfrak{z}:=(\bar{T},\underline{\bar{\omega}},\underline{\bar{u}},\bar{\gamma},\bar{x})$ be a feasible relaxed process, *at* $\mathfrak{z}$ *there is a local infimum* gap if for any continuous function $\Phi:\mathbb{R}^{1+n}\to\mathbb{R}$ there is some $\delta>0$ such that

$$\Phi(\bar{T},\bar{x}(T)) < \inf\left\{\Phi(T,x(T)):\quad (T,\omega,u,x)\in\Sigma^*,\quad d_\infty\big((T,x),(\bar{T},\bar{x})\big)<\delta\right\},$$

while $\mathfrak{z}$ is an *isolated process* if, for some $\delta>0$, one has

$$\left\{(T,\omega,u,x)\in\Sigma^*:\quad d_\infty\big((T,x),(\bar{T},\bar{x})\big)<\delta\right\}=\emptyset.$$

We say that *the constrained control system* (5.1.1)-(5.1.2) *is controllable to* $\mathfrak{z}$ when $\mathfrak{z}$ is not isolated.

As in Chapter 3, *at* $\mathfrak{z}$ *there is a local infimum gap if and only if* $\mathfrak{z}$ *is isolated.*

Throughout this section, we strengthen Hypotheses 4.1.3-4.1.4 treating time as a state variable. We shall consider the following hypotheses, in which $(\bar{T},\underline{\bar{\omega}},\underline{\bar{u}},\bar{\gamma},\bar{\zeta},\bar{x})$ is a given feasible relaxed process and, for some $\eta>0$, we set

$$\mathcal{S}^*_\eta := \left\{(t,z)\in\mathbb{R}\times\mathbb{R}^n:\quad (t,z)\in(t,\bar{x}(t))+\eta\,\mathbb{B}_{1+n},\quad t\in[0,\bar{T}]\right\}.$$

Hypothesis 5.1.3. The target $\mathcal{C}^* \subset \mathbb{R}^{1+n}$ is closed and the constraint function ψ is L_ψ-Lipschitz continuous on $\mathcal{S}^*_\eta$.

Hypothesis 5.1.4. (i) For any $(t,z,w) \in \mathcal{S}^*_\eta \times W$ the function $U \ni v \mapsto \mathcal{F}(t,z,v,w)$ is $\mathscr{B}^q$-measurable. Furthermore, there is some $c \in L^1([0,\bar{T}+\eta],\mathbb{R}_{\geq 0})$ such that, for all (t',z'), $(t'',z'') \in \{(t,\bar{x}(t))\} + \eta\mathbb{B}_{1+n}$ and any $(w,v) \in W \times U$ one has:

$$|\mathcal{F}(t',z',w,v) - \mathcal{F}(t'',z'',w,v)| \leq c(t)|(t',z') - (t'',z'')|.$$

(ii) There exists some continuous increasing function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\rho(0) = 0$ such that for any $(t,z,v) \in \mathcal{S}^*_\eta \times U$, we have

$$\begin{aligned} &|\mathcal{F}(t,x,w',v) - \mathcal{F}(t,x,w,v)| \leq \rho(|w'-w|) \qquad \forall w',\, w \in W, \\ &D_{t,x}\mathcal{F}(t,z,w',v) \subseteq D_{t,x}\mathcal{F}(t,z,w,v) + \rho(|w'-w|)\,\mathbb{B} \qquad \forall w',\, w \in W. \end{aligned}$$

Definition 5.1.5. Let $\mathfrak{Z} := (\bar{T},\underline{\bar{\omega}},\underline{\bar{u}},\bar{\gamma},\bar{x})$ be a feasible relaxed process and assume that Hypotheses 5.1.3- 5.1.4 are verified. Given a function $\Phi : \mathbb{R}^{1+n} \to \mathbb{R}$ which is Lipschitz continuous on a neighborhood of $(\bar{T},\bar{x}(\bar{T}))$, we say that $\mathfrak{Z}$ is a Φ-*extremal* if there exist a pair of paths $(p_*,p) \in W^{1,1}([0,\bar{T}],\mathbb{R}^{1+n})$, $\lambda \geq 0$, $\mu \in C^{\oplus}([0,\bar{T}])$, $(m_*,m) : [0,\bar{T}] \to \mathbb{R}^{1+n}$ Borel measurable and μ-integrable functions, verifying the following conditions:

$$\|p\|_{L^\infty} + \mu([0,\bar{T}]) + \lambda \neq 0;$$

$$\big(\dot{p}_*,-\dot{p}\big)(t) \in \sum_{j=0}^{n} \bar{\gamma}^j(t)\,\mathrm{co}\,\partial_{t,x}\left(q(t)\cdot\mathcal{F}(t,\bar{x}(t),\bar{\omega}^j(t),\bar{u}^j(t))\right) \quad \text{a.e. } t \in [0,\bar{T}]; \tag{5.1.5}$$

$$\big(q_*(\bar{T}),-q(\bar{T})\big) \in \lambda\partial\Phi\left(T,\bar{x}(T)\right) + N_{\mathcal{C}}(\bar{T},\bar{x}(\bar{T}));$$

for every $j = 0,\dots,n$, for a.e. $t \in [0,\bar{T}]$, one has

$$q(t)\cdot\mathcal{F}\Big(t,\bar{x}(t),\bar{\omega}^j(t),\bar{u}^j(t)\Big) = \max_{(w,v)\in W\times U} q(t)\cdot\mathcal{F}\Big(t,\bar{x}(t),w,v\Big); \tag{5.1.6}$$

$$\sum_{j=0}^{n} \bar{\gamma}^j(t)q(t)\cdot\mathcal{F}(t,\bar{x}(t),\bar{\omega}^j(t),\bar{u}^j(t)) = q_*(t) \qquad \text{a.e. } t \in [0,\bar{T}]; \tag{5.1.7}$$

$$(m_*,m)(t) \in \partial^{>}_{t,x}\,\psi\left(t,\bar{x}(t)\right) \qquad \mu\text{-a.e. } t \in [0,\bar{T}]; \tag{5.1.8}$$

$$\mathrm{spt}(\mu) \subseteq \{t \in [0, \bar{T}] : \psi(t, \bar{x}(t)) = 0\}, \tag{5.1.9}$$

where $(q_*, q) : [0, \bar{T}] \to \mathbb{R}^{1+n}$ is defined by

$$(q_*, q)(t) := \begin{cases} (p_*, p)(t) + \int_{[0,t[} (m_*, m)(t')\mu(dt') & t \in [0, \bar{T}[, \\ (p_*, p)(\bar{T}) + \int_{[0,\bar{T}]} (m_*, m)(t')\mu(dt') & t = \bar{T}. \end{cases} \tag{5.1.10}$$

A Φ-extremal is *normal* if all possible choices of $(p_*, p, \lambda, \mu, m_*, m)$ as above have $\lambda > 0$, and *abnormal* when it is not normal. Given a Φ-extremal $\mathfrak{Z}$, we call *nondegenerate multiplier* any set of multipliers $(p_*, p, \lambda, \mu, m_*, m)$ and (q_*, q) as above, that also verify

$$\mu(]0, T]) + \|q\|_{L^\infty} + \lambda \neq 0.$$

A Φ-extremal is *nondegenerate normal* if all choices of nondegenerate multipliers have $\lambda > 0$, and it is *nondegenerate abnormal* when there exists a nondegenerate multiplier with $\lambda = 0$. In the following, abnormal [nondegenerate abnormal] Φ-extremals will be simply called *abnormal [nondegenerate abnormal] extremals.*

5.1.1 An useful time rescaling

Theorem 4.1.7 extends to free end-time optimization problems as follows.

Theorem 5.1.6. *Let* $\mathfrak{Z} := (\bar{T}, \underline{\bar{\omega}}, \underline{\bar{u}}, \bar{\gamma}, \bar{\zeta}, \bar{x}) \in \Sigma_r^*$ *and suppose that at* $\mathfrak{Z}$ *there is a local infimum gap.*

(i) *If Hypotheses 4.1.2, 5.1.3, 5.1.4 hold, then* $\mathfrak{Z}$ *is an abnormal extremal.*

(ii) *If, in addition, also Hypotheses 3.2.6 and 4.1.5 are verified, then* $\mathfrak{Z}$ *is a nondegenerate abnormal extremal.*

Proof. Let $\mathfrak{Z} := (\bar{T}, \underline{\bar{\omega}}, \underline{\bar{u}}, \bar{\gamma}, \bar{\zeta}, \bar{x})$ be a feasible relaxed process at which there is a local infimum gap. By the above considerations, this is equivalent to suppose that $\mathfrak{Z}$ is an isolated process. Adapting a standard time-rescaling procedure (see for instance [69, Thm. 8.7.1]), we embed the extended control system (5.1.1)-(5.1.2) and the relaxed control system (5.1.3)-(5.1.2) into higher dimensional fixed end-time *rescaled* control systems, and we show that $\mathfrak{Z}$ is a process for the (fixed end-time) relaxed rescaled control system, which is also isolated with respect to feasible strict sense rescaled processes. At this point, the thesis follows by applying Theorem 4.1.7.

From the fact that $\mathfrak{Z}$ is isolated, it follows that there exists some $\delta > 0$ (we can take $\delta \leq \eta$) such that

$$\big\{(T, \omega, u, x) \in \Sigma^* : \quad d_\infty\big((T, x), (\bar{T}, \bar{x})\big) < 3\delta\big\} = \emptyset. \tag{5.1.11}$$

Hence, let $\hat{\delta} > 0$ be such that $\int_I 2\, c(t)dt \leq \delta$ for any interval I such that $\ell(I) \leq 3\bar{T}\hat{\delta}$ and set

$$\bar{\delta} := \min\left\{\hat{\delta}, \frac{1}{2}\right\}. \tag{5.1.12}$$

We say that $(\omega, u, \alpha, x^*, x)$ is an extended *rescaled* process if $\omega \in \mathcal{W}(\bar{T})$, $u \in \mathcal{U}(\bar{T})$, $\alpha \in \mathcal{M}([0, \bar{T}], [-\bar{\delta}, \bar{\delta}])$, $(x^*, x) \in W^{1,1}([0, \bar{T}], \mathbb{R} \times \mathbb{R}^n)$ and it satisfies

$$(\dot{x}^*, \dot{x})(t) = (1 + \alpha(t))\big(1, \mathcal{F}(x^*(t), x(t), \omega(t), u(t))\big) \qquad \text{a.e. } t \in [0, \bar{T}], \tag{5.1.13}$$

and we call it feasible if it satisfies the constraints

$$\psi(x^*(t), x(t)) \leq 0 \quad \forall t \in [0, \bar{T}], \qquad (x^*(0), x(0), x^*(\bar{T}), x(\bar{T})) \in \{(0, \check{z}_0)\} \times \mathcal{C}^*. \tag{5.1.14}$$

We say that $(\omega, u, \alpha, x^*, x)$ is strict sense if $\omega \in \mathcal{V}(\bar{T})$. Moreover, we say that $(\underline{\omega}, \underline{u}, \underline{\alpha}, \gamma, \zeta, x^*, x)$ is a relaxed *rescaled* process if $\underline{\omega} \in \mathcal{W}^{2+n}(\bar{T})$, $\underline{u} \in \mathcal{U}^{2+n}(\bar{T})$, $\underline{\alpha} \in \mathcal{M}^{2+n}([0, \bar{T}], [-\bar{\delta}, \bar{\delta}])$, $\gamma \in \Gamma_{n+1}(\bar{T})$ and $(\zeta, x^*, x) \in W^{1,1}([0, \bar{T}], \mathbb{R}^{2+n} \times \mathbb{R} \times \mathbb{R}^n)$ and it satisfies

$$\begin{cases} \dot{\zeta}(t) = \gamma(t) \qquad \text{a.e. } t \in [0, \bar{T}], \\ \dot{x}^*(t) = \sum_{j=0}^{n+1} \gamma^j(t)(1 + \alpha^j(t)) \qquad \text{a.e. } t \in [0, \bar{T}], \\ \dot{x}(t) = \sum_{j=0}^{n+1} \gamma^j(t)(1 + \alpha^j(t))\mathcal{F}(x^*(t), x(t), \omega^j(t), u^j(t)) \qquad \text{a.e. } t \in [0, \bar{T}], \end{cases} \tag{5.1.15}$$

and we say that it is feasible if it satisfies the constraints in (5.1.14). Finally, we denote by $\hat{\Sigma}^*$, $\hat{\Sigma}^*_e$ and $\hat{\Sigma}^*_r$ the set of strict sense, extended and relaxed rescaled processes which are feasible, respectively. We point out that we can identify $\mathfrak{Z} = (\bar{T}, \underline{\bar{\omega}}, \underline{\bar{u}}, \bar{\gamma}, \bar{\zeta}, \bar{x}) \in \Sigma^*_r$

with $\check{\mathfrak{Z}} := (\underline{\check{\omega}}, \underline{\check{u}}, \underline{\check{\alpha}}, \check{\gamma}, \check{\zeta}, \check{x}^*, \check{x}) \in \hat{\Sigma}^*_r$ by setting

$$\begin{aligned} &\underline{\check{\omega}} := (w, \underline{\bar{\omega}}), \quad \underline{\check{u}} := (v, \underline{\bar{u}}), \quad \underline{\check{\alpha}} := 0, \\ &\check{\gamma} := (0, \bar{\gamma}), \quad \check{\zeta} = (0, \bar{\zeta}), \quad \check{x}^* := id, \quad \check{x} := \bar{x}, \end{aligned} \tag{5.1.16}$$

for arbitrary $w \in W$ and $v \in U$. Since $\mathfrak{Z}$ is isolated, then $\check{\mathfrak{Z}}$ is isolated in $\hat{\Sigma}^*$. In particular, we claim that

$$\left\{(\omega, u, \alpha, x^*, x) \in \hat{\Sigma}^* : \quad \|(x^*, x) - (\check{x}^*, \check{x})\|_{L^\infty([0,\bar{T}])} < \delta\right\} = \emptyset. \tag{5.1.17}$$

Indeed, let $(\omega, u, \alpha, x^*, x) \in \hat{\Sigma}^*$ verify

$$\|(x^*, x) - (\check{x}^*, \check{x})\|_{L^\infty([0,\bar{T}])} < \delta \tag{5.1.18}$$

and consider the time-transformation $x^* : [0, \bar{T}] \to [0, T]$, where $T := x^*(\bar{T})$. Observe that x^* is a strictly increasing, Lipschitz continuous function, with Lipschitz continuous inverse, $(x^*)^{-1}$. It can be deduced that $(T, \hat{\omega}, \hat{u}, \hat{x}) \in \Sigma^*$, where

$$(\hat{\omega}, \hat{u}, \hat{x}) := (\omega, u, x) \circ (x^*)^{-1} \quad \text{in } [0, T], \tag{5.1.19}$$

Now we show that[20]

$$\theta := \sup_{t \in [0, T \vee \bar{T}]} |\check{x}((x^*)^{-1}(t \wedge T)) - \check{x}(t \wedge \bar{T})| \leq \delta. \tag{5.1.20}$$

Let us first suppose $\bar{T} \leq T$, so that

$$\theta = \max\left\{ \sup_{t \in [0,\bar{T}]} |\check{x}((x^*)^{-1}(t)) - \check{x}(t)|, \quad \sup_{t \in [\bar{T}, T]} |\check{x}((x^*)^{-1}(t)) - \check{x}((x^*)^{-1}(T))| \right\}.$$

Now, using (5.1.12), we get

$$|(x^*)^{-1}(t) - t| = \Big| \int_0^t \Big[\frac{1}{1 + \alpha((x^*)^{-1}(t'))} - 1\Big] dt' \Big| \leq \int_0^t \Big| \frac{\alpha((x^*)^{-1}(t'))}{1 + \alpha((x^*)^{-1}(t'))} \Big|$$

[20] By definition, $\hat{x}$ and $\check{x}$ are replaced with their constant, continuous extensions to $\mathbb{R}$.

$$\leq 2\hat{\delta}T = 2\hat{\delta}\int_0^{\bar{T}}(1+\alpha(t'))dt' \leq 3\hat{\delta}\bar{T}.$$

In view of the very definition of $\hat{\delta}$, this implies

$$\sup_{t\in[0,\bar{T}]}|\check{x}((x^*)^{-1}(t))-\check{x}(t)| \leq \sup_{t\in[0,\bar{T}]}\Big|\int_t^{(x^*)^{-1}(t)} 2\,c(t')dt'\Big| \leq \delta.$$

Similarly, for $t\in[\bar{T},T]$ we have

$$|(x^*)^{-1}(t)-(x^*)^{-1}(T)| \leq \int_t^T \Big|\frac{1}{1+\alpha((x^*)^{-1}(t'))}\Big|dt' \leq 2(T-\bar{T}) \leq 2\bar{T}\hat{\delta}$$

and, consequently

$$\sup_{t\in[\bar{T},T]}|\check{x}((x^*)^{-1}(t))-\check{x}((x^*)^{-1}(T))| \leq \sup_{t\in[\bar{T},T]}\int_{(x^*)^{-1}(t)}^{(x^*)^{-1}(T)} 2\,c(t')dt' \leq \delta.$$

Reasoning in the same way for the case $T\leq\bar{T}$, we deduce the validity of (5.1.20). Therefore, taking account of (5.1.16) and (5.1.19), in view of (5.1.18) and (5.1.20), by adding and subtracting '$\check{x}((x^*)^{-1}(t\wedge T))$' we can compute

$$\begin{aligned} d_\infty\big((T,\hat{x}),(\bar{T},\bar{x})\big) &= |T-\bar{T}|+\|\hat{x}-\bar{x}\|_\infty \leq |x^*(\bar{T})-\check{x}^*(\bar{T})| \\ &+\sup_{t\in[0,T\vee\bar{T}]}\Big[|x((x^*)^{-1}(t\wedge T))-\check{x}((x^*)^{-1}(t\wedge T))|+|\check{x}((x^*)^{-1}(t\wedge T))-\check{x}(t\wedge\bar{T})|\Big] \\ &\leq \|x^*-\check{x}^*\|_{L^\infty([0,\bar{T}])}+\|x-\check{x}\|_{L^\infty([0,\bar{T}])}+\delta<3\delta, \end{aligned}$$

Therefore, (5.1.11) yields (5.1.17), and the feasible rescaled relaxed process $\check{\mathfrak{Z}}$ is isolated in $\hat{\Sigma}^*$, as claimed. In order to apply the results of Theorem 4.1.7 with reference to the constrained control system (5.1.15)-(5.1.14) and to the process $\check{\mathfrak{Z}}$, it remains to show that, if we consider (w,v,γ,α) as control variables and $\tilde{z}:=(t,z)$ as the state variable for the (now, time-independent) control system (5.1.13)-(5.1.14), all the hypotheses assumed in its statement are fulfilled. To this aim, observe that Hypothesis 4.1.3 trivially follows from Hypothesis 5.1.3, while Hypothesis 5.1.4 easily implies Hypothesis 4.1.4. Finally, recalling that $\underline{\check{\alpha}}\equiv 0$ and $\check{x}^*(t)=t$ for all $t\in[0,\bar{T}]$, Hypothesis 3.2.6 can be reformulated as an hypothesis on the rescaled process $\check{\mathfrak{Z}}$. At this point, we can apply Theorem 4.1.7 to the rescaled relaxed process $\check{\mathfrak{Z}}$ and, taking account of (5.1.16), we deduce the existence of $(p_*, p) \in W^{1,1}$, $\mu \in C^\oplus$ and Borel-measurable and μ-integrable functions (m_*, m) satisfying conditions (5.1.5), (5.1.8), (5.1.9) and

$$\|p_*\|_{L^\infty} + \|p\|_{L^\infty} + \mu([0, \bar{T}]) \neq 0; \tag{5.1.21}$$

$$\big(q_*(\bar{T}), -q(\bar{T})\big) \in N_{\mathcal{C}}(\bar{T}, \bar{x}(\bar{T}));$$

for any $j = 0, \dots, n$, for a.e. $t \in [0, \bar{T}]$ one has for all

$$\begin{aligned} q(t) \cdot \mathcal{F}\big(t, \bar{x}(t), \bar{\omega}^j(t), \bar{u}^j(t)\big) - q_*(t) \\ = \max_{(w,v,b) \in W \times U \times [-\bar{\delta}, \bar{\delta}]} \Big((1+b)\big(q(t) \cdot \mathcal{F}(t, \bar{x}(t), w, v) - q_*(t)\big)\Big), \end{aligned} \tag{5.1.22}$$

where (q_*, q) is as in (5.1.10). Moreover, when Hypothesis 3.2.6 is fulfilled, then it also holds

$$\|q_*\|_{L^\infty} + \|q\|_{L^\infty} + \mu(]0, \bar{T}]) \neq 0. \tag{5.1.23}$$

Of course one has

$$\begin{aligned} &\max_{(w,v,b) \in W \times U \times [-\bar{\delta}, \bar{\delta}]} \Big((1+b)\big(q(t) \cdot \mathcal{F}(t, \bar{x}(t), w, v) - q_*(t)\big)\Big) \\ &\qquad \geq \max_{(w,v) \in W \times U} \big(q(t) \cdot \mathcal{F}(t, \bar{x}(t), w, v) - q_*(t)\big), \end{aligned}$$

thus (5.1.22) immediately implies (5.1.6). Moreover, since the maximum of the right hand side of (5.1.22) is obtained at $(\bar{w}, \bar{v}, \bar{b}) = (\bar{\omega}^j(t), \bar{u}^j(t), 0)$ and $0 \in \mathrm{Int}([-\bar{\delta}, \bar{\delta}])$, then the partial derivative with respect to b evaluated at $(\bar{w}, \bar{v}, \bar{b})$ is equal to 0, so that to obtain

$$q_*(t) = q(t) \cdot \mathcal{F}\big(t, \bar{x}(t), \bar{\omega}^j(t), \bar{u}^j(t)\big)$$

for any $j = 0, \dots, n$ and for a.e. $t \in [0, \bar{T}]$. Since $\sum_{j=0}^n \bar{\gamma}^j(t) = 1$ a.e. $t \in [0, \bar{T}]$, we get (5.1.7). In order to conclude, it remains to prove

$$\|p\|_{L^\infty} + \mu([0, \bar{T}]) \neq 0.$$

If we assumed by contradiction $\|p\|_{L^\infty} + \mu([0, \bar{T}]) = 0$, then (5.1.7) would imply $\|p_*\|_{L^\infty} = \|q_*\|_{L^\infty} = 0$, so that to contradict (5.1.21). In the case Hypothesis 3.2.6 is fulfilled, similar arguments and (5.1.23) yield $\|q\|_{L^\infty} + \mu(]0, \bar{T}]) \neq 0$. □

Again, from Theorem 5.1.6 we can get normality tests for gap avoidance and sufficient controllability conditions for the free end-time problem with Lipschitz continuous time dependence completely analogous to Theorem 4.1.8 and Theorem 4.1.9, respectively.

5.2 Impulsive extension of control-polynomial systems

In this section we focus on nonlinear control-polynomial systems where the controls belong to an unbounded cone. Among applications for which the polynomial dependence is relevant let us mention Lagrangian mechanical systems, possibly with friction forces, in which inputs are identified with the derivatives of some Lagrangian coordinates (see [17, 25]).

We first consider an original problem and we embed it into an extended one where we allow discontinuous trajectories with jumps. Afterwards, since the set of the velocities of the extended problem is in general not convex, so that the existence of minimizers is not guaranteed, we also introduce the relaxed impulsive extension of the original problem.

For some integer $\kappa \geq 1$, we consider the free end-time optimal control problem:

$$
(\mathcal{P}) \begin{cases} \qquad \text{Minimize } \Phi(S, y(S), \mathfrak{v}(S)) \\ \text{over } S > 0, \mathfrak{w} \in L^\kappa([0,S], \mathfrak{W}), \mathfrak{u} \in \mathcal{M}([0,S], U), (y, \mathfrak{v}) \in W^{1,1}([0,S], \mathbb{R}^{n+1}) \text{ s.t.} \\ (\dot{y}, \dot{\mathfrak{v}})(s) = \left(f(s,y,\mathfrak{u}) + \displaystyle\sum_{l=1}^{\kappa} \left(\sum_{1 \leq j_1 \leq \cdots \leq j_l \leq m} g^l_{j_1,\dots,j_l}(t,y)\ \mathfrak{w}^{j_1} \cdots \mathfrak{w}^{j_l} \right), |\mathfrak{w}|^\kappa \right) \text{ a.e. } s, \qquad (5.2.1) \\ (y(0), \mathfrak{v}(0), S, y(S), \mathfrak{v}(S)) \in \{(\check{z}_0, 0)\} \times \mathcal{C}^* \times]-\infty, K], \qquad (5.2.2) \\ \psi(s, y(s)) \leq 0 \quad \text{ for all } s \in [0,S]. \qquad (5.2.3) \end{cases}
$$

Here, $\mathfrak{W} \subseteq \mathbb{R}^m$ is a closed cone, $U \subseteq \mathbb{R}^q$ is a compact subset, $K > 0$ is a fixed constant, possibly equal to $+\infty$, and the target set $\mathcal{C}^* \subseteq \mathbb{R}^{1+n}$ is closed. Notice that $\mathfrak{v}(s)$ is simply the L^κ-norm to the power κ of the control function $\mathfrak{w}$ on $[0, s[$. The variable $\mathfrak{v}$ is sometimes called *fuel* or *energy* and $\mathrm{v} \mapsto \Phi(s, z, \mathrm{v})$ is usually assumed monotone nondecreasing for every $(s,z) \in \mathbb{R}^{1+n}$ (see e.g. [53, 54]). The integer $\kappa \geq 1$ will be called the *degree* of the control system. Problem $(\mathcal{P})$ is referred to as the *original problem* and we call $(S, \mathfrak{w}, \mathfrak{u}, y, \mathfrak{v})$ an *original process* if $S > 0$, $\mathfrak{w} \in L^\kappa([0,S], \mathfrak{W})$, $\mathfrak{u} \in \mathcal{M}([0,S], U)$, and $(y, \mathfrak{v}) \in W^{1,1}([0,S], \mathbb{R}^n \times \mathbb{R})$ fulfills (5.2.1), and we say that it is *feasible* if it satisfies the constraints (5.2.2) and (5.2.3).

We do not assume customary coercivity hypotheses on the cost, so that minimizing sequences of original trajectories may have larger and larger velocities and converge to discontinuous paths. Accordingly, minimizers for problem $(\mathcal{P})$ do not exist in general. As a consequence, following the so called *graph completion* approach, we reformulate problem $(\mathcal{P})$ into a free end-time *extended* problem (P_e) with bounded controls, where the graphs of absolutely continuous maps are embedded in a larger set of space-time trajectories. The new state variables of problem (P_e) are time, original state, and energy, and extended trajectories are (reparameterized) L^∞-limits of graphs of original trajectories. This compactification procedure, usually adopted to obtain an impulsive extension of control-affine systems with unbounded controls (see [22, 48, 51, 65, 75]), has been generalized to control-polynomial systems (see [64, 54]).

Let us choose

$$\begin{aligned} W &:= \left\{(w^0, w) \in \mathbb{R}_{\geq 0} \times \mathfrak{W} : \quad (w^0)^\kappa + |w|^\kappa = 1\right\}, \\ V &:= \left\{(w^0, w) \in W : \quad w^0 > 0\right\}. \end{aligned} \tag{5.2.4}$$

For every $T > 0$, we set $\mathcal{W}(T) := L^1([0,T], W)$,[21] $\mathcal{V}(T) := L^1([0,T], V)$, and $\mathcal{U}(T) := L^1([0,T], U)$, and introduce the *space-time* or *extended* problem: [22]

$$(P_e) \begin{cases} \qquad \text{Minimize } \Phi(x^0(T), x(T), \nu(T)) \\ \text{over } T > 0,\ (\omega^0, \omega) \in \mathcal{W}(T),\ u \in \mathcal{U}(T),\ (x^0, x, \nu) \in W^{1,1}([0,T], \mathbb{R}^{1+n+1}) \text{ s.t.} \\ (\dot{x}^0, \dot{x}, \dot{\nu})(t) = \Big((\omega^0)^\kappa(t), F(x^0(t), x(t), \omega^0(t), \omega(t), u(t)), |\omega(t)|^\kappa\Big) \text{ a.e. } t \in [0,T], \quad (5.2.5) \\ (x^0(0), x(0), \nu(0), x^0(T), x(T), \nu(T)) = \{(0, \check{z}_0, 0)\} \times \mathcal{C}^* \times]-\infty, K], \quad (5.2.6) \\ \psi(x^0(t), x(t)) \leq 0 \ \text{ for all } t \in [0,T] \quad (5.2.7) \end{cases}$$

[21] The controls $(\omega^0, \omega) \in \mathcal{W}(T)$ actually belong to $L^\infty \cap L^1$, since W is compact.

[22] The original time s coincides with the state arc x^0, while t is the new 'pseudo-time' variable.

where, for any $(s, z, w^0, w, v) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}_{\geq 0} \times \mathfrak{W} \times U$, we have set

$$F(s, z, w^0, w, v) := f(s, z, v)(w^0)^\kappa + \sum_{l=1}^{\kappa} \Big(\sum_{1 \leq j_1 \leq \cdots \leq j_l \leq m} g^l_{j_1,\dots,j_l}(s, z)\; w^{j_1} \cdot \, \dots \, \cdot w^{j_l}\, (w^0)^{\kappa - l} \Big). \tag{5.2.8}$$

We say that $(T, \omega^0, \omega, u, x^0, x, \nu)$ is an *extended process* if $T > 0$, $(\omega^0, \omega) \in \mathcal{W}(T)$, $u \in \mathcal{U}(T)$, $(x^0, x, \nu) \in W^{1,1}([0, T], \mathbb{R} \times \mathbb{R}^n \times \mathbb{R})$ and it fulfills (5.2.5), and we say that it is *feasible* if it satisfies the constraints (5.2.6) and (5.2.7). When $\omega^0 > 0$ almost everywhere, namely $(\omega^0, \omega) \in \mathcal{V}(T)$, $(T, \omega^0, \omega, u, x^0, x, \nu)$ is called a *strict sense process*. The problem of minimizing $\Phi(x^0(T), x(T), \nu(T))$ over feasible strict sense processes is denoted by (P).

The associated *relaxed problem* is

$$(P_r)\begin{cases} \qquad\text{Minimize } \Phi(x^0(T), x(T), \nu(T)) \\ \text{over } T > 0,\ (\underline{\omega}^0, \underline{\omega}) \in \mathcal{W}^{3+n}(T),\ \underline{u} \in \mathcal{U}^{3+n}(T),\ \gamma \in \Gamma_{2+n}(T), \\ \qquad (x^0, x, \nu) \in W^{1,1}([0, T], \mathbb{R}^{1+n+1}) \text{ s.t.} \\ (\dot{x}^0, \dot{x}, \dot{\nu})(t) = \displaystyle\sum_{j=0}^{2+n} \gamma^j(t) \Big((\omega^{0,j}(t))^\kappa, F((x^0, x, \omega^{0,j}, \omega^j, u^j)(t)), \big|\omega^j(t)\big|^\kappa \Big) \ \text{ a.e. } t, \\ (x^0(0), x(0), \nu(0), x^0(T), x(T), \nu(T)) = \{(0, \check{z}_0, 0)\} \times \mathcal{C}^* \times]-\infty, K], \\ \psi(t, x(t)) \leq 0 \quad \forall t \in [0, T], \end{cases} \tag{5.2.9}$$

We say that $(T, \underline{\omega}^0, \underline{\omega}, \underline{u}, \gamma, x^0, x, \nu)$ is a *relaxed process* if $T > 0$, $(\underline{\omega}^0, \underline{\omega}) \in \mathcal{W}^{3+n}(T)$, $\underline{u} \in \mathcal{U}^{3+n}(T)$, $\gamma \in \Gamma_{2+n}(T)$, $(x^0, x, \nu) \in W^{1,1}([0, T], \mathbb{R} \times \mathbb{R}^n \times \mathbb{R})$ and it fulfills (5.2.9), and we say that it is feasible if it satisfies (5.2.6) and (5.2.7). We will use Σ^*, Σ^*_e, Σ^*_r to denote the sets of strict sense, extended, and relaxed processes which are feasible, respectively.

The control-polynomial extension described in this section allows us to treat higher order, not only first order, impulse inputs, which occur, for instance, in some applications to Lagrangian Mechanics (see e.g. [17, 25]). In particular, the results we will state include and extend those obtained in [37, 58], where just control-affine impulsive problems and no convex relaxation are considered. The need to consider the relaxed impulsive problem (P_r) relies on the fact that the set of velocities of the extended problem (P_e) is in general not convex, so that the existence of minimizers is not guaranteed.

Throughout this section, we shall consider the following structural hypotheses:

Hypothesis 5.2.1. We assume $K > 0$, $U \subset \mathbb{R}^q$ compact, $\mathfrak{W} \subset \mathbb{R}^m$ closed convex cone, $\mathcal{C}^* \subset \mathbb{R}^{1+n+1+n}$ closed. The functions $f : \mathbb{R}^{1+n} \times U \to \mathbb{R}^n$, $g^l_{j_1 \dots j_l} : \mathbb{R}^{1+n} \to \mathbb{R}^n$ are continuous, all $g^l_{j_1,\dots,j_l}$ are locally Lipschitz continuous, and $f(\cdot,\cdot,v)$ is locally Lipschitz continuous uniformly w.r.t. $v \in U$. Furthermore, the constraint function $\psi : \mathbb{R}^{1+n} \to \mathbb{R}$ is locally Lipschitz continuous.

The original problem $(\mathcal{P})$ can be identified with problem (P), as established by the following lemma, immediate consequence of the chain rule.

Lemma 5.2.2 (Embedding)**.** *Assume Hypothesis 5.2.1. Then the map*

$$\mathcal{I} : \{(S, \mathfrak{w}, \mathfrak{u}, y, \mathfrak{v}),\ \textit{original processes}\} \to \{(T, \omega^0, \omega, u, x^0, x, \nu),\ \textit{extended processes}\}$$

defined as

$$\mathcal{I}(S, \mathfrak{w}, \mathfrak{u}, y, \mathfrak{v}) := (T, \omega^0, \omega, u, x^0, x, \nu),$$

where, setting $\tau(s) := s + \mathfrak{v}(s)$ for all $s \in [0, S]$,

$$\begin{aligned} &T := \tau(S), \quad (x^0, x, \nu)(t) := (id, y, \mathfrak{v}) \circ \tau^{-1}(t) \quad \forall t \in [0, T], \\ &(\omega^0, \omega)(t) := (1 + |\mathfrak{w}|^\kappa)^{-\frac{1}{\kappa}}\ (1, \mathfrak{w}) \circ \tau^{-1}(t), \quad u(t) := \mathfrak{u} \circ \tau^{-1}(t) \quad a.e.\ t \in [0, T],^{23} \end{aligned}$$

is injective and has as image the subset of strict sense processes. Moreover, $\mathcal{I}$ maps any feasible *original process into a* feasible *strict sense process, with the same cost. Conversely, if $(T, \omega^0, \omega, u, x^0, x, \nu)$ is a (feasible) strict sense process, the absolutely continuous, increasing and surjective inverse $\tau : [0, S] \to [0, T]$ of x^0, allows us to define the (feasible, with same cost) original process $(S, \mathfrak{w}, \mathfrak{u}, y, \mathfrak{v})$ as*

$$(S, \mathfrak{w}, \mathfrak{u}, y, \mathfrak{v}) := \Big(x^0(T),\ \ (\omega \circ \tau) \cdot (\omega^0 \circ \tau)^{-1},\ \ u \circ \tau,\ \ x \circ \tau,\ \ \nu \circ \tau\Big).$$

The extended problem (P_e) consists in considering processes $(T, \omega^0, \omega, u, x^0, x, \nu)$, where ω^0 may be zero on nondegenerate subintervals of $[0, T]$. On these intervals, the

[23] Since every L^κ-equivalence class contains Borel measurable representatives, we are tacitly assuming that all L^κ-maps we are considering are Borel measurable.

time variable $s = x^0$ is constant – i.e. the time stops –, while the state variable x evolves instantaneously, according to

$$F(t, x, 0, \omega, u) = \sum_{1 \leq j_1 \leq \cdots \leq j_\kappa \leq m} g^{\kappa}_{j_1,\ldots,j_\kappa}(t, x)\ \omega^{j_1} \cdots \omega^{j_\kappa},$$

which can be called *fast dynamics*. For this reason, problem (P_e) is often referred to as the *impulsive extension* of the original problem $(\mathcal{P})$, although it is a conventional optimization problem with bounded controls. In fact, it is well-known that the extended problem (P_e) is equivalent to an *impulsive* problem (P_{im}) where the control $\mathfrak{w}$ is replaced by a measure ϑ of bounded variation combined with a family of ordinary controls (usually called *attached controls*) that come into play at the discontinuity points of ϑ, and the trajectory is a discontinuous map of bounded variation [6, 9, 10, 43, 56, 57].

Let us introduce the *unmaximized Hamiltonian*, defined by

$$H(s, z, p_0, p, \pi, \omega^0, \omega, v) := p_0(\omega^0)^\kappa + p \cdot F(s, z, w^0, w, v) + \pi|\omega|^\kappa,$$

for all $(s, z, p_0, p, \pi, \omega^0, \omega, v) \in \mathbb{R}^{1+n+1+n+1} \times W \times U$.

The concepts of *extremal* and *nondegenerate extremal* read now as follows:

Definition 5.2.3. Assume Hypothesis 5.2.1 and let $\mathfrak{Z} := (\bar{T}, \underline{\bar{\omega}}^0, \underline{\bar{\omega}}, \underline{\bar{u}}, \bar{\gamma}, \bar{x}^0, \bar{x}, \bar{\nu})$ be a feasible relaxed process. Given a cost function Φ which is Lipschitz continuous on a neighborhood of $(\bar{x}^0(\bar{T}), \bar{x}(T), \bar{\nu}(\bar{T}))$, we say that $\mathfrak{Z}$ is a Φ-*extremal* if there exist a path $(p_0, p) \in W^{1,1}([0, \bar{T}], \mathbb{R} \times \mathbb{R}^n)$, $\lambda \geq 0$, $\pi \leq 0$, $\mu \in C^{\oplus}([0, \bar{T}])$, $(m_0, m) : [0, \bar{T}] \to \mathbb{R}^{1+n}$ Borel measurable and μ-integrable functions, verifying the following conditions:

$$\|p_0\|_{L^\infty} + \|p\|_{L^\infty} + \mu([0, \bar{T}]) + \lambda \neq 0; \tag{5.2.10}$$

$$(-\dot{p}_0, -\dot{p})(t) \in \sum_{j=0}^{2+n} \bar{\gamma}^j(t)\ \mathrm{co}\, \partial_{t,x} H\big(\bar{x}^0(t), \bar{x}(t), q_0(t), q(t), \pi, \bar{\omega}^{0,j}(t), \bar{\omega}^j(t), \bar{u}^j(t)\big) \text{ a.e. } t; \tag{5.2.11}$$

$$\big(-q_0(\bar{T}), -q(\bar{T}), -\pi\big) \in \lambda \partial \Phi \left(\bar{x}^0(\bar{T}), \bar{x}(\bar{T}), \bar{\nu}(\bar{T})\right) + N_{\mathcal{C}^* \times]-\infty, K]} \left(\bar{x}^0(\bar{T}), \bar{x}(\bar{T}), \bar{\nu}(\bar{T})\right); \tag{5.2.12}$$

for every $j = 0, \dots 2+n$, for a.e. $t \in [0, T]$, one has

$$H\Big(\bar{x}^0(t), \bar{x}(t), q_0(t), q(t), \pi, \bar{\omega}^{0,j}(t), \bar{\omega}^j(t), \bar{u}^j(t)\Big) \\ = \max_{(w^0,w,v)\in W\times U} H\Big(\bar{x}^0(t), \bar{x}(t), q_0(t), q(t), \pi, w^0, w, v\Big) = 0; \tag{5.2.13}$$

$$(m_0, m)(t) \in \partial^{>}_{t,x} \psi\left(\bar{x}^0(t), \bar{x}(t)\right) \ \mu\text{-a.e.}; \tag{5.2.14}$$

$$\operatorname{spt}(\mu) \subseteq \{t \in [0, \bar{T}] : \psi\left(\bar{x}^0(t), \bar{x}(t)\right) = 0\}, \tag{5.2.15}$$

where $(q_0, q) : [0, \bar{T}] \to \mathbb{R}^{1+n}$ is defined by

$$(q_0, q)(t) := \begin{cases} (p_0, p)(t) + \int_{[0,t[}(m_0, m)(t')\mu(dt') & t \in [0, \bar{T}[, \\ (p_0, p)(\bar{T}) + \int_{[0,\bar{T}]}(m_0, m)(t')\mu(dt') & t = \bar{T}. \end{cases} \tag{5.2.16}$$

We will call a Φ-extremal *normal* if all $(p_0, p, \pi, \lambda, \mu, m_0, m)$ as above have $\lambda > 0$, and *abnormal* when it is not normal. Given a Φ-extremal $\mathfrak{Z}$ we call *nondegenerate multipliers* all $(p_0, p, \pi, \lambda, \mu, m_0, m)$ and (q_0, q) as above, that also verify

$$\mu(]0, \bar{T}]) + \|q_0\|_{L^\infty} + \|q\|_{L^\infty} + \lambda \neq 0. \tag{5.2.17}$$

If $\lambda \partial_\nu \Phi\left(\bar{x}^0(\bar{T}), \bar{x}(\bar{T}), \bar{\nu}(\bar{T})\right) = 0$ and $\bar{\nu}(\bar{T}) < K$, then $\pi = 0$. Furthermore, if $\bar{x}^0(0) < \bar{x}^0(\bar{T})$, (5.2.10) [resp. (5.2.17)] can be strengthened to

$$\|p\|_{L^\infty} + \mu([0, \bar{T}]) + \lambda \neq 0 \quad [\text{resp. } \mu(]0, \bar{T}]) + \|q\|_{L^\infty} + \lambda \neq 0]. \tag{5.2.18}$$

We will call a Φ-extremal *nondegenerate normal* if all $(p_0, p, \pi, \lambda, \mu, m_0, m)$ and (q_0, q) as above and verifying (5.2.17), have $\lambda > 0$, and *nondegenerate abnormal* when it is not nondegenerate normal.

The notions of isolation, controllability, and infimum gap for the processes in Σ_r^* can be promptly deduced by Definition 5.1.2. We explicitly write it because it is slightly different from that considered in [37].

Definition 5.2.4. Let $\mathfrak{Z} := (\bar{T}, \underline{\bar{\omega}}^0, \underline{\bar{\omega}}, \underline{\bar{u}}, \bar{\gamma}, \bar{x}^0, \bar{x}, \bar{\nu})$ be a feasible relaxed process, *at $\mathfrak{Z}$ there is a local infimum* gap if for any continuous function $\Phi : \mathbb{R}^{1+n+1} \to \mathbb{R}$ there is some $\delta > 0$ such that

$$\Phi(\bar{x}^0(\bar{T}), \bar{x}(T), \bar{\nu}(\bar{T})) < \inf \big\{ \Phi(x^0(T), x(T), \nu(T)) : \quad (T, \omega^0, \omega, u, x^0, x, \nu) \in \Sigma^* \\ \text{such that} \quad d_\infty\big((T,(x^0,x,\nu)),(\bar{T},(\bar{x}^0,\bar{x},\bar{\nu}))\big) < \delta \big\},$$

while $\mathfrak{Z}$ is an *isolated process* if, for some $\delta > 0$, one has

$$\big\{ (T, \omega^0, \omega, u, x^0, x, \nu) \in \Sigma^* : \quad d_\infty\big((T,(x^0,x,\nu)),(\bar{T},(\bar{x}^0,\bar{x},\bar{\nu}))\big) < \delta \big\} = \emptyset.$$

We say that the constrained control system (5.2.5)-(5.2.6)-(5.2.7) is *controllable* to $\mathfrak{Z}$ when $\mathfrak{Z}$ is not isolated.

We are now ready to establish a gap-abnormality relation.

Theorem 5.2.5. *Let $\mathfrak{Z} := (\bar{T}, \underline{\bar{\omega}}^0, \underline{\bar{\omega}}, \underline{\bar{u}}, \bar{\gamma}, \bar{x}^0, \bar{x}, \bar{\nu})$ be a feasible process for the relaxed impulsive extension (P_r), and suppose that at $\mathfrak{Z}$ there is a local infimum gap. If Hypothesis 5.2.1 is verified, then $\mathfrak{Z}$ is an abnormal extremal. If, in addition, also Hypothesis 3.2.6 is satisfied, then $\mathfrak{Z}$ is a nondegenerate abnormal extremal.*

Proof. First of all we show that Hypothesis 5.2.1 allows the application of Theorem 5.1.6. To this aim, we observe that Hypothesis 4.1.2 is trivially verified, by choosing, e.g., $V_i := \{(w^0, w) \in V : \quad w^0 \geq \frac{1}{i+1}\}$ for every $i \in \mathbb{N}$, while Hypothesis 5.2.1 yields Hypothesis 5.1.3 directly.

Finally, Hypothesis 5.1.4 easily follows from Hypothesis 5.2.1, taking into account the control-polynomial structure of the dynamics. In particular, we show the validity of the first condition in Hypothesis 5.1.4,(ii), as the second one can be deduced in a similar way.

First of all we notice that, given a^j, $b^j \in [0,1]$ for $j = 1, \dots, \kappa$, then one has

$$\begin{aligned} &|a^1 \cdot \, \dots \, \cdot a^\kappa - b^1 \cdot \, \dots \, \cdot b^\kappa| \\ &\qquad = |a^2 \cdot \, \dots \, \cdot a^\kappa (a^1 - b^1) + b^1 a^3 \cdot \, \dots \, \cdot a^\kappa (a^2 - b^2) + \dots + b^1 \cdot \, \dots \, \cdot b^{\kappa-1}(a^\kappa - b^\kappa)| \\ &\qquad \leq \sum_{j=1}^{\kappa} |a^j - b^j| \leq |(a^1, \dots, a^\kappa) - (b^1, \dots, b^\kappa)|. \end{aligned} \tag{5.2.19}$$

Now fix and arbitrary $r > 0$ and, in view of Hypothesis 5.2.1, let $L_F > 0$ be a common bound for f in $r\mathbb{B}_{1+n} \times U$ and all the $g^l_{j_1,\dots,j_l}$ in $r\mathbb{B}_{1+n}$. Hence, for any $(s,z,v) \in r\mathbb{B}_{1+n} \times U$ and any (w^0,w), $(\tilde{w}^0,\tilde{w}) \in W$ one has

$$\begin{aligned}
&|F(s,z,w^0,w,v) - F(s,z,\tilde{w}^0,\tilde{w},v)| \le |f(s,z,u)||(w^0)^\kappa - (\tilde{w}^0)^\kappa| \\
&\quad + \sum_{l=1}^{\kappa} \sum_{1\le j_1 \le \dots \le j_l \le m} |g^l_{j_1,\dots,j_l}(s,z)||w^{j_1}\cdot\ \dots\ \cdot w^{j_l}\,(w^0)^{\kappa-l} - \tilde{w}^{j_1}\cdot\ \dots\ \cdot \tilde{w}^{j_l}\,(\tilde{w}^0)^{\kappa-l}| \\
&\quad \le L_{F}\tilde{K}(|w^0 - \tilde{w}^0| + |w - \tilde{w}|),
\end{aligned} \tag{5.2.20}$$

where $\tilde{K}$ is an integer depending only on κ and m. Indeed, notice that by (5.2.19) we can estimate

$$\begin{aligned}
|w^{j_1}\cdot\ \dots\ \cdot w^{j_l}\,(w^0)^{\kappa-l} - \tilde{w}^{j_1}\cdot\ \dots\ \cdot \tilde{w}^{j_l}\,(\tilde{w}^0)^{\kappa-l}| &\le \sum_{h=1}^{l} |w^{j_h} - \tilde{w}^{j_h}| + (\kappa - l)|w^0 - \tilde{w}^0| \\
&\le l|w - \tilde{w}| + (\kappa - l)|w^0 - \tilde{w}^0|.
\end{aligned}$$

In order to conclude, it remains to prove that $\mathfrak{Z}$ is an abnormal extremal, nondegenerate abnormal if also Hypothesis 3.2.6 is satisfied, as in Definition 5.2.3.

In view of the above arguments, we can apply Theorem 5.1.6 with reference to $\mathfrak{Z}$. Primarily, we observe that the costate arc associated with the time variable t (namely, p_* of Definition 5.1.5) is constantly equal to 0 because the final time is unconstrained and neither the dynamics, nor the state constraint depend explicitly on t. Accordingly, from (5.1.7) we deduce the ‘constancy of the Hamiltonian’ condition (5.2.13).

Therefore, from Theorem 5.1.6 we deduce the existence of absolutely continuous functions p_0, p, π (associated with the state variable x^0, x, ν, respectively), of a measure $\mu \in C^\oplus$ and suitable functions (m_0, m) satisfying conditions (5.2.11)–(5.2.15) and, in addition, the nontriviality condition

$$\|p_0\|_{L^\infty} + \|p\|_{L^\infty} + \mu([0,\bar{T}]) + |\pi| \ne 0 \tag{5.2.21}$$

and the transversality condition

$$\big(-q_0(\bar{T}), -q(\bar{T}), -\pi\big) \in N_{\mathcal{C}^*\times]-\infty,K]}\left(\bar{x}^0(\bar{T}), \bar{x}(\bar{T}), \bar{\nu}(\bar{T})\right), \tag{5.2.22}$$

for (q_0, q) as in (5.2.16) [24]. Moreover, if Hypothesis 3.2.6 is verified, it also holds

$$\mu(]0,\bar{T}]) + \|q_0\|_{L^\infty} + \|q\|_{L^\infty} + |\pi| \neq 0. \tag{5.2.23}$$

From (5.2.22), it is straightforward to deduce that, if $\bar{\nu}(\bar{T}) < 0$, then $\pi = 0$. Now we prove that it holds

$$\|p_0\|_{L^\infty} + \|p\|_{L^\infty} + \mu([0,\bar{T}]) \neq 0. \tag{5.2.24}$$

Suppose by contradiction that (5.2.24) is not true. Then $q_0 \equiv 0$, $q \equiv 0$ a.e., and by (5.2.21) we deduce that $\pi \neq 0$, which in turn implies $\bar{\nu}(\bar{T}) = K > 0$. Thanks to these information and integrating (5.2.13) in $[0,\bar{T}]$ we find that

$$0 = \int_0^{\bar{T}} \pi\, |\bar{\omega}(t)|^\kappa dt = \pi\, \bar{\nu}(\bar{T}) = \pi\, K \neq 0.$$

Hence, (5.2.24) holds. With similar arguments one can deduce that, if Hypothesis 3.2.6 is verified, then (5.2.23) implies

$$\mu(]0,\bar{T}]) + \|q_0\|_{L^\infty} + \|q\|_{L^\infty} \neq 0. \tag{5.2.25}$$

It remains to prove that, if $\bar{x}^0(\bar{T}) > \bar{x}^0(0)$, then (5.2.24) can be replaced by

$$\|p\|_{L^\infty} + \mu([0,\bar{T}]) \neq 0. \tag{5.2.26}$$

Indeed, if we suppose by contradiction that (5.2.26) is not true, then by (5.2.11) and (5.2.13) one obtains that $0 \neq q_0 = p_0$ is constant and

$$\max_{(w^0,w)\in W} \{p_0(w^0)^\kappa + \pi|w|^\kappa\} = 0, \qquad p_0(\bar{\omega}^0(t))^\kappa + \pi|\bar{\omega}(t)|^k = 0 \ \text{ a.e. } t \in [0,\bar{T}]. \tag{5.2.27}$$

If $\pi < 0$, then by choosing $w^0 = 1$ in the first one in (5.2.27) one deduces $p_0 \leq 0$, while from the second one in (5.2.21) one gets $p_0 = -\frac{\pi|\bar{\omega}(t)|^k}{(\bar{\omega}^0(t))^\kappa} \geq 0$ [25]. Hence, it follows $p_0 = 0$, but this contradicts (5.2.24).

[24]Notice that π is constant because the dynamics does not depend on ν. In particular, $\pi \leq 0$ is a consequence of the transversality condition (5.2.12), together with the fact that the state constraint function does not depend on ν.

[25]Of course, $\bar{\omega}^0 > 0$ a.e., otherwise $|\bar{\omega}| = 1$ on a subset of $[0,\bar{T}]$ of positive measure, and from the second one in (5.2.27) one gets $\pi = 0$, in contradiction with the assumption $\pi < 0$.

However, if $\pi = 0$, then the second one in (5.2.27) implies that $\bar{\omega}^0(t) = 0$ for a.e. $t \in [0, \bar{T}]$ (recall $p_0 \neq 0$ by (5.2.24)). But this contradicts the assumption $\bar{x}^0(\bar{T}) > \bar{x}^0(0)$.

With similar one gets that, if Hypothesis 3.2.6 is verified and $\bar{x}^0(\bar{T}) > \bar{x}^0(0)$, then (5.2.25) can be strengthened with

$$\mu(]0, \bar{T}]) + \|q\|_{L^\infty} \neq 0.$$

This concludes the proof. □

As corollaries, we have:

Theorem 5.2.6. *Assume Hypothesis 5.2.1 and let Φ be locally Lipschitz continuous.*

(i) *Let $\mathfrak{z}$ be a local Φ-minimizer for (P_e) or (P_r) which is a normal Φ-extremal. Then, at $\mathfrak{z}$ there is no local infimum gap. If in addition $\mathfrak{z}$ minimizes Φ over Σ_r^* or Σ_e^*, then it realizes the infimum of Φ over Σ^*.*

(ii) *Let $\mathfrak{z}$ be a local Φ-minimizer for (P_e) or (P_r), at which Hypothesis 3.2.6 is verified and which is a nondegenerate normal Φ-extremal. Then, at $\mathfrak{z}$ there is no local infimum gap. If in addition $\mathfrak{z}$ minimizes Φ over Σ_r^* or Σ_e^*, then it realizes the infimum of Φ over Σ^*.*

Theorem 5.2.7. *Assume Hypothesis 5.2.1. Then, either control system* (5.2.5)-(5.2.6)-(5.2.7) *is controllable to $\mathfrak{z}$, or $\mathfrak{z}$ is an abnormal extremal. If in addition also Hypothesis 3.2.6 is verified then either control system* (5.2.5)-(5.2.6)-(5.2.7) *is controllable to $\mathfrak{z}$, or $\mathfrak{z}$ is a nondegenerate abnormal extremal.*

5.3 Simplified hypotheses for nondegeneracy

It is worth to point out that, when the feasible reference process $\mathfrak{z}$ belongs to the subclass of extended processes, namely $\mathfrak{z} := (\bar{T}, \bar{\omega}^0, \bar{\omega}, \bar{u}, \bar{x}^0, \bar{x}, \bar{\nu}) \in \Sigma_e^*$, in the previous theorems Hypothesis 3.2.6 can be replaced by the following, simpler assumption.

Hypothesis 5.3.1. If $(0, \check{z}_0) \in \partial\mathfrak{Q}$ ($\mathfrak{Q}$ as in (3.2.1)), there exist $\tilde{\delta} > 0$, $\bar{t} \in]0, \bar{T}]$, some sequence $(\bar{t}, \tilde{\omega}_i^0, \tilde{\omega}_i, \tilde{u}_i, \tilde{x}_i^0, \tilde{x}_i, \tilde{\nu}_i)$ of strict sense processes (i.e. $(\tilde{\omega}_i^0, \tilde{\omega}_i) \in \mathcal{V}(\bar{t})$), a sequence of Lebesgue measurable subsets $\tilde{\Omega}_i \subset [0,\bar{t}\,]$ with $\ell(\tilde{\Omega}_i) \to \bar{t}$, some sequences $(\hat{\omega}_i^0, \hat{\omega}_i, \hat{u}_i) \subset \mathcal{W}(\bar{t}\,) \times \mathcal{U}(\bar{t}\,)$, and $(\tilde{\varphi}_i) \subset L^1([0,\bar{t}\,], \mathbb{R}_{\geq 0})$ with $\lim_{i\to+\infty} \|\tilde{\varphi}_i\|_{L^1([0,\bar{t}])} = 0$, such that for any i the following properties are fulfilled:

(i) one has

$$\psi(\tilde{x}_i^0(t), \tilde{x}_i(t)) \leq 0 \qquad \forall t \in [0,\bar{t}\,];$$

(ii) for a.e. $t \in \tilde{\Omega}_i$ one has

$$(\tilde{\omega}_i^0, \tilde{\omega}_i)(t) \in (\bar{\omega}^0, \bar{\omega})(t) + \tilde{\varphi}_i(t)\mathbb{B}, \quad \tilde{u}_i(t) = \bar{u}(t), \quad \text{a.e. } t \in \tilde{\Omega}_i;$$

(iii) for all $(\xi_0, \xi) \in \partial^*\psi(0, \check{z}_0)$, and for a.e. $t \in [0,\bar{t}\,]$, one has

$$\begin{aligned}\xi_0 \cdot &[(\hat{\omega}_i^0(t))^\kappa - (\tilde{\omega}_i^0(t))^\kappa] \\ &+ \xi \cdot \left[F(0, \check{z}_0, (\hat{\omega}_i^0, \hat{\omega}_i, \hat{u}_i)(t)) - F(0, \check{z}_0, (\tilde{\omega}_i^0, \tilde{\omega}_i, \tilde{u}_i)(t))\right] \leq -\tilde{\delta}.\end{aligned}$$

Lemma 5.3.2. *Assume Hypothesis 5.2.1 and let* $\mathfrak{z} := (\bar{T}, \bar{\omega}^0, \bar{\omega}, \bar{u}, \bar{x}^0, \bar{x}, \bar{\nu}) \in \Sigma_e^*$. *Then, Hypothesis 5.3.1 implies Hypothesis 3.2.6.*

Proof. Consider a sequence $(\delta_i) \subset \mathbb{R}_{>0}$ such that $\delta_i \downarrow 0$ and, for every i, define the strict sense control

$$(\check{\omega}_i^0, \check{\omega}_i, \check{u}_i)(t) := \begin{cases} (\tilde{\omega}_i^0, \tilde{\omega}_i, \tilde{u}_i)(t) & \text{if } t \in [0,\bar{t}\,], \\ (\bar{\omega}^0, \bar{\omega}, \bar{u})(t) & \text{if } t \in]\bar{t}, \bar{T}] \text{ and } \bar{\omega}^0(t) > 0, \\ (\delta_i, \sqrt[\kappa]{1-\delta_i^\kappa}\, \bar{\omega}(t), \bar{u}(t)) & \text{if } t \in]\bar{t}, \bar{T}] \text{ and } \bar{\omega}^0(t) = 0, \end{cases}$$

where $(\tilde{\omega}_i^0, \tilde{\omega}_i, \tilde{u}_i)$ is as in Hypothesis 5.3.1. Consider the sequence of strict sense processes $\mathfrak{z}_i := (\bar{T}, \check{\omega}_i^0, \check{\omega}_i, \check{u}_i, \check{x}_i^0, \check{x}_i, \check{\nu}_i)$ of (P_e) corresponding to $(\check{\omega}_i^0, \check{\omega}_i, \check{u}_i)$ and define $\check{\Omega}_i := \tilde{\Omega}_i \cup]\bar{t}, \bar{T}]$. Of course, $(\check{x}_i^0, \check{x}_i, \check{\nu}_i) \equiv (\tilde{x}_i^0, \tilde{x}_i, \tilde{\nu}_i)$ in $[0,\bar{t}\,]$. We have just observed in the proof of Theorem 5.2.5 that the control-polynomial structure of the dynamics together with Hypothesis 5.2.1 imply the validity of Hypothesis 5.1.4, so that from Hypothesis 5.3.1,(ii) and Theorem 2.5.2 (see also Remark 2.5.3), one has

$$\|(\check{x}_i^0, \check{x}_i, \check{\nu}_i) - (\bar{x}^0, \bar{x}, \bar{\nu})\|_{L^\infty(0,\bar{T})} \to 0.$$

Therefore, for i large, it is possible to consider a sequence $(\varepsilon_i)_i$ decreasing to 0 such that $2\sqrt[4]{\varepsilon_i} \le \bar{t}$ and $\|(\check{x}_i^0, \check{x}_i, \check{\nu}_i) - (\bar{x}^0, \bar{x}, \bar{\nu})\|_{L^\infty(0,\bar{T})} \le \varepsilon_i$ for any i. Since $(\mathfrak{Z}_i)_i \subset \Sigma_e^*$ and $\mathfrak{Z} \in \Sigma_e^*$ we can take $\check{\gamma}_i \equiv \bar{\gamma} \equiv (1, 0, \dots, 0)$, so that $\check{\zeta}_i \equiv \bar{\zeta}$. In view of this observations, Hypothesis 3.2.6 follows straightforward from the conditions in Hypothesis 5.3.1. $\square$

In some situations, Hypothesis 5.3.1 simplifies considerably.

Lemma 5.3.3. *Assume Hypothesis 5.2.1. Let $(0, \check{z}_0) \in \partial\Omega$ and fix a feasible extended process $\mathfrak{Z} := (\bar{T}, \bar{\omega}^0, \bar{\omega}, \bar{u}, \bar{x}^0, \bar{x}, \bar{\nu})$. If there are some $\tilde{\delta} > 0$, $\bar{t} \in]0, \bar{T}]$, and an extend control $(\hat{\omega}^0, \hat{\omega}, \hat{u}) \in \mathcal{W}(\bar{t}) \times \mathcal{U}(\bar{t})$ such that, for all $(\xi_0, \xi) \in \partial^*\psi(0, \check{z}_0)$ and for a.e. $t \in [0, \bar{t}]$,*

$$\xi_0 \cdot [(\hat{\omega}^0(t))^\kappa - (\bar{\omega}^0(t))^\kappa] + \xi \cdot \left[F(0, \check{z}_0, (\hat{\omega}^0, \hat{\omega}, \hat{u})(t)) - F(0, \check{z}_0, (\bar{\omega}^0, \bar{\omega}, \bar{u})(t))\right] \le -\tilde{\delta}. \tag{5.3.1}$$

and either $\bar{\omega}^0 > 0$ a.e. in $[0, \bar{t}]$, or there is some $\tilde{\delta}_1 > 0$ such that, for a.e. $t \in [0, \bar{t}]$,

$$\sup_{(\xi_0,\xi) \in \partial^*\psi(0,\check{z}_0)} \left[\xi_0(\bar{\omega}^0(t))^\kappa + \xi \cdot F(0, \check{z}_0, (\bar{\omega}^0, \bar{\omega}, \bar{u})(t))\right] \le -\tilde{\delta}_1, \tag{5.3.2}$$

then Hypothesis 5.3.1 is satisfied.

Proof. First of all we make the following observation. Reasoning as in Remark 3.2.7, (5), we deduce that if condition (5.3.2) holds, then there exist $\bar{\delta}$, $\bar{\varepsilon} > 0$ such that for any $(\xi_0, \xi) \in \operatorname{co} \partial^*\psi(\tau, z)$ with $(\tau, z) \in \{(0, \check{z}_0)\} + \bar{\varepsilon}\mathbb{B}_{1+n}$, for any $t \le \bar{t}$, for any continuous path $(x^0, x) : [0, t] \to \{(0, \check{z}_0)\} + \bar{\varepsilon}\mathbb{B}_{1+n}$ one has

$$\int_0^t \left[\xi_0(\bar{\omega}^0(t'))^\kappa + \xi \cdot F((x^0, x, \bar{\omega}^0, \bar{\omega}, \bar{u})(t'))\right] dt' \le -\bar{\delta}t. \tag{5.3.3}$$

Now, let us first suppose that $\bar{\omega}^0 > 0$ a.e. in $[0, \bar{t}]$. Then, Hypothesis 5.3.1,(i),(ii) are verified by choosing, for every i, $(\tilde{\omega}_i^0, \tilde{\omega}_i, \tilde{u}_i) = (\bar{\omega}^0, \bar{\omega}, \bar{u})$, while Hypothesis 5.3.1,(iii) follows directly from (5.3.1), taking $(\hat{\omega}_i^0, \hat{\omega}_i, \hat{u}_i) \equiv (\hat{\omega}^0, \hat{\omega}, \hat{u})$ for any i.

If instead (5.3.2) is assumed, let us consider a sequence $\delta_i \downarrow 0$ and for every i and set

$$(\tilde{\omega}_i^0, \tilde{\omega}_i, \tilde{u}_i)(t) := \begin{cases} (\bar{\omega}^0, \bar{\omega}, \bar{u})(t) & \text{if } \bar{\omega}^0(t) > 0, \\ (\delta_i, \sqrt[\kappa]{1 - \delta_i^\kappa}\, \bar{\omega}(t), \bar{u}(t)) & \text{if } \bar{\omega}^0(t) = 0, \end{cases}$$

for a.e. $t \in [0,\bar{t}\,]$. Then, $(\tilde{\omega}_i^0, \tilde{\omega}_i, \tilde{u}_i) \in \mathcal{V}(\bar{t}\,) \times \mathcal{U}(\bar{t}\,)$ and it verifies Hypothesis 5.3.1,(ii) with $\tilde{\Omega}_i = [0,\bar{t}\,]$ and $\tilde{\varphi}_i \equiv 2\delta_i$. Indeed, by construction, for any t we get

$$|\tilde{\omega}_i^0(t) - \bar{\omega}^0(t)| + |\tilde{\omega}_i(t) - \bar{\omega}(t)| \leq \delta_i + 1 - \sqrt[\kappa]{1-\delta_i^\kappa} \leq 2\delta_i. \tag{5.3.4}$$

Let $(\tilde{x}_i^0, \tilde{x}_i, \tilde{\nu}_i)$ be the solution in $[0,\bar{t}\,]$ of the extended control system (P_e) associated with $(\tilde{\omega}_i^0, \tilde{\omega}_i, \tilde{u}_i)$ and with initial condition $(\tilde{x}_i^0, \tilde{x}_i, \tilde{\nu}_i)(0) = (0, \check{z}_0, 0)$. By the Lebourg Mean Value Theorem 2.4.7 and (2.4.1) we deduce that for any $t \in [0,\bar{t}\,]$ there exists $(\xi_{0_i}^t, \xi_i^t) \in \mathrm{co}\,\partial^*\psi(y_i^0(t), y_i(t))$ for some $(y_i^0(t), y_i(t))$ belonging to the segment $\{s(\tilde{x}_i^0(t), \tilde{x}_i(t)) + (1-s)(0,\check{z}_0) : \, s \in [0,1]\}$ such that

$$\psi(\tilde{x}_i^0(t), \tilde{x}_i(t)) - \psi(0,\check{z}_0) = \langle (\xi_{0_i}^t, \xi_i^t), (\tilde{x}_i^0(t), \tilde{x}_i(t) - \check{z}_0)\rangle. \tag{5.3.5}$$

Possibly reducing $\bar{t}$, Hypothesis 5.2.1 implies that $(y_i^0(t), y_i(t)) \in \{(0,\check{z}_0)\} + \bar{\varepsilon}\mathbb{B}_{1+n}$ and $(\tilde{x}_i^0(t), \tilde{x}_i(t)) \in \{(0,\check{z}_0)\} + \bar{\varepsilon}\mathbb{B}_{1+n}$ for any $t \in [0,\bar{t}\,]$ ($\bar{\varepsilon} > 0$ as above). Accordingly, from (5.2.20), (5.3.3), (5.3.4) and (5.3.5), we get that for any $t \in [0,\bar{t}\,]$ it holds

$$\begin{aligned}
\psi(\tilde{x}_i^0(t), \tilde{x}_i(t)) &= \psi(\tilde{x}_i^0(t), \tilde{x}_i(t)) - \psi(0,\check{z}_0) \\
&= \int_0^t \xi_{0_i}^t \tilde{\omega}_i^0(t')\,dt' + \int_0^t \xi_i^t \cdot F((\tilde{x}_i^0, \tilde{x}_i, \tilde{\omega}_i^0, \tilde{\omega}_i, \tilde{u}_i)(t'))\,dt' \\
&\leq -\bar{\delta}t + \int_0^t \xi_{0_i}^t (\tilde{\omega}_i^0(t') - \bar{\omega}^0(t))\,dt' \\
&\qquad + \int_0^t \xi_i^t \cdot \big[F((\tilde{x}_i^0, \tilde{x}_i, \tilde{\omega}_i^0, \tilde{\omega}_i, \tilde{u}_i)(t')) - F((\tilde{x}_i^0, \tilde{x}_i, \bar{\omega}^0, \bar{\omega}, \bar{u})(t'))\big]\,dt' \\
&\leq t(-\bar{\delta} + 2L_\psi\delta_i + 2L_\psi L_F \tilde{K}\delta_i) \leq 0
\end{aligned}$$

for i large enough, where L_ψ is the Lipschitz constant of ψ in $\bar{\varepsilon}\mathbb{B}_{1+n}$ and L_F and $\tilde{K}$ are as in (5.2.20) (in particular, L_F is referred to the compact $\bar{\varepsilon}\mathbb{B}_{1+n} \times U$). This proves the validity of Hypothesis 5.3.1,(i).

Finally, by taking $(\hat{\omega}_i^0, \hat{\omega}_i, \hat{u}_i) \equiv (\hat{\omega}^0, \hat{\omega}, \hat{u})$, adding and subtracting '$\xi_0 \cdot (\bar{\omega}^0(t))^\kappa + \xi \cdot F(0, \check{z}_0, (\bar{\omega}^0, \bar{\omega}, \bar{u})(t))$', and using (5.2.20), (5.3.1) and (5.3.4), similarly as above we get Hypothesis 5.3.1,(iii), possibly reducing $\tilde{\delta}$, for all i large enough. □

The following example illustrates how Theorem 5.2.6,(ii), in which the normality hypothesis is understood in its nondegenerate form, can be used to exclude the occurence of an infimum gap. Of course, Theorem 5.2.6, (i) is of no use here, because for problems of this nature, in which the initial state lies in the boundary of the state constraint set, extremals are never normal in the sense of Definition 5.2.3.

Example 5.3.4. Consider the problem

$$\begin{cases} \qquad \text{Minimize} \quad -y(1) \\ \text{over } (\mathfrak{w}, y, \mathfrak{v}) \in L^1([0,1],\mathbb{R}^2) \times W^{1,1}([0,1],\mathbb{R}^3 \times \mathbb{R}) \text{ satisfying} \\ (\dot{y}, \dot{\mathfrak{v}})(t) = \Big(f(y(t)) + g_1(y(t))\,\mathfrak{w}^1(t) + g_2(y(t))\,\mathfrak{w}^2(t),\ |\mathfrak{w}(t)| \Big) \\ (y, \mathfrak{v})(0) = ((1,0,0),0), \\ y(t) \in \mathcal{Q} \ \ \forall t \in [0,1],\ \mathfrak{v}(1) \le 2,\ y(1) \in \mathcal{C}, \end{cases} \tag{5.3.6}$$

in which $\mathcal{Q} := [-1,1]^3$, $\mathcal{C} := [-1,0] \times [0,1]^2$, and

$$g_1(z) := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad g_2(z) := \begin{pmatrix} 0 \\ -1 \\ -z^1 \end{pmatrix}, \quad f(z) := \begin{pmatrix} 0 \\ z^2 z^3 \\ 0 \end{pmatrix} \qquad \forall z \in \mathbb{R}^3 .$$

Here, $W = \{(\omega^0, w) \in \mathbb{R}_{\ge 0} \times \mathbb{R}^2 : \quad w^0 + |w| = 1\}$, $V = \{(\omega^0, w) \in W : \quad w^0 > 0\}$, and the associated extended problem is

$$\begin{cases} \qquad \text{Minimize} \quad -x^1(T) \\ \text{over } T > 0,\ (\omega^0, \omega^1, \omega^2, x^0, x, \nu) \in \mathcal{W}(T) \times W^{1,1}([0,T],\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}) \quad \text{satisfying} \\ (\dot{x}^0, \dot{x}, \dot{\nu})(t) = \Big(\omega^0(t),\ f(x(t))\omega^0(t) + g_1(x(t))\,\omega^1(t) + g_2(x(t))\,\omega^2(t),\ |\omega(t)| \Big) \\ (x^0, x, \nu)(0) = (0,(1,0,0),0) \\ x(t) \in \mathcal{Q} \ \ \forall t \in [0,T], \quad (x^0(T), x(T), \nu(T)) \in \{1\} \times \mathcal{C} \times]-\infty, 2]. \end{cases}$$

As it is easy to see, an extended minimizer is given by the following feasible extended process $\mathfrak{Z} := (\bar{T}, \bar{\omega}^0, \bar{\omega}, \bar{x}^0, \bar{x}, \bar{\nu})$, where

$$\bar{T} = 2, \qquad (\bar{\omega}^0, \bar{\omega}) = (\bar{\omega}^0, \bar{\omega}^1, \bar{\omega}^2) = (1,0,0)\chi_{[0,1]} + (0,-1,0)\chi_{]1,2]},$$
$$(\bar{x}^0, \bar{x}, \bar{\nu}) = (\bar{x}^0, \bar{x}^1, \bar{x}^2, \bar{x}^3, \bar{\nu}) = (t,1,0,0,0)\chi_{[0,1]} + (1, 2-t, 0, 0, t-1)\chi_{[1,2]} .$$

From the maximum principle [37, Thm. 1.1], $\mathfrak{Z}$ is a Φ-extremal accordingly to Definition 5.2.3. Hence, there exist a set of multipliers $(p_0, p, \pi, \lambda, \mu)$ and functions (m_0, m) with $\pi = 0$, since $\nabla_\nu \Phi \equiv 0$ and $\bar{\nu}(2) = 1 < 2$, $m_0 \equiv 0$, as the state constraint does not depend on time, and $\mu([0,2]) = \mu([0,1])$. Moreover, for every $t \in [0,1]$ the fact that $\bar{x}(t) \in \mathfrak{Q}$ is equivalent to $\psi(\bar{x}(t)) \leq 0$, with $\psi(z^1, z^2, z^3) := z^1 - 1$, so that the condition $m(t) \in \partial_x^{>}\psi(\bar{x}(0))$ μ-a.e. yields $m(t) = (1,0,0)$ μ-a.e. in $[0,1]$. By the adjoint equation, it follows that the path $(p_0, p) = (p_0, p_1, p_2, p_3) \equiv (\bar{p}_0, \bar{p}_1, \bar{p}_2, \bar{p}_3)$ is constant. From the transversality condition

$$-(q_0, q_1, q_2, q_3)(2) \in \lambda\{(0,-1,0,0)\} + \mathbb{R} \times N_{\mathfrak{C}}(0,0,0),$$

where $q_0 \equiv \bar{p}_0$, and $q(t) = (\bar{p}_1 + \mu([0,1]), \bar{p}_2, \bar{p}_3)$ for all $t \in]1,2]$, we derive that $\bar{p}_0$, $\bar{p}_1 \in \mathbb{R}$, $\bar{p}_2$, $\bar{p}_3 \geq 0$, and $q_1(2) = \bar{p}_1 + \mu([0,1]) = \lambda - \alpha$ with $\alpha \geq 0$. The maximality condition in $]1,2]$ implies that $\bar{p}_2 = \bar{p}_3 = 0$. In particular, from the relations

$$\max_{w^1 \in [-1,1]} \left\{ q_1(t) w^1 \right\} \chi_{[0,1]}(t) = \bar{p}_0 \chi_{[0,1]}(t) = 0, \quad -q_1(t)\chi_{]1,2]}(t) = 0,$$

we also deduce that $\bar{p}_0 = 0$, $q_1(t) = \bar{p}_1 + \mu([0,t[) = 0$ for a.e. $t \in [0,1[$, and $q_1(t) = \bar{p}_1 + \mu([0,1]) = \lambda - \alpha = 0$ for every $t \in]1,2]$. In particular, $q(t) = 0$ for a.e. $t \in [0,2]$, $\mu([0,t[) = -\bar{p}_1$ for a.e. $t \in [0,1]$ implies that ($\bar{p}_1 \leq 0$ and) $\mu = -\bar{p}_1 \mu(\{0\})$, while the last relation yields that $\lambda = \alpha$.

It is immediate to see that the set of degenerate multipliers (p_0, p, λ, μ) with $p_0 = p_2 = p_3 = 0$, $p_1 = -1$, $\mu = \delta_{\{0\}}$, and $\lambda = 0$ meets all the conditions of the maximum principle. So, $\mathfrak{Z}$ is an abnormal extremal. However, since $\bar{\omega}^0 > 0$ for a.e. $t \in [0,1]$ and the control $(\hat{\omega}^0, \hat{\omega}) = (\hat{\omega}^0, \hat{\omega}^1, \hat{\omega}^2) \equiv (0,-1,0)$ verifies (5.3.1), from Lemma 5.3.3 it follows that Hypothesis 5.3.1 is satisfied. Therefore, in view of Lemma 5.3.2 and Theorem 5.2.6, (ii), to deduce that there is no infimum gap it is enough to observe that $\mathfrak{Z}$ is nondegenerate normal, namely, that $\lambda \neq 0$ for all sets of multipliers as above, which in addition verify

$$\mu(]0,2]) + \|q\|_{L^\infty} + \lambda \neq 0.$$

This is true, since the previous calculations imply that $\|q\|_{L^\infty} = 0$ and $\mu(]0,2]) = 0$.

5.4 Verifiable conditions for normality

Some sufficient conditions for the absence of an infimum gap that have been stated in this chapter rely on the normality of the extremals. However, the normality criterium for the absence of an infimum gap has some disadvantages. First of all, it requires to know a priori a minimizer, information that is not always available. Then, it is necessary to verify that all sets of multipliers associated to the minimizer that meet the conditions of the Maximum Principle have $\lambda > 0$. In addition, in the presence of state constraints, the normality condition may never be met, making the criterium in fact useless, as observed in Section 3.2.

In the literature on conventional, non-impulsive problems with state constraints, a variety of constraint qualifications to avoid degeneracy as well as to ensure normality are known (see e.g. [1, 2, 11, 12, 32, 33, 34, 35, 47, 62, 66] and the references therein). In impulsive control, instead, some nondegenerate Maximum Principles have been obtained in [8, 10, 41, 57], while a Maximum Principle in normal form has only recently been introduced in [57].

Based on the above considerations, in this subsection we introduce some sufficient conditions for nondegenerate normality in the special case of control-affine systems, i.e. when $\kappa = 1$ in problem (P_e). The results of this subsection are base on [37].

In particular, from now on we shall consider the following optimal control problem

$$
(\mathscr{P}_e)\begin{cases}
\qquad\qquad \text{Minimize} \;\; \Phi(x^0(0), x(0), x^0(T), x(T), \nu(T)) \\
\text{over } T>0,\ (\omega^0,\omega)\in\mathcal{W}(T),\ u\in\mathcal{U}(T),\ (x^0,x,\nu)\in W^{1,1}([0,T],\mathbb{R}^{1+n+1}) \quad \text{s.t.} \\
(\dot{x}^0,\dot{x},\dot{\nu})(t) = (\omega^0, F((x^0,x,\omega^0,\omega,u)(t)), |\omega(t)|) \qquad \text{a.e. } t\in[0,T], \\
(\nu(0),\nu(T)) \in \{0\}\times]-\infty, K], \qquad (x^0(0),x(0),x^0(T),x(T))\in\mathcal{C}, \\
\psi(x^0(t),x(t)) \le 0 \qquad \text{for all } t\in[0,T],
\end{cases}
$$

where $\mathcal{C} \subset \mathbb{R}^{1+n+1+n}$ is closed and, for any $(s, z, w^0, w, v) \in \mathbb{R}^{1+n+1+m+q}$ we set

$$
F(s,z,w^0,w,v) := f(s,z,v)w^0 + \sum_{j=1}^{m} g_j(s,z)w^j. \tag{5.4.1}
$$

In particular, $\mathcal{U}(T)$ and $\mathcal{W}(T)$ are as in the previous section with $\kappa = 1$.

We omit to specify what we mean by *feasible process* and *extremal* for $(\mathscr{P}_e)$, since these notions can be trivially inferred by the analogous ones given in Section 5.2 taking account that $\kappa = 1$ and that feasible processes for $(\mathscr{P}_e)$ actually are feasible relaxed processes (i.e. feasible process for (P_r)) with $\gamma \equiv \frac{1}{n+3}(1,\dots,1)$. The only remarkable modification regards the transversality condition (5.2.12), that from now will be replaced by

$$\begin{aligned}\big(p_0(\bar T), p(\bar T), -q_0(\bar T), -q(\bar T), -\pi\big) \in \lambda\partial\Phi\left(\bar x^0(0), \bar x(0), \bar x^0(\bar T), \bar x(\bar T), \bar\nu(\bar T)\right)\\ + N_{\mathcal{C}\times]-\infty,K]}\left(\bar x^0(0), \bar x(0), \bar x^0(\bar T), \bar x(\bar T), \bar\nu(\bar T)\right),\end{aligned} \tag{5.4.2}$$

in view of the new endpoint constraint which is not a singleton at $x(0)$. Accordingly, when in the following we will refer to Hypothesis 5.2.1 we will implicitly assume $\kappa = 1$.

Hypothesis 5.4.1. We say that a feasible process $(\bar T, \bar\omega^0, \bar\omega, \bar u, \bar x^0, \bar x, \bar\nu)$ satisfies the condition for nondegeneracy if

$$\partial^> h(\bar x^0(0), \bar x(0)) \cap \left(-\mathrm{proj}_{(t_1,x_1)}(N_{\mathcal{C}}(\bar x^0(0), \bar x(0), \bar x^0(\bar T), \bar x(\bar T)))\right) = \emptyset. \tag{5.4.3}$$

Remark 5.4.2. To clarify the geometrical meaning of Hypothesis 5.4.1, let us notice that, if $(\bar x^0(0), \bar x(0)) \in \mathrm{Int}(\mathcal{Q})$, condition (5.4.3) is trivially satisfied, since

$$\partial^> h(\bar x^0(0), \bar x(0)) = \emptyset.$$

When instead $(\bar x^0(0), \bar x(0)) \in \partial\mathcal{Q}$, (5.4.3) implies that $0 \notin \partial^> h(\bar x^0(0), \bar x(0))$. If $h \in C^2$ in a neighborhood of $(\bar x^0(0), \bar x(0)) \in \partial\mathcal{Q}$, (5.4.3) simply reads ($\nabla h(\bar x^0(0), \bar x(0)) \neq 0$ and)

$$\nabla h(\bar x^0(0), \bar x(0)) \notin -\mathrm{proj}_{(t_1,x_1)}(N_{\mathcal{C}}(\bar x^0(0), \bar x(0), \bar x^0(\bar T), \bar x(\bar T))). \tag{5.4.4}$$

Condition (5.4.4) is satisfied at a point $(\bar x^0(0), \bar x(0))$ such that $\psi(\bar x^0(0), \bar x(0)) = 0$ and $\nabla h(\bar x^0(0), \bar x(0)) \neq 0$, when, for instance, $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$ with $\mathcal{C}_1$, $\mathcal{C}_2$ closed subsets of $\mathbb{R}^{1+n}$, $\mathcal{C}_1 \subseteq \mathcal{Q}$, and $N_{\mathcal{C}_1}(\bar x^0(0), \bar x(0))$ is pointed (see Subsection 2.4.1). In this case, indeed, (5.4.4) can be derived by the following relations

$$\begin{aligned}\partial^>\psi(\bar x^0(0), \bar x(0)) = \{\nabla\psi(\bar x^0(0), \bar x(0))\} \subseteq N^P_{\mathcal{Q}}(\bar x^0(0), \bar x(0))\\ \subseteq N^P_{\mathcal{C}_1}(\bar x^0(0), \bar x(0)) \subseteq N_{\mathcal{C}_1}(\bar x^0(0), \bar x(0)).\end{aligned}$$

If we consider the quite customary situation where initial and final time are fixed and the state constraint is time independent, namely $\mathcal{C} = \{\bar t_1\} \times \mathcal{C}_1 \times \mathcal{C}_2$ with $\mathcal{C}_1 \subseteq \mathbb{R}^n$, $\mathcal{C}_2 \subseteq \mathbb{R}^{1+n}$ closed subsets, and $\psi(s,z) = \bar\psi(z)$. In this case, $\partial^>\psi(s,z) = \{0\} \times \partial^>\bar\psi(z)$ and $N_{\{\bar t_1\}\times\mathcal{C}_1}(s,z) = \mathbb{R} \times N_{\mathcal{C}_1}(z)$ for all $(s,z) \in \mathbb{R}^{1+n}$. Hence, condition (5.4.3) reduces to

$$\partial^{>}\bar{\psi}(\bar{x}(0)) \cap (-N_{\mathfrak{C}_1}(\bar{x}(0))) = \emptyset. \tag{5.4.5}$$

Proposition 5.4.3. *Assume Hypothesis 5.2.1 and let* $\mathfrak{Z} := (\bar{T}, \bar{\omega}^0, \bar{\omega}, \bar{u}, \bar{x}^0, \bar{x}, \bar{\nu})$ *be a feasible process which is an extremal for* $(\mathscr{P}_e)$*. If* $\mathfrak{Z}$ *satisfies Hypothesis 5.4.1, then any set of multipliers* $(p_0, p, \pi, \lambda, \mu, m_0, m)$ *that meets the conditions of Definition 5.2.3 satisfies the following strengthened non-triviality condition*

$$\begin{aligned} &\|q_0\|_{L^\infty} + \|q\|_{L^\infty} + \mu(]0, \bar{T}]) + \lambda \neq 0 \qquad && \textit{if } \bar{x}^0(\bar{T}) = \bar{x}^0(0), \\ &\|q\|_{L^\infty} + \mu(]0, \bar{T}]) + \lambda \neq 0 && \textit{if } \bar{x}^0(\bar{T}) > \bar{x}^0(0), \end{aligned} \tag{5.4.6}$$

for (q_0, q) *as in* (5.2.16).

We omit the proof of Proposition 5.4.3, as it is very similar to that of Proposition 3.2.2.

5.4.1 Constraint qualification for normality

We now provide some sufficient conditions in the form of constraint and endpoint qualifications to guarantee normality. Given a feasible process $(\bar{T}, \bar{\omega}^0, \bar{\omega}, \bar{u}, \bar{x}^0, \bar{x}, \bar{\nu})$, from now on we set

$$\bar{F}(t) := F(\bar{x}^0(t), \bar{x}(t), \bar{\omega}^0(t), \bar{\omega}(t), \bar{u}(t)) \qquad \forall t \in [0, \bar{T}]. \tag{5.4.7}$$

Hypothesis 5.4.4. We say that a feasible process $(\bar{T}, \bar{\omega}^0, \bar{\omega}, \bar{u}, \bar{x}^0, \bar{x}, \bar{\nu})$ satisfies the *first constraint qualification for normality* if for every $t \in]0, \bar{T}]$ such that $\psi(\bar{x}^0(t), \bar{x}(t)) = 0$ there exist ε, $\delta > 0$ and a measurable control $(\tilde{\omega}_t, \hat{u}_t) := (\tilde{\omega}, \hat{u})$ taking values in $(\mathfrak{W} \cap \partial\mathbb{B}) \times U$, satisfying

$$\max_{(\xi_0, \xi) \in \partial^* \psi(\bar{x}^0(t), \bar{x}(t))} \xi \cdot \Big[F((\bar{x}^0, \bar{x})(t), (\bar{\omega}^0, \hat{\omega}, \hat{u})(t')) - F((\bar{x}^0, \bar{x})(t), (\bar{\omega}^0, \bar{\omega}, \bar{u})(t')) \Big] < -\delta, \tag{5.4.8}$$

where $\hat{\omega} := (1 - \bar{\omega}^0)\tilde{\omega}$, for a.e. $t' \in E(t, \varepsilon)$ defined as follows

$$E(t, \varepsilon) := \Big\{ t' \in [t - \varepsilon, t] \cap [0, \bar{T}] : \max_{(\xi_0, \xi) \in \partial^* \psi(\bar{x}^0(t'), \bar{x}(t'))} [\xi_0 \bar{\omega}^0(t') + \xi \cdot \bar{F}(t')] \geq 0 \Big\}.$$

Hypothesis 5.4.5. We say that a feasible process $(\bar{T}, \bar{\omega}^0, \bar{\omega}, \bar{u}, \bar{x}^0, \bar{x}, \bar{\nu})$ satisfies the *second constraint qualification for normality* if for any $t \in [0, \bar{T}[$ such that $\psi(\bar{x}^0(t), \bar{x}(t)) = 0$ there exist ε, $\delta > 0$ and a measurable control $(\tilde{\omega}, \hat{u}) := (\tilde{\omega}_t, \hat{u}_t)$ taking values in $(\mathfrak{W} \cap \partial\mathbb{B}) \times U$, satisfying

$$\min_{(\xi^0,\xi)\in\partial^*\psi(\bar{x}^0(t),\bar{x}(t))} \xi \cdot \Big[F((\bar{x}^0,\bar{x})(t),(\bar{\omega}^0,\hat{\omega},\hat{u})(t')) - F((\bar{x}^0,\bar{x})(t),(\bar{\omega}^0,\bar{\omega},\bar{u})(t'))\Big] > \delta, \tag{5.4.9}$$

where $\hat{\omega} := (1 - \bar{\omega}^0)\tilde{\omega}$, for a.e. $t' \in \mathcal{E}(t, \varepsilon)$ defined as as follows

$$\mathcal{E}(t,\varepsilon) := \Big\{t' \in [t, t+\varepsilon] \cap [0,\bar{T}] : \min_{(\xi_0,\xi)\in\partial^*\psi(\bar{x}^0(t'),\bar{x}(t'))} [\xi_0\bar{\omega}^0(t') + \xi \cdot \bar{F}(t')] \le 0\Big\}.$$

Hypothesis 5.4.5 is at our knowledge new, while a version of Hypothesis 5.4.4 was first introduced in [57], as an adaptation to impulsive optimal control of a condition due to [34]. The first and the second constraint qualifications for normality are respectively inward/outward pointing conditions at the boundary of the state constraint which involve the minimizer but have to be satisfied just on a subset of instants at which the optimal trajectory has an outward/inward pointing velocity.

We can now establish a first sufficient condition for the normality of the extremals.

Theorem 5.4.6. *Assume Hypothesis 5.2.1, let* $\mathfrak{Z} := (\bar{T}, \bar{\omega}^0, \bar{\omega}, \bar{u}, \bar{x}^0, \bar{x}, \bar{\nu})$ *be a feasible process which is an extremal for* $(\mathscr{P}_e)$, *and let* $(p_0, p, \lambda, \pi, \mu, m_0, m)$ *be a multiplier that meets the conditions of Definition 5.2.3 and the strengthened nontriviality condition* (5.4.6). *Then,*

(i) *if* $\mathfrak{Z}$ *satisfies Hypothesis 5.4.4, one has*

$$\begin{aligned} &|q_0(\bar{T})| + |q(\bar{T})| + \lambda \neq 0 && \textit{if } \bar{x}^0(\bar{T}) = \bar{x}^0(0), \\ &|q(\bar{T})| + \lambda \neq 0 && \textit{if } \bar{x}^0(\bar{T}) > \bar{x}^0(0) \end{aligned} \tag{5.4.10}$$

In particular, if $\mathrm{proj}_{(t_2,x_2)} N_{\mathfrak{C}}(\bar{x}^0(0), \bar{x}(0), \bar{x}^0(\bar{T}), \bar{x}(\bar{T})) = \{(\xi_{t_2}, 0)\}$ *and* $\xi_{t_2} = 0$ *whenever* $\bar{x}^0(\bar{T}) = \bar{x}^0(0)$, *then* $\lambda \neq 0$;

(ii) *if* $\mathfrak{Z}$ *satisfies Hypothesis 5.4.5, one has*

$$\begin{aligned} &|q_0(0)| + |q(0)| + \lambda \neq 0 && \textit{if } \bar{x}^0(\bar{T}) = \bar{x}^0(0), \\ &|q(0)| + \lambda \neq 0 && \textit{if } \bar{x}^0(\bar{T}) > \bar{x}^0(0). \end{aligned} \tag{5.4.11}$$

In particular, if $\mathrm{proj}_{(t_1,x_1)} N_{\mathcal{C}}(\bar{x}^0(0), \bar{x}(0), \bar{x}^0(\bar{T}), \bar{x}(\bar{T})) = \{(\xi_{t_1}, 0)\}$ *and* $\xi_{t_1} = 0$ *whenever* $\bar{x}^0(\bar{T}) = \bar{x}^0(0)$*, then* $\lambda \neq 0$.

Proof. We limit ourselves to give the proof of Theorem 5.4.6,(ii), as the proof of Theorem 5.4.6,(i) can be deduced by making obvious changes. First of all we observe that for a feasible process $\mathfrak{Z} = (\bar{T}, \bar{\omega}^0, \bar{\omega}, \bar{u}, \bar{x}^0, \bar{x}, \bar{\nu})$ with $\psi(\bar{x}^0(\bar{t}\,), \bar{x}(\bar{t}\,)) = 0$ for some $\bar{t} \in]0, \bar{T}]$, we can assume that there exists some $\bar{\varepsilon}_1 > 0$ sufficiently small such that $\ell(\mathcal{E}(\bar{t}, \varepsilon)) > 0$ for all $\varepsilon \in]0, \bar{\varepsilon}_1]$. Indeed, if $\ell(\mathcal{E}(\bar{t}, \varepsilon)) = 0$ for some small $\varepsilon > 0$, then $\min_{(\xi_0,\xi) \in \partial^* \psi(\bar{x}^0(t'), \bar{x}(t'))} \left[\xi_0 \bar{\omega}^0(t') + \xi \cdot \bar{F}(t')\right] > 0$ for a.e. $t' \in [\,\bar{t}, \bar{t} + \varepsilon]$. But then, the function $\mathcal{G} := \psi \circ (\bar{x}^0, \bar{x})$ is differentiable a.e. $t \in [\,\bar{t}, \bar{t} + \varepsilon]$ and verifies

$$\begin{aligned} \frac{d\mathcal{G}}{dt}(t) &\geq \min_{(\xi_0,\xi) \in \partial^C \psi(\bar{x}^0(t), \bar{x}(t))} \left[\xi_0 \frac{d\bar{x}^0}{dt}(t) + \xi \cdot \frac{d\bar{x}}{dt}(t)\right] \\ &= \min_{(\xi_0,\xi) \in \partial^* \psi(\bar{x}^0(t), \bar{x}(t))} \left[\xi_0 \bar{\omega}^0(t) + \xi \cdot \bar{F}(t)\right] > 0, \end{aligned}$$

since the scalar product is bilinear. Thus, for all $t \in]\bar{t}, \bar{t} + \varepsilon]$ one has

$$\psi(\bar{x}^0(t), \bar{x}(t)) = \psi(\bar{x}^0(t), \bar{x}(t)) - \psi(\bar{x}^0(\bar{t}\,), \bar{x}(\bar{t}\,)) = \int_{\bar{t}}^{t} \frac{d\mathcal{G}}{dt}(t')\, dt' > 0,$$

in contradiction with the feasibility of $\mathfrak{Z}$. Furthermore, by the bilinearity of the scalar product, in Hypothesis 5.4.5 one can replace $\partial^* \psi(\bar{x}^0(t), \bar{x}(t))$ with $\partial^C \psi(\bar{x}^0(t), \bar{x}(t))$. In particular, all the conditions in it are satisfied by any $(\xi_0, \xi) \in \partial^> \psi(\bar{x}^0(t), \bar{x}(t))$, since $\partial^> \psi(\bar{x}^0(t), \bar{x}(t)) \subseteq \partial^C \psi(\bar{x}^0(t), \bar{x}(t))$.

By standard truncation and mollification arguments, we can assume that ψ is Lipschitz continuous, with Lipschitz constant $L_{\psi} > 0$, and that F and its limiting subdifferentials are bounded by some constant $L_F > 0$.
By assumption, $(\bar{T}, \bar{\omega}^0, \bar{\omega}, \bar{u}, \bar{x}^0, \bar{x}, \bar{\nu})$ has a set of multipliers $(p_0, p, \pi, \lambda, \mu, m_0, m)$ verifying the strengthened non-triviality condition (5.4.6). Let us first assume that $\bar{x}^0(\bar{T}) > \bar{x}^0(0)$ and suppose by contradiction that

$$q(0) = 0, \qquad \lambda = 0. \tag{5.4.12}$$

Set

$$\bar{t} := \sup\{t \in [0, \bar{T}] : \quad \mu(]0, t]) = 0\}.$$

Obviously, one has $\psi(\bar{x}^0(\bar{t}\,), \bar{x}(\bar{t}\,)) = 0$. Observe that $\bar{t} < \bar{T}$. Indeed, if not, $\mu(]0, \bar{T}]) = 0$. But in this case $q(t) = p(t) + \mu(\{0\})m(0)$, so that it is absolutely continuous and by the adjoint equation with initial condition $q(0) = 0$ it follows that $q \equiv 0$. Precisely, by (2.4.3), we have

$$|q(t)| \leq \int_0^t |\dot{q}(t')|\; dt' = \int_0^t |\dot{p}(t')|\; dt' \leq L_F \int_0^t |q(t')|\, dt', \qquad q(0) = 0,$$

which implies that $q \equiv 0$ by Gronwall's Lemma 2.2.3. Since $\lambda = 0$ by (5.4.12), this is in contradiction with the first relation in (5.4.6). When $\bar{x}^0(\bar{T}) = \bar{x}^0(0)$ and we assume by contradiction that

$$q_0(0) = 0, \qquad q(0) = 0, \qquad \lambda = 0, \tag{5.4.13}$$

the value $\bar{t}$ defined as above is still strictly smaller than $\bar{T}$, since otherwise $\mu(]0, \bar{T}]) = 0$, so that $(q_0, q) \equiv 0$, again by the adjoint equation. In view of (5.4.13), this yields contradiction with the second relation in (5.4.6).

From now on, the proof is the same for both cases. Introduce

$$(y_0, y)(t) := \big(p_0(t) + m_0(0)\mu(\{0\}), p(t) + m(0)\mu(\{0\})\big),$$

so that, for any $t \in [0, \bar{T}[$,

$$\begin{aligned}(q_0(t), q(t)) &= \Big(p_0(t) + \int_{[0,t[} m_0(t')\mu(dt')\,,\; p(t) + \int_{[0,t[} m(t')\mu(dt')\Big) \\ &= \Big(y_0(t) + \int_{]0,t[} m_0(t')\mu(dt')\,,\; y(t) + \int_{]0,t[} m(t')\mu(dt')\Big).\end{aligned} \tag{5.4.14}$$

By the adjoint equation, (y_0, y) verifies

$$\begin{cases} -(\dot{y}_0(t), \dot{y}(t)) \in \operatorname{co}\partial_{t,x}(q(t)\cdot \bar{F}(t)) = \operatorname{co}\partial_{t,x}\Big(y(t)\cdot\bar{F}(t) + \int_{]0,t[} m(t')\mu(dt')\cdot \bar{F}(t)\Big) \\ y(0) = 0, \quad (\text{and } y_0(0) = 0, \;\; \text{if } \bar{x}^0(\bar{T}) = \bar{x}^0(0)). \end{cases} \tag{5.4.15}$$

Since the integral on the right hand side is identically zero in $]0, \bar{t}\,[$, arguing as above we derive that $y(t) = 0$ and therefore $q(t) = 0$ for all $t \in [0, \bar{t}[$, by continuity. Moreover, Gronwall's Lemma 2.2.3 implies that $|y(t)| \leq C\,\mu([\bar{t},t[)$ for all $t \in [\bar{t},\bar{T}[$, for some $C > 0$, so that

$$|q(t)| \leq |y(t)| + L_{\psi}\mu([\bar{t},t[) \leq (C + L_{\psi})\mu([\bar{t},t[) \qquad \forall t \in [\bar{t},\bar{T}[. \tag{5.4.16}$$

As a consequence of (5.4.14), for every $t \in [\bar{t},\bar{T}]$ one gets $q(t) = y(t) + \int_{[\bar{t},t[} m(t')\mu(dt')$, and (5.4.15), (5.4.16) imply

$$\Big|q(t) - \int_{[\bar{t},t[} m(t')\mu(dt')\Big| \leq L_{F} \int_{\bar{t}}^{t} |q(t')|\,dt' \leq \bar{C}\,\mu([\bar{t},t[)(t - \bar{t}\,), \tag{5.4.17}$$

where $\bar{C} := L_{F}(L_{\psi} + C)$. By the upper semicontinuity of $\partial^{*}\psi(\cdot,\cdot)$ and Hypothesis 5.2.1, if we add and subtract '$F((\bar{x}^{0},\bar{x},\hat{\omega}^{0},\hat{\omega},\hat{u})(t'))$' and '$\bar{F}(t')$' in (5.4.9), we deduce that Hypothesis 5.4.5 implies that there exist $\bar{\varepsilon}$, $\bar{\delta} > 0$ and a measurable control $(\tilde{\omega},\hat{u}) : [0,\bar{T}] \to (\mathfrak{W} \cap \partial\mathbb{B}) \times W$, verifying for all $(\xi_0,\xi) \in \partial^{*}\psi(\bar{x}^{0}(t),\bar{x}(t))$ with $t \in]\bar{t},\bar{t}+\bar{\varepsilon}[\cap[0,\bar{T}]$:

$$\xi \cdot \left[F((\bar{x}^{0},\bar{x},\hat{\omega}^{0},\hat{\omega},\hat{u})(t')) - \bar{F}(t')\right] > \bar{\delta}, \quad \text{for a.e. } t' \in \mathcal{E}(\bar{t},\bar{\varepsilon}), \tag{5.4.18}$$

where $(\hat{\omega}^{0}(t),\hat{\omega}(t),\hat{u}(t)) := (\bar{\omega}^{0}(t),(1-\bar{\omega}^{0}(t))\tilde{\omega}(t),\hat{u}(t))$ for a.e. $t \in [0,\bar{T}]$. Observe that, being $\hat{\omega}^{0} \equiv \bar{\omega}^{0}$, one has $|\hat{\omega}| = 1 - \bar{\omega}^{0} = |\bar{\omega}|$ a.e. As observed at the beginning of the proof, $\ell(\mathcal{E}(\bar{t},\bar{\varepsilon})) > 0$ for any $\bar{\varepsilon} > 0$ sufficiently small, and (5.4.18) is valid for any $(\xi_0,\xi) \in \partial^{>}\psi(\bar{x}^{0}(t),\bar{x}(t))$. On the other hand, by the maximality condition (5.2.13) it follows that, for a.e. $t \in]\bar{t},\bar{t}+\bar{\varepsilon}[\cap[0,\bar{T}]$, it holds

$$\begin{aligned} q_0(t)\left(\hat{\omega}^{0}(t) - \bar{\omega}^{0}(t)\right) + \pi(|\hat{\omega}(t)| - |\bar{\omega}(t)|) + q(t)\left[F((\bar{x}^{0},\bar{x},\hat{\omega}^{0},\hat{\omega},\hat{u})(t)) - \bar{F}(t)\right] \\ = q(t)\left[F((\bar{x}^{0},\bar{x},\hat{\omega}^{0},\hat{\omega},\hat{u})(t)) - \bar{F}(t)\right] \leq 0. \end{aligned} \tag{5.4.19}$$

Putting together (5.4.17), (5.4.18), and (5.4.19) we get the desired contradiction. Indeed, for $\bar{\varepsilon} > 0$ small enough, for any $t' \in \mathcal{E}(\bar{t},\bar{\varepsilon})$, one has

$$0 \geq q(t')\left[F((\bar{x}^{0},\bar{x},\hat{\omega}^{0},\hat{\omega},\hat{u})(t')) - \bar{F}(t')\right]$$

$$
\begin{aligned}
&= \Big(q(t') - \int_{[\bar t, t'[} m(\tau)\mu(d\tau) + \int_{[\bar t, t'[} m(\tau)\mu(d\tau)\Big) \left[F((\bar x^0, \bar x, \hat\omega^0, \hat\omega, \hat u)(t')) - \bar F(t')\right] \\
&\geq \int_{[\bar t, t'[} m(\tau) \left[F((\bar x^0, \bar x, \hat\omega^0, \hat\omega, \hat u)(t')) - \bar F(t')\right] \mu(d\tau) - 2L_F \bar C\, \mu([\bar t, t'[)(t' - \bar t\,) \\
&\geq \mu([\bar t, t'[) \left[\bar\delta - 2L_F\, \bar C\,(t' - \bar t\,)\right] > 0
\end{aligned}
$$

for $\bar\varepsilon > 0$ sufficiently small. This concludes the proof. □

Remark 5.4.7. As discussed in [57, Remark 4.4], the statement of Theorem 5.4.6 holds true even if Hypotheses 5.4.4 and 5.4.5 are replaced, respectively, with the following simpler conditions:

> **(IPFCn)**$_b$ for every $t \in]0, \bar T]$ such that $(\bar x^0(t), \bar x(t)) \in \partial\Omega$, one has $\psi \in C^1$ on a neighborhood of $(\bar x^0(t), \bar x(t))$ and there exists $\delta > 0$ satisfying
>
> $$\inf_{v\in U} \nabla_x\psi(\bar x^0(t), \bar x(t)) \cdot f(\bar x^0(t), \bar x(t), v) < -\delta,$$
> $$\inf_{w\in\mathfrak{W}\cap\partial\mathbb{B}} \nabla_x\psi(\bar x^0(t), \bar x(t)) \cdot \sum_{i=1}^m g_i(\bar x^0(t), \bar x(t)) w^i < -\delta;$$
>
> **(IPFCn)**$_f$ for every $t \in [0, \bar T[$ such that $(\bar x^0(t), \bar x(t)) \in \partial\Omega$, one has $\psi \in C^1$ on a neighborhood of $(\bar x^0(t), \bar x(t))$ and there exists $\delta > 0$ satisfying
>
> $$\sup_{v\in U} \nabla_x\psi(\bar x^0(t), \bar x(t)) \cdot f(\bar x^0(t), \bar x(t), v) > \delta,$$
> $$\sup_{w\in\mathfrak{W}\cap\partial\mathbb{B}} \nabla_x\psi(\bar x^0(t), \bar x(t)) \cdot \sum_{i=1}^m g_i(\bar x^0(t), \bar x(t)) w^i > \delta.$$

Remark 5.4.8. It is clear from the proof that Theorem 5.4.6 still holds in the special case the cost function Φ does not depend on $\nu(T)$, $\bar\nu(\bar T) < K$ (so that $\pi = 0$ for any set of multiplier $(p_0, p, \lambda, \pi, \mu, m_0, m)$ for the extremal $\mathfrak{Z}$) and Hypothesis 5.4.4 [resp. Hypothesis 5.4.5] is replaced by the following condition: *for every* $t \in]0, \bar T]$ [resp. $t \in [0, \bar T[$] *such that* $\psi(\bar x^0(t), \bar x(t)) = 0$ *there exist* ε, $\delta > 0$ *and controls* $(\hat\omega^0_t, \hat\omega_t, \hat u_t) := (\hat\omega^0, \hat\omega, \hat u) \in \mathcal{W}(\bar T) \times \mathcal{U}(\bar T)$ *such that, for any* $(\xi_0, \xi) \in \partial^*\psi(\bar x^0(t), \bar x(t))$ *and a.e.* $t' \in E(t, \varepsilon)$ [resp. $t' \in \mathcal{E}(t, \varepsilon)$], *it holds* [resp. $> \delta$]

$$\xi_0[\hat\omega^0(t') - \bar\omega^0(t')] + \xi \cdot \Big[F((\bar x^0, \bar x)(t), (\hat\omega^0, \hat\omega, \hat u)(t')) - F((\bar x^0, \bar x)(t), (\bar\omega^0, \bar\omega, \bar u)(t'))\Big] < -\delta.$$

5.4.2 Target qualification for normality

Theorem 5.4.6 implies nondegenerate normality when essentially the endpoint constraint either at the final or at the initial position is inactive. We provide below some sufficient conditions to guarantee normality even in some situations where initial and final positions lay on the boundary of the endpoint constraint.

Hypothesis 5.4.9. We say that a feasible process $(\bar{T}, \bar{\omega}^0, \bar{\omega}, \bar{u}, \bar{x}^0, \bar{x}, \bar{\nu})$ satisfies the *first endpoint qualification for normality* if there exists $\varepsilon > 0$ such that $\psi(\bar{x}^0(t), \bar{x}(t)) < 0$ for each $t \in [\bar{T} - \varepsilon, \bar{T}[$ and one among the following conditions (a), (b) holds true:

(a) The following condition holds

$$\big(-\mathrm{proj}_{(t_2,x_2)} N_{\mathfrak{C}}(\bar{x}^0(0), \bar{x}(0), \bar{x}^0(\bar{T}), \bar{x}(\bar{T})) \setminus \{0_{1+n}\}\big) \cap \partial^{>}\psi(\bar{x}^0(\bar{T}), \bar{x}(\bar{T})) = \emptyset \tag{5.4.20}$$

and for any

$$(\xi_{t_2}, \xi_{x_2}) \in \mathrm{proj}_{(t_2,x_2)} N_{\mathfrak{C}}(\bar{x}^0(0), \bar{x}(0), \bar{x}^0(\bar{T}), \bar{x}(\bar{T})) + [0, +\infty[\cdot \partial^{>}\psi(\bar{x}^0(\bar{T}), \bar{x}(\bar{T}))$$

one has

$$\min_{v \in U} \big[\xi_{x_2} \cdot f(\bar{x}^0(\bar{T}), \bar{x}(\bar{T}), v) + \xi_{t_2}\big] < 0 \qquad \text{if } (\xi_{t_2}, \xi_{x_2}) \neq (0,0); \tag{5.4.21}$$

(b) One has $\bar{x}^0(\bar{T}) > \bar{x}^0(0)$, $\bar{\nu}(\bar{T}) < K$, and the following condition holds

$$\Big(- \mathrm{proj}_{x_2} N_{\mathfrak{C}}(\bar{x}^0(0), \bar{x}(0), \bar{x}^0(\bar{T}), \bar{x}(\bar{T})) \setminus \{0_n\}\Big) \cap \mathrm{proj}_x \partial^{>}\psi(\bar{x}^0(\bar{T}), \bar{x}(\bar{T})) = \emptyset. \tag{5.4.22}$$

Moreover, for any

$$(\xi_{t_2}, \xi_{x_2}) \in \mathrm{proj}_{(t_2,x_2)} N_{\mathfrak{C}}(\bar{x}^0(0), \bar{x}(0), \bar{x}^0(\bar{T}), \bar{x}(\bar{T})) + [0, +\infty[\cdot \partial^{>}\psi(\bar{x}^0(\bar{T}), \bar{x}(\bar{T}))$$

with $\xi_{x_2} \neq 0$, one has

$$\min_{w \in \mathfrak{W} \cap \partial \mathbb{B}} \left[\xi_{x_2} \cdot \left(\sum_{j=1}^{m} g_j(\bar{x}^0(\bar{T}), \bar{x}(\bar{T})) w^j\right)\right] < 0. \tag{5.4.23}$$

Hypothesis 5.4.10. We say that a feasible process $(\bar{T}, \bar{\omega}^0, \bar{\omega}, \bar{u}, \bar{x}^0, \bar{x}, \bar{\nu})$ satisfies the *second endpoint qualification for normality* if there exists $\varepsilon > 0$ such that $\psi(\bar{x}^0(t), \bar{x}(t)) < 0$ for each $t \in]0, \varepsilon]$ and one among the following conditions (a), (b) holds true:

(a) The following condition holds

$$\big(-\mathrm{proj}_{(t_1,x_1)} N_{\mathcal{C}}(\bar{x}^0(0), \bar{x}(0), \bar{x}^0(\bar{T}), \bar{x}(\bar{T})) \setminus \{0_{1+n}\}\big) \cap \partial^{>}\psi(\bar{x}^0(0), \bar{x}(0)) = \emptyset \tag{5.4.24}$$

and for any

$$(\xi_{t_1}, \xi_{x_1}) \in \mathrm{proj}_{(t_1,x_1)} N_{\mathcal{C}}(\bar{x}^0(0), \bar{x}(0), \bar{x}^0(\bar{T}), \bar{x}(\bar{T})) + [0, +\infty[\cdot \partial^{>}\psi(\bar{x}^0(0), \bar{x}(0))$$

one has

$$\max_{v \in U} \big[\xi_{x_1} \cdot f(\bar{x}^0(0), \bar{x}(0), v) + \xi_{t_1}\big] > 0 \qquad \text{if } (\xi_{t_1}, \xi_{x_1}) \neq (0, 0); \tag{5.4.25}$$

(b) One has $\bar{x}^0(\bar{T}) > \bar{x}^0(0)$, $\bar{\nu}(\bar{T}) < K$, and the following condition holds

$$\Big(- \mathrm{proj}_{x_1} N_{\mathcal{C}}(\bar{x}^0(0), \bar{x}(0), \bar{x}^0(\bar{T}), \bar{x}(\bar{T})) \setminus \{0_n\}\Big) \cap \mathrm{proj}_x \partial^{>}\psi(\bar{x}^0(0), \bar{x}(0)) = \emptyset. \tag{5.4.26}$$

Moreover, for any

$$(\xi_{t_1}, \xi_{x_1}) \in \mathrm{proj}_{(t_1,x_1)} N_{\mathcal{C}}(\bar{x}^0(0), \bar{x}(0), \bar{x}^0(\bar{T}), \bar{x}(\bar{T})) + [0, +\infty[\cdot \partial^{>}\psi(\bar{x}^0(0), \bar{x}(0))$$

with $\xi_{x_1} \neq 0$, one has

$$\max_{w \in \mathfrak{W} \cap \partial \mathbb{B}} \left[\xi_{x_1} \cdot \sum_{j=1}^{m} g_j(\bar{x}^0(0), \bar{x}(0)) w^j \right] > 0. \tag{5.4.27}$$

Hypothesis 5.4.9 generalizes endpoint constraint qualifications considered in [57] for the case with fixed initial point, which were in turn inspired by no gap conditions in [58, 5]. Instead, Hypothesis 5.4.10 has been introduced for the first time in [37].

We do observe that both conditions (5.4.20), (5.4.22) [resp., (5.4.24), (5.4.26)] are trivially fulfilled whenever $(\bar{x}^0(\bar{T}), \bar{x}(\bar{T})) \in \mathrm{Int}(\mathcal{Q})$ [resp., $(\bar{x}^0(0), \bar{x}(0)) \in \mathrm{Int}(\mathcal{Q})$], as

$\partial^{>}\psi(\bar{x}^0(\bar{T}),\bar{x}(\bar{T}))=\emptyset$ [resp., $\partial^{>}\psi(\bar{x}^0(0),\bar{x}(0))=\emptyset$].

Proposition 5.4.11. *Assume Hypothesis 5.2.1, let* $\mathfrak{Z}:=(\bar{T},\bar{\omega}^0,\bar{\omega},\bar{u},\bar{x}^0,\bar{x},\bar{\nu})$ *be a feasible process which is an extremal for* ($\mathscr{P}_e$)*, and let* $(p_0,p,\lambda,\pi,\mu,m_0,m)$ *be a multiplier that meets the conditions of Definition 5.2.3. Then, if either* (i) *or* (ii) *below holds true, one has* $\lambda\neq 0$*:*

(i) *Hypothesis 5.4.9 is satisfied and the multiplier* $(p_0,p,\lambda,\pi,\mu,m_0,m)$ *fulfills the strengthened non-triviality condition* (5.4.10)*;*

(ii) *Hypothesis 5.4.10 is satisfied and the multiplier* $(p_0,p,\lambda,\pi,\mu,m_0,m)$ *fulfills the strengthened non-triviality condition* (5.4.11)*.*

Proof. Let us prove (i). Assume by contradiction $\lambda=0$. Then the transversality condition (5.2.12) implies that

$$(-q_0(\bar{T}),-q(\bar{T}))=(\xi_{t_2},\xi_{x_2})\in \mathrm{proj}_{(t_2,x_2)}N_{\mathfrak{C}}(\bar{x}^0(0),\bar{x}(0),\bar{x}^0(\bar{T}),\bar{x}(\bar{T})),$$

where $(\xi_{t_2},\xi_{x_2})\neq(0,0)$ and, in particular, $\xi_{x_2}\neq 0$ if $\bar{x}^0(\bar{T})>\bar{x}^0(0)$ by (5.4.10). By Hypothesis 5.4.9, there is some $\varepsilon>0$ such that $\psi(\bar{x}^0(t),\bar{x}(t))<0$ for all $t\in[\bar{T}-\varepsilon,\bar{T}[$. Hence $\mu([\bar{T}-\varepsilon,\bar{T}[)=0$, so that, for any $t\in]\bar{T}-\varepsilon,\bar{T}[$, (q_0,q) is continuous at t and

$$(q_0(t),q(t))=\Big(p_0(t)+\int_{[0,\bar{T}-\varepsilon]}m_0(t')\mu(dt')\,,\,p(t)+\int_{[0,\bar{T}-\varepsilon]}m(t')\mu(dt')\Big).$$

Set $(q_0(\bar{T}^-),q(\bar{T}^-)):=\lim_{s\to\bar{T}^-}(q_0(t),q(t))=(p_0,p)(\bar{T})+\int_{[0,\bar{T}-\varepsilon]}(m_0,m)(t')\mu(dt')$. We get

$$(q_0(\bar{T}^-),q(\bar{T}^-))=\Big(q_0(\bar{T})-m_0(\bar{T})\mu(\{\bar{T}\})\,,\,q(\bar{T})-m(\bar{T})\mu(\{\bar{T}\})\Big)=(-\tilde{\xi}_{t_2},-\tilde{\xi}_{x_2}),$$

where $(\tilde{\xi}_{t_2},\tilde{\xi}_{x_2}):=\Big(\xi_{t_2}+\mu(\{\bar{T}\})m_0(\bar{T}),\xi_{x_2}+\mu(\{\bar{T}\})m(\bar{T})\Big)$. Thus, in particular, the pair $(\tilde{\xi}_{t_2},\tilde{\xi}_{x_2})$ verifies

$$(\tilde{\xi}_{t_2},\tilde{\xi}_{x_2})\in \mathrm{proj}_{(t_2,x_2)}N_{\mathfrak{C}}(\bar{x}^0(0),\bar{x}(0),\bar{x}^0(\bar{T}),\bar{x}(\bar{T}))+[0,+\infty[\cdot\partial^{>}\psi(\bar{x}^0(\bar{T}),\bar{x}(\bar{T})). \tag{5.4.28}$$

The continuity of (q_0,q) on $]\bar{T}-\varepsilon,\bar{T}[$ also implies that the equality (5.2.13) is verified for all $t\in]\bar{T}-\varepsilon,\bar{T}[$. Hence, passing to the limit in it as t tends to $\bar{T}^-$, for any

$(w^0, w, v) \in (\mathfrak{W} \cap \partial\mathbb{B}) \times U$ we obtain

$$\left(\tilde{\xi}_{x_2} \cdot f(\bar{x}^0(\bar{T}), \bar{x}(\bar{T}), v) + \tilde{\xi}_{t_2}\right) w^0 + \tilde{\xi}_{x_2} \cdot \sum_{j=i}^{m} g_j(\bar{x}^0(\bar{T}), \bar{x}(\bar{T})) w^j - \pi|w| \geq 0. \quad (5.4.29)$$

Suppose first that condition (a) in Hypothesis 5.4.9 is satisfied. Then, from (5.4.20) we deduce that $(\tilde{\xi}_{t_2}, \tilde{\xi}_{x_2}) \neq (0,0)$ and choosing $w = 0$ in (5.4.29) we obtain a contradiction to (5.4.21).
If instead condition (b) in Hypothesis 5.4.9 is valid, $\pi = 0$ and (5.4.10) implies that $\xi_{x_2} \neq 0$. In view of (5.4.28) and Hypothesis (5.4.22), this yields $\tilde{\xi}_{x_2} \neq 0$. At this point, we get a contradiction to (5.4.23) by choosing $w^0 = 0$ in (5.4.29). The proof of (ii) is very similar, hence we omit it. □

From Propositions 5.4.3, 5.4.11, and Theorem 5.4.6 we deduce as a corollary the main result of this subsection.

Theorem 5.4.12. *Assume Hypothesis 5.2.1 and let $\mathfrak{Z} := (\bar{T}, \bar{\omega}^0, \bar{\omega}, \bar{u}, \bar{x}^0, \bar{x}, \bar{\nu})$ be a feasible process which is an extremal for $(\mathscr{P}_e)$. If $\mathfrak{Z}$ fulfills either Hypotheses 5.4.1, 5.4.4 and 5.4.9 or Hypotheses 5.4.1, 5.4.5 and 5.4.10, then it is a nondegenerate normal extremal.*

Let us illustrate the preceding theory through some examples.

Example 5.4.13. Consider the linear impulsive optimization problem given by

$$\begin{cases} \qquad\qquad \text{Minimize} \;\; \Phi(x(T)) \\ \text{over } T > 0,\ (\omega^0, \omega) \in L^1([0,T], \mathbb{R}^{1+2}),\ (x^0, x, \nu) \in W^{1,1}([0,T], \mathbb{R}^{1+3+1}) \text{ satisfying} \\ \dot{x}^0(t) = \omega^0(t) \\ \dot{x}(t) = f(x(t))\omega^0(t) + g_1(x(t))\,\omega^1(t) + g_2(x(t))\,\omega^2(t) \\ \dot{\nu}(t) = |\omega(t)| \\ (\omega^0, \omega)(t) \in W \qquad \text{a.e. } t \in [0,T], \\ (x^0(t), x(t)) \in \mathcal{Q} \qquad \forall t \in [0,T], \\ \nu(0) = 0,\ \nu(T) \leq 2,\ (x^0(0), x(0)) = \{0\} \times \mathcal{C}_1,\ (x^0(T), x(T)) \in \{1\} \times \mathcal{C}_2, \end{cases} \quad (5.4.30)$$

in which $\Phi(z) := -z^1$ for any $z = (z^1, z^2, z^3) \in \mathbb{R}^3$, W is as in (5.2.4) for $\kappa = 1$,

$$\mathcal{Q} := \{(s, z) \in \mathbb{R} \times \mathbb{R}^3 : \ -1 \leq z^1 \leq 1 + s, \ -1 \leq z^2 \leq 1, \ -1 \leq z^3 \leq 1\},$$
$$\mathcal{C}_1 := \{z \in \mathbb{R}^3 : \ (z^1 - 1)^2 + (z^2)^2 + (z^3)^2 \leq 1/9, \ z^1 \leq 1\},$$
$$\mathcal{C}_2 := \{z \in \mathbb{R}^3 : \ (z^1 + 1)^2 + (z^2)^2 + (z^3)^2 \leq 1, \ z^1 \geq -1\},$$

and

$$f(z) := \begin{pmatrix} 0 \\ z^2 z^3 \\ 0 \end{pmatrix}, \quad g_1(z) := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad g_2(z) := \begin{pmatrix} 0 \\ -1 \\ -z^1 \end{pmatrix}, \quad \forall z \in \mathbb{R}^3 .$$

A minimizer for (5.4.30) is clearly given by the feasible process $(\bar{T}, \bar{\omega}^0, \bar{\omega}, \bar{x}^0, \bar{x}, \bar{\nu})$, where

$$\bar{T} = 2, \qquad (\bar{\omega}^0, \bar{\omega}) = (\bar{\omega}^0, \bar{\omega}^1, \bar{\omega}^2) = (1, 0, 0)\chi_{[0,1]} + (0, -1, 0)\chi_{]1,2]} , \tag{5.4.31}$$

and one considers the corresponding trajectory with initial state condition $x(0) = (1, 0, 0)$, namely,

$$(\bar{x}^0, \bar{x}, \bar{\nu}) = (\bar{x}^0, \bar{x}^1, \bar{x}^2, \bar{x}^3, \bar{\nu}) = (t, 1, 0, 0, 0)\chi_{[0,1]} + (1, 2 - t, 0, 0, t - 1)\chi_{[1,2]} . \tag{5.4.32}$$

Indeed, for all points $(z^1, z^2, z^3) \in \mathcal{C}_2$ one has $z^1 \in [-1, 0]$, so that $\bar{x}^1(2) = 0$ is the minimum admissible value of the cost function Φ. It is not difficult to check that this process fulfills Hypotheses 5.4.1, 5.4.4 and 5.4.9. In fact Hypothesis 5.4.1 is fulfilled since in a neighborhood of $(\bar{x}^0(0), \bar{x}(0)) = (0, 1, 0, 0)$ the state constraint function is represented by $\psi(s, z) = z^1 - s - 1$, so that $\partial^{>}\psi(0, 1, 0, 0) = \{(-1, 1, 0, 0)\}$, while $N_{\{0\}\times\mathcal{C}_1}(1, 0, 0) = \mathbb{R} \times \mathbb{R}_{\geq 0} \times \{0\} \times \{0\}$; Hypothesis 5.4.4 is trivially satisfied as the minimizer lays in the interior of the state constraints whenever $t \in]0, T]$; while Hypothesis 5.4.9,(b) is met since $(\bar{x}^0(\bar{T}), \bar{x}(\bar{T})) \in \mathrm{Int}(\mathcal{Q})$ and (5.4.23) is satisfied if we choose $w = (-1, 0)$, as $N_{\mathcal{C}_2}(\bar{x}(\bar{T})) = \mathbb{R}_{\geq 0} \times \{0\} \times \{0\}$. Accordingly, it is a nondegenerate normal extremal for problem (5.4.30).

Next example shows that an extremal can be normal even if the sufficient conditions presented in this subsection are not met.

Example 5.4.14. Consider again the minimization problem (5.4.30), where $\mathcal{C}_1$ is as above, while the state constraint and the final-point constraint are replaced with

$$\mathcal{Q} := \{(z^1, z^2, z^3) : \ -1 \le z^1 \le 1,\ -1 \le z^2 \le 1,\ -1 \le z^3 \le 1\},$$
$$\mathcal{C}_2 := \{(z^1, z^2, z^3) : \ -1 \le z^1 \le 0,\ 0 \le z^2 \le 1,\ 0 \le z^3 \le 1\},$$

respectively. Then the feasible process $(\bar{T}, \bar{\omega}^0, \bar{\omega}, \bar{x}^0, \bar{x}, \bar{\nu})$ given by (5.4.31), (5.4.32) is still a minimizer for (5.4.30). Nevertheless, as it is easy to check, the minimizer meets Hypothesis 5.4.1, but it does not satisfy any of Hypotheses 5.4.4, 5.4.5, 5.4.9 and 5.4.10 (even if it fulfills the conditions in Remark 5.4.8).

Despite this, $(\bar{T}, \bar{\omega}^0, \bar{\omega}, \bar{x}^0, \bar{x}, \bar{\nu})$ is an extremal since it is a minimizer. Therefore, there exists a set of multipliers $(p_0, p, \pi, \lambda, \mu, m_0, m)$ with $\pi = 0$, since $\nabla_\nu \Phi \equiv 0$ and $\bar{\nu}(2) = 1 < 2$. Also, $m_0 \equiv 0$ as the state constraint does not depend on time, $\mu([0,2]) = \mu([0,1])$, and $m(t) \in \partial_x^{>} \psi(\bar{x}(0))$ μ-a.e. yields $m(t) = (1,0,0)$ μ-a.e. in $[0,1]$. By the adjoint equation it follows that the path $(p_0, p) = (p_0, p_1, p_2, p_3) \equiv (\bar{p}_0, \bar{p}_1, \bar{p}_2, \bar{p}_3)$ is constant. From the transversality conditions

$$\begin{aligned} &(p_0, p_1, p_2, p_3)(0) \in \mathbb{R} \times N_{\mathcal{C}_1}(1,0,0), \\ &-(q_0, q_1, q_2, q_3)(2) \in \lambda\{(0,-1,0,0)\} + \mathbb{R} \times N_{\mathcal{C}_2}(0,0,0), \end{aligned} \tag{5.4.33}$$

where $q_0 \equiv \bar{p}_0$, and $q(t) = (\bar{p}_1 + \mu([0,1]), \bar{p}_2, \bar{p}_3)$ for all $t \in]1,2]$, we derive that $\bar{p}_0 \in \mathbb{R}$, $\bar{p}_1 \ge 0$, $\bar{p}_2 = \bar{p}_3 = 0$, $q_1(2) = \bar{p}_1 + \mu([0,1]) = \lambda - \alpha$ with $\alpha \ge 0$. The maximality condition implies the relations

$$\bar{p}_0 \chi_{[0,1]}(t) = 0, \quad -q_1(t)\chi_{]1,2]}(t) = 0, \tag{5.4.34}$$

from which we deduce that $\bar{p}_0 = 0$ and $\bar{p}_1 + \mu([0,1]) = \lambda - \alpha = 0$. Hence, recalling that $\bar{p}_1 \ge 0$, we get $\bar{p}_1 = \mu([0,1]) = 0$, $\lambda = \alpha \ge 0$. So, the strengthened non-triviality condition $\|p\|_{L^\infty} + \mu([0,2]) + \lambda \ne 0$ implies that $\lambda \ne 0$ and this shows that $(\bar{T}, \bar{\omega}^0, \bar{\omega}, \bar{x}^0, \bar{x}, \bar{\nu})$ is a normal extremal.

Chapter 6

Conclusions and perspectives

In this thesis we have considered a general optimal control problem with endpoint and state constraints for which the existence of minimizers is not guaranteed. Therefore, we have firstly embedded this *strict sense* original problem into an *extended* optimization problem where the (possibly non-closed) control set has been replaced by its closure. Secondly, we have constructed the *relaxed* auxiliary problem by convexifying the sets of velocities of the extended problem. Hence, we have established an equivalence between the occurrence of gap phenomena and the topological and dynamical property of *isolation* of strict sense trajectories, thanks to which we have proved that, if there is a local infimum gap at a feasible relaxed process, then it turns out to be an abnormal extremal. As a consequence, we have deduced that the *normality* of local minimizers – that is, the fact that every set of multipliers has nonzero cost multiplier – is a sufficient condition for no local infimum gap. Moreover, since the notion of *controllability* is exactly the negation of that of isolation, we have also inferred sufficient conditions for the controllability of the original constrained control system to feasible relaxed trajectories.

However, we have explained that, in case the initial state is fixed and the state constraint is active at it, sets of degenerate multipliers – hence, abnormal – always exist. Accordingly, we have provided a suitable nondegeneracy assumption under which we have enhanced our previous results by showing that a sufficient condition for no local infimum gap is represented by the *nondegenerate normality* of local minimizers – namely, the fact that every set of nondegenerate multipliers has nonzero cost multiplier.

At the beginning, we have dealt with free end-time problems with measurable time dependence, hence we have analyzed the special case of fixed end-time problems and, finally, by means of a reparameterization procedure, we have investigated free end-time problems with Lipschitz time dependence, where additional conditions on the extremals can be proved as, for instance, the constancy of the Hamiltonian for autonomous dynamics. Finally, we have discussed the special case of the impulsive extension (and relaxation) of control-polynomial systems with unbounded controls, providing a simpler nondegeneracy assumption and easily verifiable conditions for nondegenerate normality of extremals.

At this point, it is natural to ask what are possible future directions of research on these topics. With regard to this question, we are planning to address a pair of issues in the next future.

On the one hand, we are interested in figuring out wether is possible to improve the *averaged* adjoint inclusion (4.2.1) in the derivation of necessary conditions that have to be fulfilled by strict sense minimizers that are not extended, or relaxed, minimizers as well. This is an open question left in [62], for which we are convinced to have a positive answer when we consider the extended auxiliary problem only, as announced in [39]. In particular, by defining an appropriate distance between processes, we intend to prove that, if a strict sense minimizer is not also an extended minimizer, then there exists a set of multipliers (p, λ, μ, m) that satisfies conditions (4.1.4), (4.1.6), (4.1.8), (4.1.9), (4.1.10), (4.1.12), and (4.1.13), with $\lambda = 0$.

On the other hand, we aim to deal with sufficient conditions for no gap in relation to optimal control problems with delays in the dynamics. Maximum Principles for this kind of problem are already available in the literature (see e.g. [19, 70, 71]), so that the identification of gap-abnormality relations might appear quite straightforward. What has not yet been done is to give a suitable notion of solution and necessary conditions of optimality for truly nonlinear optimal impulse control problems with time delays, and this is precisely what we first aim to do. We point out that this is not in general an easy task, in particular in the case delays appear not only in the drift term and the fast dynamic depends on the state variable. Indeed, some properties of (non-delayed) impulsive control systems do not hold anymore in presence of delays. For instance, in the case of nonnegative scalar controls, it is well known that all sequences of state trajectories, corresponding to sequences of ordinary controls approximating a given impulse control, have the same limit, while this is not longer true for vector-valued controls. Nevertheless, thanks to Richard Vinter, there are examples of impulsive control systems with delays and scalar controls for which the above mentioned property is not valid anymore. Furthermore, the time reparameterization techniques employed in Section 5.2 cannot be adapted directly to the delay setting, since the same reparameterization does not generate a standard time-delay optimal control problem.

Bibliography

[1] Arutyunov A.V., *Optimality conditions. Abnormal and degenerate problems*, Mathematics and its Applications, Kluwer Academic Publishers, Dordrecht, 2000.

[2] Arutyunov A.V., Aseev S.M., *Investigation of the degeneracy phenomenon of the Maximum Principle for optimal control problems with state constraints.* SIAM J. Control Optim., vol. 35, no. 3, pp. 930–952, 1997.

[3] Arutyunov A.V., Aseev S.M., Blagodat-Skikh V.I., *Necessary conditions of the first order in a problem of the optimal control of a differential inclusion with phase constraints*, (russian), translation in Russian Acad. Sci. Sb. Math., vol. 79, pp. 117–139, 1994

[4] Ambrosio L., Fusco N., Pallara D., *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[5] Aronna M.S., Motta M., Rampazzo F., *Infimum gaps for limit solutions*, Set-Valued Var. Anal., vol. 23, no. 1, pp. 3–22, 2015.

[6] Aronna M.S., Rampazzo F., L^1 *limit solutions for control systems*, J. Differential Equations, vol. 258, pp. 954–979, 2015.

[7] Arutyunov A.V., Karamzin D.Y., *A survey on regularity conditions for state-constrained optimal control problems and the non-degenerate maximum principle*, J. Optim. Theory Appl., vol. 184, no. 3, pp. 697–723, 2020.

[8] Arutyunov A., Karamzin D., Pereira F., *A nondegenerate maximum principle for the impulse control problem with state constraints*, SIAM J. Control Optim., vol. 43, no. 5, pp. 1812–1843, 2005.

[9] Arutyunov A., Karamzin D., Pereira F., *Pontryagin's maximum principle for constrained impulsive control problems*, Nonlinear Anal. vol. 75, no. 3, pp. 1045–1057, 2012.

[10] Arutyunov A., Karamzin D., Pereira F., *State constraints in impulsive control problems: Gamkrelidze-like conditions of optimality*, J. Optim. Theory Appl., vol. 166, no. 2, pp. 440–459, 2015.

[11] Arutyunov A.V., Karamzin D.Y., *Non-degenerate necessary optimality conditions for the optimal control problem with equality-type state constraints*, J. Global Optim., vol. 64, no. 4, pp. 623–647, 2016.

[12] Arutyunov A.V., Karamzin D.Y., Pereira F.L., *Investigation of Controllability and Regularity Conditions for State Constrained Problems*, IFAC-PapersOnline, Proceedings of the IFAC Congress in Toulouse, France, pp. 6295–6302, 2017.

[13] Aubin J.P., Cellina A., *Differential inclusions. Set-valued maps and viability theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 264. Springer-Verlag, Berlin, (1984).

[14] Aubin J.P., Frankowska H. *Set-valued analysis*, Systems & Control: Foundations & Applications, 2. Birkhäuser Boston, Inc., Boston, MA, 1990.

[15] Bellman R.E., *Dynamic Programming*, Courier Dover Publications, 1957.

[16] Berkovitz L.D., Medhin N.G., *Nonlinear Optimal Control Theory*, CRC Press, Boca Raton, 2013.

[17] Bressan A., *Hyper-impulsive motions and controllizable coordinates for Lagrangean systems*, Atti Accad. Naz. Lincei, Memorie, Serie VIII, vol. 19, 1991, pp. 197–246, 1991.

[18] Bacciotti A., Rosier L., *Lyapunov functions and stability in control theory*, Springer, 2005.

[19] Boccia A., Vinter R.B., *The Maximum Principle for optimal control problems with time delays*, SIAM Journal of Control, vol. 55, no. 5, pp. 2905–2935, 2017.

[20] Bressan A., Piccoli B., *Introduction to the Mathematical Theory of Control*, Series in Applied Mathematics, American Institute of Mathematical Science, 2007.

[21] Bressan A., Rampazzo F., *On differential systems with impulsive controls*, Rendiconti del Seminario Matematico della Università di Padova, vol. 78, pp. 227–235, 1987.

[22] Bressan A., Rampazzo F., *On differential systems with vector-valued impulsive controls*, Boll. Un. Mat. Ital. B, vol. 7, no. 2, pp. 641–656, 1988.

[23] Bressan A., Rampazzo F., *Impulsive control systems with commutative vector fields*, Journal of Opt. Th. and Appl., vol. 71, no. 1, pp. 67–83, 1991.

[24] Bressan A., Rampazzo F., *On differential systems with quadratic impulses and their applications to Lagrangian mechanics*, SIAM J. Contr. Optim., vol. 31, no. 5, pp. 1205–1220, 1993.

[25] Bressan A., Rampazzo F., *Moving constraints as stabilizing controls in classical mechanics*, Arch. Ration. Mech. Anal., vol. 196, pp. 97–141, 2010.

[26] Brezis H., *Functional analysis, Sobolev spaces and partial differential equations.*, Springer, New York, 2011.

[27] Clarke F.H., *Optimization and Nonsmooth Analysis*, Wiley-Interscience, New York, 1983, reprinted as vol. 5 of Classics in Applied Mathematics, SIAM, Philadelphia, 1990.

[28] Clarke F.H., *Necessary conditions in dynamic optimization*, Mem. Amer. Math. Soc., vol. 173, 2005.

[29] Clarke F., Ledyaev Y., Sontag E., Subbotin A., *Asymptotic controllability implies feedback stabilization*, IEEE Trans. Automat. Control, vol. 42, no. 10, pp. 1394–1407, 1997.

[30] Clarke F., Ledyaev Y., Stern R.J., Wolenski P., *Nonsmooth analysis and control theory*, Graduate Texts in Mathematics, 178. Springer-Verlag, New York.

[31] Clarke F.H., Vinter, R.B., *Applications of optimal multiprocesses*, SIAM J. Control Optim., vol. 27, no. 5, pp. 1048–1071, 1989.

[32] Ferreira M.M.A., Vinter R.B., *When is the Maximum Principle for state constrained problems nondegenerate?*, J. Math. Anal. Appl., vol. 187 , no. 2, pp. 438–467, 1994.

[33] Ferreira M.M.A., Fontes F.A.C.C., Vinter R.B., *Nondegenerate necessary conditions for nonconvex optimal control problems with state constraints*, J. Math. Anal. Appl., vol. 233, no. 1, pp. 116–129, 1999.

[34] Fontes F.A.C.C., Frankowska H., *Normality and nondegeneracy for optimal control problems with state contraints*, J. Opt. Theory. Appl., vol. 166, no. 1, pp. 115–136, 2015.

[35] Frankowska H., Tonon D., *Inward pointing trajectories, normality of the Maximum Principle and the non occurrence of the Lavrentieff phenomenon in optimal control under state constraints*, Journal of Convex Analysis, vol. 20, no. 4, pp. 1147–1180, 2013.

[36] Fusco G., *Robust feedback stabilization by means of Lyapunov-like functions determined by Lie brackets*, J. Differential Equations, vol. 287, pp. 88–112, 2021.

[37] Fusco G., Motta M., *No Infimum Gap and Normality in Optimal Impulsive Control Under State Constraints*, Set-Valued Var. Anal., vol. 29, no. 2, pp. 519–550, 2021.

[38] Fusco G., Motta M., *Nondegenerate abnormality, controllability, and gap phenomena in optimal control with state constraints*, SIAM J. Control Optim., vol. 60, no. 1, pp. 280–309, 2022.

[39] Fusco G., Motta M., *Strict sense minimizers which are relaxed extended minimizers in general optimal control problems*, Proceedings of the 60th IEEE Conference on Decision and Control, CDC 2021, December 13-15. Austin, Texas, pp. 6000–6005.

[40] Fusco G., Motta M., *Gap phenomena and controllability in free end-time problems with active state constraints*, J. Math. Anal. Appl., vol. 510, no. 2, 2022.

[41] Karamazin D.Y., *Necessary conditions for the minimum in an impulsive optimal control problem*, J.Math. Sci. (N.Y.), vol. 139, no. 6, pp. 7087–7150, 2006.

[42] Karamzin D.Y., de Oliveira V.A., Pereira F., Silva G.N., *On some extension of optimal control theory*, Eur. J. Control, vol. 20, no. 6, pp. 284–291, 2014.

[43] Karamzin D.Y., de Oliveira V.A., Pereira F.L., Silva G.N., *On the properness of an impulsive control extension of dynamic optimization problems.* ESAIM Control Optim. Calc. Var., vol. 21, no. 3, pp. 857–875, 2015.

[44] Kaśkosz B., *Optimal trajectories of generalized control systems with state constraints*, Nonlinear Anal., vol. 10, no. 10, pp. 1105–1121, 1986.

[45] Kaśkosz B., *Extremality, controllability, and abundant subsets of generalized control systems*, J. Optim. Theory Appl., vol. 101, no. 1, pp. 73–108, 1999.

[46] Loewen P.D., Clarke F.H., Vinter B.V., *Differential inclusions with free time*, Ann. de l'I.H.P, sec. C, vol. 5, no. 6, 573–593, 1988.

[47] Lopes S.O., Fontes F.A.C.C., de Pinho M.d.R., *On constraint qualifications for nondegenerate necessary conditions of optimality applied to optimal control problems*, Discrete Contin. Dyn. Syst., vol. 29, no. 2, pp. 559–575, 2011.

[48] Miller B.M., *The method of discontinuous time substitution in problems of the optimal control of impulse and discrete-continuous systems*, (Russian) Avtomat. i Telemekh. no. 12, pp. 3–32, 1993; Translation in *Automat. Remote Control* vol. 54, no. 12, pp. 1727–1750, 1993.

[49] Mitrinovic D.S., Pecaric J.E., Fink A.M., *Inequalities involving functions and their integrals and derivatives*, Kluwer Academic Publishers, Dordrecht-Boston-London, 1991.

[50] Motta M., *Minimum time problem with impulsive and ordinary controls*, Discrete Contin. Dyn. Syst., vol. 38, no. 11, pp. 5781–5809, 2018.

[51] Motta M., Rampazzo F., *Space-time trajectories of non linear systems driven by ordinary and impulsive controls*, Differ. Int. Eq., vol. 8, pp.269–288, 1995.

[52] Motta M., Palladino M., Rampazzo F., *Unbounded control, infimum gaps, and higher order normality*, SIAM J. Control Optim., vol. 60, no. 3, pp. 1436–1462, 2022.

[53] Motta M. and Sartori C., *Minimum time with bounded energy, minimum energy with bounded time*, SIAM J. Control Optim., vol. 42, pp. 789–809, 2003.

[54] Motta M., Sartori C., *On asymptotic exit-time control problems lacking coercivity*, ESAIM Control Optim. Calc. Var. vol. 20, no. 4, pp. 957–982, 2014.

[55] Motta M., Sartori C., *The value function of an asymptotic exit-time optimal control problem*, Non. Diff. Eq. Appl., vol. 22, no. 1, pp. 21–44, 2015.

[56] Motta M., Sartori C., *On L^1 limit solutions in impulsive control*, Discrete Contin. Dyn. Syst. Ser. S, vol. 11, pp. 1201–1218, 2018.

[57] Motta M., Sartori C., *Normality and nondegeneracy of the Maximum Principle in optimal impulsive control under state constraints*, Journal of Optimization Theory and Applications, vol. 185, pp. 44–71, 2020.

[58] Motta M., Rampazzo F., Vinter R.B., *Normality and gap phenomena in optimal unbounded control*, ESAIM: Control, Optimisation and Calculus of Variations, vol. 24, no. 4, pp. 1645–1673, 2018.

[59] Palladino M., Rampazzo F., *A geometrically based criterion to avoid infimum gaps in optimal control*, J. Differential Equations vol. 269, no. 11, pp. 10107–10142, 2020.

[60] Palladino M., Vinter R.B., *Minimizers that are not also relaxed minimizers*, SIAM J. Control. Optim., vol. 52, no. 4, pp. 2164–2179, 2014.

[61] Palladino M., Vinter R.B., *When are minimizing controls also minimizing extended controls?*, Discrete Continuous Dynamical System, vol. 35, no. 9, pp. 4573–4592, 2015.

[62] Palladino M., Vinter R.B., *Regularity of the Hamiltonian along optimal trajectories*, SIAM J. Control Optim., vol. 53, no. 4, pp. 1892–1919, 2015.

[63] Pontryagin L.S., Boltyanskii V.G., Gamkrelidze R.V., Mischenko E.F., *The Mathematical Theory of Optimal Processes*, Wiley, New York, 1962.

[64] Rampazzo F., Sartori C., *Hamilton-Jacobi-Bellman equations with fast gradient-dependence*, Indiana Univ. Math. J., vol. 49, no. 3, pp. 1043–1078, 2000.

[65] Rishel R.W., *An extended Pontryagin principle for control systems whose control laws contain measures*, SIAM Journal of Control, vol. 3, no. 2, pp. 191–2051, 1965.

[66] Rampazzo F., Vinter R.B., *A theorem on existence of neighbouring trajectories satisfying a state constraint, with applications to optimal control*, IMA J. Math. Control Inform., vol. 16, no. 4, pp. 335–351 1999.

[67] Rowland J.D.L., Vinter R.B., *Dynamic optimization problems with free-time and active state constraints*, SIAM J. Control Optim., vol. 31, no. 3, pp. 677–697, 1993.

[68] Sussmann H.J., *Generalized differentials, variational generators, and the maximum principle with state constraints*, Mathematical Control Theory, J. Baillieul and J. C. Willems, Eds., Springer-Verlag, New York, pp.140–198, 1998.

[69] Vinter R.B., *Optimal control*, Birkhäuser, Boston, 2000.

[70] Vinter R.B., *State constrained optimal control problems with time delays*, J. Math. Anal. Appl., vol. 457, no. 2, pp. 1696–1712, 2018.

[71] Vinter R.B., *Optimal control problems with time delays: constancy of the hamiltonian*, SIAM Journal of Control, vol. 57, no. 4, pp. 2574–2602, 2019.

[72] Vinter R.B., *Free end-time optimal control problems: conditions for the absence of an infimum gap*, Vietnam Journal of Mathematics, vol. 47, pp. 757–768, 2019.

[73] Vinter R.B., Pereira F.M.F.L., *A Maximum Principle for optimal processes with discontinuous trajectories*, SIAM Journal of Control, vol. 26, no. 1, pp. 205–229, 1988.

[74] Vinter R.B., Zheng H., *Necessary conditions for free end-time measurably time dependent optimal control problems with state constraints*, vol. 8, pp. 11–29, 2000.

[75] Warga J., *Variational problems with unbounded controls*, J. Soc. Indust. Appl. Math. Ser. A Control, vol. 3, pp. 424–438, 1965.

[76] Warga J., *Normal control problems have no minimizing strictly original solutions*, Bull. Amer. Math. Soc., vol. 77, pp. 625–628, 1971.

[77] Warga J., *Optimal Control of Differential and Functional Equations*, Academic Press, New York, 1972.

[78] Warga, J., *Optimization and controllability without differentiability assumptions*, SIAM J. Control and Optimization, vol. 21, pp. 837–855, 1983.

[79] Warga, J., *Controllability, extremality, and abnormality in nonsmooth optimal control*, J. Optim. Theory Appl., vol. 41, no. 1, pp. 239–260, 1983.