The B.E. Journal of Theoretical Economics

Topics

Volume 9, *Issue* 1 2009 *Article* 6

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Recommended Citation

Diego A. Dominguez and Antonio Nicolo (2009) "Envy-Free and Efficient Minimal Rights: Recursive No-Envy," *The B.E. Journal of Theoretical Economics*: Vol. 9: Iss. 1 (Topics), Article 6.

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Abstract

In economics the main efficiency criterion is that of Pareto-optimality. For problems of distributing a social endowment a central notion of fairness is no-envy (each agent should receive a bundle at least as good, according to her own preferences, as any of the other agent's bundle). For most economies there are multiple allocations satisfying these two properties. We provide a procedure, based on distributional implications of these two properties, which selects a single allocation which is Pareto-optimal and satisfies no-envy in two-agent exchange economies. There is no straightforward generalization of our procedure to more than two-agents.

KEYWORDS: no-envy, fair allocation, recursive methods, exchange economies

*Diego Domínguez acknowledges financial support from the Asociación Mexicana de Cultura. Antonio Nicolo acknowledges financial support from MIUR under grant 2007 EENEAX, and is grateful for the hospitality of the University of Rochester. We are grateful for comments by seminar participants at SCW (Montreal) and at ASSET 2008 (European University Institute). We thank an anonymous referee for providing constructive comments and help in improving the contents of this paper. The usual disclaimer applies.

1 Introduction

In the economics literature on fair allocation, a central equity concept is no-envy (Foley, 1967); each agent should receive a bundle at least as good, according to her own preferences, as any of the other agents' bundles.[1] Starting with Varian (1974) different allocation rules have been proposed to select envy-free and efficient allocations in exchange economies, the most prominent being the competitive equilibrium from an equal endowment rule.[2] This rule selects the set of allocations obtained by distributing the aggregate endowment equally between all agents and allowing them to exchange according to competitive equilibrium prices.[3] Nevertheless, there exist many initial distributions which after trading at competitive equilibrium prices lead to envy-free and efficient allocations, and starting from an equal division there exist many possible trades different from the competitive equilibrium trade which lead to envy-free and efficient allocations. Is there a way to select a single allocation based solely on the concept of no-envy and efficiency?

One way to select an envy-free allocation is by means of an iterative procedure, which strengthens the no-envy condition recursively, based on partial assignment of most preferred allocations within the set of envy-free allocations (Baumol, 1982). Unfortunately, this procedure may fail to select an efficient allocation (Philpotts, 1983). In this paper we propose a modification to Baumol's (1982) proposal, based on partial assignment of most preferred allocations within the set of efficient and envy-free allocations, and show that under mild conditions it selects an efficient and envy-free allocation for two-agent exchange economies. Our proposal is also based on strengthening the no-envy condition in a recursive manner, but when doing so we take into account the efficiency requirement. We first find the *"minimal right of each agent"* consisting of the minimal amounts of commodities that she receives at any envy-free and efficient allocation. Then we assign to each agent her minimal right, and define a reduced economy where preferences over the remaining resources are defined taking into account the amount of goods already assigned. Finally, we find the *minimal rights* in the reduced economy, assign them and proceed recursively. We show that, for two-agent economies, (i) assigning to each agent her minimal right guarantees that the final allocation will be envy-free, and (ii) iterated assignment of minimal rights process leads to an (envy-free and) efficient allocation.

Our procedure can be interpreted in the following manner: If all members of society agree that an envy-free and efficient allocation should be selected, they im-

[1] When an allocation satisfies no-envy we say that it is an envy-free allocation.

[2] It is worth mentioning the work of Kolm (1972), who not only looks for envy-free and efficient allocations but also studies the properties of the set of envy-free allocations of an economy.

[3] For a survey of the fair allocation literature, see Thomson (2007).

plicitly agree that each agent has to receive at least her minimal right; thus we can assign to each agent her minimal right. But once minimal rights have been assigned it is only natural to distribute the remaining resources in an envy-free and efficient way, and we can find the minimal rights over the remaining resources, assign them and proceed recursively until minimal rights are zero.

Our procedure can also be interpreted as a non-manipulability property of a distribution rule with respect to assignment of minimal rights. Since each agent agrees that the other should receive her minimal rights, when faced with a specific distribution problem an arbitrator can decide to apply a distribution rule directly, or do it in two steps: (i) assign to each agent her minimal right (over which there is no conflict of interest) and (ii) apply the rule to the remaining resources. In order to avoid manipulability of the procedure it is desirable to ask that the rule select the same allocation no matter which of these two options the arbitrator chooses. It is easy to see that our rule is the only rule which satisfies this non-manipulability property. For bankruptcy problems (O'Neill, 1982), this invariance was introduced by (Curiel, Maschler, and Tijs, 1987). In such problems, an agent's minimal right is the amount of the social endowment not claimed by the other agents, that is, the minimal amount that an agent receives at any efficient distribution of the social endowment (Aumann and Maschler, 1985).[4] The property of *minimal rights first* is an invariance property with respect to the assignment of minimal rights; it is widely accepted and it appears in several characterizations of the main distribution rules in bankruptcy. Formally, minimal rights first states that a rule should recommend the same awards when applied directly to a problem, or when applied to the reduced problem after minimal rights have been assigned. For bankruptcy problems, invariance under minimal rights does not select a single award vector since at the second step minimal rights are not positive.

In contrast to Baumol (1982), our rule selects an efficient allocation because it only considers efficient allocations when defining minimal rights. It selects an envy-free allocations because for two-agent economies assigning minimal rights in the original economy guarantees no-envy of the final allocation.[5]

Our result also contrasts with some previous results in the fair allocation literature where a society may start from a fair (envy-free) allocation, engage in a fair ("envy-free trade") transition path, and end up in an unfair allocation (an allocation where one agent envies another) (Feldman and Kirman, 1974). In our procedure this is not the case; our results show that recursively applying the same fairness transition (assignment of minimal rights) leads to a fair final outcome.

[4]For bankruptcy problems no-envy is not a meaningful concept since agents differ in the claim that they hold over the social endowment.

[5]Efficiency implies that all the resources are distributed among the agents.

The idea of distributing each agent's minimal rights and iterating this procedure until the entire endowment is assigned to the agents resembles the idea of a gradual process which is already present in the bargaining literature. The idea of gradualism in bargaining first appears in the seminal paper of Admati and Perry (1991). More recently, Compte and Jehiel (2003) present a model in which gradualism derives from reciprocal concessions that agents make under the threat of an inefficient termination option. For cake division problems Nicolò and Yu (2008) propose, in a fair division game, an iterated version of the divide and choose rule, previously analyzed by Crawford (1977), in which an envy-free and efficient allocation is reached in equilibrium.

The paper is organized as follows. The next section contains the model. In Section 3 we define minimal rights and some of their properties, Section 4 contains the results for two-agent economies, and in Section 5 we conclude and discuss some possible extensions for economies with more than two agents as well as the competitive equilibrium from equal endowments rule.

2 The model

There is a **social endowment** $\Omega \in \mathbb{R}^m_{++}$ of $M = \{1, ..., m\}$ commodities to be distributed among a set $N = \{A, B\}$ of agents.[6] Each agent can consume non-negative amounts of each commodity; for each agent $i \in N$ her **consumption set** is $X_i = \mathbb{R}^m_+$. We refer to agent i's vector of consumption $x_i \in X_i$ as her **bundle**. Each agent $i \in N$ has a continuous, strictly monotonic, and strictly convex preference relation R_i over her consumption set.[7] For simplicity we assume that preferences satisfy the following property:

Definition 1. Strong decreasing marginal rate of substitution: Let $x, x' \in X_i$ such that $x \neq x'$, and $x \ngeq x'$ and $x \nleq x'$, let p and p' denote any supporting prices of the upper contour sets of R at x and x' respectively. Then, for each commodity $j, k \in M$, such that $x_j \geq x'_j$ and $x_k \leq x'_k$, the relative prices between commodity j and k satisfy $\frac{p_j}{p_k} \leq \frac{p'_j}{p'_k}$.

Even though strong decreasing marginal rates of substitution imply that the greater-than relation between relative prices of commodity j in terms of commodity k is independent of the consumption levels of other commodities, the level of

[6] The set $\mathbb{R}_+$ is the set of non-negative reals and the set $\mathbb{R}_{++}$ is the set of positive reals. Throughout the paper use the following vector inequalities: $x \geqq y \Leftrightarrow$ for each $i \in N$, $x_i \geq y_i$; $x \geq y \Leftrightarrow x \geqq y$ and $x \neq y$; $x > y \Leftrightarrow$ for each $i \in N$, $x_i > y_i$.

[7] Given a preference relation R, we denote strict preference by P and indifference by I.

the relative prices may depend on the consumption levels of other commodities. A sufficient condition for preferences to satisfy strong decreasing marginal rates of substitution is that preferences are both convex and homothetic; but other preferences also satisfy this property, such as strictly convex quasi-linear preferences in the two-commodity case. It is worth noting that this condition is not necessary for our results to hold but, as we later show, it implies that the set of *efficient* allocations defines an increasing curve in the Edgeworth box (see Remark 2), and this latter condition is used in the proof of our main theorem.

The set of all preferences satisfying the above conditions is denoted by $\mathcal{R}$. A **profile of preferences** is $R = (R_A, R_B) \in \mathcal{R}^2$, an **economy** consists of a profile of preferences and a social endowment $(R, \Omega) \in \mathcal{E} \equiv \mathcal{R}^2 \times \mathbb{R}^m_{++}$.

An **allocation** $x = (x_A, x_B) \in \prod_{i\in N} X_i$ assigns to each agent a bundle $x_i \in X_i$. The set of all possible allocations is denoted $X \equiv \prod_{i\in N} X_i$. An allocation $x \in X$ is **feasible** for the economy (R, Ω) if $\sum_{i\in N} x_i \leq \Omega$, that is, the aggregate endowment is sufficient to assign to each agent her bundle. For each economy $(R, \Omega) \in \mathcal{E}$ we denote its set of feasible allocations $Z(R, \Omega)$. A feasible allocation $x \in Z(R, \Omega)$ is **efficient** for the economy (R, Ω) if there is no feasible allocation $x' \in Z(R, \Omega)$ such that, for each $i \in N$, $x'_i \; R_i \; x_i$, and for some $i \in N$, $x'_i \; P_i \; x_i$. For each economy $(R, \Omega) \in \mathcal{E}$ we denote its set of efficient allocations $P(R, \Omega)$. For each agent $i \in N$ the projection of the set $P(R, \Omega)$ onto her consumption space is denoted $P_i(R, \Omega)$; it consists of all bundles $x_i \in X_i$ such that $(x_i, \Omega - x_i) \in P(R, \Omega)$.[8] Throughout the paper we use the following notation: Given a set of allocations $S \subset X$, the set $S_i \subset X_i$ denotes the projection of S onto agent i's consumption space.

An allocation $x = (x_A, x_B)$ satisfies **no-envy** (or is **envy-free**) for the economy (R, Ω) if each agent is at least as well off consuming her bundle as consuming the other agent's bundle, that is, for each $i, j \in N$, $x_i \; R_i \; x_j$. For each economy $(R, \Omega) \in \mathcal{E}$ we denote its set of envy-free allocations $F(R, \Omega)$. The set of envy-free and efficient allocations is denoted $PF(R, \Omega)$ and consists of the intersection of the sets $P(R, \Omega)$ and $F(R, \Omega)$.

We are interested in recommending an allocation to each economy. An **allocation rule** (or just a **rule**) $\varphi : \mathcal{E} \to Z(R, \omega)$,[9] is a function from the set of economies into its set of feasible allocations. A rule is efficient if for each economy it recommends an efficient allocation; it satisfies no-envy if for each economy it recommends an envy-free allocation.

[8]Note that, by strict monotonicity of preferences, if an allocation is efficient it distributes the social endowment fully.

[9]We ask that a rule selects a single feasible allocation; one possibility for generalizing our results to more than two agents is to consider allocation correspondences.

2.1 Preliminary results

We now provide some preliminary results of envy-free and efficient allocations. Our first result is well-known in the literature and was shown by Varian (1974).

Remark 1. For each $(R,\Omega) \in \mathcal{E}$ the set of envy-free and efficient allocations is a non-empty and closed set.

Given strong decreasing marginal rates of substitution, starting from an efficient allocation and moving to another feasible allocation in which each agent obtains a bundle which is no larger nor smaller than her original bundle, each of the agents' supporting prices changes in opposite directions. Thus, the new allocation cannot be efficient, and each agent's set of efficient bundles defines an increasing curve in consumption space.

Remark 2. For each $(R,\Omega) \in \mathcal{E}$, the efficient set defines an increasing curve in the Edgeworth box, that is, if two distinct allocations $(x_A, x_B), (x'_A, x'_B) \in P(R,\Omega)$, then we have (i) $x_A \geq x'_A$ or (ii) $x_A \leq x'_A$.

Proof. Let $(R,\Omega) \in \mathcal{E}$, by contradiction assume that there exist two distinct allocations $(x_A, x_B), (x'_A, x'_B) \in P(R,\Omega)$ such that $x_A \ngeq x'_A$ and $x_A \nleq x'_A$. Let $p, p' \in \mathbb{R}^m_{++}$ denote some supporting prices of R_A at x_A and x'_A respectively, and $q, q' \in \mathbb{R}^m_{++}$ be supporting prices of R_B at x_B and x'_B respectively. Since (x_A, x_B) and (x'_A, x'_B) are efficient, there exists prices such that $p = q$ and $p' = q'$.

First we show that the relative prices between any two commodities that move in opposite directions from x_A to x'_A remain the same. Let $j,k \in M$ be such that $x_{Aj} \geq x'_{Aj}$ and $x_{Ak} \leq x'_{Ak}$. By efficiency the full endowment is consumed, hence we have $x_{Bj} \leq x'_{Bj}$ and $x_{Bk} \geq x'_{Bk}$. By strong decreasing marginal rates of substitution, $\frac{p_j}{p_k} \leq \frac{p'_j}{p'_k}$ and $\frac{q'_j}{q'_k} \leq \frac{q_j}{q_k}$; by efficiency we know that $\frac{p_j}{p_k} = \frac{q_j}{q_k}$ and $\frac{p'_j}{p'_k} = \frac{q'_j}{q'_k}$. Thus, it has to be that $\frac{p_j}{p_k} = \frac{p'_j}{p'_k} = \frac{q'_j}{q'_k} = \frac{q_j}{q_k}$.

Now we show that the relative prices between any two commodities that move in the same direction from x_A to x'_A remain the same. Let $k,l \in M$ be such that $x_{Ak} \geq x'_{Ak}$ and $x_{Al} \geq x'_{Al}$. Let j be such that $x_{Aj} \leq x'_{Aj}$ (since $x_A \ngeq x'_A$ and $x'_A \ngeq x_A$ such j exists). From the previous step, $\frac{p_j}{p_k} = \frac{p'_j}{p'_k}$ and $\frac{p_l}{p_j} = \frac{p'_l}{p'_j}$. Multiplying both equations, we get $\frac{p_l}{p_k} = \frac{p'_l}{p'_k}$.

Hence we have that $p = p' = q = q'$, thus there exists some supporting prices of R_A at x_A which are also supporting prices of R_A at x'_A.

Finally, let $x''_A \in box(x_A, x'_A)$ be such that $x''_A \neq x_A$ and $x''_A \; I_A \; x_A$.[10] Such x''_A exists

[10]For each pair $x, x' \in \mathbb{R}^M M$ the set $box(x,x') = \{y \in \mathbb{R}^M : \text{for each} j \in M,\ \min\{x_j, x'_j\} \leq y_j \leq \max\{x_j, x'_j\}\}$.

by continuity and strict monotonicity of preferences. Let $p'' \in \mathbb{R}^m_{++}$ be supporting prices of R_A at x''_a. By strong decreasing marginal rates of substitution and using the same argument as above, moving from x''_A to x_A and x'_A respectively leads to changes in supporting prices in weakly opposite directions; but given that p supports R_A at both x_A and x'_A then $p'' = p$. Thus $x_A \; I_A \; x''_A$ and the supporting prices of R_A at x''_A also support R_A at x_A, which contradicts strict convexity of preferences. $\square$

As noted earlier, strong decreasing marginal rates of substitution is a sufficient condition for Remark 2 to hold, but it is not a necessary condition, and all our results hold if we only require that the efficient set satisfies this property.

3 Envy-free and efficient minimal rights

Given an economy $(R,\Omega) \in \mathcal{E}$, no-envy and efficiency restrict the bundles that each agent can receive; for each agent $i \in N$, the set of *potential envy-free and efficient bundles*, $PF_i(R,\Omega)$, consists of all bundles $x_i \in X_i$ such that $(x_i, \Omega - x_i) \in PF(R,\Omega)$. The *minimal rights* of agent i are given by the minimum amount of each commodity that she receives at any $x_i \in PF_i(R,\Omega)$ (see Figure 1).

Definition 2. For each economy $(R,\Omega) \in \mathcal{E}$, each agent $i \in N$ and each commodity $k \in M$,

(i) Agent **i's minimal right of commodity k** is given by:
$m_{ik}(R,\Omega) = \inf \{ \, x_k \mid \text{there exists } x_{-k} \in \mathbb{R}^{M-1}_{+}, \; (x_k, x_{-k}) \in PF_i(R,\Omega)\}$

(ii) Agent **i's minimal rights** are: $m_i(R,\Omega) = (m_{ik}(R,\Omega))_{k \in M}$

(iii) The economy's **minimal rights** are: $m(R,\Omega) = (m_i(R,\Omega))_{i \in N}$

It is easy to see that for each economy its vector of minimal rights defines an envy-free and feasible allocation. As the next proposition shows, each allocation which assigns to one agent her minimal rights and the remainder to the other agent is an envy-free and efficient allocation (see Figure 1).

Proposition 1. *For each economy $(R,\Omega) \in \mathcal{E}$ and each agent $i \in N$, the allocation $(m_i(R,\Omega), \Omega - m_i(R,\Omega))$, is envy-free and efficient.*

Proof. Let $(R,\Omega) \in \mathcal{E}$ and $i \in N$. We need to show that $m_i(R,\Omega) \in PF_i(R,\Omega)$.

For each commodity $k \in M$ there exists a sequence of potential envy-free bundles $\{x^n_{ik}\}_{n \in \mathbb{N}} \in PF_i(R,\Omega)$ such that $x^n_{ik} \to m_{ik}(R,\Omega)$. Let $X(k)$ be the set of elements of such sequence and let $X = \bigcup_{k \in M} X(k)$. By Remark 2 the efficient

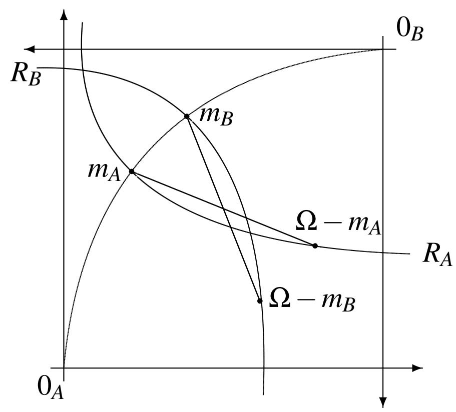

Figure 1: **Minimal rights.** The minimal rights of each agent $i \in N$ are denoted m_i. Given strong decreasing marginal rates of substitution, the Pareto set defines an increasing curve in the Edgeworth box. Each agent is indifferent between consuming her minimal right m_i and the remaining resources $(\Omega - m_i)$. Each agent's minimal right is part of an efficient allocation.

set defines an increasing curve. Therefore, we can order the elements of X in a decreasing sequence; let $\{x^n\}_{n\in\mathbb{N}}$ be such decreasing sequence. Then, for each $n \in \mathbb{N}$, $x^n \in PF_i(R,\Omega)$ and $x^n \to m_i(R,\Omega)$. Since the set $PF_i(R,\Omega)$ is closed, then $m_i(R,\Omega) \in PF_i(R,\Omega)$. $\square$

The intuition behind Proposition 1 is the following: since the efficient set defines an increasing curve in the Edgeworth box, agent i's minimal rights of each good are obtained from the same allocation, and therefore are part of an efficient and envy-free allocation.

The next proposition is one of the main results of the paper. If we assign to each agent her minimal rights, then regardless of how we distribute the remaining resources, we obtain an envy-free allocation.

Proposition 2. *For each $(R,\Omega) \in \mathcal{E}$ and each $x \in Z(R,\Omega)$, if $x \geq m(R,\Omega)$ then $x \in F(R,\Omega)$.*

Proof. Let $(R,\Omega) \in \mathcal{E}$, $i \in N$, and $x \in Z(R,\Omega)$ such that $x \geq m(R,\Omega)$; then for $j \neq i$ we have $x_j \leq \Omega - x_i$. Moreover, since $x_i \geq m_i(R,\Omega)$, then $x_j \leq \Omega - m_i(R,\Omega)$. Hence,

$$x_i \; R_i \; m_i(R,\Omega) \; R_i \; (\Omega - m_i(R,\Omega)) \; R_i \; x_j.$$

Thus, for each $i \in N$ we have $x_i \; R_i \; x_j$, and therefore $x \in F(R,\Omega)$. $\square$

The next corollary shows that, if there is a positive amount of resources in the economy, each agent's minimal rights are positive.

Corollary 1. *Let $\Omega \geq (0,...,0)$ and $R \in \mathcal{R}^2$, for each $i \in N$, $m_i(R,\Omega) \geq 0$. Moreover, for each $i \in N$ there exists a commodity $k \in M$ such that $m_{ik}(R,\Omega) \geq \frac{\Omega_k}{2} > 0$.*

Proof. First we show that $m_i(R,\Omega) = 0 \Rightarrow \Omega = 0$.

Let $(R,\Omega) \in \mathcal{E}$, and $i \in N$. By Proposition 1 we have $m_i(R,\Omega)\ R_i\ (\Omega - m_i(R,\Omega))$. Suppose that $m_i(R,\Omega) = 0$; then $0\ R_i\ \Omega$. By strict monotonicity of preferences $\Omega = 0$.

To show that there exists a commodity $k \in M$ such that $m_{ik}(R,\Omega) \geq \frac{\Omega_k}{2} > 0$, assume by contradiction that for each $k \in M$ with $\Omega_k > 0$, $m_{ik}(R,\Omega) < \frac{\Omega_k}{2}$. Then $m_i(R,\Omega) \leq \frac{\Omega}{2}$ and, by strict monotonicity of preferences, the agent would prefer consuming the remaining resources $(\Omega - m_i(R,\Omega))\ P_i\ m_i(R,\Omega)$, contradicting the conclusions of Proposition 1. □

If no-envy and efficiency are normative criteria shared by both agents, then in order to satisfy these two normative criteria, each agent must receive at least her minimal rights. Moreover, no matter how we distribute the remaining resources, assigning to each agent her minimal rights guarantees an envy-free distribution. Hence, a natural way to select an allocation is to assign minimal rights and then distribute the remaining resources. The remaining resources, along with the agent's preferences over the remaining resources, define a new economy; if the minimal rights of this economy are positive it is only natural to assign them and iterate the process until minimal rights are zero. In the next section we study such a procedure.

4 A selection from the no-envy and efficient set

After assigning to each agent her minimal rights, the remaining resources are given by $(\Omega - \sum_{i \in N} m_i(R,\Omega)) \in \mathbb{R}^m_+$, and the preferences of each agent over these remaining resources are the restriction of her preferences over bundles dominating her minimal rights; the remaining resources and the implied preferences over them define a new economy (see Figure 2).

Definition 3. For each economy $(R,\Omega) \in \mathcal{E}$ the **minimal rights reduced economy** (or just *reduced economy*), $r^m(R,\Omega)$, is the economy $(R',\Omega') \in \mathcal{E}$, where:

(i) For each $i \in N$ and each $x_i, x'_i \in X_i$, $x_i\ R'_i\ x'_i \Leftrightarrow (x_i + m_i(R,\Omega))\ R_i\ (x'_i + m_i(R,\Omega))$

(ii) $\Omega' = \Omega - \sum_{i \in N} m_i(R,\Omega)$

After assigning minimal rights, any efficient allocation in the reduced economy will also be an efficient allocation in the original economy. That is, if an allocation is efficient for the reduced economy, then after summing to each agent's bundle her minimal rights, we obtain an efficient allocation for the original economy.

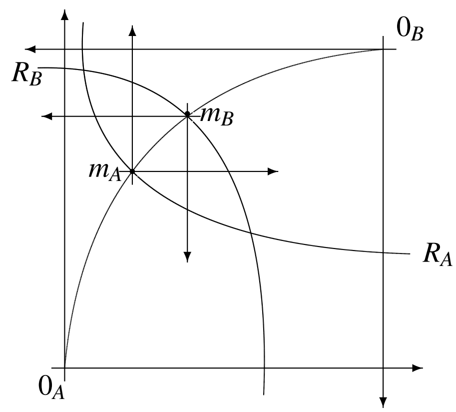

Figure 2: **Minimal rights reduced economy.** The economy's minimal rights are given by (m_A, m_B). After assigning minimal rights, we define the reduced economy given by the box(m_A, m_B) and the implied preferences over the remaining resources.

Proposition 3. *For each* $(R, \Omega) \in \mathcal{E}$ *an allocation* $x \in P(r^m(R, \Omega))$ *if and only if the allocation* $(m(R, \Omega) + x) \in P(R, \Omega)$.

Proof. The fact that $(m(R,\Omega)+x) \in P(R,\Omega)$ implies that $x \in P(r^m(R,\Omega))$ is straightforward. We show that the converse holds.

Let $(R, \Omega) \in \mathcal{E}$, $(R', \Omega') = r^m(R, \Omega)$, and $x \in P(R', \Omega')$. By contradiction assume that $(x + m(R, \Omega)) \notin P(R, \Omega)$. Then there exists an allocation $x' \in P(R, \Omega)$ which Pareto-dominates $(x + m(R, \Omega))$ according to the preference profile R. By definition of the reduced economy we have $\sum_{i \in N}(x'_i - m_i(R, \Omega)) = \Omega'$. We have two cases:

(i) For each $i \in N$, $(x'_i - m_i(R, \Omega)) \geqq 0$. In this case the allocation $(x' - m(R, \Omega))$ is feasible in the reduced economy. But by assumption, x' Pareto-dominates $(x + m(R, \Omega))$ according to the preference profile R, but by the definition of preferences in the reduced economy, $(x' - m(R, \Omega))$ Pareto-dominates x according to the preference profile R'. This contradicts the fact that $x \in P(R', \Omega')$.

(ii) There exists an agent $i \in N$ and a commodity $k \in M$ such that $x'_{ik} < m_{ik}(R, \Omega)$. Without loss of generality let A be such agent. By Proposition 1 we know that $m_A(R, \Omega) \in P_A(R, \Omega)$, and by assumption $x'_A \in P_A(R, \Omega)$. Then, by Remark 2 we have that $m_A(R, \Omega) > x'_A$, and by monotonicity of preferences $(x_A + m_A(R, \Omega))\ P_A\ x'_A$ contradicting the assumption that the allocation x' Pareto-dominates the allocation $(x + m(R, \Omega))$.

Since these two cases are exhaustive we must have $(x + m(R, \Omega)) \in P(R, \Omega)$. $\square$

The intuition behind the proof is the following: if an allocation x_0 is efficient in the reduced economy and the sum of the allocation and the minimal rights (x_0+m) is not efficient in the original economy, it must be Pareto-dominated by an allocation x_1 outside the Edgeworth box of the reduced economy. Since the efficient set defines an increasing curve in the Edgeworth box, the allocation x_1 gives to at least one agent a bundle smaller than her minimal rights, but this contradicts that x_1 Pareto-dominates (x_0+m).

Once we assign to each agent her minimal rights, how should we assign the remaining resources? If there is a positive amount of social endowment still available, minimal rights of the reduced economy are positive, and a natural way to proceed is to assign to each agent new minimal rights, further reduce the economy, and iterate this procedure.

Definition 4. For each $(R,\Omega)\in\mathcal{E}$ and each $k\in\mathbb{N}$ the **k-envy-free and efficient minimal rights** (or *k-minimal rights*) are defined recursively by:

$$\begin{array}{ll} \text{For } k=1 & m^1(R,\Omega)=m(R,\Omega)+m(r^m(R,\Omega)) \\ \text{For } k>1 & m^k(R,\Omega)=m^{k-1}(R,\Omega)+m(r^{m^{k-1}}(R,\Omega)) \end{array}$$

where $r^{m^{k-1}}(R,\Omega)$ denotes the reduction of the economy (R,Ω) according to its m^{k-1} minimal rights.

Since minimal rights in each of the reduced economies are positive, by Proposition 2 each allocation assigning to one agent her k-minimal rights and to the other the remainder is envy-free and as we show next it is also efficient.

Proposition 4. *For each economy $(R,\Omega)\in\mathcal{E}$, each agent $i\in N$, and each $k\in\mathbb{N}$, the allocation $(m_i^k(R,\Omega),\Omega-m_i^k(R,\Omega))$ is envy-free and efficient.*

Proof. For each $j\in\mathbb{N}$ let $(R^j,\Omega^j)=r^{m^j}(R,\Omega)$. Let $k\in\mathbb{N}$, by applying Proposition 1 to the economy (R^k,Ω^k), we know that $(m_i(R^k,\Omega^k),(\Omega^k-m_i(R^k,\Omega^k)))$ is efficient for the economy (R^k,Ω^k). Then, by Proposition 3 the allocation defined by $((m_i(R^{k-1},\Omega^{k-1})+m_i(R^k,\Omega^k)),(m_j(R^{k-1},\Omega^{k-1})+\Omega^k-m_i(R^k,\Omega^k)))$ is efficient for the economy (R^{k-1},Ω^{k-1}). Repeated application of Proposition 3 obtains that $\left(\sum_{j=1}^k m_i(R^j,\Omega^j),\Omega-\sum_{j=1}^k m_i(R^j,\Omega^j)\right)$ is efficient for the economy (R,Ω). By definition $\sum_{j=1}^k m_i(R^j,\Omega^j)=m_i^k(R,\Omega)$, hence $(m_i^k(R,\Omega),\Omega-m_i^k(R,\Omega))$ is efficient.

Since for each $k\in\mathbb{N}$ minimal rights are positive and feasible we have that, $m_i(R,\Omega)\leqq m_i^k(R,\Omega)\leqq(\Omega-m_j(R,\Omega))$. Hence assigning to one agent her k-minimal rights and the remainder to the other dominates the original minimal rights, that is, $(m_i^k(R,\Omega),\Omega-m_i^k(R,\Omega))\geq(m_i(R,\Omega),m_j(R,\Omega))$. Thus, by Proposition 2 the allocation $(m_i^K(R,\Omega),\Omega-m_i^K(R,\Omega))$ is envy-free. $\square$

Iterating the process of assigning minimal rights leads to an increasing sequence of feasible allocations. This sequence has a limit, and it defines the recursive minimal rights:

Definition 5. For each $(R,\Omega) \in \mathcal{E}$, the **recursive envy-free and efficient minimal rights** (or just *recursive minimal rights*) are given by:

$$M(R,\Omega) = \lim_{k\to\infty} m^k(R,\Omega).$$

Our main theorem shows that not only does Proposition 4 hold in the limit, but the recursive minimal rights define an efficient allocation.

Theorem 1. *Recursive assignment of minimal rights selects an efficient and envy-free allocation.*

Proof. Let $(R,\Omega) \in \mathcal{E}$. First, we show that at the allocation $M(R,\Omega)$ all the resources are distributed, that is, $\sum_{i\in N} M_i(R,\Omega) = \Omega$. Let $m^0 = m(R,\omega)$, $\Omega^0 = \Omega$, and for each $k \in \mathbb{N}$, recursively define $(R^k,\Omega^k) = r^{m^{k-1}}(R,\Omega)$, we need to show that in the limit we exhaust resources, that is, $\Omega^k \to 0$.

Since $\Omega^k = \Omega^{k-1} - \sum_{j\in N} m_j(r^{m^{k-1}}(R,\Omega))$ and minimal rights are non-negative, the sequence $\{\Omega^k\}_{k\in\mathbb{N}}$ defines a decreasing sequence of non-negative vectors and it has a limit. Moreover, by a recursive application of the definition of Ω^k we obtain that $\Omega^k = \Omega - \sum_{l=1}^{k}\sum_{j\in N} m_j(r^{m^{l-1}}(R,\Omega))$; since $\{\Omega^k\}_{k\in\mathbb{N}}$ is converging, then $\lim_{k\to\infty}\sum_{j\in N} m_j(r^{m^{k-1}}(R,\Omega)) = 0$. Since for each $i \in N$ minimal rights are non-negative we must have $\lim_{k\to\infty} m_i(r^{m^{k-1}}(R,\Omega)) \to 0$, and minimal rights in the limit economy are equal to zero. But then, by Corollary 1 the limit endowment is zero, that is $\Omega^k \to 0$.

Now we show that $M(R,\Omega)$ is envy-free and efficient. Let $i \in N$, by the previous step $M(R,\Omega) = \lim_{k\to\infty}(m_i^k(R,\Omega), \Omega - m_i^k(R,\Omega))$. By Proposition 4 for each $k \in \mathbb{N}$ we have $(m_i^k(R,\Omega), \Omega - m_i^k(R,\Omega)) \in PF(R,\Omega)$. Since the set of envy-free and efficient allocation is closed, $M(R,\Omega) \in PF(R,\Omega)$. □

The intuition behind the proof of the theorem is straightforward: since by definition the k-minimal rights are converging to the recursive minimal rights, then the series is convergent and the minimal rights of the k-reduced economy must be converging to zero. Then, by Corollary 1 the endowment of the reduced economy must be going to zero; thus the recursive minimal rights distribute the entire resources. Now, since for each $k \in \mathbb{N}$ assigning to one agent her k-minimal rights and the remaining resources to the other agent is envy-free and efficient, then so is the limit.

5 Conclusions

For most economies there are many envy-free and efficient allocations. We proposed a procedure which selects a single such allocation by strengthening the concept of envy-freeness in a recursive manner. Our focus is not strategic and we do not deal with the implementation problem, but we offer a solution concept in which the step by step procedure is justified on normative grounds. Our solution is based only on the logical consequences of envy-free and efficiency; hence if agents agree that the properties of efficiency and envy-free are socially desirable, they have no dispute over the minimal amounts of goods that each of them receives at any allocation satisfying these properties. Therefore, they should also agree on distributing this minimum amount and then negotiate how to distribute the remaining resources. However, once these minimal rights are distributed, the economy has changed and either the agents may agree on solving this new problem as they did before or an arbitrator can choose to solve the reduced problem in the same manner.

An alternative solution to the problem of selecting an envy-free and efficient allocation is to assign to each agent half of the endowment and allow them to trade at competitive equilibrium prices. Since each agent has the same initial endowment, and both face the same prices, they are constrained by the same budget set. Hence, each agent could choose to consume the other agent's bundle and the (efficient) equilibrium allocation will be envy-free. However, there are many other initial distributions which also lead to envy-free allocations, and choosing an equal distribution as a starting point is not an implication of either no-envy or efficiency. Moreover, even if agents agree on using an equal distribution as a starting point there are many possible trades which lead to different efficient and envy-free allocations. This also holds even if we restrict to "envy-free" trades where no agent envies the trade of the other, and the fact that agents trade at competitive equilibrium prices is not an implication of either no-envy or efficiency. Furthermore, there is no compelling reason why agents should act as price-takers in a two-agent economy, and an arbitrator could be needed to uphold such prices, but once an arbitrator is needed to distribute the resources our rule seems to arise naturally based solely on the implications of no-envy and efficiency.

Given that equal division also seems like a good reference point, agents may also agree that for each economy the solution guarantees to each agent the welfare level obtained from an equal division of the endowment. In this situation our definition of minimal rights should be properly modified to incorporate this requirement. By defining minimal rights as the minimal amount of goods in the set of efficient and envy-free allocations in which each agent obtains a welfare level at least as large as from an equal division of the endowment, under the same assumptions made in the paper, the recursive procedure applied to the modified minimal rights obtains an efficient and envy-free allocation which assigns to each agent a welfare level at least as large as equal division of the endowment.

Generalizing our solution to economies with more than two agents faces two difficulties. First, envy-free is no longer defined by comparing what an agent consumes to the remaining resources, but it depends on the distribution of the remaining resources among the rest of the agents. This problem can be solved by asking for a weaker no-envy requirement based on agent's not envying the average amount of resources that the others receive. Second, for more than two agents the efficient set no longer describes an increasing curve, and when defining minimal rights we can proceed in two ways: (i) by selecting bundles which are minimal in terms of welfare (but then there may be multiplicity problems), or (ii) by using the same definition of minimal rights in terms of commodities (but we cannot guarantee the efficiency or envy-freeness of the selected allocation). In many economic applications, such as economies with quasi-linear preferences, division of indivisible goods with monetary transfers (i.e., auction settings), or cake division problems, the efficient set is well structured for any number of agents; in these settings our procedure can be generalized to accommodate more agents.

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